SPHERES AND PROJECTIVE SPACES DETERMINED BY THE ONE-FORM SPECTRUM

ANASS NIFA

Abstract. The ordinary one-form spectrum determines the standard spheres, real, complex and quaternionic projective spaces, and the Cayley plane among closed connected Riemannian manifolds. The same conclusion holds in the complementary degree without an orientability assumption; for the complex projective plane it holds in each degree $0 \le p \le 4$. Above dimension four, partial spectral matchings suffice in degree one and in the complementary degree under quantitative bounds on omitted eigenvalue copies and matching cost. In sufficiently large dimension, a matching after independent finite omissions with error $o(\nu^{-1})$, uniformly as the model eigenvalue $\nu \to \infty$, determines every d-dimensional model in a common set of form degrees. As $d \to \infty$, the lower limit of their proportion among $0, \dots, d$ is at least $\sqrt{8/21}$. Heat identities and geodesic returns give averaged curvature bounds; for nonreal projective targets of dimension greater than four, equality in a periodic index estimate then determines the Jacobi operators.

1. Introduction

Let (M, g) be a closed connected Riemannian manifold. For an integer $0 \le p \le \dim M$, the Hodge Laplacian

$$\Delta_p = d\delta + \delta d, \qquad \delta = d^*,$$

acts on ordinary complexified p-forms. Its spectrum, denoted by $\mathrm{Spec}_p(M)$, is counted with multiplicity; zero is included when present. The model spaces are

$$S^d(1), \qquad \mathbb{R}\mathrm{P}^d(1), \qquad \mathbb{C}\mathrm{P}^n, \qquad \mathbb{H}\mathrm{P}^n, \qquad \mathbb{O}\mathrm{P}^2. \tag{1.1}$$

Here $d \ge 1$ and $n \ge 1$. The sphere has radius one and the real projective metric is its quotient. The nonreal projective metrics have maximal sectional curvature four. In particular, $\mathrm{Vol}(\mathbb{C}\mathrm{P}^n) = \pi^n/n!$ [5, Table 1]. The projective lines $\mathbb{C}\mathrm{P}^1$ and $\mathbb{H}\mathrm{P}^1$ are round spheres of radius $1/2$.

Theorem A (One-forms). *Let X be one of the spaces* (1.1)*, and put $d = \dim X$. If a closed connected Riemannian manifold M satisfies*

$$\mathrm{Spec}_p(M) = \mathrm{Spec}_p(X)$$

for $p = 1$ or $p = d - 1$, then M is isometric to X.

No orientability or additional geometric structure is prescribed on M. The spectrum determines its dimension. In degree $d - 1$, Hodge star intertwines the ordinary Laplacian with the one-form Laplacian with coefficients in the complexified orientation line. For $\mathbb{C}\mathrm{P}^2$, the conclusion holds in each ordinary degree $0 \le p \le 4$ (Corollary 5.5).

The comparison can be weakened in dimensions greater than four. A partial matching is a bijection between submultisets of the two spectra; thus individual eigenvalue copies are matched. Write (λ, ν) for a matched pair, with ν an eigenvalue of the model. Let $D(R)$ count the unmatched copies on both sides with eigenvalue at most R^2. For $\eta > 0$, put

$$c_\eta(\lambda, \nu) = \begin{cases} |\lambda - \nu|, & |\lambda - \nu| < \eta, \\ 1 + \min(\lambda, \nu), & |\lambda - \nu| \ge \eta. \end{cases} \tag{1.2}$$

2020 *Mathematics Subject Classification.* Primary 58J53; Secondary 53C22, 53C26, 53C55.

Key words and phrases. Hodge Laplacian, inverse spectral geometry, compact rank-one symmetric space, periodic geodesic flow.

Theorem B (Partial spectra). *Let X be one of the spaces* (1.1)*, with $d = \dim X > 4$, and let $p \in \{1, d-1\}$. Suppose that the ordinary p-spectra of a closed connected Riemannian manifold M and X admit a partial matching such that*

$$D(R) = o(R^{d-4}), \qquad \sum_{\substack{(\lambda,\nu) \text{ matched} \\ \min(\lambda,\nu)\le R^2}} c_{1/4}(\lambda,\nu) = o(R^{d-2}) \qquad (R \to \infty). \tag{1.3}$$

Then M is isometric to X.

Exact equality satisfies (1.3). Thus Theorem B implies Theorem A for $d > 4$; the lower-dimensional cases are given in Section 5.

Fix now an ordinary form degree p. Consider a bijection between the two spectra after independent finite omissions, with

$$\lim_{R\to\infty} \sup_{\substack{(\lambda,\nu) \text{ matched} \\ \nu\ge R}} (1+\nu)|\lambda-\nu| = 0. \tag{1.4}$$

Set

$$\rho_- = \frac{1}{2}\left(1 - \sqrt{\frac{11+6\sqrt{2}}{21}}\right), \qquad \rho_+ = \frac{1}{2}\left(1 - \sqrt{\frac{11-6\sqrt{2}}{21}}\right), \qquad \rho_0 = \frac{1-1/\sqrt{3}}{2}. \tag{1.5}$$

Theorem C (Proportional degrees). *For every compact set $K \subset (\rho_-, \rho_+) \setminus \{\rho_0\}$ there is an integer $d_K \ge 17$ with the following property. Let $d \ge d_K$, let X be one of the spaces* (1.1) *of dimension d, and let $p \in \{0, \dots, d\}$ satisfy*

$$\frac{\min(p, d-p)}{d} \in K.$$

If the ordinary p-spectra of a closed connected Riemannian manifold M and X admit a comparison (1.4)*, then M is isometric to X.*

Let $D_d^* \subset \{0, \dots, d\}$ consist of the degrees for which (1.4) determines every model in (1.1) of dimension d, among closed connected Riemannian manifolds. Theorem C gives

$$\liminf_{d\to\infty} \frac{|D_d^*|}{d+1} \ge 2(\rho_+ - \rho_-) = \sqrt{\frac{8}{21}} = 0.617213399848\ldots. \tag{1.6}$$

The Cayley plane, of dimension sixteen, occurs in Theorems A and B.

Earlier results. The first three heat coefficients of the Hodge Laplacians are integrals of curvature polynomials [39, pp. 270, 277, Proposition 2.1]; see also [14]. Patodi also proved that equality of the function and one-form spectra preserves constant sectional curvature and the Einstein condition [39, Proposition 4.1(II), p. 281]. From equality of the function and one-form spectra with those of an Einstein locally symmetric manifold, Ii deduced local symmetry [23, p. 558]. Tanno obtained one-form recognition of round spheres among orientable competitors in dimensions $2, 3$ and 16 through 93, and recognition of $\mathbb{CP}^n$ among Kähler competitors for $8 \le n \le 51$ [46, Theorems A and C and their corollaries]. Tsagas obtained recognition of the round sphere among orientable competitors in degree $p = \lfloor d/3 \rfloor$ for $d \ge 6$, and in degree zero for $2 \le d \le 5$ [48, Theorem 3.1 and Corollary 3.2]. Bettiol–Lauret–Piccione proved scalar spectral uniqueness among homogeneous metrics on compact rank-one symmetric spaces [5, Theorem C].

Within the Kähler class, Li proved recognition of $\mathbb{CP}^n$ for even degrees $2 \le p \le 2n-2$ satisfying $p^2 - 2np + n(2n-1)/3 \ne 0$ [32, arXiv version, Theorem 1.2]. Huang–Liu–Xu–Zhi obtained further degree ranges, including proportional ranges as $n \to \infty$ [22, Corollary 2.4 and p. 280]. Liu–Xu proved recognition in every degree within the Kähler class [35, Theorem 1]. They also treated comparisons $|\lambda_j - \nu_j| = o(\nu_j^{-1})$ of the ordered spectra for $n \ge 2$, and exact equality of the spectral multisets above a finite cutoff for $n \ge 3$ [35, Theorem 12 and Proposition 13]. Here the eigenvalues are listed in nondecreasing order with multiplicity, and ν_j belongs to the model.

Wu introduced power bounds for differences of ordered scalar eigenvalues [50, p. 1374]. Li–Sun–Zhu obtained curvature characterizations from ordered p-spectra satisfying $|\lambda_j - \nu_j| = O(j^{-\alpha})$, $\alpha > 2/d$, under specified degree and dimension restrictions [33, arXiv version, Definition 2.7 and Theorem 2.10]. For a real space form reference, the competing manifold is Einstein or a gradient shrinking Ricci soliton; for a complex space form reference, it is Kähler and its first Chern class is proportional, in real cohomology, to its Kähler class. The competitors in Theorems A–C are arbitrary closed connected Riemannian manifolds. In Theorem B, the unmatched copies need not be finite, and (1.3) bounds the total matching cost.

Curvature and geodesic flow. In Theorems B and C, the spectral hypotheses imply $\dim M = d > 4$. Let s, Ric and Rm denote the scalar, Ricci and Riemann curvatures of g, and let s_X be the scalar curvature of the model metric g_X. Norms are full tensor norms. Write $\mathrm{Ric}^\circ = \mathrm{Ric} - (s/d)g$, and put

$$J(g) = \int_M \left(|\mathrm{Ric}|^2 + \frac{3}{2}|\mathrm{Rm}|^2 \right) dV_g. \tag{1.7}$$

For round targets with $16 \le d \le 93$, Tanno's curvature argument and the volume give Theorem B from the first three one-form heat coefficients; see the proof of Proposition 4.9. In the remaining cases of that theorem, spectral multiplicities and half-wave returns give equality of the first three scalar heat coefficients (Section 4). Together with the one-form heat coefficients, these equalities imply

$$\mathrm{Vol}(M) = \mathrm{Vol}(X), \qquad \int_M s\, dV_g = s_X \,\mathrm{Vol}(X),$$
$$J(g) - J(g_X) = -4 \int_M (s - s_X)^2\, dV_g \le 0$$

by (3.7). For the degrees in Theorem C, the same volume and total scalar curvature are obtained, and Lemma 6.2 gives

$$J(g) - J(g_X) = u \int_M |\mathrm{Ric}^\circ|^2\, dV_g + v \int_M (s - s_X)^2\, dV_g \le 0,$$

where $u, v \le 0$ are given by (6.1).

For a unit tangent vector v, let $K_v : v^\perp \to v^\perp$ be defined by $\langle K_v w, z\rangle = \mathrm{Rm}(w, v, v, z)$. The angular identities (3.1) bound the first two moments of K_v. For round targets they give $K_v = I$. For the nonreal projective targets, the unit-speed geodesic flow G_t of M satisfies $G_\pi = \mathrm{id}$. The normal parallel return $H_v : v^\perp \to v^\perp$ over $[0, \pi]$ satisfies

$$H_v^2 = I, \qquad \dim \ker(H_v - I) = r, \qquad r = 1, 3, 7$$

for $\mathbb{CP}^n$, $\mathbb{HP}^n$ and $\mathbb{OP}^2$, respectively. Let $K_v(t)$ denote the Jacobi operator along the geodesic with initial velocity v, viewed on $v^\perp$ by parallel transport, and let P_v be the orthogonal projection onto $\ker(H_v - I)$. The equality case of the periodic index form gives $K_v(t) = I + 3P_v$ (Lemma 3.3). Spherical-rank rigidity and the volume then give isometry with X (Proposition 3.4).

Contents

1. Introduction 1
2. Spectral data and geodesic returns 3
3. Curvature moments and the periodic index 10
4. One-form recovery 14
5. Dimensions at most four 26
6. Proportional degrees 29

References 34

2. Spectral data and geodesic returns

2.1. **Curvature and heat coefficients.** Let SM be the unit tangent bundle of M, let G_t be its unit-speed geodesic flow, and let $d\ell$ be the Liouville probability measure. Thus $d\ell = \mathrm{Vol}(M)^{-1} dV_g\, d\sigma_x$, where

$d\sigma_x$ is the probability measure on S_xM. For $w \in SM$, define K_w on $w^\perp$ by $\langle K_w u, z\rangle = \mathrm{Rm}(u, w, w, z)$; the curvature convention is such that $K_w = I$ on the unit sphere. On $\mathbb{CP}^n$, $\mathbb{HP}^n$ and $\mathbb{OP}^2$, set $r = 1, 3, 7$, respectively. In the normalization of Section 1, K_w has eigenvalue 4 with multiplicity r and eigenvalue 1 with multiplicity $d - 1 - r$, with the latter eigenspace allowed to be zero [6, pp. 31, 33, 35 and 41]. Consequently

$$\mathrm{Ric}(w, w) = \operatorname{tr} K_w = d - 1 + 3r, \qquad s = d(d - 1 + 3r).$$

The projective lines $\mathbb{CP}^1$, $\mathbb{HP}^1$ and $\mathbb{OP}^1$ are round spheres of radius $1/2$ and dimensions $2, 4, 8$, respectively. Every unit-speed geodesic of a nonreal projective model lies in a totally geodesic projective line [6, p. 31], and hence has prime period π. On $S^d(1)$ and $\mathbb{RP}^d(1)$ the prime periods are 2π and π, respectively. For $d = 1$, these are the lengths of the respective circles. For every $u > 0$,

$$\Delta_{p,u^2 g} = u^{-2}\Delta_{p,g}.$$

For comparison with a d-dimensional model, fix $0 \le p \le d$. If $m = \dim M$, write

$$\mathrm{Tr}(e^{-t\Delta_p}) \sim (4\pi t)^{-m/2} \sum_{j\ge 0} a_{j,p} t^j \qquad (t \downarrow 0). \tag{2.1}$$

The coefficients are integrals of local invariants of Δ_p [14, Lemma 1.7.4]. Tensoring with a flat Hermitian line equipped with its compatible flat connection leaves them unchanged: in a local parallel unit frame, the twisted and ordinary operators coincide. Equality of spectra determines the dimension whenever the form bundle has positive rank, and then every $a_{j,p}$. All curvature norms are real tensor norms; in particular, $|\mathrm{Rm}|^2 = \sum_{i,j,k,l} R_{ijkl}^2$ in an orthonormal frame.

For $m \ge 4$, set $C_0 = \binom{m}{p}$, $C_1 = \binom{m-2}{p-1}$ and $C_2 = \binom{m-4}{p-2}$, with out-of-range binomial coefficients equal to zero. Patodi's formulas [39], with the coefficients written in [33, arXiv version, Lemma 3.3, (3.7)–(3.8)], are

$$a_{0,p} = C_0 \operatorname{Vol}(M), \tag{2.2}$$

$$a_{1,p} = (C_0/6 - C_1) \int_M s\, dV, \tag{2.3}$$

$$a_{2,p} = \int_M \left(\alpha|\mathrm{Rm}|^2 + \beta|\mathrm{Ric}|^2 + \gamma s^2\right) dV, \tag{2.4}$$

where

$$\begin{aligned} \alpha &= C_0/180 - C_1/12 + C_2/2, \\ \beta &= -C_0/180 + C_1/2 - 2C_2, \\ \gamma &= C_0/72 - C_1/6 + C_2/2. \end{aligned} \tag{2.5}$$

If M is nonorientable, apply these formulas to its orientation double cover; locality divides both the heat coefficients and the curvature integrals by two.

2.2. **Partial spectral matching.**

Lemma 2.1. *Let X be a model of dimension $d > 4$, and fix $0 \le p \le d$. Let the spectra of the p-form Laplacians on M and X, each possibly tensored with a Hermitian line carrying a compatible flat connection, admit a partial matching. Fix $\eta > 0$, define c_η as in Section 1 with this threshold, and assume* (1.3) *for this cost. Discard the matched pairs with $|\lambda - \nu| \ge \eta$. Let $D_0(y)$ count all unmatched eigenvalue copies on both sides through y after this deletion, and let*

$$S(y) = \sum_{\substack{\text{retained pairs} \\ \min(\lambda,\nu)\le y}} |\lambda - \nu|.$$

Then

$$D_0(y) = o(y^{d/2-2}), \qquad S(y) = o(y^{d/2-1}).$$

The total heat mass of the unmatched copies and the sum of the absolute differences of the retained heat weights are each $o(t^{2-d/2})$. Moreover,

$$\dim M = d, \qquad \mathrm{Vol}(M) = \mathrm{Vol}(X), \qquad a_{j,p}(M) = a_{j,p}(X) \quad (0 \le j \le 2).$$

Proof. Set $a = d/2 - 2 > 0$ and $x = \min(\lambda, \nu)$. Let $C(y)$ be the sum of $1 + x$ over the discarded pairs with $x \le y$. By (1.3), $C(y) = o(y^{a+1})$. Stieltjes integration by parts gives

$$\#\{\text{discarded pairs} : x \le y\} = \int_{[0,y]} \frac{dC(s)}{1+s} = \frac{C(y)}{1+y} + \int_0^y \frac{C(s)}{(1+s)^2}\, ds = o(y^a).$$

Here $C(0-) = 0$; the last estimate follows from $a > 0$. Each discarded pair contributes at most two copies through y, and any such pair has $x \le y$. Hence

$$D_0(y) \le D(\sqrt{y}) + 2\#\{\text{discarded pairs} : x \le y\} = o(y^a).$$

On the retained pairs the cost is $|\lambda - \nu|$, so $S(y) = o(y^{a+1})$.

For a nondecreasing function F with $F(0-) = 0$ and $F(y) = o(y^b)$, where $b > 0$, integration by parts yields

$$\int_{[0,\infty)} e^{-ty}\, dF(y) = t \int_0^\infty e^{-ty} F(y)\, dy = o(t^{-b}).$$

Indeed, if $F(y) \le \varepsilon y^b$ for $y \ge Y$, the middle expression is at most $F(Y) + \varepsilon\Gamma(b+1)t^{-b}$. Apply this estimate to D_0 with $b = a$ and to S with $b = a + 1$. The unmatched heat mass is $\int_{[0,\infty)} e^{-ty}\, dD_0(y)$, while

$$\sum_{\text{retained pairs}} |e^{-t\lambda} - e^{-t\nu}| \le t \int_{[0,\infty)} e^{-ty}\, dS(y),$$

since $|e^{-t\lambda} - e^{-t\nu}| \le t|\lambda - \nu| e^{-t\min(\lambda,\nu)}$. Both expressions are $o(t^{-a})$.

Thus $(4\pi t)^{d/2}$ times the difference of the heat traces is $o(t^2)$. The normalized model trace tends to $\binom{d}{p}\mathrm{Vol}(X) > 0$. The same holds for the normalized trace on M, so $p \le m$: otherwise its form bundle would have rank zero. Its leading term is $(4\pi t)^{(d-m)/2}\binom{m}{p}\mathrm{Vol}(M)$. It follows that $m = d$ and $\mathrm{Vol}(M) = \mathrm{Vol}(X)$. Comparison of the coefficients of t and t^2 in (2.1) gives the remaining equalities. □

Corollary 2.2. *Let X be a model of dimension $d > 4$, and fix $0 \le p \le d$. A comparison* (1.4) *in degree p, with Hermitian lines carrying compatible flat connections permitted on either manifold, satisfies* (1.3) *for every fixed $\eta > 0$. Consequently the dimension, volume and heat coefficients through order two agree with those of X.*

Proof. The unmatched count is bounded. For any $\varepsilon > 0$, all but finitely many matched pairs satisfy $|\lambda - \nu| \le \varepsilon(1+\nu)^{-1} < \eta$ and $|\lambda - \nu| < 1$. If $\min(\lambda, \nu) \le y$, such a pair has $\nu \le y + 1$. Weyl's bound $N_X(y) = O(y^{d/2})$ and Stieltjes integration give

$$\sum_{\nu \le y+1} \frac{1}{1+\nu} = O(y^{d/2-1}).$$

The sum of c_η through y is therefore $O_\varepsilon(1) + O(\varepsilon y^{d/2-1})$. Divide by $y^{d/2-1}$, let $y \to \infty$ and then $\varepsilon \downarrow 0$. Together with $D(R) = O(1) = o(R^{d-4})$, this proves (1.3). Apply Lemma 2.1. □

2.3. **Flat lines and half-wave operators.** All flat Hermitian lines below carry compatible flat connections. For such a line F, put $\delta_F = d_F^*$ and $\Delta_{q,F} = d_F\delta_F + \delta_F d_F$ on F-valued q-forms. For $\lambda > 0$, Hodge decomposition [14, Theorem 1.5.2] gives the isometry

$$\lambda^{-1/2} d_F : \ker(\Delta_{0,F} - \lambda) \longrightarrow \ker(\Delta_{1,F} - \lambda) \cap \mathrm{im}\, d_F,$$

with inverse $\lambda^{-1/2}\delta_F$. The orthogonal complement in $\ker(\Delta_{1,F}-\lambda)$ is the coexact summand. If $m = \dim M$, Hodge star intertwines the ordinary Laplacian in degree $m-1$ with the one-form Laplacian with coefficients in the complexified orientation line.

Let M^m be closed and connected, with $m \geq 2$. For a nonnegative self-adjoint Laplace-type operator L on a Hermitian vector bundle $E \to M$ of positive rank, set

$$P_{L,c} = (L+c)^{1/2}, \qquad U_{L,c}(t) = e^{-itP_{L,c}}, \qquad c \geq 0.$$

For the scalar Laplacian these operators are denoted by $P_{c,M}$ and $U_{c,M}(t)$. The map $u \mapsto u\,(d\,\mathrm{vol}_g)^{1/2}$ identifies sections with half-density-valued sections. The half-density bundle $|\Lambda^m T^*M|^{1/2}$ and the metric density are defined on nonorientable manifolds as well. Write $\Psi^q_{\mathrm{cl}}(M;E)$ for the classical pseudodifferential operators of order q on these sections. In this realization $P_{L,c}$ is classical of order one, with principal symbol $|\xi|_g I$ by [26, arXiv version, §3.1, p. 10]. For $c=0$, the orthogonal projection Π_0 onto $\ker L$ is smoothing by elliptic regularity, and

$$P_{L,0} = (L+\Pi_0)^{1/2} - \Pi_0.$$

For $c>0$,

$$P_{L,c} - P_{L,0} = c(P_{L,c}+P_{L,0})^{-1} \in \Psi^{-1}_{\mathrm{cl}}(M;E).$$

For the scalar Laplacian, the subprincipal symbol of $P_{c,M}$ is zero. Write G_t for the Hamiltonian flow of $|\xi|_g$, on S^*M or its homogeneous extension to $T^*M\setminus 0$. The metric identifies SM with S^*M.

The operator L has compact resolvent and finite-dimensional eigenspaces. For every bounded Borel function $F:[0,\infty)\to\mathbb{C}$, the spectral theorem gives

$$F(L) \text{ is compact} \quad\Longleftrightarrow\quad \lim_{\substack{\lambda\in\mathrm{Spec}(L)\\ \lambda\to\infty}} F(\lambda) = 0. \tag{2.6}$$

Proposition 2.3. *Let ∇ be the unitary connection and V the Hermitian endomorphism determined by $L=\nabla^*\nabla+V$. Let $T>0$. If $G_T=\mathrm{id}$, let $\tau_T^E(x,\xi):E_x\to E_x$ be parallel transport along the geodesic with initial covector $(x,\xi)\in S^*M$. Then $U_{L,c}(T)\in\Psi^0_{\mathrm{cl}}(M;E)$ and*

$$\sigma_0\big(U_{L,c}(T)\big)(x,\xi) = \mu_T\,\tau_T^E(x,\xi), \qquad |\mu_T|=1. \tag{2.7}$$

Here μ_T is the constant principal symbol of the scalar operator $U_{c,M}(T)$ and is independent of $c\geq 0$.

Proof. For $L=\nabla^*\nabla$ and $c>0$, the return-symbol formula is given by [29, Proposition 2.1 and its proof, pp. 242–245]. The change $\widehat{g}=(2\pi/T)^2 g$ and $\widehat{c}=(T/2\pi)^2 c$ replaces the time-T propagator by the time-2π propagator. The formula requires only $G_T=\mathrm{id}$ and permits smaller prime periods.

For $L=\nabla^*\nabla+V$, the subprincipal partial connection of $P_{L,c}$ is the pullback of ∇ along $H_{|\xi|}$ [26, arXiv version, Proposition 3.2]. It is independent of $V+c$. The propagator is a classical Fourier integral operator of order zero with canonical relation the graph of G_t [11, Theorem 1.1], [29, §2, pp. 242–244]. At $t=T$ this graph is the diagonal. By [12, Theorem 5.3.1], its principal transport equation is the same as for $(\nabla^*\nabla+1)^{1/2}$, and its bundle factor is τ_T^E. These local symbol identities agree under changes of coordinates and hold also when M is nonorientable. The kernel modification by Π_0 changes the propagator by a smoothing operator.

The scalar operator $(T/2\pi)P_{c,M}$ has zero subprincipal symbol and Hamiltonian flow of period 2π. The scalar return formula [11, proof of Theorem 3.1, p. 53], [10, proof of Theorem 1.1, p. 510] gives

$$U_{c,M}(T) = e^{-i\pi\alpha_T/2} I \pmod{\Psi^{-1}_{\mathrm{cl}}},$$

where α_T is the Maslov index of the period-T Hamiltonian flow. This index is locally constant on S^*M, which is connected because M is connected and $m\geq 2$. Its value is independent of c, and (2.7) follows. □

For the Hodge Laplacian on p-forms, the connection is induced by the Levi-Civita connection. At a positive common period T, let $H_T(v)$ be tangent parallel transport, and identify tangent and cotangent spaces by the metric. Then

$$\sigma_0\Big(e^{-iT\sqrt{\Delta_p+c}}\Big)(v) = \mu_T \Lambda^p H_T(v). \tag{2.8}$$

Proposition 2.4. *Let L be a nonnegative self-adjoint Laplace-type operator on a Hermitian bundle of positive rank, let $T \in \mathbb{R}$, and put $U = U_{L,c}(T)$. Let $|z| = 1$. Suppose*

$$t^{m/2}\operatorname{Tr}\big((U - zI)^*(U - zI)e^{-tL}\big) \longrightarrow 0.$$

Then $G_T = \mathrm{id}$ and $U - zI \in \Psi^{-1}_{\mathrm{cl}}(M; E)$.

Proof. If $T = 0$, the leading heat coefficient gives $z = 1$. If $T < 0$, taking adjoints replaces (T, z) by $(-T, \bar z)$, with the squared distances unchanged on every eigenspace. It therefore suffices to take $T > 0$. Enumerate the eigenvalues λ_j of L with multiplicity. Put $\mu_j = \sqrt{\lambda_j + c}$ and $\nu_j = \sqrt{\lambda_j + c + 1}$. For $\mu_j \le R$ one has $e^{-\lambda_j/R^2} \ge e^{-1}$, whence

$$\sum_{\mu_j \le R} |e^{-iT\mu_j} - z|^2 = o(R^m).$$

For any $\delta > 0$ the number of terms satisfying $|e^{-iT\mu_j} - z| \ge \delta$ is $o(R^m)$. Also $\nu_j - \mu_j = (\nu_j + \mu_j)^{-1} \to 0$. Given $0 < \kappa < 1$, all but $o(R^m)$ of the $\nu_j \le R$ thus satisfy $|e^{iT\nu_j} - \bar z|^2 < 2(1 - \kappa)$, equivalently $\operatorname{Re}(ze^{iT\nu_j}) > \kappa$. Their total number is asymptotic to a positive multiple of R^m. Apply [37, Theorem 3.1, author's version] to the strictly positive Laplace-type operator $L + c + 1$ with $e^{-i\beta} = z$. It gives $G_T = \mathrm{id}$ and $\operatorname{Re}(zv) \ge \kappa I$ for the unitary principal symbol v of $e^{iTP_{L,c+1}}$. Letting $\kappa \uparrow 1$ gives $\operatorname{Re}(zv) \ge I$. Since v is unitary,

$$0 \le (v - \bar z I)^*(v - \bar z I) = 2I - 2\operatorname{Re}(zv) \le 0,$$

so $v = \bar z I$. By Proposition 2.3, U is classical of order zero and its principal symbol is unchanged when c is replaced by $c + 1$. Thus $\sigma_0(U) = v^* = zI$, and $U - zI \in \Psi^{-1}_{\mathrm{cl}}(M; E)$. □

Proposition 2.5. *Let M^m be closed and connected, with $m \ge 2$, and let L be a nonnegative self-adjoint Laplace-type operator on a Hermitian bundle $E \to M$ of positive rank. Fix $c \ge 0$. For $T > 0$ and $z \in \mathbb{C}$, the condition*

$$U_{L,c}(T) - zI \quad \text{is compact on } L^2(M; E) \tag{2.9}$$

implies $|z| = 1$ and $G_T = \mathrm{id}$. If $G_T = \mathrm{id}$, the universal Riemannian cover $p : X \to M$ has finite degree, X is compact, and $\widetilde G_{2T} = \mathrm{id}$. The integral cohomology ring of X is that of a simply connected compact rank-one symmetric space.

Proof. By (2.6), $e^{-iT\sqrt{\lambda+c}} \to z$ as $\lambda \to \infty$ in $\operatorname{Spec}(L)$. Hence $|z| = 1$. Given $\varepsilon > 0$, the sum of $|e^{-iT\sqrt{\lambda+c}} - z|^2$ against $e^{-t\lambda}$, with multiplicities, is bounded by $O_\varepsilon(1) + \varepsilon^2 \operatorname{Tr}(e^{-tL})$. After multiplication by $t^{m/2}$, first let $t \downarrow 0$ and then $\varepsilon \downarrow 0$. Since $\operatorname{Tr}(e^{-tL}) = O(t^{-m/2})$, Proposition 2.4 gives $G_T = \mathrm{id}$.

Suppose $G_T = \mathrm{id}$. The manifold is Besse. By [2, Corollary 4.7 and its proof], the universal cover is compact of finite degree and a based geodesic traversed for the least common period represents an element of order at most two in $\pi_1(M)$. The common periods, together with zero and their negatives, form a subgroup of $\mathbb{R}$. No nonzero element has absolute value less than $\operatorname{inj} M$, so this subgroup is $T_0\mathbb{Z}$ for some $T_0 > 0$. Thus $T = kT_0$ for an integer $k \ge 1$, and the based class at time $2T$ is trivial. Its lift closes, and the lifted velocity returns since the covering is a local isometry. By the Bott–Samelson–McCleary theorem for simply connected Besse manifolds [2, Theorem 1.1], [41, p. 912], the ring $H^*(X; \mathbb{Z})$ is isomorphic to the integral cohomology ring of a simply connected compact rank-one symmetric space. □

2.4. **Index and Maslov phase.** At a positive common period T, the normal initial data $(J(0), \nabla_t J(0)) \in v^\perp \oplus v^\perp$ define tangent vectors to SM. Since $dG_T = I$, all $2(m-1)$ normal Jacobi fields are periodic together with their first covariant derivatives. The tangential fields are $f\dot\gamma$ with periodic scalar index form $\int_0^T |f'|^2\,dt$. This summand has index zero and nullity one. Thus the full periodic nullity is $2m-1$ and the normal and full indices agree. For nearby closed geodesics, parallel transport along the short minimizing segments joining corresponding points identifies their pullback tangent bundles over $\mathbb{R}/T\mathbb{Z}$ and hence their spaces of H^1 periodic fields. The full index forms vary continuously in these identifications. Constant nullity gives locally constant index by the min–max principle [41, p. 922]. Hence the index is constant on SM. The splitting by $\dot\gamma$ is defined without a choice of orientation.

Lemma 2.6. *Let M^m be closed and connected, $m \geq 2$. Let $T > 0$, suppose $G_T = \mathrm{id}$, and let b be the common Morse index of these geodesics for the energy on $H^1(\mathbb{R}/T\mathbb{Z}, M)$. For the scalar half-wave with negative time exponent,*

$$\mu_T = \exp\bigl(-i\pi(b+m-1)/2\bigr). \tag{2.10}$$

Proof. The fixed manifold SM is clean since $dG_T = I$, and has dimension $2m-1$. The period T is isolated: $G_{T+u} = G_u$ has no fixed point for $0 < |u| < \operatorname{inj} M$. For $\mathcal{F}_+ f(\rho) = \int e^{iu\rho} f(u)\,du$, choose a real $\chi \in C_c^\infty((-\delta,\delta))$ equal to one near zero, where $0 < \delta < \min\{T, \operatorname{inj} M/2\}$. Let A_T be the coefficient of ρ^{m-1} in $\mathcal{F}_+[\chi(u)\operatorname{Tr} U_{0,M}(T+u)](\rho)$ as $\rho \to +\infty$. Replacing Δ_0 by $\Delta_0 + \Pi_0$ changes its half-wave trace by the smooth function $(e^{-it}-1)\dim\ker\Delta_0$ and leaves the principal and subprincipal symbols unchanged. The resulting operator is strictly positive. For $q(x,\xi) = |\xi|_g$,

$$d_\xi^2 q(\eta,\eta) = \frac{|\eta|_g^2}{|\xi|_g} > 0 \qquad (0 \neq \eta \perp \xi).$$

Hence the integer σ in the clean trace formula is the periodic Morse index b [11, pp. 69–70]. With dv_T the canonical positive density on S^*M in [11, Theorem 4.5, Eqs. (4.6)–(4.8)], that formula gives

$$A_T = (2\pi i)^{-(m-1)} i^{-b} \int_{S^*M} dv_T, \qquad \frac{A_T}{|A_T|} = i^{-(b+m-1)};$$

cf. [44, Eqs. (1.5)–(1.6)]. The factor 2π from $\mathcal{F}_+$ cancels the factor $(2\pi)^{-1}$ in [11, Eq. (4.8)]. On the other hand, Proposition 2.3 gives

$$U_{0,M}(T+u) = (\mu_T I + R)U_{0,M}(u), \qquad R \in \Psi_{\mathrm{cl}}^{-1}(M).$$

Let A_0 be the coefficient of ρ^{m-1} in $\mathcal{F}_+[\chi(u)\operatorname{Tr} U_{0,M}(u)](\rho)$. The mollified counting expansion [31, Eqs. (1.18) and (1.23)] gives $A_0 > 0$ and

$$\mathcal{F}_+[\chi(u)\operatorname{Tr}(RU_{0,M}(u))](\rho) = O(\rho^{m-2}).$$

Thus $A_T = \mu_T A_0$. Since $|\mu_T| = 1$, $\mu_T = i^{-(b+m-1)}$. □

Lemma 2.7. *Suppose $G_T = \mathrm{id}$, where $T > 0$. Let $H_T(v)$ be tangent parallel transport around the geodesic with initial vector v. Then*

$$\det H_T(v) = \mu_T^2. \tag{2.11}$$

Proof. Let V denote the vertical Lagrangian distribution in $T(T^*M \setminus 0)$. Along a closed geodesic γ, the Lagrangian loop $(dG_t)^{-1}V_{G_t(x,\xi)}$ has tautological bundle isomorphic to γ^*T^*M. The mod-two reduction of its Maslov class is the first Stiefel–Whitney class [4, §§1.2–1.4]. If α_T is its Maslov index,

$$\det H_T = (-1)^{\langle w_1(TM),[\gamma]\rangle} = (-1)^{\alpha_T} = \mu_T^2,$$

by the phase convention in Proposition 2.3. □

2.5. **The complex-projective model return.**

Lemma 2.8. *For every $0 \le p \le 2n$ and $c > 0$, on $(\mathbb{CP}^n, g_{\mathrm{FS}})$,*

$$e^{-2\pi i(\Delta_p + c)^{1/2}} - I \in \Psi_{\mathrm{cl}}^{-1}.$$

Proof. Let J be the complex structure. A unit-speed projective geodesic γ is the Hopf projection of $z(t) = \cos(t)z_0 + \sin(t)z_1$, where z_0, z_1 are complex orthonormal. A constant vector W complex-orthogonal to z_0, z_1 is horizontal along z and satisfies $\nabla^S_{\dot z} W = 0$. For the Hopf projection h, the connection formula for a Riemannian submersion [30, Problem 5-6(b), Eq. (5.27)] gives

$$\nabla_{\dot\gamma}\big(dh_{z(t)} W\big) = dh_{z(t)}\big(\nabla^S_{\dot z} W\big) = 0.$$

Since $z(\pi) = -z_0$ and $h(-z) = h(z)$, $dh_{-z_0}(W) = -dh_{z_0}(W)$. Moreover $\nabla J = 0$ and γ is a geodesic, so $\dot\gamma$ and $J\dot\gamma$ are parallel and return at time π. Thus $H_\pi(v)$ is I on $\operatorname{span}_{\mathbb{R}}\{v, Jv\}$ and $-I$ on its orthogonal complement. In particular, $H_{2\pi} = I$.

The scalar spectrum is $\lambda_k = 4k(k+n)$, $k \ge 0$, by [24, Theorem 4.2(b) and §7, p. 542], in the present curvature normalization. Since $\sqrt{\lambda_k + n^2} = 2k + n$, one has $U_{n^2,\mathbb{CP}^n}(2\pi) = I$ and hence $\mu_{2\pi} = 1$. Formula (2.8) gives the identity principal symbol in every degree and for every $c > 0$. Therefore $e^{-2\pi i(\Delta_p + c)^{1/2}} - I \in \Psi_{\mathrm{cl}}^{-1}$. □

2.6. **A trace estimate.**

Proposition 2.9. *Let D be a nonnegative self-adjoint Laplace-type operator on a Hermitian bundle E over a closed Riemannian m-manifold, $m \ge 1$. Let F be a unitary classical Fourier integral operator of order zero whose canonical relation is the graph of a homogeneous canonical transformation χ preserving S^*M. If $\operatorname{Fix}(\chi) \cap S^*M$ has Liouville measure zero, then*

$$\lim_{t \downarrow 0} t^{m/2} \operatorname{Tr}(F e^{-tD}) = 0. \tag{2.12}$$

Proof. Choose a smooth Hermitian complement E' and a unitary identification $E \oplus E' \simeq M \times \mathbb{C}^N$ [18, Proposition 1.4]. The construction there applies with smooth complex trivializations and a smooth partition of unity. Choose a unitary connection ∇' on E', and put

$$\widetilde D = D \oplus \nabla'^{*}\nabla', \qquad D_0 = \Delta_{0,M} \otimes I_N, \qquad \widetilde F = F \oplus 0.$$

In the chosen trivialization, $B = \widetilde D - D_0$ has order at most one. Thus $(1 + \widetilde D)^{-1/2} B$ and $B(1 + D_0)^{-1/2}$ are bounded. Weyl's law gives

$$\|(1 + A)^{1/2} e^{-vA}\|_1 = O(v^{-(m+1)/2}), \qquad A \in \{\widetilde D, D_0\}, \quad 0 < v \le 1.$$

In Duhamel's identity

$$e^{-t\widetilde D} - e^{-tD_0} = -\int_0^t e^{-(t-u)\widetilde D} B e^{-uD_0}\, du,$$

the two bounded products above give

$$\begin{aligned} \|e^{-t\widetilde D} - e^{-tD_0}\|_1 &\le C \int_0^{t/2} (t-u)^{-(m+1)/2}\, du + C \int_{t/2}^t u^{-(m+1)/2}\, du \\ &= O(t^{(1-m)/2}). \end{aligned}$$

Each diagonal entry $\widetilde F_{aa}$ is a scalar classical Fourier integral operator of order zero associated with χ. Since M has no boundary, it belongs to the class $\mathcal{A}_0$ of [25, §4.5]. Let $\Delta_{0,M}\varphi_j = \lambda_j \varphi_j$ be an orthonormal scalar eigenbasis. The zero-measure case of [25, Theorem 6.2] gives

$$A_a(y) := \sum_{\lambda_j \le y} \langle \widetilde F_{aa} \varphi_j, \varphi_j \rangle = o(y^{m/2}).$$

Stieltjes integration by parts, with $A_a(0-) = 0$, yields

$$\left|\operatorname{Tr}(\widetilde{F}_{aa}e^{-t\Delta_{0,M}})\right| \le t\int_0^\infty e^{-ty}|A_a(y)|\,dy = o(t^{-m/2}).$$

Summing over a and applying the trace-norm estimate gives

$$\operatorname{Tr}(Fe^{-tD}) = \operatorname{Tr}(\widetilde{F}e^{-t\widetilde{D}}) = \sum_{a=1}^{N}\operatorname{Tr}(\widetilde{F}_{aa}e^{-t\Delta_{0,M}}) + O(t^{(1-m)/2}),$$

which proves (2.12). □

3. Curvature moments and the periodic index

3.1. **Curvature moments.** For a closed Riemannian manifold (M, g), put

$$J(g) = \int_M \left(|\operatorname{Ric}|^2 + \tfrac{3}{2}|\operatorname{Rm}|^2\right) dV_g.$$

Angular brackets denote integration over SM against the Liouville probability measure $d\ell$.

Lemma 3.1. *Let (M^d, g) be closed, with $d \ge 2$. Then*

$$\left\langle \operatorname{tr} K_w \right\rangle = \frac{1}{d\operatorname{Vol}(M)}\int_M s\,dV_g, \qquad \left\langle \operatorname{tr} K_w^2 \right\rangle = \frac{J(g)}{d(d+2)\operatorname{Vol}(M)}. \tag{3.1}$$

Let X be a normalized model of dimension d, and put $r = 0$ for $S^d(1)$ and $\mathbb{RP}^d(1)$, and $r = 1, 3, 7$ for $\mathbb{CP}^{d/2}$, $\mathbb{HP}^{d/4}$ and $\mathbb{OP}^2$, respectively. If

$$\operatorname{Vol}(M) = \operatorname{Vol}(X), \qquad \int_M s\,dV_g = s_X\operatorname{Vol}(X), \qquad J(g) \le J(g_X), \tag{3.2}$$

then

$$\left\langle \operatorname{tr}(K_w - I)\right\rangle = 3r, \qquad \left\langle \|K_w - I\|_{\mathrm{HS}}^2 \right\rangle \le 9r. \tag{3.3}$$

Proof. At $x \in M$, choose an orthonormal frame and set $R_{iabj} = \langle R(e_i, e_a)e_b, e_j\rangle$. Extend K_w to T_xM by $K_w w = 0$. Its matrix entries are $\sum_{a,b} R_{iabj}w_a w_b$. For the probability measure on S_xM, the sphere integrals in [16, proof of Theorem 3.3, pp. 167–168] give

$$\int_{S_xM} w_a w_b\,d\sigma_x = \frac{\delta_{ab}}{d}, \qquad \int_{S_xM} w_a w_b w_c w_e\,d\sigma_x = \frac{\delta_{ab}\delta_{ce} + \delta_{ac}\delta_{be} + \delta_{ae}\delta_{bc}}{d(d+2)}.$$

By [16, Lemma 2.2, (2.7), p. 161],

$$\sum_{i,a,b,j} R_{iabj}R_{ibaj} = \tfrac{1}{2}|\operatorname{Rm}|^2.$$

The second moment gives s/d for the fibre mean of $\operatorname{tr} K_w$. The three pairings in the fourth moment give, respectively, $|\operatorname{Ric}|^2$, $|\operatorname{Rm}|^2$ and $|\operatorname{Rm}|^2/2$ for the numerator of the fibre mean of $\operatorname{tr} K_w^2$. Integration against $dV_g/\operatorname{Vol}(M)$ proves (3.1).

On X, the eigenvalues of $K_w - I$ are 3 and 0, with multiplicities r and $d-1-r$. On $w^\perp$,

$$\|K_w - I\|_{\mathrm{HS}}^2 = \operatorname{tr} K_w^2 - 2\operatorname{tr} K_w + d - 1.$$

Thus (3.2) and (3.1) give (3.3). □

For $d \ge 2$, integration of Patodi's scalar and one-form formulas [39, Proposition 2.1, (21)–(23), p. 277] gives

$$\begin{aligned} 360\,a_{2,0} &= \int_M (5s^2 - 2|\operatorname{Ric}|^2 + 2|\operatorname{Rm}|^2)\,dV_g,\\ 360\,a_{2,1} &= \int_M \left((5d-60)s^2 + (180-2d)|\operatorname{Ric}|^2 + (2d-30)|\operatorname{Rm}|^2\right) dV_g. \end{aligned} \tag{3.4}$$

The divergence terms integrate to zero since M is closed. Consequently

$$J(g) + 4\int_M s^2\, dV_g = 6\big((60-d)a_{2,0}(M) + a_{2,1}(M)\big). \tag{3.5}$$

Suppose that X^d is closed, has constant scalar curvature s_X, and satisfies

$$a_{j,0}(M) = a_{j,0}(X) \quad (0 \le j \le 2), \qquad a_{2,1}(M) = a_{2,1}(X). \tag{3.6}$$

Since $a_{0,0} = \mathrm{Vol}$ and $a_{1,0} = \frac{1}{6}\int s\, dV$, we obtain the two equalities in (3.2). Subtracting (3.5) for the two metrics gives

$$J(g) - J(g_X) = -4\int_M (s - s_X)^2\, dV_g \le 0. \tag{3.7}$$

3.2. **The index estimate.** Along the unit-speed geodesic γ_v, identify the normal spaces with $v^\perp$ by parallel transport. Write $K_v(t)$ for the matrix of $R(\,\cdot\,,\dot\gamma_v)\dot\gamma_v$ in this frame; the normal Jacobi equation is $u'' + K_v u = 0$. Put

$$\overline{f} = \frac{1}{\pi}\int_0^\pi f(t)\, dt,$$

and take all L^2 inner products on $[0,\pi]$ with respect to dt/π. Set

$$V_v(t) = K_v(t) - I, \qquad e(v) = \overline{\mathrm{tr}(V_v^2)}.$$

Lemma 3.2. *Let M^d be closed and connected, with $d \ge 2$ and $G_\pi = \mathrm{id}$. Let H_v be normal parallel return through π. Suppose that $H_v^2 = I$ and $\dim\ker(H_v - I) = q$ for every $v \in SM$. Put $m = d-1$. The normal Morse index i is constant, and*

$$i + q \le \frac{32}{135}e(v) \qquad (v \in SM). \tag{3.8}$$

Proof. Fix $v \in SM$, and let H be the normal parallel return. Since $H^{-1} = H$, the normal index operator is

$$\begin{aligned} L &= -\partial_t^2 - K_v(t), \\ \mathcal{D}(L) &= \{u \in H^2([0,\pi];\mathbb{R}^m) \,:\, u(\pi) = Hu(0),\ u'(\pi) = Hu'(0)\}. \end{aligned}$$

On the two eigenspaces of H, $-\partial_t^2$ is the periodic or antiperiodic Laplacian. Bounded symmetric perturbation by $-K_v$ gives a self-adjoint operator L, bounded below and with compact resolvent. Since $dG_\pi = I$, every normal Jacobi field satisfies these boundary conditions, so $\dim\ker L = 2m$. The constant-nullity argument preceding Lemma 2.6 makes its index constant on the connected bundle SM.

Let $D = -\partial_t^2 - I$ have the same domain. Its nonpositive eigenspace W consists of the constant fields in $\ker(H-I)$ and the frequency-one sine and cosine fields in $\ker(H+I)$. Thus $\dim W = 2m - q$. Let P be the orthogonal projection onto $W^\perp$ and put $D_+ = D|_{W^\perp}$. Then $D_+ \ge 3I$, and

$$E = \mathrm{ran}\,\mathbf{1}_{(-\infty,0]}(L) \cap W^\perp \quad \text{satisfies} \quad \dim E \ge (i+2m) - (2m-q) = i + q.$$

The operator

$$C = D_+^{-1/2} P V_v P D_+^{-1/2} \,:\, W^\perp \longrightarrow W^\perp$$

is compact and self-adjoint. For $u \in E \subset \mathcal{D}(D)$, put $z = D_+^{1/2}u$. Then

$$\langle Cz, z\rangle - \|z\|^2 = -\langle Lu, u\rangle \ge 0.$$

The map $u \mapsto z$ is injective. The min–max principle [47, Theorem 4.10, p. 119], applied to $-C$, therefore gives at least $i+q$ eigenvalues of C in $[1,\infty)$, counted with multiplicity.

Let (e_j) be an orthonormal eigenbasis of D_+, with $D_+ e_j = \lambda_j e_j$. Symmetry of $|\langle V_v e_j, e_k\rangle|^2$ in j,k and $2/(xy) \le x^{-2} + y^{-2}$ give

$$\begin{aligned}\sum_{j,k} \frac{|\langle V_v e_j, e_k\rangle|^2}{\lambda_j \lambda_k} &\le \sum_j \lambda_j^{-2} \|P V_v e_j\|^2 \\ &\le \sum_j \lambda_j^{-2} \|V_v e_j\|^2.\end{aligned}$$

The sums are increasing limits of nonnegative finite partial sums. Set $P_+ = (I+H)/2$ and $P_- = I - P_+$. The sum of the squares of the normalized sine and cosine functions of any positive integral frequency is 2. Hence the last sum equals

$$\kappa_+ \overline{\operatorname{tr}(P_+ V_v^2)} + \kappa_- \overline{\operatorname{tr}(P_- V_v^2)},$$

where

$$\kappa_+ = 2\sum_{k\ge 1} (4k^2-1)^{-2}, \qquad \kappa_- = 2\sum_{k\ge 1} \big((2k+1)^2 - 1\big)^{-2}.$$

Termwise comparison gives $\kappa_- < \kappa_+$. Since $4k^2 - 1 \ge 15k^2/4$ for $k \ge 2$,

$$\begin{aligned}\kappa_+ &\le \frac{2}{9} + \frac{32}{225}\sum_{k\ge 2} k^{-4} \\ &\le \frac{2}{9} + \frac{32}{225}\left(2^{-4} + \int_2^\infty x^{-4}\,dx\right) = \frac{32}{135}.\end{aligned}$$

Since $\operatorname{tr}(P_\pm V_v^2) \ge 0$, the double sum is finite and at most $(32/135)e(v)$. Thus C is Hilbert–Schmidt and $i + q \le \|C\|_{\mathrm{HS}}^2 \le (32/135)e(v)$. □

Lemma 3.3. *Let M^d be closed and connected, with $d \ge 2$ and $G_\pi = \mathrm{id}$. Suppose that its normal parallel return H_v satisfies $H_v^2 = I$ and $\dim\ker(H_v - I) = r \in \{1,3,7\}$ for every $v \in SM$. Put $m = d-1$, and suppose that $m - r$ is even. If*

$$\int_{SM} \overline{\operatorname{tr} V_v}\, d\ell = 3r, \qquad \int_{SM} e(v)\, d\ell(v) \le 9r,$$

then, on every geodesic,

$$K_v(t) = I + 3P_+,$$

where $P_+ = (I + H_v)/2$ in the parallel frame.

Proof. Let i be the normal Morse index. Integrating (3.8) gives $i + r \le (32/135)9r$, hence

$$i \le 17r/15 < r + 1.$$

Since i is an integer, $i \le r$.

Fix $v \in SM$, and write $H = H_v$. Choose an orthogonal complex structure A on $\ker(H+I)$ and set $A = 0$ on $\ker(H - I)$. Then $A^*A = I - P_+$, and $F(t) = \exp(tA)$ is orthogonal with $F(\pi) = H$. Put $B(t) = F(t)^* V_v(t) F(t)$. Since $AH = HA$, the fields $u_x(t) = F(t)x$ satisfy both boundary conditions of $\mathcal{D}(L)$. Their index form is

$$\overline{|u_x'|^2 - \langle K_v u_x, u_x\rangle} = -\langle (P_+ + \overline{B})x, x\rangle.$$

The map $x \mapsto u_x$ is injective, so $P_+ + \overline{B}$ has at most i strictly positive eigenvalues. Since $\overline{B} \le P_+ + \overline{B}$, the same is true of $\overline{B}$ by the min–max principle.

Let Q be the orthogonal projection onto r orthonormal eigenvectors for the largest eigenvalues of $\overline{B}$. All remaining eigenvalues are nonpositive, whence

$$\operatorname{tr}(Q\overline{B}) \ge \operatorname{tr}\overline{B} = \overline{\operatorname{tr} V_v} =: \tau(v).$$

It follows that

$$0 \le \overline{\|B - 3Q\|_{\mathrm{HS}}^2} = e(v) - 6\,\mathrm{tr}(Q\overline{B}) + 9r \le e(v) - 6\tau(v) + 9r.$$

Thus $h(v) = e(v) - 6\tau(v) + 9r$ is nonnegative. The functions e and τ are time averages of the smooth functions $\|K_w - I\|_{\mathrm{HS}}^2$ and $\mathrm{tr}(K_w - I)$ on SM; hence h is continuous. The hypotheses imply $\int_{SM} h\, d\ell \le 0$. Liouville measure has full support, so $h = 0$ on SM. On the fixed orbit, $\overline{\|B - 3Q\|_{\mathrm{HS}}^2} = 0$; continuity gives $B(t) = 3Q$ for every t.

The positive semidefinite matrix $P_+ + 3Q$ has rank at most $i \le r$ and kernel $\ker P_+ \cap \ker Q$. Both projections have rank r, so their ranges coincide and $Q = P_+$. Since $F(t)$ fixes $\mathrm{ran}\, P_+$ pointwise,

$$V_v(t) = F(t)(3P_+)F(t)^* = 3P_+.$$

Since $K_v(t + \pi) = HK_v(t)H$, the identity holds for all $t \in \mathbb{R}$. □

3.3. **The metric.** Let M^d be closed and connected, with $d \ge 2$ and $\sec_g \equiv 1$. Then, up to isometry,

$$\begin{aligned} \mathrm{Vol}(M) = \mathrm{Vol}(S^d(1)) \quad &\Longrightarrow \quad M \cong S^d(1), \\ \mathrm{Vol}(M) = \tfrac{1}{2}\,\mathrm{Vol}(S^d(1)) \quad &\Longrightarrow \quad M \cong \mathbb{RP}^d(1). \end{aligned} \tag{3.9}$$

Indeed, the complete simply connected Riemannian cover is $S^d(1)$ by the Killing–Hopf theorem [30, Theorem 12.4, p. 348]. The volume ratio gives one or two sheets. In the second case, the nonidentity deck transformation is the restriction of an orthogonal involution A of $\mathbb{R}^{d+1}$. Its action on $S^d(1)$ is free, so $\ker(A - I) = 0$ and $A = -I$.

Proposition 3.4. *Let X^d be a normalized model, with $d > 4$, and let (M^d, g) be closed and connected. Suppose that* (3.2) *holds. If X is nonreal projective, suppose also that $G_\pi = \mathrm{id}$ and that normal parallel return H_v through π satisfies $H_v^2 = I$ and $\dim\ker(H_v - I) = r$ for every $v \in SM$, where $r = 1, 3, 7$ for the complex, quaternionic and octonionic models, respectively. Then (M, g) is isometric to X.*

Proof. Lemma 3.1 gives (3.3). For a round model, $r = 0$, so the continuous nonnegative function $\|K_w - I\|_{\mathrm{HS}}^2$ has zero Liouville integral. Hence $K_w = I$ for all $w \in SM$. Thus M has constant sectional curvature one, and (3.9) gives the isometry.

Now let X be nonreal projective. Its value of $d - 1 - r$ is even. Invariance of Liouville measure under G_t gives

$$\int_{SM} \overline{\mathrm{tr}\, V_v}\, d\ell = \Big\langle \mathrm{tr}(K_w - I) \Big\rangle, \qquad \int_{SM} e(v)\, d\ell(v) = \Big\langle \|K_w - I\|_{\mathrm{HS}}^2 \Big\rangle.$$

Lemma 3.3 therefore gives $K_v(t) = I + 3P_+$ in a parallel frame along every geodesic. Let $(Y, \widetilde{g})$ be the Riemannian universal cover of (M, g). It is complete and simply connected. For the metric $4\widetilde{g}$, the normal Jacobi eigenvalues are 1 and $1/4$, with parallel eigenspaces of multiplicities r and $d - 1 - r$. In particular, $\sec \le 1$, and every geodesic admits a nonzero parallel normal field spanning with its velocity a plane of sectional curvature one. By [42, Corollary 2, p. 67], $(Y, 4\widetilde{g})$ is a compact rank-one symmetric space. Since $0 < r < d - 1$, the dimension and the two Jacobi multiplicities identify $(Y, \widetilde{g})$ with X in the original normalization [6, p. 41]. The cover $Y \longrightarrow M$ has a finite number N of sheets because Y is compact, and

$$N = \frac{\mathrm{Vol}(Y, \widetilde{g})}{\mathrm{Vol}(M, g)} = 1.$$

□

4. One-form recovery

4.1. Commuting operators.

Lemma 4.1. *Let $P = (L + c)^{1/2}$, where L is a nonnegative self-adjoint Laplace-type operator on a Hermitian bundle over a closed d-manifold and $c \geq 0$. Suppose that $Z \in \Psi^0_{\mathrm{cl}}$ and P have a common orthonormal eigenbasis, with eigenvalues (ω_j, z_j). For an integer $q \geq 0$, the condition*

$$\#\{j \,:\, \omega_j \leq R,\ \omega_j^q |z_j| > u\} = o(R^d) \qquad \text{for every } u > 0 \tag{4.1}$$

implies $Z \in \Psi^{-q-1}_{\mathrm{cl}}$.

Proof. Suppose $0 \leq k \leq q$ and $Z \in \Psi^{-k}_{\mathrm{cl}}$. The operator $B = P^k Z \in \Psi^0_{\mathrm{cl}}$ has eigenvalues $\omega_j^k z_j$. Put $F_B(R) = \operatorname{Tr}(B^* B \mathbf{1}_{[0,R]}(P))$. For $\omega_j \geq 1$, the inequality $\omega_j^k |z_j| > u$ implies $\omega_j^q |z_j| > u$. The frequencies below one form a finite set. Thus, for every $u > 0$,

$$F_B(R) \leq u^2 N_P(R) + \|B\|^2 o(R^d).$$

Weyl's law and then $u \downarrow 0$ give $F_B(R) = o(R^d)$. Stieltjes integration and the substitution $x = \sqrt{t}R$ give

$$t^{d/2} \operatorname{Tr}(B^* B e^{-tP^2}) = 2t^{1+d/2} \int_0^\infty R e^{-tR^2} F_B(R)\, dR \longrightarrow 0.$$

The weighted heat formula [31, Eqs. (1.22)–(1.23)] identifies this limit with

$$(4\pi)^{-d/2} \operatorname{Vol}(M) \int_{S^*M} \operatorname{tr}\big(\sigma_0(B)^* \sigma_0(B)\big)\, d\ell.$$

The integrand is continuous and nonnegative, so $\sigma_0(B) = |\xi|^k \sigma_{-k}(Z) = 0$. Consequently $Z \in \Psi^{-k-1}_{\mathrm{cl}}$. Iteration for $k = 0, \dots, q$ gives $Z \in \Psi^{-q-1}_{\mathrm{cl}}$. □

An elliptic $A \in \Psi^0_{\mathrm{cl}}$ which is invertible on L^2 has inverse in Ψ^0_{cl}, and inversion is continuous in the classical topology [3, Example 2, Theorem 5.4, and proof of Theorem 7.6, arXiv version]. The bundle case follows by adding a complementary bundle. For $A \in \Psi^0_{\mathrm{cl}}$, a separating contour in its resolvent set defines a classical order-zero Riesz projection whenever $zI - \sigma_0(A)$ is invertible on S^*M for every point z of the contour. If A is normal, this projection is orthogonal.

Lemma 4.2. *Let $P \in \Psi^1_{\mathrm{cl}}$ be positive and self-adjoint on $H^1(E)$, with principal symbol $|\xi| I$. Let $\Pi \in \Psi^0_{\mathrm{cl}}$ be an orthogonal projection commuting with P. Fix $T > 0$, $\beta \in \mathbb{R}$ and an integer $s \geq 1$. Suppose*

$$W := e^{iT\beta} \Pi e^{-iTP} \Pi + (I - \Pi), \qquad W - I \in \Psi^{-s}_{\mathrm{cl}}.$$

On $\operatorname{ran}\Pi$, round every sufficiently high eigenvalue of P to the nearest member of $\beta + (2\pi/T)\mathbb{Z}$, and assign positive weights in this lattice to the remaining eigenspaces. The resulting self-adjoint operator N has domain $\Pi H^1(E)$ and satisfies

$$N = P|_{\operatorname{ran}\Pi} + K|_{\operatorname{ran}\Pi}, \qquad K \in \Psi^{-s}_{\mathrm{cl}}, \qquad K = \Pi K \Pi, \qquad [K, P] = 0.$$

Any modification on finitely many eigenspaces changes K by a smoothing operator.

Proof. The operator W is unitary and commutes with P and Π. Since $W - I$ is compact, for a sufficiently large R the smoothing projection $F = \mathbf{1}_{[0,R]}(P)$ satisfies

$$\widehat{W} = W(I - F) + F, \qquad \|\widehat{W} - I\| < 1.$$

The operator $\widehat{W}$ is unitary, commutes with P and Π, and equals the identity on $\ker\Pi$. Put $A = \widehat{W} - I$. The logarithmic construction of [29, Lemma 3.4 and proof of Theorem 5.1] gives

$$K_0 = -\frac{i}{T} \log \widehat{W}, \qquad \log(I + A) = A f(A), \qquad f(z) = \frac{\log(1+z)}{z}, \quad f(0) = 1.$$

Here $A \in \Psi_{\mathrm{cl}}^{-s}$. On the circle $|z| = r$, with $\|A\| < r < 1$, the operator $zI - A$ is elliptic and invertible. Holomorphic functional calculus therefore gives $f(A) \in \Psi_{\mathrm{cl}}^{0}$ and $K_0 \in \Psi_{\mathrm{cl}}^{-s}$. The principal logarithm of $\widehat{W}$ is skew-adjoint, so $K_0 = K_0^*$, $K_0 = \Pi K_0 \Pi$, and $[K_0, P] = 0$. On a high eigenvalue ω of $P|_{\operatorname{ran}\Pi}$, the logarithm is $-iT(\omega - h)$ for the nearest lattice point h; hence $K_0 = h - \omega$ there. For the prescribed low weights h_ω, put

$$S = \sum_{\omega \le R} (h_\omega - \omega)\Pi \mathbf{1}_{\{\omega\}}(P), \qquad K = K_0 + S.$$

Since $K_0 F = 0$, the operator $P + K$ has the prescribed eigenvalues on $\operatorname{ran}\Pi$. The operator S is self-adjoint and smoothing, satisfies $S = \Pi S \Pi$, and commutes with P. Bounded self-adjoint perturbation gives $\mathcal{D}(N) = \Pi H^1(E)$. □

Lemma 4.3. *Let N_0, N_1 be self-adjoint operators with compact resolvent on Hilbert spaces $\mathcal{H}_0, \mathcal{H}_1$, with spectra in the same lattice $\beta + \delta\mathbb{Z}$, where $\delta > 0$. Let $B : \mathcal{H}_0 \to \mathcal{H}_1$ be bounded and Fredholm, suppose that $B\mathcal{D}(N_0) \subset \mathcal{D}(N_1)$, and suppose that $N_1 B - B N_0$ extends to a compact operator. Then*

$$\dim\ker(N_0 - h) = \dim\ker(N_1 - h)$$

for every lattice weight h outside a finite set.

Proof. Let K be the compact extension of $N_1 B - B N_0$. The identity $N_1 B u = B N_0 u + K u$ shows that B is bounded for the graph norms. Differentiation on $\mathcal{D}(N_0)$ and integration give

$$e^{itN_1} B e^{-itN_0} - B = i \int_0^t e^{isN_1} K e^{-isN_0}\, ds.$$

The integrand is norm-continuous because K is compact. Thus the displayed difference is compact and norm-continuous in t. Set $T = 2\pi/\delta$. The operators $e^{it(N_j - \beta I)}$ define unitary actions of $\mathbb{R}/T\mathbb{Z}$. Compact-group Fredholm averaging [19, §8, pp. 643–644] gives an intertwining Fredholm compact perturbation $\overline{B}$ of B. Its kernel and cokernel are finite-dimensional circle representations, so contain only finitely many weights. For every other weight h, $\overline{B}$ is injective on $\ker(N_0 - h)$ and $\ker(N_1 - h) \subset \operatorname{ran}\overline{B}$. Writing $E_{j,h}$ for the weight projections, the identity $\overline{B} E_{0,h} = E_{1,h}\overline{B}$ gives surjectivity on the weight spaces as well. □

Lemma 4.4. *Let M be connected of dimension $d \ge 3$, and let $\pi : S^*M \to M$. Suppose that a real metric line subbundle $\mathcal{L} \subset \pi^*E$ is preserved by geodesic parallel transport for a real metric connection on E. There are a real metric line ℓ on M and an isometric inclusion $j : \pi^*\ell \to \pi^*E$ with image $\mathcal{L}$. For the unique metric connection on ℓ, which is flat, $\nabla_{X_g} j = 0$.*

Proof. The fiber S^{d-1} is connected and simply connected. The homotopy sequence of the sphere bundle gives $\pi_1(S^*M) \simeq \pi_1(M)$ [17, Theorem 4.41]. Hence pullback is an isomorphism on $H^1(\cdot\,; \mathbb{F}_2)$. Classification by the first Stiefel–Whitney class [18, Proposition 3.10] gives a smooth real line ℓ with $\pi^*\ell \simeq \mathcal{L}$. Choose a metric on ℓ and normalize a smooth isomorphism to obtain j. The maps $\tau_t^E j_v$ and $j_{G_t v}\tau_t^\ell$ are isometries between the same real lines. Their quotient is a continuous sign equal to one at $t = 0$, so they agree whenever defined. Differentiation at $t = 0$ gives $\nabla_{X_g} j = 0$. □

Lemma 4.5. *Let $Y \to M$ be a finite connected normal Riemannian cover of a closed manifold. Suppose that the geodesic flow on Y has period T. Fix $c \ge 0$. If the pullbacks of flat Hermitian lines F_1, F_2 admit parallel unit sections on Y, the operators $e^{-iT\sqrt{\Delta_{0,F_j} + c}}$ have the same complete local symbols modulo $S^{-\infty}$ in parallel unit frames.*

Proof. Let Γ be the deck group and let K_Y be the kernel of $e^{-iT\sqrt{\Delta_Y + c}}$. By Proposition 2.3, this operator belongs to Ψ_{cl}^0. Choose an evenly covered ball U and a lift $s : U \to Y$ such that $\overline{s(U)} \cap \overline{\gamma\, s(U)} = \emptyset$ for $\gamma \ne 1$. In parallel unit frames on the cover, write $f(\gamma y) = \chi_j(\gamma) f(y)$ for the equivariance condition of sections of F_j. Decomposing the integral over the sheets gives, relative to the Riemannian density on U,

$$K_{F_j}(x, z) = \sum_{\gamma \in \Gamma} \chi_j(\gamma) K_Y(s(x), \gamma\, s(z)).$$

The identity term has coefficient $\chi_j(1) = 1$. Every other term is smooth on $U \times U$, since K_Y is smooth off the diagonal. Thus the two localized kernels differ by a smooth kernel, and their complete symbols agree modulo $S^{-\infty}$. □

Lemma 4.6. *Let L be a nonnegative self-adjoint Laplace-type operator on a bundle over a closed d-manifold, where $d > 4$. Let $H \in \Psi^{-3}_{\mathrm{cl}}$ be self-adjoint. Then*

$$\operatorname{Tr}(e^{-t(L+H)}) - \operatorname{Tr}(e^{-tL}) = O(t^{(5-d)/2}) = o(t^{2-d/2}).$$

Proof. Choose $a > \|H\| + 1$ and put $A_0 = L + aI$, $A_1 = L + H + aI$. These are positive self-adjoint classical operators of order two on H^2, with principal symbol $|\xi|^2 I$. Their homogeneous symbols of degrees $2, 1, 0, -1, -2$ agree. With common local coordinates, frames, and symbol extensions for small ξ, the resolvent construction [15, §§1–2, (1.5)–(1.7) and (2.2), arXiv v4] therefore gives the same first five local heat-kernel terms. The remainder estimate [15, Theorem 2.1(2), (2.6), arXiv v4], with operator order 2 and $M = 5$, is $O(t^{(5-d)/2})$ for both kernels, since $2 - 5 > -d$. Integrating their fiber traces on the diagonal and multiplying by e^{at} gives the trace estimate. Finally, $t^{(5-d)/2} = t^{1/2} t^{2-d/2}$. □

4.2. **Complex projective space.**

Lemma 4.7. *Let M^{2n} be a closed connected Riemannian manifold homotopy equivalent to $\mathbb{CP}^n$, where $n \geq 2$ is even. If its geodesic flow has a positive common period, the scalar return at every positive common period has principal symbol $+1$.*

Proof. Put $d = 2n$. Let T_0 be the least positive common period, b_0 the common free-loop index at T_0, and $B = b_0 + d - 1 > 0$. Degrees in loop homology are shifted by d. Every closed geodesic has index $> 3 - d$, since $d \geq 4$. Corollary 2.14(a),(c) of [7, §2.5, arXiv v3], with coefficients in $\mathbb{F}_2$, gives an invertible period class θ of shifted degree B in extended loop homology. The sign local systems are trivial over $\mathbb{F}_2$. In the splitting of [7, Theorem 1.5(a), arXiv v3], the complementary summand in unshifted degree $d + B$ is reduced cohomology of degree $1 - d - B < 0$, hence zero. Thus θ belongs to the reduced ordinary loop algebra, which is a subalgebra, and $\theta^k \neq 0$ for every $k \geq 1$.

Orient M so that a fixed homotopy equivalence to $\mathbb{CP}^n$ has degree one. Homotopy invariance of the loop product [9, Theorem 1] and its value on $\mathbb{CP}^n$ [8, Theorem 3] give the integral shifted algebra

$$\big(\Lambda(w) \otimes \mathbb{Z}[c, z]\big)/(c^{n+1}, (n+1)c^n z, wc^n), \qquad |w| = -1, \quad |c| = -2, \quad |z| = 2n.$$

Its additive torsion is annihilated by the odd integer $n + 1$. The coefficient map respects the loop product, and the Tor term in the universal coefficient theorem [17, Theorem 3A.3] vanishes. Thus reduction modulo two gives the $\mathbb{F}_2$ loop algebra. The reduced algebra is the quotient by $\chi(M)[\mathrm{pt}] = c^n$ over $\mathbb{F}_2$ [7, Example 2.10(a), arXiv v3]; it is

$$\mathbb{F}_2[c, w, z]/(c^n, w^2).$$

The ideal (c, w) is nilpotent and the quotient is $\mathbb{F}_2[z]$. The nonnilpotent homogeneous element θ has nonzero image there, so $2n \mid B$. Since $4 \mid 2n$, Lemma 2.6 gives $\mu_{T_0} = i^{-B} = 1$. Every positive common period is an integer multiple of T_0, and its phase is the corresponding power of μ_{T_0}. □

Proposition 4.8. *Let $X = \mathbb{CP}^n$ have holomorphic sectional curvature four, with $n \geq 3$ and $d = 2n$. Let M be closed and connected, and let E be a flat Hermitian line on M. If the spectra of $\Delta_{1,E}$ on M and Δ_1 on X admit a partial matching satisfying* (1.3) *with $\eta = 1/4$, then M is isometric to X and E has a global parallel unit section.*

Proof. Lemma 2.1 gives $\dim M = d$, $\mathrm{Vol}(M) = V = \pi^n/n!$ and equality of the first three one-form heat coefficients. After discarding pairs of discrepancy at least $1/4$, write $D_0(y)$ for the unmatched count on both sides through eigenvalue y, and

$$S(y) = \sum_{\substack{\text{retained pairs} \\ \min(\lambda,\nu) \leq y}} |\lambda - \nu|.$$

Then

$$D_0(y) = o(y^{d/2-2}), \qquad S(y) = o(y^{d/2-1}). \tag{4.2}$$

Put $\epsilon = (-1)^n$.

4.2.1. *The two spectral families.* For type $(1,0)$, the table preceding [24, Theorem 5.2] has parameters $(r,s) = (1,0)$ and $(2,0)$, each with representation multiplicity one. Replacing its index by $k-1$ and multiplying eigenvalues by four gives the normalization fixed by the scalar spectrum $4k(k+n)$ of Section 2. Complex conjugation gives type $(0,1)$. The one-form families are therefore

$$\begin{aligned} A_k &= 4k(k+n), & \sqrt{A_k + n^2} &= 2k+n, \\ B_k &= 4(k+1)(k+n), & \sqrt{B_k + (n-1)^2} &= 2k+n+1 \qquad (k \geq 1). \end{aligned}$$

The scalar representation at A_k has dimension

$$m_k = \binom{k+n}{n}^2 - \binom{k+n-1}{n}^2,$$

the dimension of harmonic polynomials of bidegree (k,k) on $\mathbb{C}^{n+1}$ [24, §7]. It occurs once in each complex one-form type, so its one-form multiplicity is $2m_k$. The identities $B_k - A_k = 4(k+n)$ and $A_{k+1} - B_k = 4(k+1)$ show that the families are disjoint.

Let H_A, H_B be their model heat traces. Then

$$H_A(t) = 2\big(\operatorname{Tr}(e^{-t\Delta_{0,X}}) - 1\big), \qquad H_B(t) = \operatorname{Tr}(e^{-t\Delta_{1,X}}) - H_A(t),$$

and

$$(4\pi t)^{d/2} H_A(t) \longrightarrow 2V, \qquad (4\pi t)^{d/2} H_B(t) \longrightarrow (d-2)V.$$

The unmatched heat mass and the sum of the retained heat-weight discrepancies are $o(t^{2-d/2})$ by Lemma 2.1. Thus these limits also hold for the retained copies on M assigned to the respective model families.

Put

$$P_1 = (\Delta_{1,E} + n^2)^{1/2}, \qquad U(t) = e^{-itP_1}, \qquad \omega = \sqrt{\lambda + n^2}, \quad \omega_* = \sqrt{\nu + n^2}.$$

For every retained pair, $|\omega - \omega_*| < 1/[4(\omega + \omega_*)]$. On the A-family $\omega_* = 2k+n$; on the B-family,

$$\omega_* - (2k+n+1) = \frac{2n-1}{\omega_* + 2k + n + 1}.$$

Hence $e^{-i\pi\omega}$ tends uniformly at high retained frequencies to ϵ on the A-family and to $-\epsilon$ on the B-family. In both families $e^{-2\pi i\omega} \to 1$. If $H_{\mathrm{miss}}(t)$ is the unmatched heat mass on M and

$$q_K = \sup_{\substack{\text{retained pairs} \\ \nu \geq K}} |e^{-2\pi i\omega} - 1|,$$

then $q_K \to 0$ and

$$\operatorname{Tr}\big((U(2\pi) - I)^*(U(2\pi) - I)e^{-t\Delta_{1,E}}\big) \leq C_K + q_K^2 \operatorname{Tr}(e^{-t\Delta_{1,E}}) + 4H_{\mathrm{miss}}(t).$$

First let $t \downarrow 0$ after multiplication by $t^{d/2}$, then let $K \to \infty$. Proposition 2.4 gives $G_{2\pi} = \mathrm{id}$ and $\sigma_0(U(2\pi)) = I$. By (2.7),

$$\mu_{2\pi} \operatorname{Hol}_{2\pi}(E) H_{2\pi} = I.$$

Tangent transport fixes the velocity, so its scalar factor is one and $H_{2\pi} = I$.

The two heat masses at time π give

$$\lim_{t \downarrow 0} (4\pi t)^{d/2} \operatorname{Tr}\big(U(\pi)e^{-t\Delta_{1,E}}\big) = \epsilon(4-d)V \neq 0.$$

The map G_π is an involution. Averaging any metric on SM over this involution makes it an isometry. Its fixed set is a finite union of submanifolds [14, Lemma 4.5.1]. A fixed component of full dimension makes the isometry the identity on connected SM. Otherwise the fixed set has Liouville measure zero, contradicting Proposition 2.9 and the displayed limit. Thus $G_\pi = \mathrm{id}$.

Write $H = H_\pi$ and $\alpha = \mu_\pi \operatorname{Hol}_\pi(E)$. The group law gives $H^2 = I$ and $\alpha^2 = 1$. The positive tangent rank a and $\alpha \in \{1, -1\}$ are constant on SM. If τ_t denotes tangent parallel transport along γ_v, periodicity gives

$$H_{G_t v} = \tau_t H_v \tau_t^{-1}.$$

Thus both eigenspace bundles of H are preserved by geodesic parallel transport. The weighted heat formula [31, Eqs. (1.22)–(1.23)] gives

$$\epsilon(4-d) = \int_{SM} \operatorname{tr}(\alpha H_v)\, d\ell(v) = \alpha(2a-d).$$

Hence

$$(\alpha, a) = (\epsilon, 2) \quad \text{or} \quad (\alpha, a) = (-\epsilon, d-2). \tag{4.3}$$

Both alternatives have $\det H = 1$.

The finite universal cover Y has the integral cohomology of a simply connected compact rank-one space, by Proposition 2.5. Write

$$H^*(Y;\mathbb{Q}) = \mathbb{Q}[u]/(u^{h+1}), \qquad |u| \text{ even}, \quad h|u| = d.$$

A deck transformation D has $D^* u = \zeta u$, $\zeta = \pm 1$. If it preserves orientation, then $\zeta^h = 1$ and

$$L(D) = \sum_{j=0}^{h} \zeta^j = \begin{cases} h+1, & \zeta = 1, \\ 1, & \zeta = -1,\ h \text{ even}. \end{cases}$$

The Lefschetz theorem [17, Theorem 2C.3] makes D the identity, since deck transformations act freely. Thus the orientation character of $\pi_1(M)$ is injective. Every period-π geodesic has orientation character $\det H = 1$, so is nullhomotopic. Its lift to Y is closed with its initial velocity. In particular, $G_\pi^Y = \mathrm{id}$, $\operatorname{Hol}_\pi(E) = 1$ and $\alpha = \mu_\pi$.

4.2.2. *The scalar return sign.* Suppose the second alternative of (4.3) holds. The bundle $\mathcal{F}_v = \ker(H_v + I)$ is a real plane in $v^\perp = T_v S_x Y$. Its restriction to $S_x Y = S^{2n-1}$ is orientable, since $H^1(S^{2n-1};\mathbb{F}_2) = 0$. Regarded as a complex line, it is trivial, since $H^2(S^{2n-1};\mathbb{Z}) = 0$ [18, Propositions 3.10 and 3.11]. It would give two independent tangent vector fields. For odd n, the Radon–Hurwitz number of $2n$ is two; Adams's theorem [1, Theorem 1.1] excludes this.

Let n be even. The sphere fibration gives $\pi_1(SY) = 0$, so choose an orientation of $\mathcal{F}$. Reversal $\rho(v) = -v$ satisfies $H_{-v} = H_v^{-1} = H_v$. The resulting identification $\mathcal{F}_v = \mathcal{F}_{-v}$ has constant orientation sign on SY. If it reversed orientation, its restriction to a tangent fibre would descend to a plane bundle over $\mathbb{RP}^{d-1}$ with orientation system $\mathbb{Z}_{\mathrm{sign}}$. In the cellular cochain complex, the differential from degree two to degree three is multiplication by two, so

$$H^2(\mathbb{RP}^{d-1};\mathbb{Z}_{\mathrm{sign}}) = 0.$$

This group contains the only obstruction to a section of its unit circle bundle, since $\pi_j(S^1) = 0$ for $j > 1$ [18, §3.3, pp. 101–102]. A unit section would lift to an even map $f : S^{d-1} \to S^{d-1}$ with $f(v) \perp v$. The homotopy $\cos(t)v + \sin(t)f(v)$, $0 \le t \le \pi/2$, gives $\deg f = 1$. But f factors through the oriented double cover $S^{d-1} \to \mathbb{RP}^{d-1}$, so its degree is even. Therefore reversal preserves the orientation of $\mathcal{F}$.

Suppose $H^2(Y;\mathbb{Z}) = 0$. Since $2 < d-1$, the Gysin sequence gives $H^2(SY;\mathbb{Z}) = 0$ [17, p. 438]. Regarded as a Hermitian line, $\mathcal{F}$ has a global unit section σ. Let $\varpi : SY \to Y$ and give $\mathcal{F}$ the projected connection $\nabla^{\mathcal{F}} = P_{\mathcal{F}} \varpi^* \nabla$. Write $\nabla^{\mathcal{F}} \sigma = iA \otimes \sigma$. Geodesic parallel transport preserves $\mathcal{F}$ and has holonomy -1 on it through π. Hence

$$I(v) := \int_{c_v} A \in \pi + 2\pi\mathbb{Z}, \qquad c_v(t) = G_t v, \quad 0 \le t \le \pi.$$

The function I is continuous on connected SY, so is constant.

Reversal intertwines the projected connections because $\varpi\rho = \varpi$ and $P_{\mathcal{F}_{-v}} = P_{\mathcal{F}_v}$. It is complex linear by the orientation conclusion. Write $\sigma(\rho v) = e^{i\theta(v)}\sigma(v)$. A real-valued lift θ exists since SY is simply connected, and $\rho^* A = A + d\theta$. Since $\rho G_t = G_{-t}\rho$,

$$I(v) = \int_{c_v} \rho^* A = -I(\rho v).$$

This contradicts constancy and $I(v) \in \pi + 2\pi\mathbb{Z}$.

Thus Y has the integral cohomology ring $\mathbb{Z}[u]/(u^{n+1})$, $|u| = 2$. Since n is even, every deck transformation preserves orientation; the preceding Lefschetz argument gives $M = Y$. Let $x \in H^2(\mathbb{CP}^n;\mathbb{Z})$ be the hyperplane class. Represent u by a map $M \to \mathbb{CP}^\infty$. Since $\dim M = 2n$, cellular approximation gives $f : M \to \mathbb{CP}^n$ with $f^*(x^j) = u^j$ for $0 \le j \le n$ [17, Example 4.50 and Theorems 4.57 and 4.8]. Both spaces have free integral homology, so f is an integral homology equivalence. They are simply connected, so it is a homotopy equivalence [17, Corollary 4.33]. Lemma 4.7 gives $\mu_\pi = 1$, whereas the second alternative gives $\mu_\pi = -\epsilon = -1$. Consequently

$$\mu_\pi = \epsilon, \qquad \ker(H_v - I) = \mathbb{R}v \oplus \mathcal{L}_v, \qquad \dim_{\mathbb{R}} \mathcal{L}_v = 1. \tag{4.4}$$

4.2.3. *The scalar coefficients.* Identify $\mathcal{L}_v$ with a cotangent line by the metric. Lemma 4.4 gives a real flat metric line ℓ and an isometry

$$j : \pi^*\ell \longrightarrow \mathcal{L} \subset \pi^* T^* M, \qquad \nabla_{X_g} j = 0.$$

For a flat Hermitian line F, put

$$P_F = (\Delta_{0,F} + n^2)^{1/2}, \qquad U_F = e^{-\pi i P_F}.$$

The period loops are nullhomotopic; hence $\sigma_0(U_F) = \epsilon$ by (2.7). At each positive scalar E-eigenvalue λ, choose a one-form eigenbasis containing the image of a scalar eigenbasis under $d_E/\sqrt{\lambda}$, and assign the matching labels to it. At most $D_0(y) + O(1)$ scalar copies through eigenvalue y have no retained label. Since $U_E - \epsilon I$ is compact, no sufficiently high scalar copy has a retained B_k label: its phase would tend to $-\epsilon$. For a retained A_k copy, put $h_k = 2k + n$ and $\omega = \sqrt{\lambda + n^2}$. Then

$$\omega^3 |e^{-\pi i\omega} - \epsilon| \le \pi \frac{\omega^3}{\omega + h_k} |\lambda - A_k| \le \pi R^2 |\lambda - A_k| \qquad (\omega \le R).$$

For every $u > 0$, therefore,

$$\#\{\omega \le R : \omega^3 |e^{-\pi i\omega} - \epsilon| > u\} \le O(1) + D_0(R^2) + \pi u^{-1} R^2 S(R^2) = o(R^d). \tag{4.5}$$

Lemma 4.1, with $q = 3$, gives $U_E - \epsilon I \in \Psi^{-4}_{\mathrm{cl}}$.

The finite universal cover Y has period π, and every flat Hermitian line becomes parallel trivial on Y. Lemma 4.5 consequently gives

$$U_F - \epsilon I \in \Psi^{-4}_{\mathrm{cl}} \qquad \text{for every flat Hermitian line } F. \tag{4.6}$$

The essential spectrum of $U(\pi)$ is contained in $\{\epsilon, -\epsilon\}$. Choose a circle about ϵ, of radius less than one, in its resolvent set, and let Π be the Riesz projection. Then $\Pi \in \Psi^0_{\mathrm{cl}}$ is orthogonal, commutes with P_1, and has principal symbol

$$(v \otimes v + jj^*) \otimes I_E.$$

The exact projection

$$\Pi_e = d_E \Delta^\dagger_{0,E} \delta_E,$$

where the dagger denotes the inverse off the kernel, commutes with P_1 and Π. Hence $\Pi_\ell = \Pi(I - \Pi_e)$ is an orthogonal projection with symbol $jj^* \otimes I_E$.

The operator $Z_A = \Pi(U(\pi) - \epsilon I)\Pi$ has order -1 and vanishes on $\ker\Pi$. At sufficiently high frequency, Π is the identity on every retained A_k copy and zero on every retained B_k copy. Thus its eigenvalues satisfy (4.5), with $|e^{-\pi i\omega} - \epsilon|$ replaced by $|z|$. Lemma 4.1 gives $Z_A \in \Psi_{\mathrm{cl}}^{-4}$.

Round the high eigenvalues of P_F and $P_1|_{\operatorname{ran}\Pi}$ to $h_k = 2k + n$, $k \geq 0$. Assign the finitely many remaining eigenspaces weights in the same set. By Lemma 4.2, with $T = \pi$ and $s = 4$, the resulting operators satisfy

$$N_F = P_F + K_F, \qquad K_F \in \Psi_{\mathrm{cl}}^{-4},$$
$$N_A = P_1|_{\operatorname{ran}\Pi} + K_A|_{\operatorname{ran}\Pi}, \qquad K_A \in \Psi_{\mathrm{cl}}^{-4}, \qquad K_A = \Pi K_A \Pi.$$

The operators N_A, K_A commute with Π_ℓ. Their domains are respectively $\Pi H^1(T^*M \otimes E)$ and $L^2(T^*M \otimes E)$; the domain of N_F is $H^1(F)$. In particular,

$$N_F^2 - n^2 - \Delta_{0,F} \in \Psi_{\mathrm{cl}}^{-3}. \tag{4.7}$$

Put $E' = E \otimes \ell_{\mathbb{C}}$. Quantize $j \otimes I_E$ as $C \in \Psi_{\mathrm{cl}}^0(E', T^*M \otimes E)$. Then

$$B = \Pi_\ell C \,:\, L^2(E') \longrightarrow \operatorname{ran}\Pi_\ell$$

is Fredholm, since $B^*B - I$ and $BB^* - \Pi_\ell$ have order -1. The subprincipal connection formula [26, arXiv version, Proposition 3.2 and (12), (14)], applied on half-densities to $(T^*M \otimes E) \oplus E'$, gives

$$\sigma_0(P_1 B - B P_{E'}) = \frac{1}{i}\nabla_{X_g}(j \otimes I_E) = 0.$$

Consequently $P_1 B - B P_{E'} \in \Psi_{\mathrm{cl}}^{-1}$ and $N_A B - B N_{E'}$ is compact. Moreover, B maps $H^1(E')$ into $\Pi_\ell H^1(T^*M \otimes E)$. Lemma 4.3, for the lattice $n + 2\mathbb{Z}$, gives

$$\dim\ker(N_{E'} - h_k) = \dim\big(\operatorname{ran}\Pi_\ell \cap \ker(N_A - h_k)\big) \qquad (k \gg 1).$$

Put $m_0 = 1$ and $s_k = \dim\ker(N_A - h_k)$. Since $U_E - \epsilon I$ is compact, all sufficiently high exact forms belong to $\operatorname{ran}\Pi$. The isometry $d_E/\sqrt{\lambda}$ intertwines the square roots and their nearest rounded weights. Thus

$$\dim\ker(N_E - h_k) + \dim\ker(N_{E'} - h_k) = s_k \qquad (k \gg 1). \tag{4.8}$$

Every retained high A_k pair has the same rounded weight h_k on both sides, since $|\omega - h_k| \leq 1/[4(\omega + h_k)]$ and consecutive weights differ by two. Retained high B_k copies lie outside $\operatorname{ran}\Pi$. Let u_k count the unmatched copies on M in $\operatorname{ran}\Pi$ rounded to h_k, and let v_k count the unmatched model copies at A_k. Then

$$s_k - 2m_k = u_k - v_k \qquad (k \gg 1).$$

For $h_k \leq R$, every copy counted by u_k has frequency at most $R + \|K_A\|$, and every copy counted by v_k has eigenvalue at most R^2. Thus, for a fixed $C > \|K_A\|$,

$$\sum_{h_k \leq R} |s_k - 2m_k| \leq O(1) + D_0((R + C)^2) = o(R^{d-4}).$$

With $y = h_k^2 - n^2$, (4.8) gives a cumulative absolute multiplicity discrepancy $o(y^{d/2-2})$ between the two rounded scalar spectra and two copies of the scalar model spectrum. The finitely many remaining weights contribute $O(1)$, and $d/2 - 2 > 0$. Laplace–Stieltjes integration, as in Lemma 2.1, gives

$$\operatorname{Tr} e^{-t(N_E^2 - n^2)} + \operatorname{Tr} e^{-t(N_{E'}^2 - n^2)} - 2\operatorname{Tr} e^{-t\Delta_{0,X}} = o(t^{2-d/2}).$$

For $F = E, E'$, the operator in (4.7) is self-adjoint and commutes with $\Delta_{0,F}$. Lemma 4.6 removes both perturbations. The heat coefficients of a flat line twist are the ordinary scalar coefficients. Hence

$$a_{j,0}(M) = a_{j,0}(X) \qquad (0 \leq j \leq 2).$$

The one-form equalities were obtained at the start of the proof. Equation (3.7) gives $J(M) \leq J(X)$. Together with (4.4), these equalities satisfy the hypotheses of Proposition 3.4, with positive normal rank one. Thus M is isometric to X. Since X is simply connected, E has a parallel unit section. □

4.3. **The remaining models.**

4.3.1. *Eigenvalue families.* For a compact symmetric pair G/K with G connected and semisimple, the form Laplacian for the metric induced by minus the Killing form is minus the Casimir [24, Propositions 2.2–2.3]. Rescaling the metric multiplies its eigenvalues by the same factor in every degree. At maximal sectional curvature four, the first positive scalar eigenvalues of $\mathbb{HP}^n$ and $\mathbb{OP}^2$ are $8(n+1)$ and 48 [5, Table 1].

For the models other than complex projective spaces, the one-form eigenvalues have the form

$$h^2 - \beta_X^2 + \delta, \qquad h \in \beta_X + \mathbb{Z}, \tag{4.9}$$

where

X	β_X	δ
S^d, $\mathbb{RP}^d$ $(d \geq 3)$	$(d-1)/2$	$0,\ d-2$
$\mathbb{HP}^n$ $(n \geq 2)$	$2n+1$	$0,\ 8,\ 4n+1$
$\mathbb{OP}^2$	11	$0,\ 21,\ 24.$

The ranges and parity restrictions are as follows.

On the unit sphere the exact and coexact families are

$$j(j+d-1), \qquad (j+1)(j+d-2), \qquad j \geq 1,$$

respectively [24, Theorem 4.2]. Both have $h = j + \beta_X$. The antipodal map acts by $(-1)^j$ on the exact family, since differentiation commutes with pullback. On the coexact family it acts by $(-1)^{j+1}$: the coefficients of a tangential polynomial one-form have homogeneous degree j [24, Theorem 6.8(3),(4), p. 537], and pullback changes the sign of each coordinate differential. On $\mathbb{RP}^d$, the trivial line retains character 1, and the orientation line retains character $(-1)^{d+1}$. The exact family has the same parity restriction as the corresponding twisted scalar spectrum.

For $\mathbb{HP}^n$, the three one-form series are given in [43, §2, p. 97]. In the minus-Killing metric the scalar eigenvalues are $j(j+2n+1)/(2(n+2))$. Comparing the first positive eigenvalue with $8(n+1)$ gives the scale factor $8(n+2)$. Put $A_j = 4j(j+2n+1)$. The series are

eigenvalue	$\mathrm{Sp}(n+1)$ highest weight	range
A_j	$(j, j, 0, \dots, 0)$	$j \geq 1$
$A_{j+1} + 8$	$(j+2, j, 0, \dots, 0)$	$j \geq 0$
$4(j+1)(j+2n+1)$	$(j+1, j, 1, 0, \dots, 0)$	$j \geq 1.$

By [27, Theorem 1.3] and [28, p. 144, case CII(i), with $p = 1$ and $q = n$], each representation occurs once in one-forms. The scalar type $(j, j, 0, \dots, 0)$ occurs once on functions [49, Theorem 1.3]. The identities

$$\begin{aligned} A_j + \beta_X^2 &= (2j+2n+1)^2, \\ A_{j+1} + 8 + \beta_X^2 &= (2j+2n+3)^2 + 8, \\ 4(j+1)(j+2n+1) + \beta_X^2 &= (2j+2n+2)^2 + 4n + 1 \end{aligned}$$

give (4.9).

For $\mathbb{OP}^2 = F_4/\mathrm{Spin}(9)$, the scalar and one-form spectra are given in [36, §4.2, Table 2, p. 393]. The first positive scalar eigenvalue there is 12; hence multiplication by four gives the normalization fixed above. With $\beta_X = 11$, the three one-form families are

eigenvalue	eigenvalue $+ \beta_X^2$	range
$4k(k+11)$	$(2k+11)^2$	$k \geq 1$
$4(k^2+13k+18)$	$(2k+13)^2 + 24$	$k \geq 0$
$4(k^2+14k+24)$	$(2k+14)^2 + 21$	$k \geq 0.$

Each listed irreducible F_4-representation occurs once in one-forms [28, p. 161, case FII]. The first family consists of the nonconstant scalar representations, each occurring once on functions.

On $\mathbb{HP}^n$ and $\mathbb{OP}^2$, the spectral Hodge isometry of Section 2.3 is equivariant. It identifies each positive scalar representation with its unique copy in one-forms. Thus, for each model the zero-offset family is the image of the positive scalar spectrum, with the prescribed flat twist on $\mathbb{RP}^d$. At sufficiently large weights it has no intersection with the other families. Indeed, distinct positive weights $h, h' \in \beta_X + \mathbb{Z}$ satisfy $|h^2 - (h')^2| \geq 2\min(h, h') + 1$, whereas the offsets belong to a fixed finite set; equal weights with distinct offsets give distinct eigenvalues. Thus every sufficiently high zero-offset multiplicity is exactly the scalar multiplicity.

4.3.2. *The projective return.* On $\mathbb{CP}^n, \mathbb{HP}^n, \mathbb{OP}^2$, the shifted scalar frequencies are $2j + \beta_X$, $j \geq 0$, with $\beta_X = n, 2n+1, 11$, respectively. Thus the scalar return at π is $(-1)^{\beta_X} I$. The model has $G_\pi = \mathrm{id}$ by Section 2; (2.7) gives scalar phase $(-1)^{\beta_X}$. For $\mathbb{CP}^n$, the tangent return is positive on the velocity and one normal direction by Lemma 2.8. For $\mathbb{HP}^n$ and $\mathbb{OP}^2$, symmetry gives $\nabla \mathrm{Rm} = 0$. Their normal Jacobi operators have eigenvalues four and one of multiplicities r and $d-1-r$, where $r = 3$ and 7. For a normal eigenvector u with eigenvalue q^2, $q = 1, 2$, the Jacobi field with initial data $(u, 0)$ is $J(t) = \cos(qt)\tau_t u$. Since $dG_\pi = I$, $J(\pi) = u$, and hence $\tau_\pi u = (-1)^q u$. Thus tangent parallel transport at π is positive on the curvature-four space and the velocity, and negative on the curvature-one space. In all three cases the tangent return has trace

$$2 + 2r - d, \qquad r = 1, 3, 7. \tag{4.10}$$

Proposition 4.9. *Let X be S^d, $\mathbb{RP}^d$, $\mathbb{HP}^n$ ($n \geq 2$) or $\mathbb{OP}^2$, with $d = \dim X > 4$. Let M be a closed connected Riemannian manifold. Let F, F_* be the trivial real metric lines on M, X, or their respective orientation lines. The same symbols denote their complexifications. If the spectra of $\Delta_{1,F}$ and Δ^X_{1,F_*} admit a partial matching satisfying* (1.3)*, then M is isometric to X.*

Proof. Lemma 2.1 gives $\dim M = d$, $\mathrm{Vol}(M) = \mathrm{Vol}(X) = V$ and equality of the first three one-form coefficients; flat line twists have the same local coefficients.

If X is round and $16 \leq d \leq 93$, the proof of [46, Theorem A(i), (2.3) and (2.7), pp. 126–128] gives constant sectional curvature on M. The local curvature formulas and their integrals are valid without orientability. Equation (2.3) fixes the curvature value to one, and (3.9) gives $M \cong X$.

For the remaining cases, discard the pairs of discrepancy at least $1/4$. Write $D_0(y)$ and $S(y)$ as in (4.2), so that $D_0(y) = o(y^{d/2-2})$ and $S(y) = o(y^{d/2-1})$. Put

$$\beta = \beta_X, \qquad P_1 = (\Delta_{1,F} + \beta^2)^{1/2}, \qquad \zeta = e^{-2\pi i \beta}.$$

For a retained pair $\nu = h^2 - \beta^2 + \delta$, put $\omega = \sqrt{\lambda + \beta^2}$. Then

$$|\omega - h| = \frac{|\delta + \lambda - \nu|}{\omega + h} = O(h^{-1}).$$

The squared phase errors $|e^{-2\pi i\sqrt{\lambda+\beta^2}} - \zeta|^2$ are bounded by four and tend to zero outside a set of zero Weyl density. Their sum through eigenvalue y is $o(y^{d/2})$. Laplace–Stieltjes integration and Proposition 2.4 give

$$G_{2\pi} = \mathrm{id}, \qquad \sigma_0(e^{-2\pi i P_1}) = \zeta I.$$

If $H_{2\pi}$ denotes tangent parallel transport, then $\mu_{2\pi}\,\mathrm{Hol}_{2\pi}(F)H_{2\pi} = \zeta I$. Since $H_{2\pi}$ fixes the velocity, $H_{2\pi} = I$. The holonomy of F along this loop is one: for the orientation line it is $\det H_{2\pi} = 1$. Hence $\mu_{2\pi} = \zeta$.

4.3.3. *The zero-offset projection.* Fix a Borel function $f_\beta : [0, \infty) \to (\beta + \mathbb{Z}) \cap (0, \infty)$ which assigns to λ a nearest positive lattice point to $\sqrt{\lambda + \beta^2}$, with a fixed choice at ties. Put $N_1 = f_\beta(\Delta_{1,F})$ and use the same function for every scalar rounded operator below. Thus every rounded operator is scalar on each complete Laplace eigenspace. Lemma 4.2 gives

$$N_1 - P_1 \in \Psi^{-1}_{\mathrm{cl}}, \qquad Q_1 = P_1^2 - N_1^2 \in \Psi^0_{\mathrm{cl}}.$$

The principal symbol $q_1 = \sigma_0(Q_1)$ is real symmetric. In real orthonormal frames on half-densities, the coefficients of $1, \hbar, \hbar^2$ in the left semiclassical symbol of $\hbar^2 P_1^2$ are real, imaginary and real. The square-root recursion preserves this alternation. Its leading coefficient is $|\xi| I$; the equations for the next two coefficients have scalar coefficient $2|\xi|$, and a term with j symbol derivatives has the factor i^{-j}. For a real phase ϕ, where $d\phi \neq 0$, this gives

$$e^{-i\phi/\hbar} \hbar P_1 (e^{i\phi/\hbar} b) = |d\phi| b + i\hbar D_1 b + \hbar^2 D_2 b + O(\hbar^3),$$

with D_1, D_2 real. The first two transport equations are

$$(\partial_t - D_1) b_0 = 0, \qquad i(\partial_t - D_1) b_1 = D_2 b_0.$$

Their initial amplitudes $I, 0$ give real b_0 and imaginary b_1; see [29, §2, (2.1)–(2.3), p. 243]. Under composition and changes of real homogeneous phase, the jth stationary-phase correction is i^j times a real differential expression, after removing the common signature factor [21, Theorem 3.1.6, §3.2, (3.2.5)–(3.2.7), Lemma 3.2.3]. Intermediate cutoffs may be chosen equal to one near the ray. Thus the leading and next amplitudes retain this alternation up to their common Maslov factor. The homogeneous action is zero, since $\xi \cdot \partial_\xi |\xi| - |\xi| = 0$. At the return the first two coefficients in the diagonal phase have the form $m(B_0 + i\hbar B_1)$, with B_0, B_1 real and m the common Maslov factor. Since $mB_0 = \zeta I$, m/ζ is real. Consequently

$$\sigma_{-1}\big(\zeta^{-1} e^{-2\pi i P_1} - I\big) \quad \text{is purely imaginary.}$$

The logarithmic formula in Lemma 4.2 gives

$$q_1 = \frac{i|\xi|}{\pi} \sigma_{-1}\big(\zeta^{-1} e^{-2\pi i P_1} - I\big). \tag{4.11}$$

Thus q_1 is real, and self-adjointness makes it symmetric.

Let $\boldsymbol{D}$ be the finite model offset set and put $p(s) = \prod_{\delta \in \boldsymbol{D}} (s - \delta)$. On each sufficiently high retained copy, $N_1 = h$ and the eigenvalue q of Q_1 is $\delta + \lambda - \nu$. Hence $|p(q)| \leq C|\lambda - \nu|$ for a constant C independent of the copy. For every $u > 0$,

$$\#\{\omega \leq R \,:\, |p(q)| > u\} \leq O(1) + D_0(R^2) + Cu^{-1} S(R^2) = o(R^d).$$

Lemma 4.1, with $q = 0$, gives

$$\prod_{\delta \in \boldsymbol{D}} (Q_1 - \delta I) \in \Psi_{\mathrm{cl}}^{-1}, \qquad \prod_{\delta \in \boldsymbol{D}} (q_1 - \delta I) = 0.$$

Hence the spectrum of Q_1 accumulates only in $\boldsymbol{D}$. Choose $1/4 < \rho < 1/2$ so that $|z| = \rho$ lies in the resolvent set, and let Π be the corresponding Riesz projection. Its contour avoids the eigenvalues of q_1; thus $\Pi \in \Psi_{\mathrm{cl}}^0$ is orthogonal and commutes with P_1. The offsets are separated by at least one. Every high retained zero-offset copy lies in $\operatorname{ran} \Pi$, and every other high retained copy lies outside it. The zero-offset model multiplicities are the scalar multiplicities. The heat-trace estimates in the proof of Lemma 2.1, applied to the retained zero-offset pairs, give

$$\operatorname{Tr}(\Pi e^{-t\Delta_{1,F}}) - \operatorname{Tr} e^{-t\Delta^X_{0,F_*}} = o(t^{2-d/2}).$$

The weighted heat formula therefore gives

$$V = \lim_{t \downarrow 0} (4\pi t)^{d/2} \operatorname{Tr}(\Pi e^{-t\Delta_{1,F}}) = V \int_{S^*M} \operatorname{rank} \sigma_0(\Pi)\, d\ell.$$

Since S^*M is connected, $\sigma_0(\Pi)$ has rank one. Its range is real by (4.11).

The relation $[\Pi, P_1] = 0$ and the subprincipal connection formula imply that $\operatorname{ran} \sigma_0(\Pi)$ is preserved by geodesic parallel transport. Lemma 4.4 gives a real flat metric line L on M and an isometry

$$j \,:\, \pi^* L \longrightarrow \operatorname{ran} \sigma_0(\Pi) \subset \pi^*(T^*M \otimes F), \qquad \nabla_{X_g} j = 0.$$

At 2π the ambient parallel transport is the identity. The identity $\tau_t^{T^*M \otimes F} j_v = j_{G_t v} \tau_t^L$ shows that each period-2π loop has trivial L-holonomy. Write also L for its complexification.

4.3.4. *Scalar multiplicities and the constant offset.* For $J = F, L$, put

$$P_J = (\Delta_{0,J} + \beta^2)^{1/2}, \qquad N_J = f_\beta(\Delta_{0,J}), \qquad Q_J = P_J^2 - N_J^2.$$

The scalar return symbol at 2π is ζ for both lines. Lemma 4.2 gives $N_J - P_J \in \Psi_{\mathrm{cl}}^{-1}$. Let C be an order-zero quantization of j, and put $B = \Pi C : L^2(L) \to \operatorname{ran}\Pi$. The differences $B^*B - I$ and $BB^* - \Pi$ have order -1, so B is Fredholm. On half-densities, the subprincipal connection formula [26, Proposition 3.2, (12) and (14), arXiv version 1], applied to $P_1 \oplus P_L$, gives

$$\sigma_0(P_1 B - B P_L) = \frac{1}{i}\nabla_{X_g} j = 0.$$

Hence $P_1 B - B P_L \in \Psi_{\mathrm{cl}}^{-1}$ and $N_1 B - B N_L \in \Psi_{\mathrm{cl}}^{-1}$. The map B takes $H^1(L)$ to $\Pi H^1(T^*M \otimes F)$. Lemma 4.3, for the lattice $\beta + \mathbb{Z}$, yields

$$\dim\ker(N_L - h) = \dim\big(\Pi\ker(N_1 - h)\big) \qquad (h \gg 1). \tag{4.12}$$

For $h \in \beta + \mathbb{Z}$, put

$$m_*(h) = \dim\ker\big((\Delta_{0,F_*}^X + \beta^2)^{1/2} - h\big), \qquad s(h) = \dim\big(\Pi\ker(N_1 - h)\big).$$

Every high retained zero-offset pair has the same rounded weight on both sides. After restriction to those pairs, let u_h count the unmatched copies in $\Pi\ker(N_1 - h)$ and v_h those in the model scalar space at weight h. For all sufficiently large h,

$$s(h) - m_*(h) = u_h - v_h.$$

Since Q_1 and the model offsets are bounded, a copy at rounded weight at most R has eigenvalue at most $(R + C)^2$ for a fixed constant C. Therefore

$$\sum_{h \le R} |\dim\ker(N_L - h) - m_*(h)| \le O(1) + D_0((R + C)^2) = o(R^{d-4}). \tag{4.13}$$

On $(\ker\Delta_{0,F})^\perp$, the operator $T = d_F \Delta_{0,F}^{-1/2}$ is an isometry onto the exact one-form subspace. The common rounding function gives $N_1 T = T N_F$ on the H^1 domains. Hence, as bounded operators on $(\ker\Delta_{0,F})^\perp$,

$$Q_1 T = T Q_F, \qquad T^* \prod_{\delta \in \boldsymbol{D}} (Q_1 - \delta I) T = \prod_{\delta \in \boldsymbol{D}} (Q_F - \delta I).$$

The left side is compact. The scalar kernel is finite dimensional, so the polynomial on the right is compact on the full scalar space. Its principal symbol vanishes. Thus the smooth scalar symbol $\sigma_0(Q_F)$ takes values in $\boldsymbol{D}$. Connectedness gives a constant $c \in \boldsymbol{D}$ such that $Q_F - cI \in \Psi_{\mathrm{cl}}^{-1}$.

Assign the matching labels in one-form eigenbases adapted to the exact subspaces. On a high retained scalar copy, the model offset is c: the eigenvalues of $Q_F - cI$ tend uniformly to zero, whereas the other offsets are at least one from c and $|\lambda - \nu| < 1/4$. Write z for the eigenvalue of $Q_F - cI$ on a scalar copy. On every sufficiently high retained copy, $z = \lambda - \nu$. At most $D_0(y) + O(1)$ scalar copies through eigenvalue y have no retained label. With $\omega = \sqrt{\lambda + \beta^2}$, for every $u > 0$ we obtain

$$\#\{\omega \le R : \omega^2 |z| > u\} \le O(1) + D_0(R^2) + u^{-1} R^2 S(R^2) = o(R^d).$$

Lemma 4.1, with $q = 2$, gives

$$Q_F - cI \in \Psi_{\mathrm{cl}}^{-3}. \tag{4.14}$$

Take the connected cover corresponding to $\ker\mathrm{Hol}_F \cap \ker\mathrm{Hol}_L$. Its degree is at most four. Both lines admit parallel unit sections there, and every period-2π geodesic lifts closed with its velocity, since its two holonomies are trivial. Lemma 4.5 gives equality of the complete local return symbols for F and L. Their square-root symbols also agree in parallel unit frames. The logarithm and products preserve this equality, so

$$Q_L - cI \in \Psi_{\mathrm{cl}}^{-3}.$$

Write $P_L^2 = N_L^2 + cI + B_L$, with $B_L \in \Psi_{\mathrm{cl}}^{-3}$ commuting with P_L. Lemma 4.6, applied to $\Delta_{0,L}$ and $-B_L$, gives

$$\operatorname{Tr} e^{-t\Delta_{0,L}} - e^{-ct} \operatorname{Tr} e^{-t(N_L^2-\beta^2)} = o(t^{2-d/2}).$$

Laplace–Stieltjes integration in (4.13) compares the rounded trace with the model scalar trace to the same order. The local scalar coefficients for F and L agree. Consequently

$$\operatorname{Tr} e^{-t\Delta_{0,F}} - e^{-ct} \operatorname{Tr} e^{-t\Delta_{0,F_*}^X} = o(t^{2-d/2}). \tag{4.15}$$

Comparison of the coefficients gives

$$a_{j,0}(M) = \sum_{\ell=0}^{j} \frac{(-c)^\ell}{\ell!}\, a_{j-\ell,0}(X), \qquad 0 \le j \le 2.$$

In particular, $a_{1,0}(M) = a_{1,0}(X) - cV$. By (2.3), $a_{1,1} = (d-6)a_{1,0}$. Equality of the one-form coefficients therefore gives $(d-6)cV = 0$, and hence $c = 0$ when $d \neq 6$.

In dimension six the present models are round and $\boldsymbol{D} = \{0, 4\}$. Their scalar coefficients per unit volume at orders one and two are 5 and 12. If $c = 4$, (4.15) would give $a_{1,0}(M) = V$ and $a_{2,0}(M) = 0$. Let W denote the Weyl curvature tensor of M. Equation (3.4) and the curvature decomposition give

$$360\, a_{2,0}(M) = \int_M \left(2|W|^2 + \frac{24}{5} s^2 \right) dV \ge \frac{24}{5V} \left(\int_M s\, dV \right)^2 = \frac{864}{5} V > 0,$$

since $\int_M s\, dV = 6V$. This contradiction gives $c = 0$ also in dimension six. Thus the first three ordinary scalar and one-form coefficients agree with those of X, and (3.7) gives $J(M) \le J(X)$.

4.3.5. *The quaternionic and Cayley returns.* For a round target, these heat equalities suffice for Proposition 3.4. Suppose now that $X = \mathbb{HP}^n$, $n \ge 2$, or $X = \mathbb{OP}^2$. The relation $Q_1 d_F = d_F Q_F$ and $c = 0$ give

$$q_1(\xi \otimes F) = 0.$$

The zero eigenspace of q_1 has rank one. The exact projection $\Pi_e = d_F \Delta_{0,F}^{\dagger} \delta_F$ has principal symbol the projection onto $\mathbb{R}\xi \otimes F$. Therefore $\Pi - \Pi_e \in \Psi_{\mathrm{cl}}^{-1}$. Since Π is a spectral function of $\Delta_{1,F}$, it commutes with Π_e. The two operators $\Pi(I - \Pi_e)$ and $(I - \Pi)\Pi_e$ are compact orthogonal projections, hence have finite rank. They commute with $\Delta_{1,F}$; thus $\Pi = \Pi_e$ on all sufficiently high eigenspaces.

Restricting the retained pairs to the zero-offset families therefore gives a scalar matching with unmatched count at most $D_0(y) + O(1)$ and cost at most $S(y) + O(1)$. The shifted model scalar frequencies are $2j + \beta$, with $\beta = 2n + 1$ or 11. Their phases at π are $(-1)^\beta$. The retained scalar phase errors tend to zero, and the exceptional copies have zero Weyl density. Proposition 2.4 gives

$$G_\pi = \mathrm{id}, \qquad \sigma_0(e^{-i\pi P_F}) = (-1)^\beta.$$

Let H be tangent parallel transport through π. The one-form return symbol divided by the scalar F-return symbol is H. The equality $H_{2\pi} = I$ gives $H^2 = I$. For a retained pair, set

$$f_t(\lambda) = e^{-i\pi\sqrt{\lambda+\beta^2}} e^{-t\lambda}, \qquad x = \min(\lambda, \nu).$$

Then

$$|f_t(\lambda) - f_t(\nu)| \le \left(\pi \left| \sqrt{\lambda+\beta^2} - \sqrt{\nu+\beta^2} \right| + t|\lambda - \nu| \right) e^{-tx}.$$

At high frequency the factor in parentheses is at most $C(x^{-1/2} + t)$. Weyl summation gives

$$t^{d/2} \sum_{\text{retained}} |f_t(\lambda) - f_t(\nu)| = O(t^{1/2}) + O(t).$$

The omitted heat mass is $o(t^{-d/2})$. Thus the one-form return traces, multiplied by $t^{d/2}$, have the same limit on M and X. Put $r = 3$ for $X = \mathbb{HP}^n$ and $r = 7$ for $X = \mathbb{OP}^2$. The weighted local Weyl formula, the common scalar return symbol and (4.10) give

$$\begin{aligned}\lim_{t\downarrow 0}(4\pi t)^{d/2}\operatorname{Tr}(e^{-i\pi P_1}e^{-t\Delta_{1,F}}) &= (-1)^{\beta}V\int_{S^*M}\operatorname{tr} H\, d\ell\\ &= (-1)^{\beta}V(2+2r-d).\end{aligned}$$

Since H is a smooth involution on connected S^*M, its trace is constant. Thus $\dim\ker(H-I) = (d+\operatorname{tr} H)/2 = r+1$. It fixes the velocity, so its positive normal rank is r. Its negative rank $d-1-r$ is even. Thus $\det H = 1$, the orientation-line holonomy along each period-π geodesic is trivial, and the ordinary scalar return phase is $(-1)^{\beta}$. All hypotheses of Proposition 3.4 now hold. Thus M is isometric to X. □

Proof of Theorem B. Apply Lemma 2.1 in the measured ordinary degree. It gives $\dim M = d$ before Hodge star is applied. For $p = 1$ take the trivial lines F, F_*; for $p = d-1$ take the complexified orientation lines. Hodge star gives the corresponding comparison of one-form spectra. If $X = \mathbb{CP}^n$, then $d > 4$ gives $n \geq 3$, and Proposition 4.8 applies. Every other target of dimension greater than four occurs in Proposition 4.9. □

5. Dimensions at most four

5.1. **Round manifolds.** For a closed four-manifold M, the Gauss–Bonnet formula in the full tensor norm is

$$32\pi^2\chi(M) = \int_M \left(|\operatorname{Rm}|^2 - 4|\operatorname{Ric}|^2 + s^2\right) dV \tag{5.1}$$

[14, p. 106 and Theorem 2.4.8]. For a nonorientable M, apply the oriented formula to its orientation cover and divide by two.

Corollary 5.1. *Let $X = S^d(1)$ or $\mathbb{RP}^d(1)$, where $1 \leq d \leq 4$, and let $p \in \{1, d-1\}$. If a closed connected Riemannian manifold M satisfies $\operatorname{Spec}_p(M) = \operatorname{Spec}_p(X)$, then M is isometric to X.*

Proof. The case $X = \mathbb{RP}^4(1)$ is Lemma 5.2 below. For the other models, Weyl's law gives $\dim M = d$ and $\operatorname{Vol}(M) = \operatorname{Vol}(X)$. If $d = 1$, a closed connected one-manifold is a circle, and its length determines its metric up to isometry.

Suppose $d \geq 2$. Let F be the trivial line if $p = 1$ and the complexified orientation line otherwise. For $p \neq 1$, Hodge star intertwines Δ_p with $\Delta_{1,F}$. Thus $\operatorname{Spec}_{1,F}(M) = \operatorname{Spec}_1(X)$ in either case. The first three heat coefficients of $\Delta_{1,F}$ have the ordinary one-form curvature formulas, since F is flat of rank one. For $d = 2, 3$, the proof of Tanno's Theorem A(i) [46, p. 127] applies to these coefficients and gives constant sectional curvature one. Equation (3.9) then identifies M with X.

Let $X = S^4(1)$. The same heat coefficients and (5.1) give $\chi(M) \geq 2$ by the proof of Tanno's Theorem B [46, p. 128]. The positive one-form eigenvalues of X are $k(k+3)$ and $(k+1)(k+2)$, $k \geq 1$ [24, Theorem 4.2]. Their square roots after addition of $9/4$ are $k + 3/2 + O(k^{-1})$. Thus

$$U = \exp\left(-2\pi i\sqrt{\Delta_{1,F} + 9/4}\right) \quad \text{satisfies} \quad U + I \text{ compact.}$$

Proposition 2.5 gives $G_{2\pi} = \mathrm{id}$ and a compact simply connected universal cover Y with the integral cohomology ring of S^4 or $\mathbb{CP}^2$. Since

$$[Y : M]\chi(M) = \chi(Y) \in \{2, 3\},$$

we have $Y = M$. In particular, M is simply connected and F is trivial.

Suppose $\chi(M) = 3$. Then $H^*(M;\mathbb{Z}) = \mathbb{Z}[u]/(u^3)$, $|u| = 2$. A map $M \to \mathbb{CP}^\infty$ representing u is homotopic into the four-skeleton $\mathbb{CP}^2$. The resulting map induces an isomorphism on integral cohomology, hence on integral homology. Since both spaces are simply connected, it is a homotopy equivalence [17, Theorem 4.8, Corollary 4.33, Example 4.50 and Theorem 4.57]. Lemma 4.7 gives

$\mu_{2\pi} = 1$. On the other hand, tangent parallel transport fixes the velocity. Since F is trivial, the principal symbol of U on the metric dual of this velocity is $\mu_{2\pi}$ by Proposition 2.3. Compactness of $U + I$ therefore gives $\mu_{2\pi} = -1$. This contradiction proves $\chi(M) = 2$. Now M is orientable, $\mathrm{Spec}_1(M) = \mathrm{Spec}_1(S^4(1))$, and $\chi(M) = \chi(S^4)$. Tanno's Theorem A(ii) [46, pp. 126 and 128] gives constant sectional curvature one. Equation (3.9) gives $M \simeq S^4(1)$ isometrically. □

Let $Y \to M$ be a finite universal covering of a closed connected four-manifold. If $H^*(Y;\mathbb{Z}) = \mathbb{Z}[u]/(u^3)$, $|u| = 2$, every deck transformation fixes u^2 and preserves orientation. The Lefschetz calculation in Section 4.2.1, with $h = 2$, gives

$$H^*(Y;\mathbb{Z}) = \mathbb{Z}[u]/(u^3), \quad |u| = 2 \quad \Longrightarrow \quad [Y : M] = 1. \tag{5.2}$$

Lemma 5.2. *Let $p = 1$ or 3. A closed connected Riemannian manifold with the ordinary p-spectrum of $\mathbb{RP}^4(1)$ is isometric to $\mathbb{RP}^4(1)$.*

Proof. Weyl's law gives $\dim M = 4$ and $\mathrm{Vol}(M) = \mathrm{Vol}(S^4(1))/2$. Put $U(t) = e^{-it\sqrt{\Delta_p}}$ and $z = e^{-3\pi i/2} = i$. On $\mathbb{RP}^4(1)$, the tangent parallel return at time π is $1 \oplus (-I_3)$: a geodesic lifts to a semicircle, and the antipodal differential identifies its endpoints. The even spherical harmonics give the scalar eigenvalues $2k(2k+3)$, $k \geq 0$ [24, Theorem 4.2], and hence the scalar phase z. Proposition 2.3 and the leading local heat coefficient consequently give

$$U(2\pi) + I \text{ compact}, \qquad \lim_{t\downarrow 0} \frac{\mathrm{Tr}(U(\pi)e^{-t\Delta_p})}{\mathrm{Tr}(e^{-t\Delta_p})} = \frac{z\ \mathrm{tr}\,\Lambda^p(1 \oplus (-I_3))}{\binom{4}{p}} = z(-1)^p \frac{4-2p}{4} \neq 0.$$

Both statements hold on M by equality of spectra. Proposition 2.5 gives $G_{2\pi} = \mathrm{id}$ and a compact simply connected universal cover Y with the integral cohomology ring of S^4 or $\mathbb{CP}^2$.

The involution G_π on the connected manifold SM is an isometry for the metric $h + G_\pi^* h$, where h is any metric. If $G_\pi \neq \mathrm{id}$, its fixed locus has measure zero [14, Lemma 4.5.1], and Proposition 2.9 makes the displayed limit zero. Hence $G_\pi = \mathrm{id}$.

Let H be tangent parallel transport around a geodesic of length π, with initial velocity v, and put $\mu = \mu_\pi$. The principal symbol of $U(2\pi)$ gives

$$\mu^2 \Lambda^p(H^2) = -I.$$

The map H^2 preserves orientation and fixes v. For $p = 1$, $\Lambda^p(H^2)$ fixes $v^\flat$; for $p = 3$, it fixes $* v^\flat$, where $*$ is defined by any orientation of the initial tangent space. Hence $\mu^2 = -1$, and Lemma 2.7 gives $\det H = -1$.

The orientation line has holonomy -1 along this geodesic, so M is nonorientable. Its universal cover factors through its orientation double cover; hence $[Y : M]$ is even. Equation (5.2) therefore excludes the cohomology ring of $\mathbb{CP}^2$. Thus Y is an integral homology four-sphere. By Hurewicz and Whitehead [17, Theorem 4.32 and Corollary 4.33], it is a smooth homotopy four-sphere. The manifold M is Besse, and Lin–Schmidt's theorem [34, Theorem 2] gives constant sectional curvature K. For $p = 1, 3$, the coefficient $a_{1,p}$ equals $-\frac{1}{3}\int_M s\, dV$ by (2.3). Its equality with the model coefficient, together with equality of volumes and $s = 12K$, gives $K = 1$. Equation (3.9) gives the assertion. □

Since $\mathbb{CP}^1$ and $\mathbb{HP}^1$ are round spheres of radius $1/2$, simultaneous dilation of the two metrics in a spectral comparison also gives Theorem A for these projective lines.

5.2. **The complex projective plane.** Write $X_2 = (\mathbb{CP}^2, g_{\mathrm{FS}})$, so that $\mathrm{Vol}(X_2) = \pi^2/2$ and $s_{X_2} = 24$. On a four-manifold, let W be the Weyl tensor and put $E = \mathrm{Ric} - sg/4$. The curvature decomposition and (5.1) give

$$32\pi^2\chi(M) = \int_M \left(|W|^2 - 2|E|^2 + s^2/6\right) dV. \tag{5.3}$$

For an oriented M, let W^+ and W^- be the self-dual and anti-self-dual Weyl tensors, and let $\tau(M)$ be its signature. In the same tensor norm, the signature formula [14, §2.2, pp. 98–99, and Theorem 3.1.1] is

$$48\pi^2\tau(M) = \int_M \left(|W^+|^2 - |W^-|^2\right) dV. \tag{5.4}$$

Lemma 5.3. *Let M be a closed connected Riemannian four-manifold whose universal cover Y is compact and has the integral cohomology ring of S^4 or $\mathbb{CP}^2$. Fix $p \in \{0,1,2,3,4\}$ and suppose*

$$a_{j,p}(M) = a_{j,p}(X_2) \qquad (0 \le j \le 2).$$

If $p \in \{1,3\}$ or $H^2(Y;\mathbb{Z}) \neq 0$, then M is isometric to X_2.

Proof. The coefficients of $\int s$ in (2.3), in degrees $0,1,2,3,4$, are $1/6, -1/3, -1, -1/3, 1/6$. Hence

$$V = \mathrm{Vol}(M) = \pi^2/2, \qquad \int_M s\, dV = 24V.$$

For $p = 1,3$, the local heat coefficients coincide with the one-form coefficients. The proof of Tanno's Theorem B [46, p. 128], together with (5.3), gives $\chi(M) \ge 3$. On the other hand, $\chi(M) = \chi(Y)/[Y : M] \le 3$. Thus $\chi(M) = 3$, and M is Einstein by the equality case. Moreover, $\chi(M) = 3$ forces $[Y : M] = 1$ and the cohomology ring of $\mathbb{CP}^2$.

In any degree, if $H^2(Y;\mathbb{Z}) \neq 0$, the cohomology ring is $\mathbb{Z}[u]/(u^3)$, and (5.2) gives $[Y : M] = 1$. Thus in every case under consideration, M is simply connected with the integral cohomology ring of $\mathbb{CP}^2$. Orient it so that $\tau(M) = 1$, and set

$$A = \int_M |E|^2\, dV, \qquad B = \int_M (s-24)^2\, dV.$$

Subtracting (5.4) from (5.3) gives

$$B = 12A - 12\int_M |W^-|^2\, dV. \tag{5.5}$$

For $p = 0$, the first three scalar heat coefficients agree with those of the Einstein manifold X_2. Both manifolds are orientable and have Euler characteristic three. By Tanno's Proposition 8 [45, p. 398], M is Einstein. The case $p = 4$ reduces to this case by Hodge star. Thus M is Einstein for $p = 0,1,3,4$. Schur's theorem and $\int_M s\, dV = 24V$ give $s = 24$, whence $A = B = 0$. Equation (5.5) gives $W^- = 0$.

For $p = 2$, substitution of $|\mathrm{Ric}|^2 = |E|^2 + s^2/4$ and $|\mathrm{Rm}|^2 = |W|^2 + 2|E|^2 + s^2/6$ in (2.4)–(2.5), followed by (5.3) and $\chi(M) = 3$, gives

$$0 = 120\big(a_{2,2}(M) - a_{2,2}(X_2)\big) = 52A - B. \tag{5.6}$$

Together with (5.5), this yields

$$40A + 12\int_M |W^-|^2\, dV = 0.$$

Hence $A = B = 0$ and $W^- = 0$ in this case as well.

Consequently M is compact, simply connected, self-dual and Einstein, with $\mathrm{Ric} = 6g$. By Hitchin's positive-scalar-curvature classification [20, p. 104, proof of Theorem 13, positive-scalar case], M is homothetic to the round S^4 or to Fubini–Study $\mathbb{CP}^2$. The cohomology excludes S^4, and $s = 24$ fixes the stated Fubini–Study normalization. □

Lemma 5.4. *Let Y be a compact simply connected Besse four-manifold with the integral cohomology of S^4. All prime geodesics have a common length L. For every $c > 0$,*

$$e^{-iL\sqrt{\Delta_{0,Y}+c}} + I \in \Psi_{\mathrm{cl}}^{-1}(Y).$$

Proof. By Hurewicz and Whitehead [17, Theorem 4.32 and Corollary 4.33], Y is a homotopy sphere. By Freedman's theorem [13, Theorem 1.6, p. 371], it is homeomorphic to S^4. Radeschi–Wilking's theorem [41, Theorem A] gives a common prime length L. At this period, the periodic nullity is seven and the prime Morse index b is constant, by the argument preceding Lemma 2.6. The least nonconstant index is three [41, pp. 941–942]. Since every prime geodesic has length L, [41, Corollary 4.5, with $q = 1$] gives, for each such geodesic γ,

$$\mathrm{ind}(\gamma^m) \ge b + 6 \qquad (m \ge 2).$$

Every nonconstant closed geodesic is an iterate of a prime one. Thus $b = 3$, and Lemma 2.6 gives

$$\mu_L = \exp\bigl(-i\pi(3+4-1)/2\bigr) = -1.$$

Proposition 2.3 gives the stated remainder for every scalar shift $c > 0$. □

Corollary 5.5. *Let M be a closed connected Riemannian manifold. If, for some $p \in \{0,1,2,3,4\}$,*

$$\operatorname{Spec}_p(M) = \operatorname{Spec}_p(X_2),$$

then M is isometric to X_2.

Proof. Weyl's law gives $\dim M = 4$. If $p = 4$, the equality $b_4(M) = 1$ gives orientability [17, Theorem 3.26]; Hodge star then gives $\operatorname{Spec}_0(M) = \operatorname{Spec}_0(X_2)$. It therefore suffices to consider $p \in \{0,1,2,3\}$.

Equality of the heat traces gives $a_{j,p}(M) = a_{j,p}(X_2)$ for $0 \le j \le 2$. Lemma 2.8 and (2.6) give compactness of $e^{-2\pi i\sqrt{\Delta_p+4}} - I$ on M. Proposition 2.5 supplies a compact simply connected Besse universal cover Y with the integral cohomology ring of S^4 or $\mathbb{CP}^2$. Lemma 5.3 applies directly for $p = 1, 3$.

For $p = 2$, the zero eigenspace gives $b_2(M) = 1$. Pullback $H^2(M;\mathbb{Q}) \to H^2(Y;\mathbb{Q})$ is injective by transfer for the finite cover [17, Proposition 3G.1]. Hence $H^2(Y;\mathbb{Z}) \neq 0$, and Lemma 5.3 applies.

It remains to treat $p = 0$. Suppose that Y has the integral cohomology of S^4. Let L be its common prime geodesic length, given by Lemma 5.4. By that lemma, $e^{-iL\sqrt{\Delta_{0,Y}+4}} + I$ is compact. Pullback by the Riemannian covering $Y \to M$ is injective and intertwines the scalar Laplacians. Thus every scalar eigenvalue of M occurs on Y. These eigenvalues are $4k(k+2)$, $k \ge 0$, and their shifted square roots are $2(k+1)$. Equation (2.6) therefore gives

$$e^{-2ijL} \longrightarrow -1 \qquad (j \to \infty).$$

Consecutive quotients yield $e^{-2iL} = 1$, which would give $e^{-2ijL} = 1$ for every j. This contradiction excludes the spherical cohomology ring. Lemma 5.3 completes the proof. □

Perrone [40, Theorem 1, p. 871] proved the conclusion of Corollary 5.5 for $p = 1, 3$ when (M, g, J) is Kähler and $c_1(M,J) \in \mathbb{R}[\omega_g] \subset H^2(M;\mathbb{R})$, where $c_1(M,J)$ is the first Chern class in real cohomology and ω_g is the Kähler form.

Proof of Theorem A. For $d > 4$, the identity matching satisfies (1.3) with no omissions and zero cost, so Theorem B applies. For $d \le 4$, the round spheres and real projective spaces are covered by Corollary 5.1. The projective lines follow by dilation from that corollary. The only remaining model is $\mathbb{CP}^2$, to which Corollary 5.5 applies. □

6. Proportional degrees

6.1. **A finite criterion.** Fix integers $d > 4$ and $1 \le p \le d-1$. Put

$$x = \frac{p(d-p)}{d(d-1)}, \qquad y = \frac{p(p-1)(d-p)(d-p-1)}{d(d-1)(d-2)(d-3)},$$
$$\mathsf{A} = 2 - 30x + 180y, \qquad \mathsf{B} = -2 + 180x - 720y, \qquad \mathsf{C} = 5 - 60x + 180y.$$

When $\mathsf{A} \neq 0$, define

$$u = 1 - \frac{3\mathsf{B}}{2\mathsf{A}}, \qquad v = \frac{1}{d} - \frac{3}{2\mathsf{A}}\left(\frac{\mathsf{B}}{d} + \mathsf{C}\right). \tag{6.1}$$

For an integer $0 \le j \le d$, set

$$U_p(j) = \binom{d}{p}^{-1} [t^p](1-t)^j(1+t)^{d-j}. \tag{6.2}$$

Thus $\binom{d}{p}U_p(j)$ is the trace of the pth exterior power of an orthogonal involution whose negative eigenspace has dimension j.

Theorem 6.1. *Suppose that*

$$x \neq \frac{1}{6}, \qquad \mathsf{A} \neq 0, \qquad u \leq 0, \qquad v \leq 0. \tag{6.3}$$

Let M be a closed connected Riemannian manifold. If its ordinary p-form spectrum admits a matching with that of $S^d(1)$ or $\mathbb{RP}^d(1)$ after finitely many omissions, satisfying (1.4)*, then M is isometric to that model.*

Let (X, r) be $(\mathbb{CP}^n, 1)$ with $d = 2n$, or $(\mathbb{HP}^n, 3)$ with $d = 4n$, and give X its metric of maximal sectional curvature four. Suppose also that

$$\left.\begin{aligned} j \in 2\mathbb{Z}, \qquad 1 \leq j \leq 1 + \frac{32r}{15}, \\ |U_p(j)| = |U_p(r+1)| \end{aligned}\right\} \quad \Longrightarrow \quad j = r + 1. \tag{6.4}$$

If the ordinary p-spectra of M and X admit a matching after finitely many omissions satisfying (1.4)*, then M is isometric to X. For $r = 1$, condition* (6.4) *is automatic; for $r = 3$, the possible values of j are* $2, 4, 6$.

For a d-dimensional metric g, write $\mathrm{Ric}^\circ = \mathrm{Ric} - (s/d)g$, and let J be as in (1.7).

Lemma 6.2. *Let M be a closed connected Riemannian manifold, and let X be one of the models in Theorem 6.1. Suppose that the ordinary p-spectra of M and X admit a matching after finitely many omissions satisfying* (1.4)*. If $x \neq 1/6$ and $\mathsf{A} \neq 0$, then*

$$\dim M = d, \qquad \mathrm{Vol}(M) = \mathrm{Vol}(X), \qquad \int_M s\, dV_g = s_X \,\mathrm{Vol}(X),$$

and

$$J(g) - J(g_X) = u \int_M |\mathrm{Ric}^\circ|^2 \, dV_g + v \int_M (s - s_X)^2 \, dV_g. \tag{6.5}$$

Proof. Corollary 2.2 gives the dimension, volume and equalities $a_{j,p}(M) = a_{j,p}(X)$ for $0 \leq j \leq 2$. The ratios C_1/C_0 and C_2/C_0 in (2.3)–(2.5) are x and y. Since $x \neq 1/6$, equation (2.3) gives

$$\int_M s\, dV_g = s_X \,\mathrm{Vol}(X).$$

Equations (2.4), (2.5) and (6.1) give

$$\frac{540\, a_{2,p}(M)}{\mathsf{A}\binom{d}{p}} = J(g) - u \int_M |\mathrm{Ric}^\circ|^2 \, dV_g - v \int_M s^2 \, dV_g.$$

The same identity holds on X, which is Einstein. Moreover, the equal volume and total scalar curvature give

$$\int_M s^2 \, dV_g - s_X^2 \,\mathrm{Vol}(X) = \int_M (s - s_X)^2 \, dV_g.$$

Subtraction gives (6.5). □

6.2. **The projective return.**

Lemma 6.3. *Let $X = \mathbb{CP}^n$ with $d = 2n$, or $X = \mathbb{HP}^n$ with $d = 4n$, and give X its metric of maximal sectional curvature four. Suppose $d > 4$ and $1 \leq p \leq d - 1$. Let M be a closed connected Riemannian manifold whose ordinary p-form spectrum admits a matching with that of X after finitely many omissions, with $|\lambda - \nu| \to 0$ uniformly as the matched model eigenvalue ν tends to infinity. Then*

$$\dim M = d, \qquad \mathrm{Vol}(M) = \mathrm{Vol}(X), \qquad G_\pi = \mathrm{id}\,.$$

For every $v \in SM$, tangent parallel return $H = H_\pi(v)$ satisfies $H^2 = I$ and $\det H = 1$.

Proof. The matching forces $p \le \dim M$, since the model spectrum is infinite. The matched errors are bounded. If N_M and N_X are the counting functions, including multiplicities, there are constants C, C_0 such that, for all sufficiently large y,

$$N_X(y - C) - C_0 \le N_M(y) \le N_X(y + C) + C_0.$$

Weyl's law gives $\dim M = d$. The form bundles then have the same rank $\binom{d}{p}$, so $\mathrm{Vol}(M) = \mathrm{Vol}(X)$.

For $\mathbb{CP}^n$, Hodge duality and the Lefschetz decomposition reduce all form degrees to the integral eigenvalue families in [24, Eq. (1.5), §3 and Theorem 5.2(b)]. The scalar family $k(k + n)$ in that normalization becomes $4k(k + n)$ here; the same factor four multiplies every form eigenvalue.

For $\mathbb{HP}^n$, the form Laplacian of the metric induced by minus the Killing form equals minus the Casimir [24, Propositions 2.2–2.3]. A metric dilation rescales all form Laplacians by the same factor. In orthonormal coordinates of type C_{n+1}, an occurring highest weight $(b_1, \dots, b_{n+1})$ has integral coordinates. Its eigenvalue in the present normalization is

$$2 \sum_{j=1}^{n+1} b_j\big(b_j + 2(n + 2 - j)\big) \in 2\mathbb{Z}.$$

The scale is fixed by the scalar weight $(1, 1, 0, \dots, 0)$, whose eigenvalue is $8(n + 1)$ in Section 4.3.1. Thus, in both cases, $\mathrm{Spec}_p(X) \subset 2\mathbb{Z}$.

Let $a = n$ for $\mathbb{CP}^n$ and $a = 2n + 1$ for $\mathbb{HP}^n$, and put $P_Y = (\Delta_{p,Y} + a^2)^{1/2}$ for $Y = M, X$. On X, Section 4.3.2 gives $G_\pi = \mathrm{id}$, scalar phase $(-1)^a$ and involutive tangent parallel return. Hence $e^{-2\pi i P_X} - I \in \Psi^{-1}_{\mathrm{cl}}$ by Proposition 2.3. For a matched pair,

$$\sqrt{\lambda + a^2} - \sqrt{\nu + a^2} = \frac{\lambda - \nu}{\sqrt{\lambda + a^2} + \sqrt{\nu + a^2}} \longrightarrow 0.$$

By (2.6), $e^{-2\pi i P_M} - I$ is compact. Proposition 2.5 gives $G_{2\pi} = \mathrm{id}$. By Proposition 2.3, $e^{-2\pi i P_M} \in \Psi^0_{\mathrm{cl}}$. Its compact difference from I has zero principal symbol, so $e^{-2\pi i P_M} - I \in \Psi^{-1}_{\mathrm{cl}}$.

Write $P = P_M$. Apply Lemma 4.2 with $\Pi = I$, $T = 2\pi$, $\beta = a$ and $s = 1$. Since $a \in \mathbb{Z}$, it gives a self-adjoint integer-valued rounding N commuting with P, such that

$$P - N \in \Psi^{-1}_{\mathrm{cl}}, \qquad Q = (P - N)(P + N) = P^2 - N^2 \in \Psi^0_{\mathrm{cl}}.$$

Initially defined on smooth forms, Q extends to a bounded self-adjoint operator on L^2. On a common eigenvector, write $\omega = \sqrt{\lambda + a^2}$ for the eigenvalue of P and $h \in \mathbb{Z}$ for that of N. Since integer squares have the same parity as their bases,

$$\begin{aligned} e^{-i\pi\omega} - (-1)^a e^{i\pi(\lambda + a^2 - h^2)} \\ = (-1)^h\big(e^{-i\pi(\omega - h)} - e^{i\pi\lambda}\big). \end{aligned}$$

Now $\omega - h \to 0$ and $\mathrm{dist}(\lambda, 2\mathbb{Z}) \to 0$. Equation (2.6) gives

$$e^{-i\pi P} - (-1)^a e^{i\pi Q} \quad \text{compact.}$$

For $C \in \Psi^0_{\mathrm{cl}}$ with scalar principal symbol, $[Q, C] \in \Psi^{-1}_{\mathrm{cl}}$. The identity

$$[e^{i\pi Q}, C] = i \int_0^\pi e^{isQ}[Q, C]e^{i(\pi - s)Q}\, ds$$

is a norm-convergent integral of compact operators. Consequently $e^{i\pi P} C e^{-i\pi P} - C$ is compact. Egorov's theorem [26, arXiv version, Theorem 1.4 and Proposition 3.3] gives $c \circ G_\pi = c$ for $c = \sigma_0(C)$. Smooth scalar symbols separate the points of S^*M, so $G_\pi = \mathrm{id}$.

Let μ be the scalar phase at π and fix $v \in SM$. For $H = H_\pi(v)$, the principal symbol of the squared return is

$$\mu^2 \Lambda^p(H^2) = I.$$

Since H^2 is orthogonal and $1 \le p \le d - 1$, scalarity of $\Lambda^p(H^2)$ implies scalarity of H^2 [35, §5.2, p. 11]. As H fixes the velocity, $H^2 = I$ and $\mu^2 = 1$. Lemma 2.7 gives $\det H = 1$. □

Lemma 6.4. *Let M be a closed connected Riemannian manifold. Let $(X, r) = (\mathbb{CP}^n, 1)$ with $d = 2n$, or $(X, r) = (\mathbb{HP}^n, 3)$ with $d = 4n$, and give X its metric of maximal sectional curvature four. Suppose $d > 4$ and $1 \le p \le d-1$. Suppose that* (6.4) *holds and that the ordinary p-spectra of M and X admit a matching after finitely many omissions satisfying* (1.4)*. If* (3.2) *holds, then M is isometric to X.*

Proof. Put $h = r + 1$ and $b = (d + h - 2)/2$. On X, tangent parallel return at π has positive rank h and negative rank $d - h$, and the scalar phase at π is $(-1)^b$ by Section 4.3.2. Lemma 6.3 gives $G_\pi = \mathrm{id}$, $H^2 = I$ and $\det H = 1$ on M. The positive tangent rank a is constant on the connected bundle SM, and $a \ge 1$ because H fixes the velocity. Since d is even and $\det H = (-1)^{d-a} = 1$, the integer a is even.

For $Y = M, X$, set

$$P_Y = (\Delta_{p,Y} + b^2)^{1/2}, \qquad Z_Y(t) = \mathrm{Tr}(e^{-t\Delta_{p,Y}}),$$
$$T_Y(t) = \mathrm{Tr}(e^{-i\pi P_Y} e^{-t\Delta_{p,Y}}), \quad L_Y(t) = T_Y(t)/Z_Y(t).$$

Write $\zeta_\lambda = e^{-i\pi\sqrt{\lambda+b^2}}$. For each matched pair,

$$|\zeta_\lambda e^{-t\lambda} - \zeta_\nu e^{-t\nu}| \le |e^{-t\lambda} - e^{-t\nu}| + |\zeta_\lambda - \zeta_\nu| e^{-t\nu}.$$

Corollary 2.2 and Lemma 2.1 bound the omitted heat mass and the sum of the first terms by $o(t^{2-d/2})$. The square-root identity in the proof of Lemma 6.3 gives $|\zeta_\lambda - \zeta_\nu| \to 0$ uniformly on the matched tail. Thus, for each $\varepsilon > 0$,

$$|T_M(t) - T_X(t)| \le o(t^{2-d/2}) + O_\varepsilon(1) + \varepsilon Z_X(t),$$
$$Z_M(t) \sim Z_X(t) \sim (4\pi t)^{-d/2} \binom{d}{p} \mathrm{Vol}(X).$$

Since $|T_X| \le Z_X$,

$$|L_M - L_X| \le \frac{|T_M - T_X| + |Z_M - Z_X|}{Z_M} \longrightarrow 0,$$

by first letting $t \downarrow 0$ and then $\varepsilon \downarrow 0$.

The scalar phase μ_π on M is constant. The return symbol (2.7) and the local heat Weyl formula [31, Eqs. (1.22)–(1.23)] give

$$\lim_{t\downarrow 0} L_X(t) = (-1)^b U_p(d-h), \qquad \lim_{t\downarrow 0} L_M(t) = \mu_\pi U_p(d-a).$$

Since $|\mu_\pi| = 1$ and $U_p(d-j) = (-1)^p U_p(j)$, it follows that

$$|U_p(a)| = |U_p(h)|. \tag{6.6}$$

The positive normal rank is $a - 1$. Lemma 3.1 gives (3.3). Lemma 3.2 gives $a - 1 \le i + a - 1 \le (32/135)e(w)$, where $i \ge 0$ is the normal Morse index. Invariance of Liouville measure under the geodesic flow gives

$$a - 1 \le \frac{32}{135}\left\langle \|K_w - I\|_{\mathrm{HS}}^2 \right\rangle \le \frac{32r}{15}.$$

Thus (6.4) and (6.6) force $a = h$. The normal return has positive rank r, so Proposition 3.4 applies. □

Proof of Theorem 6.1. By Lemma 6.2 and (6.3), (3.2) holds. For a round target, Proposition 3.4 gives the conclusion. For a complex or quaternionic target, apply Lemma 6.4. □

6.3. **The interval of degrees.**

Proof of Theorem C. For $K = \emptyset$, take $d_K = 17$. Fix a nonempty compact set $K \subset (\rho_-, \rho_+) \setminus \{\rho_0\}$. First suppose $p \le d/2$, and put $\rho = p/d$, $z = \rho(1-\rho)$ and $\varepsilon = d^{-1}$. Then

$$x = \frac{z}{1-\varepsilon}, \qquad y = \frac{z^2 - \varepsilon z + \varepsilon^2 z}{(1-\varepsilon)(1-2\varepsilon)(1-3\varepsilon)}.$$

Thus $x = z + O(d^{-1})$ and $y = z^2 + O(d^{-1})$, uniformly for $0 \le p \le d$. Put

$$\begin{aligned} A_\infty &= 180\left(z - \frac{1}{12}\right)^2 + \frac{3}{4}, \\ u_\infty &= \frac{10 - 600z + 2520z^2}{2A_\infty}, \quad v_\infty = -\frac{15(1-6z)^2}{2A_\infty}. \end{aligned} \tag{6.7}$$

Since $A_\infty \ge 3/4$, the definitions give $\mathsf{A} = A_\infty + O(d^{-1})$, $u = u_\infty + O(d^{-1})$ and $v = v_\infty + O(d^{-1})$, uniformly for $0 \le p \le d$. Moreover,

$$u_\infty < 0 \quad \Longleftrightarrow \quad \frac{5 - 3\sqrt{2}}{42} < z < \frac{5 + 3\sqrt{2}}{42} \quad \Longleftrightarrow \quad \rho_- < \rho < \rho_+.$$

In the range $0 \le \rho \le 1/2$, one has $v_\infty < 0$ precisely when $\rho \ne \rho_0$. Hence

$$c_K = \min_{\rho \in K} \left\{ \left| z - \frac{1}{6} \right|, -u_\infty, -v_\infty \right\} > 0.$$

For all sufficiently large d, uniformly for $\rho \in K$,

$$\mathsf{A} \ge \frac{3}{8}, \qquad \left| x - \frac{1}{6} \right| \ge \frac{c_K}{2}, \qquad u, v \le -\frac{c_K}{2}.$$

These are the conditions (6.3).

Fix an integer $j \ge 0$ and suppose $d \ge 2j$. Write $(q)_\ell = q(q-1)\cdots(q-\ell+1)$ and $(q)_0 = 1$. At binomial parameter $1/2$, the generating and hypergeometric formulae for the Krawtchouk polynomials [38, Eqs. (18.20.6) and (18.23.3)], with the normalization (6.2), give

$$U_p(j) = \sum_{\ell=0}^{j} (-2)^\ell \binom{j}{\ell} \frac{(p)_\ell}{(d)_\ell}.$$

For $0 \le p \le d$ and $0 \le \ell \le j$, all factors $(p-k)/(d-k)$ have absolute value at most one, and

$$\left| \frac{(p)_\ell}{(d)_\ell} - \rho^\ell \right| \le \sum_{k=0}^{\ell-1} \left| \frac{p-k}{d-k} - \rho \right| \le \frac{\ell(\ell-1)}{d}.$$

Consequently,

$$U_p(j) = (1 - 2\rho)^j + O_j(d^{-1}), \tag{6.8}$$

uniformly for $0 \le p \le d$. Put $s = 1 - 2\rho$. Since $0 < s < 1$ on K,

$$\delta_K = \min_{\rho \in K} s^4(1 - s^2) > 0.$$

The differences between the distinct powers s^2, s^4, s^6 are at least δ_K. Equation (6.8) therefore gives, for all sufficiently large d and $\rho \in K$,

$$\big| |U_p(j)| - |U_p(k)| \big| \ge \frac{\delta_K}{2} \qquad (j, k \in \{2, 4, 6\},\ j \ne k).$$

This gives (6.4) for $r = 3$; for $r = 1$ the condition is automatic.

The identities

$$x(d, d-p) = x(d, p), \qquad y(d, d-p) = y(d, p), \qquad U_{d-p}(j) = (-1)^j U_p(j)$$

show that (6.3) and (6.4) are invariant under $p \mapsto d - p$. They therefore hold in the measured degree p whenever $\min(p, d-p)/d \in K$. Choose a common threshold $d_K \ge 17$ for the preceding estimates. Theorem 6.1 applies to every model of dimension $d \ge d_K$ in (1.1), since the Cayley plane has dimension sixteen.

Choose closed intervals $I_- \subset (\rho_-, \rho_0)$ and $I_+ \subset (\rho_0, \rho_+)$. For all sufficiently large d, every integral degree p with p/d in $I_- \cup I_+$ or its reflection about $1/2$ lies in D_d^*. These four intervals are disjoint. For $I = [\alpha, \beta]$,

$$\#\{p \in \mathbb{Z} : p/d \in I\} = \lfloor d\beta \rfloor - \lceil d\alpha \rceil + 1 = d|I| + O(1).$$

It follows that

$$\liminf_{d\to\infty} \frac{|D_d^*|}{d+1} \geq 2\big(|I_-| + |I_+|\big).$$

Letting the endpoints of I_- and I_+ tend to the endpoints of their respective open intervals gives

$$\liminf_{d\to\infty} \frac{|D_d^*|}{d+1} \geq 2(\rho_+ - \rho_-) = \sqrt{\frac{8}{21}} = 0.617213399848\ldots. \qquad \square$$

Remark 6.5. Put

$$\sigma_- = \frac{1}{2}\left(1 - \sqrt{\frac{15+\sqrt{185}}{30}}\right), \qquad \sigma_+ = \frac{1}{2}\left(1 - \sqrt{\frac{15-\sqrt{185}}{30}}\right).$$

Let $L \subset (\sigma_-, \sigma_+) \setminus \{\rho_0\}$ be compact. For all sufficiently large integers n and every integer p with $p/(2n) \in L$, a closed connected Kähler manifold of complex dimension n with the same ordinary p-spectrum as $\mathbb{CP}^n$ is holomorphically isometric to $\mathbb{CP}^n$ [22, Corollary 2.4 and p. 280]. One has

$$0 < \sigma_- < \rho_- < \rho_0 < \rho_+ < \sigma_+ < \frac{1}{2}.$$

References

[1] J. F. Adams, Vector fields on spheres, *Ann. of Math.* (2) **75** (1962), 603–632.

[2] M. Amann, C. Lange, and M. Radeschi, Odd-dimensional orbifolds with all geodesics closed are covered by manifolds, *Math. Ann.* **380** (2021), 1355–1386.

[3] B. Ammann, R. Lauter, V. Nistor and A. Vasy, Complex powers and non-compact manifolds, *Comm. Partial Differential Equations* **29** (2004), no. 5–6, 671–705. doi:10.1081/PDE-120037329. Preprint: https://arxiv.org/abs/math/0211305v1.

[4] V. I. Arnol'd, Characteristic class entering in quantization conditions, *Funct. Anal. Appl.* **1** (1967), no. 1, 1–13. doi:10.1007/BF01075861.

[5] R. G. Bettiol, E. A. Lauret and P. Piccione, The first eigenvalue of a homogeneous CROSS, *J. Geom. Anal.* **32** (2022), article 76, 63 pp.

[6] Q.-S. Chi, Curvature characterization and classification of rank-one symmetric spaces, *Pacific J. Math.* **150** (1991), no. 1, 31–42.

[7] K. Cieliebak, N. Hingston and A. Oancea, Poincaré duality for loop spaces, *Compos. Math.* **161** (2025), 3140–3212. https://arxiv.org/abs/2008.13161v3.

[8] R. L. Cohen, J. D. S. Jones and J. Yan, The loop homology algebra of spheres and projective spaces, in *Categorical Decomposition Techniques in Algebraic Topology*, Progr. Math. **215**, Birkhäuser, 2003, 77–92.

[9] R. L. Cohen, J. R. Klein and D. Sullivan, The homotopy invariance of the string topology loop product and string bracket, *J. Topol.* **1** (2008), 391–408.

[10] Y. Colin de Verdière, Sur le spectre des opérateurs elliptiques à bicaractéristiques toutes périodiques, *Comment. Math. Helv.* **54** (1979), 508–522.

[11] J. J. Duistermaat and V. W. Guillemin, The spectrum of positive elliptic operators and periodic bicharacteristics, *Invent. Math.* **29** (1975), 39–79.

[12] J. J. Duistermaat and L. Hörmander, Fourier integral operators. II, *Acta Math.* **128** (1972), 183–269.

[13] M. H. Freedman, The topology of four-dimensional manifolds, *J. Differential Geom.* **17** (1982), 357–453.

[14] P. B. Gilkey, *Invariance Theory, the Heat Equation, and the Atiyah–Singer Index Theorem*, Publish or Perish, 1984.

[15] H. Gimperlein and G. Grubb, Heat kernel estimates for pseudodifferential operators, fractional Laplacians and Dirichlet-to-Neumann operators, *J. Evol. Equ.* **14** (2014), 49–83. doi:10.1007/s00028-013-0206-2. Preprint: arXiv:1302.6529v4.

[16] A. Gray and L. Vanhecke, Riemannian geometry as determined by the volumes of small geodesic balls, *Acta Math.* **142** (1979), 157–198. doi:10.1007/BF02395060.

[17] A. Hatcher, *Algebraic Topology*, Cambridge University Press, Cambridge, 2002. https://pi.math.cornell.edu/~hatcher/AT/AT.pdf.

[18] A. Hatcher, *Vector Bundles & K-Theory*, version 2.2, November 2017. https://pi.math.cornell.edu/~hatcher/VBKT/VB.pdf.

[19] N. Higson, The Baum–Connes conjecture, *Doc. Math.*, Extra Volume ICM 1998, II, 637–646. doi:10.4171/DMS/1-2/61.

[20] N. J. Hitchin, Twistor spaces, Einstein metrics and isomonodromic deformations, *J. Differential Geom.* **42** (1995), 30–112.
[21] L. Hörmander, Fourier integral operators. I, *Acta Math.* **127** (1971), 79–183. doi:10.1007/BF02392052.
[22] X. Huang, K. Liu, H. Xu, and Y. Zhi, Asymptotic characterization of complex projective space, *Algebraic Geometry and Physics* **2** (2025), no. 2, 271–308.
[23] K. Ii, Curvature and spectrum of Riemannian manifold, *Tohoku Math. J.* (2) **25** (1973), 557–567.
[24] A. Ikeda and Y. Taniguchi, Spectra and eigenforms of the Laplacian on S^n and $P^n(\mathbb{C})$, *Osaka J. Math.* **15** (1978), 515–546.
[25] D. Jakobson, Y. Safarov and A. Strohmaier, The semiclassical theory of discontinuous systems and ray-splitting billiards, with an appendix by Y. Colin de Verdière, *Amer. J. Math.* **137** (2015), 859–906.
[26] D. Jakobson and A. Strohmaier, High energy limits of Laplace-type and Dirac-type eigenfunctions and frame flows, *Comm. Math. Phys.* **270** (2007), 813–833. Preprint: arXiv:math/0607616v1.
[27] E. Kaneda, The spectra of 1-forms on simply connected compact irreducible Riemannian symmetric spaces, *J. Math. Kyoto Univ.* **23** (1983), no. 2, 369–395. doi:10.1215/kjm/1250521565.
[28] E. Kaneda, The spectra of 1-forms on simply connected compact irreducible Riemannian symmetric spaces II, *J. Math. Kyoto Univ.* **24** (1984), no. 1, 141–162. doi:10.1215/kjm/1250521390.
[29] R. Kuwabara, Spectrum of the Laplacian on vector bundles over $C_{2\pi}$-manifolds, *J. Differential Geom.* **27** (1988), no. 2, 241–258. doi:10.4310/jdg/1214441782.
[30] J. M. Lee, *Introduction to Riemannian Manifolds*, second edition, Graduate Texts in Mathematics 176, Springer, 2018. doi:10.1007/978-3-319-91755-9.
[31] L. Li and A. Strohmaier, The local counting function of operators of Dirac and Laplace type, *J. Geom. Phys.* **104** (2016), 204–228.
[32] P. Li, The spectral rigidity of complex projective spaces, revisited, *Math. Z.* **290** (2018), 1115–1143. doi:10.1007/s00209-018-2055-8. Preprint: arXiv:1608.02737v2.
[33] P. Li, X. Sun, and A. Zhu, The spectral rigidity of Ricci soliton and Einstein-type manifolds, *Comm. Anal. Geom.*, published online 22 October 2026, 32 pp. doi:10.4310/CAG.261018010944. Preprint: arXiv:2312.07259v1 (2023).
[34] S. Lin and B. Schmidt, Real projective spaces with all geodesics closed, *Geom. Funct. Anal.* **27** (2017), 631–636, doi:10.1007/s00039-017-0407-x.
[35] K. Liu and H. Xu, A proof of the spectral rigidity for complex projective space, arXiv:2611.03536v1, 2 November 2026.
[36] K. Mashimo, Spectra of the Laplacian on the Cayley projective plane, *Tsukuba J. Math.* **21** (1997), no. 2, 367–396. doi:10.21099/tkbjm/1496163248.
[37] A. Nifa, *Spectral resolution and rigidity of spherical space forms*, preprint, version 2, 23 October 2026. doi:10.5281/zenodo.23586002. Author's version: CVGMT 8220.
[38] *NIST Digital Library of Mathematical Functions*, Release 1.2.8 of 2026-09-15, https://dlmf.nist.gov/.
[39] V. K. Patodi, Curvature and the fundamental solution of the heat operator, *J. Indian Math. Soc.* (N.S.) **34** (1970), 269–285.
[40] D. Perrone, On the spectral rigidity of $\mathbb{CP}^n$, *Proc. Amer. Math. Soc.* **104** (1988), no. 3, 871–875. doi:10.1090/S0002-9939-1988-0964867-5.
[41] M. Radeschi and B. Wilking, On the Berger conjecture for manifolds all of whose geodesics are closed, *Invent. Math.* **210** (2017), 911–962.
[42] K. Shankar, R. Spatzier and B. Wilking, Spherical rank rigidity and Blaschke manifolds, *Duke Math. J.* **128** (2005), 65–81.
[43] N. Sthanumoorthy, Spectral invariant of the zeta function of the Laplacian on $\mathrm{Sp}(r+1)/\mathrm{Sp}(1)\times\mathrm{Sp}(r)$, *Arch. Math. (Brno)* **27** (1991), 95–104. DML-CZ 107408.
[44] C. J. Sutton, On the Poisson relation for compact Lie groups, *Ann. Global Anal. Geom.* **57** (2020), no. 4, 537–589. Preprint: arXiv:1606.07426.
[45] S. Tanno, Eigenvalues of the Laplacian of Riemannian manifolds, *Tohoku Math. J.* (2) **25** (1973), 391–403.
[46] S. Tanno, The spectrum of the Laplacian for 1-forms, *Proc. Amer. Math. Soc.* **45** (1974), 125–129.
[47] G. Teschl, *Mathematical Methods in Quantum Mechanics: With Applications to Schrödinger Operators*, Graduate Studies in Mathematics, vol. 99, Amer. Math. Soc., Providence, RI, 2009. https://www.mat.univie.ac.at/~gerald/ftp/book-schroe/schroe.pdf.
[48] G. Tsagas, The spectrum of the Laplace operator for a special Riemannian manifold, *Kodai Math. J.* **4** (1981), 377–382.
[49] C. Tsukamoto, Spectra of Laplace–Beltrami operators on $\mathrm{SO}(n+2)/\mathrm{SO}(2)\times\mathrm{SO}(n)$ and $\mathrm{Sp}(n+1)/\mathrm{Sp}(1)\times\mathrm{Sp}(n)$, *Osaka J. Math.* **18** (1981), 407–426.
[50] J.-Y. Wu, On almost isospectral manifolds, *Indiana Univ. Math. J.* **39** (1990), no. 4, 1373–1381.

Toulouse, France
Email address: nifa.anass@gmail.com