SHARP SCHRÖDINGER INEQUALITIES FOR EMBEDDED HYPERSURFACES

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Abstract. Let $\Delta = -\operatorname{div}\nabla$. For a closed connected embedded minimal hypersurface $M^n \subset \mathbb{S}^{n+1}$, $n \geq 2$, which is not totally geodesic, we prove the quadratic-form inequality $\Delta + |h|^2 \geq n$. Taking the test function equal to one gives Perdomo's average-curvature inequality, with equality precisely for minimal Clifford hypersurfaces. Let Q be a positive-definite self-adjoint endomorphism of $\mathbb{R}^{n+1}$ and let $F(X) = \frac{1}{2}\langle QX, X\rangle$. For a connected complete properly embedded hypersurface without boundary satisfying $H = \langle QX, \nu\rangle$, where $H = \operatorname{tr} h$, and which is not a hyperplane, we prove $\Delta_F + |h|^2 + \langle Q\nu, \nu\rangle \geq 2\lambda_{\min}(Q)$. Properness implies finite weighted volume. Equality in the corresponding average inequality characterizes generalized cylinders whose spherical factor lies in the least eigenspace of Q. In dimensions $n \geq 2$, the isotropic case gives the sharp Gaussian average inequalities conjectured by Li and Zhao. The proofs construct a positive supersolution from the sum of two tangent-ball curvature functions. The singular parts of their distributional Hessians are retained in the weak inequality. Consequences include compact spectral comparisons, the embedded low-index classification, and a Bernstein theorem in specified eigendirections.

Contents

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1. Introduction

Let $X : M^n \hookrightarrow \mathbb{S}^{n+1}$ be a closed connected embedded minimal hypersurface, and let $S = |h|^2$. With the convention $\Delta = -\operatorname{div}\nabla$, Simons' identity [31] gives

$$\int_M S(S-n)\,dv_g = \int_M |\nabla h|^2\,dv_g. \tag{1.1}$$

The factor S in the integrand on the left prevents deduction of a lower bound for $\int_M S\,dv_g$ from this identity alone. Perdomo's conjecture asserts that $\int_M S\,dv_g \geq n\operatorname{Vol}(M)$ unless M is totally geodesic, and that equality characterizes minimal Clifford hypersurfaces. This is Conjecture B in [29] and Conjecture 1.5 in [20]. We prove an inequality for every test function.

1.1. Minimal hypersurfaces of the sphere.

Theorem 1.1. *Let $n \geq 2$, and let $X : M^n \hookrightarrow \mathbb{S}^{n+1}$ be a closed connected smooth minimal embedding. If M is not totally geodesic, then*

$$\int_M \left(|\nabla f|^2 + Sf^2\right) dv_g \geq n\int_M f^2\,dv_g \qquad (f \in H^1(M)). \tag{1.2}$$

Moreover,

$$\int_M S\,dv_g \geq n\operatorname{Vol}(M), \tag{1.3}$$

with equality if and only if, after an ambient isometry,

$$X(M) = \mathbb{S}^p\left(\sqrt{p/n}\right) \times \mathbb{S}^{n-p}\left(\sqrt{(n-p)/n}\right), \qquad 1 \leq p \leq n-1. \tag{1.4}$$

The proof of (1.2) has the following dependencies. Separation fixes the normal at a second contact (Lemmas 3.1 and 3.4). For the envelope k of Section 3, let $B = g^{-1}h$ and $\mathscr{D} = \operatorname{div}\nabla$. At a diagonal contact, Lemma 3.5 gives $d\phi|_{\ker(k\mathrm{Id}-B)} = 0$, including multiple eigenvalues. Proposition 4.1 then yields the two one-sided inequalities; their sum satisfies $\mathscr{D}w + (S-n)w \geq 2w^{-1}|\nabla w|^2$. Lemma 2.2 retains the nonnegative singular Hessian measure, and Proposition 2.4 gives a positive supersolution $u = w^{-1}$ of $(\Delta + S - n)u \geq 0$. Theorem 4.2 proves the form identity, and Proposition 4.4 gives the equality classification.

The $S \equiv n$ equality classification is due to Chern–do Carmo–Kobayashi [14] and Lawson [25]; its global product identification is proved below. The equality assertion concerns (1.3). We do not assert that equality of the least eigenvalue of $\Delta + S$ with n alone implies (1.4).

Ge–Li [20, Theorem 1.4] proved $\int_M S\,dv_g \geq \delta(n)\operatorname{Vol}(M)$ for a positive constant $\delta(n)$ depending only on n. Wang–Wang [34, Theorem 1.2] obtained the coefficient n when the hypersurface is invariant under reflection in $n+2$ mutually perpendicular great hyperspheres. Theorem 1.1 imposes no such symmetry. In dimension two, the average inequality follows from Gauss–Bonnet and the fact that a minimal two-sphere in $\mathbb{S}^3$ is totally geodesic; the equality case is Brendle's theorem on embedded minimal tori [6]. The higher-dimensional form estimate (1.2) is the principal assertion here.

The distinction between embeddings and immersions is necessary. The immersed rotational examples discussed in [20, discussion after Conjecture 1.5] have the reverse average inequality. Injectivity makes the denominators in the contact quotients positive away from the diagonal. Properness makes bounded sets of Euclidean competitors compact. Separation of the complement determines the normal at each finite second contact.

1.2. Quadratic Gaussian densities.

The second theorem concerns a quadratic density on Euclidean space. Let Q be a positive-definite self-adjoint endomorphism of $\mathbb{R}^{n+1}$, and set

$$a = \lambda_{\min}(Q), \qquad b = \lambda_{\max}(Q), \qquad F(z) = \tfrac{1}{2}\langle Qz, z\rangle. \tag{1.5}$$

For an oriented Euclidean hypersurface, our convention is

$$h(v,z) = \langle \bar{D}_v \nu, dX(z)\rangle, \qquad H = \operatorname{tr}_g h, \qquad S = |h|^2. \tag{1.6}$$

Thus H is the trace, rather than the average, of the principal curvatures. A hypersurface stationary for the density e^{-F} satisfies

$$H = \langle QX, \nu\rangle. \tag{1.7}$$

Write

$$d\mu_F = e^{-F\circ X} dv_g, \qquad \Delta_F = -\operatorname{div}\nabla + \langle (QX)^T, \nabla\rangle, \qquad q_\nu = \langle Q\nu, \nu\rangle, \qquad U = S + q_\nu. \tag{1.8}$$

The ambient metric is Euclidean throughout. No nonorthogonal change of coordinates is used.

Theorem 1.2. *Let $n \geq 2$. Let $X : M^n \hookrightarrow \mathbb{R}^{n+1}$ be a connected complete smooth proper embedding without boundary satisfying* (1.7). *If $X(M)$ is not a hyperplane, then*

$$\int_M \left(|\nabla f|^2 + U f^2\right) d\mu_F \geq 2a \int_M f^2\, d\mu_F \qquad (f \in C_c^\infty(M)). \tag{1.9}$$

The weighted volume is finite, and

$$\int_M U\, d\mu_F \geq 2a\, \mu_F(M). \tag{1.10}$$

Equality in (1.10) *holds if and only if there are an integer $1 \leq p \leq n$ and a linear subspace $E \subset \ker(Q - a\mathrm{Id})$, $\dim E = p+1$, such that*

$$X(M) = \{x + y : x \in E,\ |x| = \sqrt{p/a},\ y \in E^\perp\}. \tag{1.11}$$

In the compact equality case $p = n$ and $Q = a\mathrm{Id}$.

No integrability assumption on S is imposed in this theorem. The left side of (1.10) may be infinite. The operator statement is for the Friedrichs realization of $\Delta_F + U$. Since $q_\nu \leq b$, it also gives $\Delta_F + S \geq 2a - b$ in quadratic-form sense.

Equation (1.7) is the weighted-minimal equation for the density e^{-F}. Its first and second variations are treated in [12, 23]; their stability form has potential $-U$. Theorem 1.2 concerns $\Delta_F + U$. The scalar Simons identity and the equation for the normal components used below are specializations of Cheng–Mejia–Zhou [13, Theorem 3 and Proposition 2]. We derive them in our conventions in Lemma 5.1.

For $Q = \frac{1}{2}\mathrm{Id}$, equation (1.7) is the self-shrinker equation. With $d\gamma = e^{-|X|^2/4} dv_g$, Theorem 1.2 gives

$$\Delta_\gamma + S \geq \tfrac{1}{2}, \qquad \int_M S\, d\gamma \geq \tfrac{1}{2}\gamma(M). \tag{1.12}$$

Average equality characterizes $\mathbb{S}^p(\sqrt{2p}) \times \mathbb{R}^{n-p}$. These models and their pointwise curvature-gap classifications precede the present integral assertion. In particular, Cao–Li [9, Theorem 1.1] and Ding–Xin [18, Proposition 3.1] classify complete self-shrinkers under the pointwise upper bound $S \leq \frac{1}{2}$ in the normalization of (1.12), with their stated growth or properness hypotheses. Cao–Li use $\widehat{S} \leq 1$ at scale $\widehat{X} = X/\sqrt{2}$. Here the bound on S is not a hypothesis: under average equality, it follows from the supersolution identity, and the proof then determines the parallel shape operator. Cheng–Zhou's equivalence theorem [15, Theorem 1.3] identifies the proper and polynomial-volume-growth formulations for complete self-shrinkers. The entropy normalization follows from the calculation in the proof of Colding–Minicozzi's Lemma 7.10 [16, Section 7.2]. We reproduce the non-strict monotonicity identity in Section 7; it does not require the nonsplitting hypothesis used for strictness in that lemma. Consequently, in dimensions $n \geq 2$, (1.12) gives the assertions of Conjectures 1.6–1.7 of Li–Zhao [26]. Their complete assertion contains the closed case. Those conjectures were stated in the June 2026 preprint, and concern curvature rather than entropy minimization.

Ding–Xin [17, Theorem 1.3] and Brendle–Tsiamis [8, Theorem 1.1] estimate the first positive eigenvalue of the scalar drift Laplacian, in the compact and properly embedded cases respectively. Their weighted Poincaré inequality has coefficient $1/4$ and mean-zero test functions. It is distinct from (1.12), which has potential S and no mean-zero restriction. Li–Zhao use those scalar estimates to obtain lower curvature bounds; their Conjectures 1.6–1.7 ask for the sharper model coefficient appearing above.

Li–Zhao [27, Theorems 1.2–1.3] also prove an upper-pinching classification for complete properly embedded self-shrinkers. Their normalization has $\widehat{H}+\langle\widehat{X},\widehat{\nu}\rangle=0$, $\widehat{h}_{ij}=\langle\bar{D}_{e_i}e_j,\widehat{\nu}\rangle$, and density $e^{-|\widehat{X}|^2/2}$. For a hypersurface in (1.12), set $\widehat{X}=X/\sqrt{2}$. Then

$$\widehat{g}=\tfrac{1}{2}g,\qquad \widehat{S}=2S,\qquad e^{-|\widehat{X}|^2/2}=e^{-|X|^2/4}. \tag{1.13}$$

Their condition $\widehat{S}<3/2$ is therefore $S<3/4$ in our convention; in dimension two they include $S\le 3/4$. Under these pointwise hypotheses the conclusion is a hyperplane or a generalized round cylinder. Neither (1.9) nor its average consequence assumes such an upper bound. We cite the third arXiv version, dated 3 August 2026.

For general Q, weighted finiteness is proved directly from

$$\mathcal{L}_F(F\circ X)=\operatorname{tr}Q-q_\nu-|QX|^2\le \operatorname{tr}Q-a-2a(F\circ X).$$

This argument does not assume that the ambient weighted Euclidean space is a gradient Ricci soliton. Such a soliton assumption occurs in [12, Proposition 4]; for the density considered here, the ambient Bakry–Émery tensor is Q, and it is a scalar multiple of the Euclidean metric only when Q is scalar.

1.3. **Graphical and spectral consequences.** The quadratic theorem also has a graphical consequence.

Theorem 1.3. *Let $n\ge 2$, let Q satisfy* (1.5), *and let $X:M^n\hookrightarrow\mathbb{R}^{n+1}$ be connected, complete, smooth, properly embedded, without boundary, and satisfy* (1.7). *Suppose a nonzero constant vector v satisfies*

$$Qv=\beta v,\qquad \beta\le 2a,\qquad \langle\nu,v\rangle>0\quad \text{on } M. \tag{1.14}$$

Then $X(M)$ is a hyperplane through the origin. In particular, an entire smooth graph in such an eigendirection is a hyperplane, without a growth or bounded-slope assumption.

The isotropic entire-graph assertion is Wang's theorem [33]. An earlier theorem of Hurtado–Palmer–Rosales [22, Theorem 4.26] applies to entire graphs for product densities $e^{\eta(x)+\mu(t)}$. Besides a radial decay condition on the logarithmic density, it assumes $\operatorname{Hess}\eta(\xi,\xi)\le c\le\mu''(t)$ for every unit horizontal vector ξ. Write $Q=\operatorname{diag}(A,\beta)$ in a unit β-eigendirection and set $\eta(x)=-\langle Ax,x\rangle/2$, $\mu(t)=-\beta t^2/2$. Then

$$\langle\nabla(\eta+\mu),\nabla|(x,t)|\rangle\le -a|(x,t)|\quad ((x,t)\ne 0),\qquad -\lambda_{\min}(A)\le c\le-\beta. \tag{1.15}$$

Such a c exists precisely when $\beta\le\lambda_{\min}(A)$, that is, when $\beta=a$. Thus [22, Theorem 4.26] already contains the entire-graph conclusion in least eigendirections. The additional range in Theorem 1.3 is $a<\beta\le 2a$; the theorem also treats complete properly embedded hypersurfaces with positive angle, without assuming that they are entire graphs. No optimality of the threshold $2a$ for this graphical assertion is claimed.

The spherical theorem, combined with Perdomo's upper-average estimate [29, Theorem 3.1], gives the embedded index-$(n+3)$ classification. This implication is already stated in [29, 20]. The dimension-two classification is Urbano's theorem [32]. In higher dimensions, Chen–Wang [11, Corollary 1.3] prove an index lower bound of $n+4$ under $\lambda_1(\Delta)<n$; that hypothesis does not occur in our corollary. The present use of Perdomo's result does not classify arbitrary immersed hypersurfaces of index $n+3$.

On compact hypersurfaces the same positive supersolution gives a comparison for every eigenvalue. Its constant is $(\min w/\max w)^2$ and depends on the embedding. No dimension-only positive lower bound for that ratio is asserted. Equality cylinders in Theorem 1.2 require $\dim\ker(Q-a\mathrm{Id})\ge 2$; their existence does not establish optimality of $2a$ for each fixed Q with a simple least eigenvalue.

1.4. **Contact inequalities and positive supersolutions.** Tangent-ball curvatures and their parabolic inequalities were developed by Andrews [1], Andrews–Langford–McCoy [4], and Andrews–Han–Li–Wei [2]. Brendle's proof of the Lawson conjecture uses a two-point function [6]; Andrews–Li extend this method to constant-mean-curvature tori in the three-sphere [5]. Proposition 3 of the third arXiv version of [5] is the dimension-two antecedent of the finite-contact calculation in Proposition 4.1; their mean curvature is $H/2$ in the trace convention used here. The higher-dimensional two-sided theory of Andrews–Langford contains the reflected tangent-space identity, the directional-diagonal contact argument, and the optimization over derivatives in the additional tangent directions [3, Lemma 2.2, Lemma 2.4 and Proposition 2.5].

Theorem 2 of Andrews–Langford–McCoy [4] already gives unnormalized viscosity inequalities for contact curvatures. Positivity of the speed is added in their quotient Corollary 3, not in that theorem. Unnormalized contact inequalities are therefore also antecedents. We use stationary equations and do not divide by the mean curvature. The shape operator may be indefinite; the diagonal calculation below uses only the strictly positive gaps $k - \kappa_i$, including when the extreme eigenvalue is multiple.

Brendle [7, Propositions 3, 5 and Corollary 6] proves local semiconvexity, the derivative term $2\sum_i \phi_i^2/(\phi - \kappa_i)$, and a distributional inequality on the strict-contact region in a mean-convex flow. The comparison is at the level of the same gap denominators and singular Hessian sign. Here both orientations and all contacts, including diagonal and noncompact limiting contacts, enter the stationary construction. The ensuing assertion is the positive supersolution for the particular potentials below; neither the tangent-ball functions nor the reciprocal ground-state principle is introduced here.

Let k and ℓ denote the two one-sided functions defined in Section 3. Their derivative estimates contain the denominators

$$A_i = k - \kappa_i, \qquad B_i = \ell + \kappa_i, \qquad A_i + B_i = w := k + \ell. \tag{1.16}$$

For $A_i, B_i > 0$,

$$\frac{k_i^2}{A_i} + \frac{\ell_i^2}{B_i} \geq \frac{(k_i + \ell_i)^2}{w}.$$

At a zero denominator, the corresponding derivative vanishes by the diagonal-contact lemma. Thus addition gives

$$\mathcal{L}_F w + V w \geq \frac{2|\nabla w|^2}{w}. \tag{1.17}$$

Here $V = S - n$ in the minimal spherical case, and $V = U - 2a$ in the quadratic Gaussian case. In Euclidean space each signed supremum is replaced by its maximum with zero. In the sphere no such replacement is made. The first convention gives $k, \ell \geq 0$; the second preserves the term $H(k^2 - \ell^2)$ in the constant-mean-curvature identity.

The reciprocal $u = w^{-1}$ satisfies $(\Delta_F + V)u \geq 0$. Passage from a positive solution to a quadratic-form identity is the ground-state transformation; see, for example, Pinchover–Tintarev [30, Lemma 2.4]. We prove the form needed for a locally Lipschitz supersolution and retain the nonnegative measure $(\Delta_F + V)u$. The geometric assertion is the construction of this supersolution for the displayed potentials, not the general ground-state principle.

1.5. **Weak inequalities and proof of the main results.** Each contact function is locally the supremum of a family with a common lower bound for its coordinate Hessians. It is therefore locally semiconvex, and the singular part of its distributional Hessian is positive semidefinite. We use this fact, together with Alexandrov differentiability, to pass from upper-test inequalities to a measure inequality. The contact-curvature use of this passage occurs in [7, Corollary 6]; for the measure-theoretic statements see [19, Chapter 6] and [28, Section 2 and the Appendix]. The reciprocal calculation and the tests against the resulting measure are written out in Section 2. The compact spectral statements use the quadratic-form min–max principle [10, Chapter I] and [24, Chapter VI].

Section 3 establishes separation, regularity, attainment, and the diagonal-contact identities. Section 4 proves the spherical inequality, its constant-mean-curvature version, and Clifford rigidity. Section 5 gives the corresponding matrix calculation for Q. The weighted exhaustion and the equality classification are proved in Section 6. Section 7 contains the graph theorem, the spectral comparisons, the index corollary, and the Gaussian normalization.

The inequalities concern $\Delta + S$ and $\Delta_F + U$, rather than the scalar Laplacians alone. In particular, neither Yau's assertion $\lambda_1(\Delta) = n$ nor an entropy-minimization conjecture is a consequence of the stated estimates. The equality assertions in Theorems 1.1 and 1.2 concern their average inequalities. The eigendirection restriction in Theorem 1.3 is not removed.

2. Positive supersolutions and quadratic forms

Let (M, g) be a smooth manifold without boundary, let $F \in C^\infty(M)$, and put

$$d\mu_F = e^{-F} dv_g, \quad \mathcal{L}_F = \operatorname{div}\nabla - \langle \nabla F, \nabla\rangle, \quad \Delta_F = -\mathcal{L}_F. \tag{2.1}$$

All function spaces are real. The space $H^1(M, d\mu_F)$ consists of functions whose distributional gradient and value are square integrable with respect to $d\mu_F$. On a closed manifold this is the usual $H^1(M)$ as a set, with an equivalent norm. The notation $K \Subset M$ means that K is compact and contained in M. A C^2 function ϕ is an upper test for a continuous function z at p if $\phi(p) = z(p)$ and $\phi \geq z$ on a neighbourhood of p.

The unweighted case is $F = 0$. For locally Lipschitz z, the distribution $\mathcal{L}_F z$ is defined, relative to $d\mu_F$, by

$$\langle \mathcal{L}_F z, \eta\rangle_F = -\int_M \langle \nabla z, \nabla \eta\rangle \, d\mu_F, \qquad \eta \in C_c^\infty(M). \tag{2.2}$$

When this distribution is a measure, the measure includes the weight. Thus $\mathcal{L}_F z = f$ for a smooth function means the measure $f\, d\mu_F$, not $f\, dv_g$.

Lemma 2.1. *Let $K \Subset M$, and let $z \geq 0$ be Lipschitz with support in K. There are nonnegative $z_j \in C_c^\infty(M)$, all supported in one compact set, such that $z_j \to z$ uniformly and in $H^1(M, d\mu_F)$.*

Proof. Choose relatively compact coordinate balls U_i, $1 \leq i \leq N$, and nonnegative $\rho_i \in C_c^\infty(U_i)$ such that $\sum_i \rho_i = 1$ near K. In the coordinates of U_i, extend $\rho_i z$ by zero to $\mathbb{R}^n$ and denote the result by a_i. Its support has positive distance from the boundary of the chart, so a_i is Lipschitz. Let $\eta \geq 0$ be smooth, supported in the unit ball, with $\int \eta = 1$, and set $\eta_\varepsilon(x) = \varepsilon^{-n}\eta(x/\varepsilon)$. For $\varepsilon > 0$ small the support of $a_i * \eta_\varepsilon$ remains in U_i, and

$$\begin{aligned} \|a_i * \eta_\varepsilon - a_i\|_\infty &\leq \varepsilon \operatorname{Lip}(a_i) \int |y| \eta(y)\, dy, \\ \|Da_i * \eta_\varepsilon - Da_i\|_{L^2} &\leq \int \eta(y) \|Da_i(\cdot - \varepsilon y) - Da_i\|_{L^2}\, dy \longrightarrow 0. \end{aligned} \tag{2.3}$$

The convergence in the second line follows from continuity of translations in L^2; its integrand is bounded by $2\|Da_i\|_{L^2}\eta(y)$. Pull back these convolutions and sum over i. Their sum is nonnegative and converges to $\sum_i \rho_i z = z$. On their common compact support, the density $e^{-F}\sqrt{\det g}$ and the eigenvalues of (g^{ij}) have positive upper and lower bounds. The coordinate estimates therefore give the asserted weighted convergence. □

A function z is locally semiconvex if, on each sufficiently small coordinate ball, $z + C|x|^2/2$ is convex for some $C \geq 0$. Such a function is locally Lipschitz. We use Alexandrov's theorem in the following precise form: a finite convex function on an open subset of $\mathbb{R}^n$ has, at almost every point p, an expansion $z(p + x) = z(p) + Dz(p)x + \frac{1}{2}x^T B(p)x + o(|x|^2)$; see [19, Chapter 6].

Lemma 2.2. *If z is locally semiconvex, its coordinate Hessian is a locally finite symmetric matrix measure. Its singular part is positive semidefinite and its absolutely continuous density is the Alexandrov Hessian almost everywhere. The singular part of $\mathcal{L}_F z$, relative to $d\mu_F$, is nonnegative.*

Proof. First suppose z is convex on a Euclidean ball. For $\xi \in \mathbb{R}^n$ and a nonnegative test function η, convexity gives

$$\int \eta(x)\frac{z(x+t\xi)-2z(x)+z(x-t\xi)}{t^2}\,dx \geq 0.$$

Changing variables in the first and last terms and letting $t \to 0$ proves $\langle \partial_{\xi\xi}z, \eta\rangle \geq 0$. Thus $\partial_{\xi\xi}z$ is a positive distribution. On a compact set K, choose $\rho \geq 0$ smooth with $\rho = 1$ near K. Positivity gives $|\langle \partial_{\xi\xi}z, \eta\rangle| \leq \|\eta\|_\infty \langle \partial_{\xi\xi}z, \rho\rangle$ for tests supported in K. The representation theorem for positive functionals identifies $\partial_{\xi\xi}z$ with a locally finite measure. Polarization, in the form

$$2\partial_{ij}z = \partial_{e_i+e_j,e_i+e_j}z - \partial_{ii}z - \partial_{jj}z,$$

gives a symmetric matrix measure $\mathcal{H} = (\partial_{ij}z)$. For every constant ξ the measure $\xi^T\mathcal{H}\xi$ is nonnegative. Its singular part is nonnegative by the Lebesgue decomposition theorem. Taking first $\xi \in \mathbb{Q}^n$ gives a common exceptional null set for all these inequalities; continuity of a finite-dimensional quadratic form then gives them for all $\xi \in \mathbb{R}^n$. For a Borel set E with compact closure, positivity on $e_i + e_j$ and $e_i - e_j$ gives $2|\mathcal{H}_{ij}(E)| \leq \mathcal{H}_{ii}(E) + \mathcal{H}_{jj}(E)$. Apply this inequality to every finite Borel partition of E and take the supremum. Then

$$2|\mathcal{H}_{ij}|(E) \leq \mathcal{H}_{ii}(E) + \mathcal{H}_{jj}(E). \tag{2.4}$$

Thus the trace measure dominates the total variation of every entry.

Write $\mathcal{H} = A(x)\,dx + \mathcal{H}^s$. Choose a point p where Alexandrov's expansion holds, A has a Lebesgue value, and $r^{-n}|\mathcal{H}^s|(B_r(p)) \to 0$. These conditions hold almost everywhere by the differentiation theorem for measures. Set

$$z_r(y) = \frac{z(p+ry)-z(p)-rDz(p)y}{r^2}.$$

For every fixed $R < \infty$, $\sup_{|y|\leq R}|z_r(y) - \frac{1}{2}y^TB(p)y| \to 0$ as $r \downarrow 0$. Thus $D^2z_r \to B(p)\,dy$ as distributions. On the other hand, $D^2z_r = A(p+ry)\,dy + r^{-n}(T_r)_*\mathcal{H}^s$, where $T_r(x) = (x-p)/r$. For every fixed $R < \infty$, the Lebesgue-point and measure-density conditions give

$$\begin{gathered}\int_{B_R}|A(p+ry)-A(p)|\,dy = r^{-n}\int_{B_{rR}(p)}|A(x)-A(p)|\,dx \longrightarrow 0,\\ r^{-n}|\mathcal{H}^s|(B_{rR}(p)) \longrightarrow 0.\end{gathered} \tag{2.5}$$

The two distributional limits therefore agree, so $A(p) = B(p)$.

For semiconvex z, apply the preceding argument to $z + C|x|^2/2$; subtracting the smooth quadratic changes only the absolutely continuous part. Put $\omega = e^{-F}\sqrt{\det g}$. The coordinate expression of the weighted measure is

$$\mathcal{L}_Fz = \omega g^{ij}\mathcal{H}_{ij} + \omega\big(-g^{ij}\Gamma^k_{ij}\partial_kz - g^{ij}(\partial_iF)(\partial_jz)\big)\,dx. \tag{2.6}$$

It follows by expanding $\partial_i(\omega g^{ij}\partial_jz)$ in distributions. The parenthesized function is locally bounded. To check the sign of the other singular term, write $\mathcal{H}^s = C(x)\,d\sigma$ with $\sigma = \operatorname{tr}\mathcal{H}^s$ and $C(x) \geq 0$ for σ-almost every x. Then $g^{ij}C_{ij} = \operatorname{tr}(g^{-1/2}Cg^{-1/2}) \geq 0$, and $\omega > 0$. This proves the assertion. No vanishing of a singular continuous part is required. □

Lemma 2.3. *Suppose z is locally semiconvex and $G \in L^1_{\mathrm{loc}}(M, d\mu_F)$. If the absolutely continuous value of $\mathcal{L}_Fz$ satisfies $\mathcal{L}_Fz \geq G$ almost everywhere, then $\mathcal{L}_Fz \geq G\,d\mu_F$ distributionally. At an Alexandrov point, any upper-test inequality whose expression is continuous in the Hessian with value and gradient fixed passes to that absolutely continuous value.*

Proof. The first assertion follows by adding the nonnegative singular part from Lemma 2.2. At an Alexandrov point, in local coordinates with origin at the point,

$$z(x) = z(0) + p\cdot x + \tfrac{1}{2}x^TBx + o(|x|^2). \tag{2.7}$$

For every $\varepsilon > 0$, the quadratic polynomial on the right with an additional $\varepsilon|x|^2$ lies above z in a sufficiently small ball and agrees at the origin. Its value and gradient are fixed, and its Hessian tends to B as $\varepsilon \downarrow 0$. Apply the upper-test inequality and pass to this limit. □

The next identity is the supersolution form of the ground-state transformation; compare [30, Lemma 2.4].

Proposition 2.4. *Let $w > 0$ be locally Lipschitz, and suppose $\mathcal{L}_F w$ is a locally finite measure. Let $V \in L^\infty_{\mathrm{loc}}(M)$ and suppose*

$$\mathcal{L}_F w + V w \, d\mu_F \geq 2w^{-1}|\nabla w|^2 \, d\mu_F. \tag{2.8}$$

Then $u = w^{-1}$ is locally Lipschitz and

$$\mathfrak{m} = (\Delta_F + V)u \tag{2.9}$$

is a nonnegative locally finite Radon measure. For every $f \in C_c^\infty(M)$,

$$\mathcal{Q}_V(f) := \int_M (|\nabla f|^2 + V f^2) \, d\mu_F = \int_M w^{-2}|\nabla(wf)|^2 \, d\mu_F + \int_M w f^2 \, d\mathfrak{m}. \tag{2.10}$$

Proof. On every compact set w has a positive lower bound. The Sobolev chain rule gives $\nabla u = -w^{-2}\nabla w$. Let $\eta \geq 0$ be smooth and compactly supported. The test η/w^2 is nonnegative Lipschitz. Lemma 2.1 permits it in (2.8): uniform convergence passes the measure term, and strong H^1 convergence passes the gradient term. Expanding its derivative gives

$$-\int \frac{\langle \nabla w, \nabla \eta \rangle}{w^2} \, d\mu_F + 2\int \frac{\eta|\nabla w|^2}{w^3} \, d\mu_F + \int \frac{V\eta}{w} \, d\mu_F \geq 2\int \frac{\eta|\nabla w|^2}{w^3} \, d\mu_F. \tag{2.11}$$

Cancellation proves $\int (\langle \nabla u, \nabla \eta \rangle + V u \eta) \, d\mu_F \geq 0$.

A positive distribution is a Radon measure: on a compact set choose a nonnegative cutoff ρ equal to one nearby; positivity bounds its value on a smooth test η by $\|\eta\|_\infty \langle \mathfrak{m}, \rho \rangle$. It therefore extends to continuous compactly supported tests, and the representation theorem for positive functionals gives the measure. This proves (2.9) with local finiteness.

Use f^2/u in the distributional identity for $\mathfrak{m}$, justified by the same approximation. Then

$$\int \frac{f^2}{u} \, d\mathfrak{m} = \int \left(2\frac{f}{u}\langle \nabla u, \nabla f \rangle - \frac{f^2}{u^2}|\nabla u|^2 + V f^2 \right) d\mu_F. \tag{2.12}$$

Since

$$u^2|\nabla(f/u)|^2 = |\nabla f|^2 - 2\frac{f}{u}\langle \nabla u, \nabla f \rangle + \frac{f^2}{u^2}|\nabla u|^2, \tag{2.13}$$

addition gives (2.10). □

Lemma 2.5. *Let $V \in L^\infty_{\mathrm{loc}}(M)$, let k, ℓ be locally semiconvex, let $w = k + \ell > 0$, and let $\kappa_1, \dots, \kappa_n$ be the eigenvalues of a smooth symmetric endomorphism. At each point where k and ℓ are differentiable, choose an orthonormal eigenbasis $e_1, \dots, e_n$, and put $k_i = dk(e_i)$ and $\ell_i = d\ell(e_i)$. No smooth choice of eigenbasis is assumed. Suppose $k \geq \kappa_i$ and $\ell \geq -\kappa_i$. At almost every point assume*

$$\mathcal{L}_F k + V k \geq 2\sum_i \frac{k_i^2}{k - \kappa_i}, \qquad \mathcal{L}_F \ell + V\ell \geq 2\sum_i \frac{\ell_i^2}{\ell + \kappa_i}. \tag{2.14}$$

A zero denominator is required to have zero numerator, and that summand is set equal to zero. Then (2.8) *holds.*

Proof. For $A, B > 0$,

$$\frac{r^2}{A} + \frac{s^2}{B} - \frac{(r+s)^2}{A+B} = \frac{(Br - As)^2}{AB(A+B)} \geq 0. \tag{2.15}$$

If $A = 0$ and $r = 0$, both sides of $r^2/A + s^2/B \geq (r+s)^2/(A+B)$, with the prescribed convention, equal s^2/B. If $B = 0$ and $s = 0$, both sides equal r^2/A. Apply this with $A = k - \kappa_i$, $B = \ell + \kappa_i$, $r = k_i$,

$s = \ell_i$. Both denominators cannot vanish because their sum is $w > 0$. Summing gives (2.8) almost everywhere. The right side is locally bounded. Lemma 2.3 adds the nonnegative singular part of $\mathcal{L}_F w$. □

Lemma 2.6. *Under Proposition 2.4, suppose $V \in L^1(M, d\mu_F)$ and there are smooth compactly supported cutoffs $0 \le \chi_j \uparrow 1$, equal to one eventually on every compact set, with $\int |\nabla \chi_j|^2 \, d\mu_F \to 0$. Then*

$$\int_M V \, d\mu_F = \int_M |\nabla \log w|^2 \, d\mu_F + \int_M w \, d\mathfrak{m}. \tag{2.16}$$

Both terms on the right are finite. On a closed manifold this identity holds with $f = 1$ directly.

Proof. Apply (2.10) to χ_j:

$$\int (|\nabla \chi_j|^2 + V\chi_j^2) \, d\mu_F = \int |\nabla \chi_j + \chi_j \nabla \log w|^2 \, d\mu_F + \int w\chi_j^2 \, d\mathfrak{m}. \tag{2.17}$$

The left side tends to $\int V \, d\mu_F$. Let $q = \int_M V \, d\mu_F$, which is finite by hypothesis. For each x, $\chi_j = 1$ in a neighbourhood of x for all sufficiently large j. Fatou's lemma for the first nonnegative integrand and monotone convergence for the second give

$$\int_M |\nabla \log w|^2 \, d\mu_F + \int_M w \, d\mathfrak{m} \le \liminf_j \mathcal{Q}_V(\chi_j) = q.$$

In particular $q \ge 0$ and $\nabla \log w \in L^2(d\mu_F)$. The mixed term in the first square is bounded in absolute value by

$$2\|\nabla \chi_j\|_{L^2(d\mu_F)} \|\nabla \log w\|_{L^2(d\mu_F)} \longrightarrow 0. \tag{2.18}$$

The energy assumption gives $\int |\nabla \chi_j|^2 \, d\mu_F \to 0$. Monotone convergence gives $\int \chi_j^2 |\nabla \log w|^2 \, d\mu_F \to \int |\nabla \log w|^2 \, d\mu_F$ and $\int w\chi_j^2 \, d\mathfrak{m} \to \int w \, d\mathfrak{m}$. Taking limits in (2.17) proves (2.16). □

Lemma 2.7. *On a closed manifold, multiplication by a Lipschitz function w is bounded on $H^1(M, d\mu_F)$. If $\min_M w > 0$, multiplication by w is invertible on that space. On any connected manifold, a locally Lipschitz function with zero weak gradient is constant.*

Proof. The weak product and chain rules give

$$\nabla(wf) = w\nabla f + f\nabla w, \qquad \nabla(w^{-1}) = -w^{-2}\nabla w.$$

These rules follow in coordinate charts by convolution, the almost-everywhere differentiability of Lipschitz functions, and the distributional definition of first derivatives. Consequently

$$\|wf\|_{L^2} \le \|w\|_\infty \|f\|_{L^2}, \qquad \|\nabla(wf)\|_{L^2} \le \|w\|_\infty \|\nabla f\|_{L^2} + \|\nabla w\|_\infty \|f\|_{L^2}. \tag{2.19}$$

The same estimates apply to w^{-1} under the positive-minimum hypothesis. For the last assertion, convolve w on a coordinate ball and restrict to a smaller ball. Its smooth convolutions have zero gradient and hence are constant there. Uniform convergence makes w constant on the smaller ball. The constants agree on intersecting balls; connectedness gives a single constant on M. □

3. Tangent-ball curvatures

We use two ambient spaces: $\mathbb{S}^{n+1} \subset \mathbb{R}^{n+2}$ and $\mathbb{R}^{n+1}$, with $n \ge 2$. In the spherical case the embedding is closed; in the Euclidean case it is proper and without boundary. All hypersurfaces in this section are connected.

Lemma 3.1. *Such an embedded hypersurface has a smooth global unit normal. Its complement has exactly two connected components. The normal can be chosen to point from either specified component to the other.*

Proof. Let N be the ambient space and $\Sigma = X(M)$. Properness implies that Σ is closed: a convergent sequence in Σ lies in a compact ambient neighbourhood; its inverse images have a convergent subsequence in M. Fix $z_0 \in N \setminus \Sigma$. For $z \in N \setminus \Sigma$, choose a smooth path γ from z_0 to z transverse to Σ, and put

$$\epsilon(z) = \#\gamma^{-1}(\Sigma) \pmod 2.$$

The intersection set is discrete and compact, and hence finite. For two such paths, their union, after smoothing at the endpoints in the complement, is a loop. The ambient space is simply connected. A filling disk may be chosen smooth and transverse to Σ relative to its boundary, by relative transversality [21, Chapter 3]. Its inverse image of Σ is a compact one-dimensional manifold with boundary. Its boundary has even cardinality, so the two paths have equal intersection parity. Thus ϵ is well-defined and is constant on each component of the complement.

A transverse segment across a hypersurface chart changes ϵ by one. The normal pointing from the local side of parity zero to that of parity one is therefore independent of the chart and is smooth. The tubular neighbourhood theorem [21, Chapter 4] supplies a positive radius function $r : M \to (0, \infty)$ for which the normal exponential is an embedding on $\{(p, t) : |t| < r(p)\}$. Its two subsets $0 < t < r(p)$ and $-r(p) < t < 0$ are connected, since each is homeomorphic to $M \times (0, 1)$. Every complementary component meets one of them: choose a path from one of its points to a point of Σ, make it transverse, and restrict it to the interval before its first intersection. That terminal interval lies in one local tubular side. The two tubular sides have different parities. They consequently lie in different components, and every component contains one of them. There are exactly two. □

Choose such a normal ν, and identify tangent vectors with their images under dX. Define, for $x \neq y$,

$$d(x, y) = X(y) - X(x), \quad A(x, y) = \tfrac{1}{2}|d(x, y)|^2, \quad c(x, y) = -\frac{\langle \nu(x), d(x, y)\rangle}{A(x, y)}. \tag{3.1}$$

For a spherical embedding $A = 1 - \langle X(x), X(y)\rangle$. Define the two envelopes by

$$\begin{aligned} \mathbb{S}^{n+1} : &\quad k = \sup\nolimits_{y\neq x} c(x, y), & \ell &= \sup\nolimits_{y\neq x}[-c(x, y)], \\ \mathbb{R}^{n+1} : &\quad k = \max\{0, \sup\nolimits_{y\neq x} c(x, y)\}, & \ell &= \max\{0, \sup\nolimits_{y\neq x}[-c(x, y)]\}. \end{aligned} \tag{3.2}$$

In both cases put $w = k + \ell$. Normal reversal interchanges k and ℓ and leaves w unchanged. The spherical envelopes are not truncated at zero. This convention is used in the constant-mean-curvature calculation.

Lemma 3.2. *The functions k, ℓ are finite, locally Lipschitz, and locally semiconvex. If $\kappa_1, \dots, \kappa_n$ are the principal curvatures, then*

$$k \geq \max_i \kappa_i, \qquad \ell \geq -\min_i \kappa_i. \tag{3.3}$$

In the spherical case a strict inequality $k(p) > \max_i \kappa_i(p)$ has an off-diagonal maximizer. In the Euclidean case this holds when also $k(p) > 0$.

Proof. Let $\gamma(t) = \exp_x(tv)$, where $|v| = 1$. The Gauss formula gives

$$A(x, \gamma(t)) = \tfrac{1}{2}t^2 + O(t^3), \qquad -\langle \nu(x), X(\gamma(t)) - X(x)\rangle = \tfrac{1}{2}h_x(v, v)t^2 + O(t^3). \tag{3.4}$$

The quotient tends to $h_x(v, v)$; taking suprema over unit vectors proves (3.3).

Choose a relatively compact geodesically convex neighbourhood of p and a smooth orthonormal frame on it. After reducing $\varepsilon > 0$, the exponential maps are defined for x in a compact smaller neighbourhood and $|tv_x| \leq \varepsilon$, and every pair with $d_M(p, x) < \varepsilon/4$ and $d_M(p, y) < \varepsilon/2$ is joined by the unique short geodesic in the larger neighbourhood. All parameters below range over the closure of the smaller neighbourhood, $v \in \mathbb{S}^{n-1}$ and $0 \leq t \leq \varepsilon$. The near-diagonal family is

$$(x, v, t) \longmapsto c(x, \exp_x(tv_x)). \tag{3.5}$$

For a smooth numerator $N(x, v, t)$ with $N(x, v, 0) = N_t(x, v, 0) = 0$,

$$\frac{N(x, v, t)}{t^2} = \int_0^1 (1 - s) N_{tt}(x, v, st)\, ds. \tag{3.6}$$

Both numerator and denominator in (3.5) have this property, and the divided denominator is $1/2$ at $t = 0$. Thus (3.5) extends smoothly to $t = 0$. On the compact parameter set its first two x derivatives are uniformly bounded.

Use the fixed far set $\{y : d_M(p, y) \geq \varepsilon/2\}$. If y is outside this far set, then $d_M(x, y) < 3\varepsilon/4$; hence it occurs in the near family. The intersection of the far set with the inverse image of a closed bounded ambient set is compact by properness. For y in this intersection, $d_M(x, y) > \varepsilon/4$. Injectivity and compactness give a positive lower bound for $A(x, y)$. Their quotient and its first two x derivatives are bounded uniformly.

Only in the Euclidean case can a sequence $X(y_j)$ be unbounded. For x in the fixed compact neighbourhood and $|X(y)|$ large, the numerator and its first two x derivatives are $O(|X(y)|)$, the denominator is bounded below by $c_0|X(y)|^2$, and its first two x derivatives are $O(|X(y)|)$. For $c = -N/A$ its coordinate derivatives are

$$\begin{aligned} \partial_i c &= -\frac{N_i}{A} + \frac{NA_i}{A^2}, \\ \partial_{ij} c &= -\frac{N_{ij}}{A} + \frac{N_i A_j + N_j A_i + NA_{ij}}{A^2} - \frac{2NA_i A_j}{A^3}. \end{aligned} \tag{3.7}$$

Here N is the numerator without its minus sign. The bounds just obtained imply

$$|c(x, y)| + |\partial_x c(x, y)| + |\partial_x^2 c(x, y)| \leq C|X(y)|^{-1}. \tag{3.8}$$

Hence all members of the near and far families have a common Lipschitz bound and a common lower coordinate Hessian bound. Every member of these families is a quotient at a point of $M \setminus \{x\}$ or a directional limit of such quotients. Their supremum equals the untruncated supremum in (3.2). In Euclidean space include the constant zero function as an additional member. On a smaller convex coordinate ball, each $f_\alpha + C|x|^2/2$ is convex. For $0 \leq t \leq 1$,

$$\sup_\alpha (f_\alpha + C|\cdot|^2/2)((1 - t)x + ty) \leq (1 - t) \sup_\alpha (f_\alpha(x) + C|x|^2/2) + t \sup_\alpha (f_\alpha(y) + C|y|^2/2).$$

Thus the supremum is semiconvex. The inequality $|\sup_\alpha f_\alpha(x) - \sup_\alpha f_\alpha(y)| \leq L|x - y|$ gives the Lipschitz bound. For ℓ, replace every member c by $-c$; the bounds for $|\partial_x c|$ and $|\partial_x^2 c|$ are unchanged. In Euclidean space retain the zero member. The same constants L, C therefore give the Lipschitz and semiconvexity bounds for ℓ.

For attainment, let $y_j \neq p$ satisfy $c(p, y_j) \to k(p)$ in either strict case of the statement. After passing to a subsequence, either $y_j \to q \neq p$, $y_j \to p$, or, in Euclidean space, $|X(y_j)| \to \infty$. Properness gives the first two alternatives whenever $X(y_j)$ is bounded. The second case has upper limit at most $\max_i \kappa_i(p)$ by (3.4). The last case is Euclidean and has limit zero by (3.8). The stated strict inequalities exclude both alternatives. Properness gives a convergent subsequence in the remaining case. □

Lemma 3.3. *In the spherical case $w > 0$ everywhere unless the image is a round totally umbilic hypersphere. In the Euclidean case $w > 0$ everywhere unless the image is an affine hyperplane.*

Proof. In the spherical case $w(p)$ is the supremum minus the infimum of $c(p, \cdot)$. If it is zero, $c(p, y) = t$ for every $y \neq p$, and continuity gives

$$\langle \nu(p) - tX(p), X(y) \rangle = -t \qquad (y \in M). \tag{3.9}$$

The vector on the left has squared norm $1 + t^2$. Its normalized affine level has magnitude less than one, so the section of the sphere in (3.9) is a round n-sphere. The immersion into that section is a local diffeomorphism, since dimensions agree. Its image is open by the inverse function theorem and closed because it is compact. Since the section is connected and the image is nonempty, they coincide.

In the Euclidean case $k, \ell \geq 0$. If $w(p) = 0$, the two defining inequalities imply $c(p, y) = 0$ for every $y \neq p$. The image lies in $X(p) + \nu(p)^{\perp}$. It is open in that hyperplane by the inverse function theorem, and closed by properness. It is the entire hyperplane. Both assertions follow. On a compact nonexceptional hypersurface, continuity gives $\min_M w > 0$. □

Lemma 3.4. *Let N be either ambient space of Lemma 3.1, and let Ω_- be the component from which ν points. Let $B \subset N \setminus X(M)$ be connected. Assume $B \cap \Omega_- \neq \emptyset$. At another point $Y = X(q) \in \partial B$, suppose B is locally $\{G < 0\}$, where G is smooth near Y, $G(Y) = 0$, $\nabla^N G(Y) \neq 0$, and $\nabla^N G(Y) \perp T_q M$. Then $\nabla^N G(Y)/|\nabla^N G(Y)| = \nu(q)$.*

Proof. Connectedness and Lemma 3.1 give $B \subset \Omega_-$. Put $V = \nabla^N G(Y)/|\nabla^N G(Y)|$. For small $t > 0$, $G(\exp_Y(-tV)) = -t|\nabla^N G(Y)| + O(t^2) < 0$. Hence $\exp_Y(-tV) \in \Omega_-$. The only unit normals are $\nu(q)$ and $-\nu(q)$. Since $\exp_Y(t\nu(q)) \in \Omega_+$ for small $t > 0$, the second choice is impossible. □

3.1. **Covariant identities.** Let $c_0 = 1$ in the sphere and $c_0 = 0$ in Euclidean space. Let B denote the shape endomorphism: $h(v, z) = g(Bv, z)$. With $\bar{D}$ the flat connection in the containing Euclidean space,

$$\bar{D}_v dX(z) = dX(\nabla_v z) - h(v, z)\nu - c_0 g(v, z)X, \qquad \bar{D}_v \nu = dX(Bv). \tag{3.10}$$

We use $R(v, z)t = \nabla_v \nabla_z t - \nabla_z \nabla_v t - \nabla_{[v,z]} t$. Commuting the flat derivatives in (3.10) gives

$$\begin{aligned} R(v, z)t &= c_0\big(g(z, t)v - g(v, t)z\big) + h(z, t)Bv - h(v, t)Bz, \\ (\nabla_v h)(z, t) &= (\nabla_z h)(v, t). \end{aligned} \tag{3.11}$$

In particular ∇h is fully symmetric. Write $\mathscr{D} = \operatorname{div} \nabla$, and use the same symbol for the trace of the covariant Hessian of a tensor. Then

$$\mathscr{D}h = \nabla^2 H + (nc_0 - S)h + Hh^2 - c_0 Hg. \tag{3.12}$$

Here $(h^2)(v, z) = g(B^2 v, z)$.

Derivation of (3.12). At a point choose an orthonormal frame with vanishing covariant derivatives. Codazzi and contraction give

$$\begin{aligned} (\mathscr{D}h)_{ij} - H_{;ij} &= \sum_k \big((\nabla_k \nabla_i - \nabla_i \nabla_k)h\big)_{kj} \\ &= -\sum_k h(R(e_k, e_i)e_k, e_j) - \sum_k h(e_k, R(e_k, e_i)e_j). \end{aligned} \tag{3.13}$$

Substitute (3.11). The two sums on the last line are respectively

$$(n-1)c_0 h - h^3 + Hh^2, \qquad c_0 h - c_0 Hg - Sh + h^3. \tag{3.14}$$

For instance, writing $h_{ij} = h(e_i, e_j)$, the first contraction equals

$$(n-1)c_0 h_{ij} - \sum_{k,m} h_{ki} h_{km} h_{mj} + H \sum_m h_{im} h_{mj},$$

whereas the second equals

$$c_0 h_{ij} - c_0 H \delta_{ij} - Sh_{ij} + \sum_{k,m} h_{km} h_{mi} h_{kj}.$$

Symmetry of h identifies the cubic contractions and they cancel. The resulting sum is $(nc_0 - S)h + Hh^2 - c_0 Hg$, proving the identity. Pairing it with h uses $\frac{1}{2}\mathscr{D}|h|^2 = \langle \mathscr{D}h, h\rangle + |\nabla h|^2$. □

3.2. **Diagonal contacts.** The cubic contact and derivative optimization have antecedents in [3, Lemma 2.4 and Proposition 2.5]. The following lemma applies to both ambient spaces and both envelope conventions.

Lemma 3.5. *Let k be the envelope for ν, let ϕ be a C^2 upper test at p, and suppose $k(p) = \kappa_{\max}(p)$. Put $E = \ker(k(p)\mathrm{Id} - B_p)$. Then*

$$d\phi|_E = 0. \tag{3.15}$$

Let Z be any smooth tangent field and let $L_Z = \mathscr{D} - \nabla_Z$. Choose an orthonormal eigenbasis with $e_1 \in E$, $|e_1| = 1$. Then

$$L_Z\phi(p) \geq (L_Zh)_{11}(p) + 2\sum_{j=1}^{n}\sum_{\kappa_i<k}\frac{h_{1i;j}^2}{k-\kappa_i} \geq (L_Zh)_{11}(p) + 2\sum_{\kappa_i<k}\frac{\phi_i^2}{k-\kappa_i}. \tag{3.16}$$

All quantities on the right are evaluated at p.

Proof. For any unit $v \in E$, put $Y(t) = X(\exp_p(tv))$. The support inequality at p gives

$$f(t) = \tfrac{k(p)}{2}|Y(t) - X(p)|^2 + \langle \nu(p), Y(t) - X(p)\rangle \geq 0 \tag{3.17}$$

for both signs of small t. With v parallel along the geodesic, (3.10) gives

$$Y'(0) = v, \quad Y''(0) = -h(v,v)\nu - c_0X, \quad \langle \nu(p), Y'''(0)\rangle = -(\nabla_v h)(v,v). \tag{3.18}$$

Writing $A(t) = |Y(t) - X(p)|^2/2$, one has $A'(t) = \langle Y - X(p), Y'\rangle$, $A''(0) = |v|^2 = 1$, and $A'''(0) = 3\langle v, Y''(0)\rangle = 0$. Differentiation of the Gauss formula also gives $Y'''(0) = -h_{vv;v}\nu - h(v,v)dX(Bv) - c_0v$. Thus

$$f(0) = f'(0) = f''(0) = 0, \qquad f'''(0) = -(\nabla_v h)(v,v) = 0. \tag{3.19}$$

Indeed, Taylor's formula would give $f(t) = f'''(0)t^3/6 + o(|t|^3)$. If $f'''(0) \neq 0$, choosing the sign of t opposite to that coefficient contradicts $f(t) \geq 0$.

For a smooth local unit extension V of v, the function $\phi - h(V,V)$ is nonnegative and vanishes at p. Its derivative vanishes. Since $B_pv = k(p)v$ and $g(\nabla_jV, V) = 0$,

$$\phi_j = h_{vv;j}. \tag{3.20}$$

Take $e_j = v$ and use (3.19). The resulting equation $d\phi(v) = 0$ holds for every unit vector of E, proving (3.15) also for a multiple eigenvalue.

Choose $v = e_1$ and prescribe

$$\nabla_{e_j}V = \sum_{\kappa_i<k} c_{ji}e_i \quad \text{at } p. \tag{3.21}$$

Let $(x^1,\dots,x^n)$ be normal coordinates at p, and extend the eigenbasis by a smooth orthonormal frame with $\nabla e_i(p) = 0$. Define

$$\widehat{V}(x) = e_1(x) + \sum_{j,\,\kappa_i<k} c_{ji}x^je_i(x), \qquad V(x) = \widehat{V}(x)/|\widehat{V}(x)|.$$

Near p the denominator is positive. At p the derivative of $|\widehat{V}|$ vanishes because all prescribed derivatives are perpendicular to e_1. Thus $V(p) = e_1$ and its derivatives are exactly (3.21). The product rule gives

$$\begin{aligned} L_Z(h(V,V)) = (L_Zh)(V,V) + 4\sum_j(\nabla_jh)(\nabla_jV, V) \\ + 2\sum_j h(\nabla_jV,\nabla_jV) + 2h(L_ZV, V). \end{aligned} \tag{3.22}$$

Moreover, $0 = L_Z|V|^2 = 2g(L_Z V, V) + 2\sum_j |\nabla_j V|^2$. At p, $h(L_Z V, V) = k\, g(L_Z V, V)$ because $V = e_1$ is a k-eigenvector. The scalar minimum therefore gives

$$\begin{aligned} L_Z\phi &\geq L_Z(h(V,V)) \\ &= (L_Z h)_{11} + 4\sum_{j,i} c_{ji}h_{1i;j} - 2\sum_{j,i}(k-\kappa_i)c_{ji}^2. \end{aligned} \tag{3.23}$$

The gradient of the scalar difference is zero, so the drift does not alter the minimum comparison. For $d_i = k - \kappa_i > 0$ and $T_{ji} = h_{1i;j}$,

$$4c_{ji}T_{ji} - 2d_i c_{ji}^2 = \frac{2T_{ji}^2}{d_i} - 2d_i\left(c_{ji} - \frac{T_{ji}}{d_i}\right)^2. \tag{3.24}$$

Choose $c_{ji} = T_{ji}/d_i$ for every $\kappa_i < k$. This is permitted by the prescribed-jet construction and uses no division in the k-eigenspace. Discarding all nonnegative summands except those with $j = 1$ gives the second inequality in (3.16). Codazzi and (3.20) give $h_{1i;1} = h_{11;i} = \phi_i$, proving (3.16). □

4. The spherical inequality

In this section $M^n \hookrightarrow \mathbb{S}^{n+1}$ is closed, connected, and of constant trace mean curvature H. Suppose it is not totally umbilic. Use the untruncated spherical envelopes in (3.2); their sum has a positive minimum by Lemma 3.3.

Tracing (3.10), and differentiating ν once more, gives

$$\mathscr{D}X = -nX - H\nu, \qquad \mathscr{D}\nu = -S\nu - HX, \qquad \mathscr{D}h = (n-S)h + H(h^2 - g). \tag{4.1}$$

For the second equation, the tangential term is $\nabla H = 0$, the normal term is $-S\nu$, and the radial term is $-HX$.

Proposition 4.1. *If ϕ is a C^2 upper test for k at p, then*

$$\mathscr{D}\phi + (S-n)k - H(k^2-1) \geq 2\sum_{\kappa_i<k}\frac{\phi_i^2}{k-\kappa_i}. \tag{4.2}$$

A component corresponding to $\kappa_i = k$ is zero.

Proof. If $k(p) = \kappa_{\max}(p)$, apply Lemma 3.5 with $Z = 0$. Equation (4.1) gives $(\mathscr{D}h)_{11} = (n-S)k + H(k^2-1)$ and proves the assertion.

Suppose $k(p) > \kappa_{\max}(p)$. By Lemma 3.2, choose a maximizing $q \neq p$. Write

$$X = X(p), \quad Y = X(q), \quad A = 1 - \langle X, Y\rangle > 0, \quad d = Y - X, \quad b_i = \langle e_i, Y\rangle, \tag{4.3}$$

where $e_1, \dots, e_n$ is an orthonormal eigenbasis of h_p. Define

$$Z(x,y) = \phi(x)(1 - \langle X(x), X(y)\rangle) + \langle \nu(x), X(y)\rangle. \tag{4.4}$$

For x near p it is nonnegative for every y, including $y = x$, and $Z(p,q) = 0$. At (p,q) one has $\partial_{x_i}(1 - \langle X(x), X(y)\rangle) = -b_i$ and $\partial_{x_i}\langle \nu(x), X(y)\rangle = \kappa_i b_i$. Thus $Z_{x_i} = A\phi_i - (k-\kappa_i)b_i = 0$, and the zero value of Z gives

$$\langle \nu(p), Y\rangle = -kA, \qquad A\phi_i = (k-\kappa_i)b_i. \tag{4.5}$$

The y derivatives imply that $\nu(p) - kX$ is perpendicular to T_qM. The vector

$$W = \nu(p) + kd \tag{4.6}$$

is perpendicular to Y and to T_qM, and $|W|^2 = 1 + 2k(-kA) + 2k^2A = 1$.

Reflection in $d^\perp$ sends the orthonormal vectors e_i to

$$\widetilde{e}_i = e_i - \frac{b_i}{A}d. \tag{4.7}$$

Their inner products with Y and W vanish. They form an orthonormal basis of T_qM. Also

$$\langle X, \widetilde{e_i}\rangle = b_i, \qquad \langle e_i, \widetilde{e_i}\rangle = 1 - b_i^2/A. \tag{4.8}$$

To fix the sign of W, consider the spherical cap

$$B = \{z \in \mathbb{S}^{n+1} \ : \ k(1 - \langle X, z\rangle) + \langle \nu(p), z\rangle < 0\}. \tag{4.9}$$

Its defining affine normal is $\nu(p) - kX$, of length $\sqrt{1+k^2}$, and its normalized level is $-k/\sqrt{1+k^2} \in (-1, 1)$. The cap is connected and nonempty. It is disjoint from $X(M)$ by the support inequality. Its side near X is the negative-ν side. Lemma 3.1 places all of it in that component. At Y the spherical gradient is $(\nu(p) - kX) - \langle \nu(p) - kX, Y\rangle Y = \nu(p) - kX + kY = W$. At X it is $\nu(p)$. Lemma 3.4 therefore gives $\nu(q) = W$. The level lies in $(-1, 1)$ for every real k, so this conclusion also holds when $k \le 0$.

Take geodesic coordinates using e_i at p and $\widetilde{e_i}$ at q. Put $C = \langle X, Y\rangle = 1 - A$. Tracing derivatives of (4.4) gives

$$\sum_i Z_{x_i x_i} = A\mathscr{D}\phi - 2\sum_i \phi_i b_i + (nk - H)C + (S - Hk)kA, \tag{4.10}$$

$$\sum_i Z_{y_i y_i} = nk - H, \tag{4.11}$$

$$\sum_i Z_{x_i y_i} = H - nk. \tag{4.12}$$

For (4.10), the three traced contributions are $A\mathscr{D}\phi - 2\sum \phi_i b_i$, $k\langle nX + H\nu, Y\rangle = k(nC - HkA)$, and $\langle -S\nu - HX, Y\rangle = SkA - HC$. For (4.11), use $\mathscr{D}Y = -nY - HW$, $\langle \nu - kX, Y\rangle = -k$, and $\langle \nu - kX, W\rangle = 1$. The ith mixed derivative is

$$-\phi_i b_i + (\kappa_i - k)(1 - b_i^2/A). \tag{4.13}$$

Equation (4.5) cancels its two terms involving b_i^2 and $\phi_i b_i$. Summing the remaining terms gives (4.12).

For each i, put $\alpha_i(t) = (\exp_p(te_i), \exp_q(t\widetilde{e_i}))$. These are product geodesics and $Z \circ \alpha_i$ has a minimum at $t = 0$. Hence $(Z \circ \alpha_i)''(0) = \operatorname{Hess} Z((e_i, \widetilde{e_i}), (e_i, \widetilde{e_i})) \ge 0$. The frames are fixed at (p, q); no derivatives of a reflection field enter this Hessian. Consequently

$$\begin{aligned} 0 &\le \sum_i (\partial_{x_i} + \partial_{y_i})^2 Z \qquad (4.14)\\ &= A\left(\mathscr{D}\phi + (S - n)k - H(k^2 - 1) - 2\sum_i \frac{\phi_i^2}{k - \kappa_i}\right). \end{aligned}$$

Here all denominators are positive and $A > 0$. This proves the off-diagonal case and the proposition. □

Theorem 4.2. *For a closed connected embedded non-totally-umbilic CMC hypersurface of* $\mathbb{S}^{n+1}$, *put*

$$V = S - n - H(k - \ell), \qquad u = (k + \ell)^{-1}. \tag{4.15}$$

Then $(\Delta + V)u = \mathfrak{m}$ *is a nonnegative finite measure and*

$$\int_M (|\nabla f|^2 + Vf^2)\, dv_g = \int_M w^{-2}|\nabla(wf)|^2\, dv_g + \int_M wf^2\, d\mathfrak{m} \tag{4.16}$$

for every smooth f. *In particular* $\Delta + V \ge 0$ *on* $H^1(M)$. *The function* V *and the measure* $\mathfrak{m}$ *are unchanged by reversing the normal.*

Proof. The intersection of the Alexandrov differentiability sets of k and ℓ has full measure. At such a point, use the second assertion of Lemma 2.3 separately for the two upper-test inequalities. Their values and gradients are fixed during the quadratic approximation, so all zero-denominator terms have their prescribed zero numerators throughout the limit. Proposition 4.1 and normal reversal give

$$\mathscr{D}k + (S-n)k - H(k^2-1) \geq 2\sum_i \frac{k_i^2}{k-\kappa_i}, \tag{4.17}$$
$$\mathscr{D}\ell + (S-n)\ell + H(\ell^2-1) \geq 2\sum_i \frac{\ell_i^2}{\ell+\kappa_i}.$$

Their sum has zeroth-order term

$$(S-n)(k+\ell) - H(k^2-\ell^2) = [S-n-H(k-\ell)]w = Vw. \tag{4.18}$$

The weighted square identity (2.15) gives $\mathscr{D}w + Vw \geq 2|\nabla w|^2/w$ almost everywhere. Its right side is bounded on the compact manifold. Lemma 2.2 supplies the nonnegative singular part, so the inequality is distributional. Proposition 2.4 proves (4.16).

Let $f_j \in C^\infty(M)$ tend to $f \in H^1(M)$. Lemma 2.7 gives $wf_j \to wf$ in H^1. Since $V, w^{-1} \in L^\infty(M)$, both quadratic integrals in the inequality obtained from (4.16) converge. Dropping the nonnegative measure term before passage to the limit proves the stated H^1 inequality; no measure integral of an arbitrary H^1 representative is asserted. Normal reversal sends (H,k,ℓ) to $(-H,\ell,k)$; the product $H(k-\ell)$, and hence $u, V, \mathfrak{m}$, is unchanged. □

Proof of Theorem 1.1. A minimal totally umbilic hypersurface has $h = (H/n)g = 0$. Thus the non-totally-geodesic hypothesis permits Theorem 4.2. Set $H = 0$ there. Then $V = S - n$, and (4.16) proves (1.2). With $f = 1$ it gives

$$\int_M (S-n)\, dv_g = \int_M |\nabla \log w|^2\, dv_g + \int_M w\, d\mathfrak{m} \geq 0. \tag{4.19}$$

Suppose equality holds. Both terms on the right vanish. Thus w is constant by Lemma 2.7. Since $w > 0$, equation (2.9) now reads $\mathfrak{m} = (S-n)w^{-1}dv_g$. It implies $S - n \geq 0$. Its integral is zero, so continuity gives $S = n$. Pairing (3.12) with h, with $c_0 = 1$ and $H = 0$, yields

$$0 = \tfrac{1}{2}\mathscr{D}S = |\nabla h|^2 + (n-S)S = |\nabla h|^2. \tag{4.20}$$

Hence h is parallel. Proposition 4.4 identifies its image with (1.4); Lemma 4.5 proves the converse. These two results complete the proof. □

Lemma 4.3. *Let $\mathcal{E}$ be a smooth rank-r subbundle of $M \times \mathbb{R}^N$, with M connected. Suppose that the Euclidean derivative of every local section of $\mathcal{E}$ belongs to $\mathcal{E}$. Then $\mathcal{E}_p$ is a fixed linear subspace of $\mathbb{R}^N$, independent of p.*

Proof. Along a smooth curve choose a smooth frame, and write its columns as an $N \times r$ matrix $E(t)$. By the hypothesis $E'(t) = E(t)C(t)$ for a smooth $r \times r$ matrix $C(t)$. Let $U'(t) = U(t)C(t)$ and $U(0) = \mathrm{Id}$. Uniqueness for the linear system gives $E(t) = E(0)U(t)$. Let $V'(t) = -C(t)V(t)$ with $V(0) = \mathrm{Id}$. Then

$$(U(t)V(t))' = U(t)C(t)V(t) - U(t)C(t)V(t) = 0, \qquad U(t)V(t) = \mathrm{Id}. \tag{4.21}$$

Thus $U(t)$ is invertible and $\operatorname{span} E(t) = \operatorname{span} E(0)$. The conclusion follows along piecewise smooth curves, which join any two points of a connected manifold. □

Proposition 4.4. *Let $M^n \hookrightarrow \mathbb{S}^{n+1}$ be closed, connected, and minimal. If $\nabla h = 0$ and $h \neq 0$, its image is a product* (1.4).

Proof. Let γ be a smooth curve and $P_t : T_{\gamma(0)}M \to T_{\gamma(t)}M$ its parallel transport. Since $\nabla B = 0$,

$$\frac{d}{dt}\left(P_t^{-1}B_{\gamma(t)}P_t\right) = P_t^{-1}(\nabla_{\dot\gamma}B)P_t = 0. \tag{4.22}$$

Hence the eigenvalues and their multiplicities are constant, and parallel transport preserves each eigenspace. The corresponding distributions are smooth: their projections are the polynomials

$\prod_{\mu\neq\lambda}(B-\mu\mathrm{Id})/(\lambda-\mu)$ in B. If v, z belong to different eigendistributions, the vector $R(v,z)z$ lies in the distribution containing z, so $g(R(v,z)z,v)=0$. For orthonormal eigenvectors the Gauss equation gives

$$1+\lambda\gamma=0 \qquad (\lambda\neq\gamma). \tag{4.23}$$

There are at most two distinct eigenvalues, because for a fixed nonzero λ equation (4.23) determines every different eigenvalue. If there were only one eigenvalue λ, then $0=\operatorname{tr} h=n\lambda$, contradicting $h\neq 0$. A zero eigenvalue together with a different eigenvalue would contradict (4.23). Thus there are exactly two, $s>0$ and $-t<0$, with multiplicities $p,q\geq 1$, satisfying

$$st=1, \qquad ps=qt, \qquad p+q=n, \qquad s^2=q/p. \tag{4.24}$$

Define ambient vector-valued maps

$$Y_1=\frac{\nu+tX}{s+t}, \qquad Y_2=\frac{sX-\nu}{s+t}. \tag{4.25}$$

They satisfy

$$X=Y_1+Y_2, \quad dY_1=P_s dX, \quad dY_2=P_{-t}dX, \quad Y_1\perp Y_2, \quad |Y_1|^2=p/n, \quad |Y_2|^2=q/n. \tag{4.26}$$

The ambient subbundles $E_s\oplus\mathbb{R}Y_1$ and $E_{-t}\oplus\mathbb{R}Y_2$ are orthogonal. If Z is a section of E_s, then

$$\bar{D}_v Z=\nabla_v Z-g(v,Z)(s\nu+X)=\nabla_v Z-(1+s^2)g(v,Z)Y_1, \tag{4.27}$$

and $dY_1(v)=P_s v$. Since E_s is parallel, $\nabla_v Z\in E_s$. For the other subbundle, $\bar{D}_v Z=\nabla_v Z-g(v,Z)(-t\nu+X)=\nabla_v Z-(1+t^2)g(v,Z)Y_2$ when $Z\in E_{-t}$, and $dY_2(v)=P_{-t}v$. Lemma 4.3 applies to each subbundle.

Thus Y_1, Y_2 take values in fixed orthogonal subspaces of dimensions $p+1, q+1$. Equation (4.26) places $X(M)$ in the product of their spheres of radii $\sqrt{p/n}, \sqrt{q/n}$. The inclusion is a local diffeomorphism and has compact image. The connected product is therefore the whole image. This proves the asserted global product identification. □

Lemma 4.5. *Let $p,q\geq 1$ be integers with $p+q=n$, and let $r,t>0$ satisfy $r^2+t^2=1$. On $\mathbb{S}^p(r)\times\mathbb{S}^q(t)\subset\mathbb{S}^{n+1}$, write $X=(rx,ty)$ with $x\in\mathbb{S}^p(1)$ and $y\in\mathbb{S}^q(1)$, and choose $\nu=(tx,-ry)$. Then*

$$k=t/r, \qquad \ell=r/t, \qquad S-n-H(k-\ell)=0. \tag{4.28}$$

In the minimal case $r^2=p/n$, $S=n$, and equality holds in (1.3).

Proof. The principal curvatures are $s=t/r$ and $-t_0=-r/t$. For a second point (rx',ty'), put $A_1=1-\langle x,x'\rangle$ and $A_2=1-\langle y,y'\rangle$. Then

$$c=\frac{rt(A_1-A_2)}{r^2A_1+t^2A_2}. \tag{4.29}$$

For $A_1,A_2\geq 0$, not both zero, this is a convex combination of t/r and $-r/t$, with weights r^2A_1 and t^2A_2. Each endpoint is attained by fixing one factor and varying the other. Hence the first two identities in (4.28) hold. Since $st_0=1$, $S=ps^2+qt_0^2$ and $H=ps-qt_0$ give $H(s-t_0)=ps^2+qt_0^2-(p+q)=S-n$. Minimality is $ps=qt_0$, which gives the stated radii and $S=n$. □

5. The quadratic Gaussian inequality

Let $Q>0$ be self-adjoint, and retain (1.5)–(1.8). Let $X: M^n\hookrightarrow\mathbb{R}^{n+1}$ be complete, connected, properly embedded, and without boundary. Choose the normal from Lemma 3.1. Assume $H=\langle QX,\nu\rangle$.

Let X_t be a smooth variation equal to X outside a fixed compact set, with $X_0=X$ and $\dot{X}_0=f\nu$. Put $g_t=X_t^*g_{\mathrm{Eucl}}$ and $J_t=dv_{g_t}/dv_g$. The relative weighted area

$$\mathcal{A}(t)=\int_M\left(e^{-F\circ X_t}J_t-e^{-F\circ X}\right)dv_g$$

is finite because its integrand has fixed compact support. Since $\dot{g}_0 = 2fh$, one has $\dot{J}_0 = fH$, whereas $\partial_t|_0 (F \circ X_t) = f\langle QX, \nu\rangle$. Differentiation on that compact set gives

$$\mathcal{A}'(0) = \int_M (H - \langle QX, \nu\rangle) f \, d\mu_F. \tag{5.1}$$

Thus stationarity is (1.7), without any prior weighted-volume assumption. This convention agrees with [12, Proposition 2 and Definition 3].

The scalar and normal identities in the next lemma specialize [13, Theorem 3 and Proposition 2] to a flat ambient space with $\overline{\mathrm{Ric}}_F = Q$ and $\bar{\nabla} Q = 0$. We also record the tensor identity and derive all three formulas with $H = \operatorname{tr} h$.

Lemma 5.1. *Put $Z = (QX)^T$ and let $Q_T = P_T Q|_{TM}$. Identifying endomorphisms with their symmetric bilinear forms,*

$$\mathcal{L}_F h = hQ_T + Q_T h - Uh, \tag{5.2}$$

$$\tfrac{1}{2}\mathcal{L}_F S = |\nabla h|^2 + 2\operatorname{tr}(h^2 Q_T) - US, \tag{5.3}$$

$$\mathcal{L}_F \nu = Q\nu - U\nu. \tag{5.4}$$

The last equation uses componentwise ambient derivatives and the scalar drift Laplacian.

Proof. At a geodesic orthonormal frame, write $Z_i = \langle QX, e_i\rangle$ and $Q_{ij} = \langle Qe_i, e_j\rangle$. Differentiating stationarity and then Z_i gives

$$H_i = \langle Qe_i, \nu\rangle + h_{ij}Z_j, \qquad \nabla_j Z_i = Q_{ij} - Hh_{ij}. \tag{5.5}$$

The first term of H_i has derivative

$$\nabla_j \langle Qe_i, \nu\rangle = -h_{ij}q_\nu + Q_{ik}h_{jk}. \tag{5.6}$$

The derivative of $h_{ik}Z_k$ is

$$\nabla_j(h_{ik}Z_k) = h_{ik;j}Z_k + h_{ik}(Q_{kj} - Hh_{kj}) = h_{ij;k}Z_k + h_{ik}Q_{jk} - Hh_{ik}h_{jk}.$$

The last equality uses $h_{ik;j} = h_{ij;k}$ from Codazzi. Consequently

$$\nabla^2 H = \nabla_Z h - q_\nu h + Q_T h + hQ_T - Hh^2. \tag{5.7}$$

Substitute this in (3.12) with $c_0 = 0$; the Hh^2 terms cancel, proving (5.2). Pairing with h gives (5.3).

The Euclidean traces of (3.10) give

$$\mathscr{D}X = -H\nu, \qquad \mathscr{D}\nu = \nabla H - S\nu. \tag{5.8}$$

Equation (5.5) says $\nabla H = P_T Q\nu + hZ$. Subtract $\bar{D}_Z \nu = hZ$ and use $P_T Q\nu = Q\nu - q_\nu \nu$ to obtain (5.4). ☐

Use the nonnegative Euclidean envelopes in (3.2). Assume M is not a hyperplane, so $w > 0$ by Lemma 3.3.

Proposition 5.2. *For every C^2 upper test ϕ of k at p,*

$$\mathcal{L}_F \phi + (U - 2a)k \geq 2 \sum_{\kappa_i < k} \frac{\phi_i^2}{k - \kappa_i}. \tag{5.9}$$

If $k = \kappa_i$, the corresponding component ϕ_i vanishes. If $k(p) = 0$, every component of $\nabla\phi(p)$ vanishes.

Proof. If $k(p) = 0$, the upper test is nonnegative near p and vanishes there. Hence $\nabla\phi(p) = 0$ and $\mathscr{D}\phi(p) \geq 0$. The drift and zeroth-order terms vanish at p, proving the assertion. This also treats a maximizing sequence y_j with $|X(y_j)| \to \infty$, for which (3.8) gives $c(p, y_j) \to 0$.

If $k(p) = \kappa_{\max}(p) > 0$, Lemma 3.5 applies with $Z = (QX)^T$. For a unit vector e_1 in the maximal-eigenvalue eigenspace, (5.2) gives

$$(\mathcal{L}_F h)_{11} = (2\langle Qe_1, e_1\rangle - U)k. \tag{5.10}$$

Thus (3.16) gives the stronger inequality

$$\mathcal{L}_F\phi + (U - 2a)k \geq 2k(\langle Qe_1, e_1\rangle - a) + 2\sum_{\kappa_i<k}\frac{\phi_i^2}{k-\kappa_i}. \tag{5.11}$$

The first term on the right is nonnegative because $k > 0$ and $Q \geq a\mathrm{Id}$.

It remains to treat $k(p) > 0$ and $k(p) > \kappa_{\max}(p)$. Choose a finite off-diagonal maximizer q by Lemma 3.2. Choose an orthonormal basis $e_1, \dots, e_n$ of T_pM such that $B_pe_i = \kappa_ie_i$, and put $\phi_i = d\phi_p(e_i)$ and $H_i = dH_p(e_i)$. Write

$$X = X(p), \quad Y = X(q), \quad d = Y - X, \quad A = |d|^2/2, \quad b_i = \langle e_i, d\rangle, \quad H_x = H(p),\ H_y = H(q). \tag{5.12}$$

Define

$$\mathcal{Z}(x, y) = \tfrac{\phi(x)}{2}|X(y) - X(x)|^2 + \langle\nu(x), X(y) - X(x)\rangle. \tag{5.13}$$

It is nonnegative for x near p and all y and has a minimum zero at (p, q). Here $A_{x_i} = -b_i$ and $\partial_{x_i}\langle\nu(x), X(y) - X(x)\rangle = \kappa_ib_i$, since $\nu \perp e_i$. The equations $\mathcal{Z} = 0$ and $\mathcal{Z}_{x_i} = 0$ therefore give

$$\langle\nu(p), d\rangle = -kA, \qquad A\phi_i = (k - \kappa_i)b_i. \tag{5.14}$$

The y derivatives make $W = \nu(p) + kd$ perpendicular to T_qM, and $|W| = 1$. The orthogonal reflection in $d^\perp$ sends $\nu(p)$ to W and sends e_i to

$$\widetilde{e_i} = e_i - \frac{b_i}{A}d. \tag{5.15}$$

Thus these vectors form an orthonormal basis of T_qM. Since $|d|^2 = 2A$,

$$\langle d, \widetilde{e_i}\rangle = -b_i, \qquad \langle e_i, \widetilde{e_i}\rangle = 1 - b_i^2/A. \tag{5.16}$$

The set $\{z : k|z-X|^2/2+\langle\nu(p), z-X\rangle < 0\}$ is the open ball of radius $1/k$ and centre $X - \nu(p)/k$. It is disjoint from M, and near X it is on the negative-normal side. By Lemma 3.1 it lies in that component of the complement. At Y the gradient of its defining function is W, of length one. Lemma 3.4 gives

$$\nu(q) = \nu(p) + kd. \tag{5.17}$$

This equality fixes the sign in the terms containing H_y.

Use geodesic coordinates with the two frames above. Tracing derivatives of (5.13) gives

$$\sum_i \mathcal{Z}_{x_ix_i} = A(\mathscr{D}\phi + Sk - H_xk^2) + nk - H_x - 2\sum_i\phi_ib_i + \sum_i H_ib_i, \tag{5.18}$$

$$\sum_i \mathcal{Z}_{y_iy_i} = nk - H_y, \tag{5.19}$$

$$\sum_i \mathcal{Z}_{x_iy_i} = H_x - nk. \tag{5.20}$$

Indeed, $A_{x_i} = -b_i$ and $\sum A_{x_ix_i} = n - H_xkA$. If $N(x, y) = \langle\nu(x), X(y) - X(x)\rangle$, then

$$\sum_i N_{x_ix_i} = \langle\nabla H_x - S\nu(p), d\rangle - 2H_x + H_x = \sum_i H_ib_i + SkA - H_x. \tag{5.21}$$

These identities give (5.18). In the y trace use $\mathscr{D}Y = -H_y\nu(q)$ and (5.17); the normal contributions sum to $-H_y$. For the mixed derivative, differentiate once in each variable and use $\bar{D}_{e_i}\nu = \kappa_ie_i$ and (5.16):

$$\begin{aligned}\mathcal{Z}_{x_iy_i} &= \phi_i\langle d, \widetilde{e_i}\rangle + \langle(B_p - k\mathrm{Id})e_i, \widetilde{e_i}\rangle\\ &= -\phi_ib_i + (\kappa_i - k)(1 - b_i^2/A).\end{aligned} \tag{5.22}$$

The terms depending on b_i cancel by (5.14), leaving $\sum_i(\kappa_i - k) = H_x - nk$.

For the product geodesics $\alpha_i(t) = (\exp_p(te_i), \exp_q(t\widetilde{e_i}))$, $(\mathcal{Z} \circ \alpha_i)''(0) \geq 0$. Their second derivatives are the product-Hessian values in the fixed directions $(e_i, \widetilde{e_i})$, so the combined trace is

$$T := \sum_i (\partial_{x_i} + \partial_{y_i})^2 \mathcal{Z} \geq 0, \tag{5.23}$$

$$T = A(\mathscr{D}\phi + Sk - H_x k^2) - 2\sum_i \phi_i b_i + \sum_i H_i b_i + H_x - H_y.$$

Put $Z_i = \langle QX, e_i \rangle$. The stationary equation and (5.17) give

$$H_y - H_x = \langle Qd, \nu(p)\rangle + k\langle QX, d\rangle + k\langle Qd, d\rangle, \tag{5.24}$$

$$\sum_i H_i b_i = \langle Qd, \nu(p)\rangle + kAq_\nu + \sum_i \kappa_i Z_i b_i. \tag{5.25}$$

For the second equation use (5.5) and the decomposition

$$d = \sum_i b_i e_i - kA\nu(p), \qquad \langle QX, d\rangle = \sum_i Z_i b_i - kAH_x. \tag{5.26}$$

Subtracting (5.24) from (5.25), and using the contact derivatives, gives

$$\sum_i H_i b_i + H_x - H_y = -A\sum_i Z_i \phi_i + k^2 AH_x + kAq_\nu - k\langle Qd, d\rangle. \tag{5.27}$$

Substitute in (5.23). The $H_x k^2$ terms cancel. With $e = d/|d|$,

$$\frac{T}{A} = \mathcal{L}_F \phi + (U - 2\langle Qe, e\rangle)k - 2\sum_i \frac{\phi_i^2}{k - \kappa_i}. \tag{5.28}$$

Rearranging (5.28) gives

$$\mathcal{L}_F \phi + (U - 2a)k = \frac{T}{A} + 2k(\langle Qe, e\rangle - a) + 2\sum_i \frac{\phi_i^2}{k - \kappa_i}. \tag{5.29}$$

Here $T \geq 0$, $A > 0$, $k > 0$ and $\langle Qe, e\rangle \geq a$. Dropping the first two nonnegative terms proves (5.9) in this case. Together with the zero and diagonal cases this proves the proposition. □

Proposition 5.3. *For every hypersurface in this section whose image is not a hyperplane, put $V = U - 2a$ and $u = w^{-1}$. Then*

$$\mathcal{L}_F w + (U - 2a)w \geq 2w^{-1}|\nabla w|^2 \tag{5.30}$$

distributionally. The distribution $\mathfrak{m} = (\Delta_F + U - 2a)u$ is a nonnegative locally finite measure, and

$$\int_M \left(|\nabla f|^2 + (U - 2a)f^2\right) d\mu_F = \int_M w^{-2}|\nabla(wf)|^2\, d\mu_F + \int_M wf^2\, d\mathfrak{m} \tag{5.31}$$

for every $f \in C_c^\infty(M)$.

Proof. Normal reversal preserves S, q_ν, U and $\mathcal{L}_F$, and interchanges k, ℓ. At common Alexandrov points, apply Proposition 5.2 in both orientations. The zero-denominator numerators vanish by that proposition. Lemmas 2.3 and 2.5 give (5.30), including the nonnegative singular Hessian measure. Apply Proposition 2.4 with $V = U - 2a$. Its measure (2.9) is $\mathfrak{m} = (\Delta_F + U - 2a)w^{-1}$, and its identity (2.10) is exactly (5.31). For every compact $K \Subset M$, continuity gives $\min_K w > 0$; these local lower bounds suffice in Proposition 2.4. □

Quadratic-form assertion of Theorem 1.2. Equation (5.31) proves (1.9). To specify its realization, consider

$$q(f) = \int_M (|\nabla f|^2 + Uf^2)\, d\mu_F, \qquad f \in C_c^\infty(M). \tag{5.32}$$

Here $U \geq a > 0$ is smooth. The form is closable. If $f_j \to 0$ in L^2 and both ∇f_j and $\sqrt{U} f_j$ are Cauchy in L^2, their limits vanish: the first by testing weak derivatives against compactly supported smooth vector fields, and the second because U is bounded on each compact set. Indeed, compact exhaustion shows that both limits vanish almost everywhere on M. Thus $\nabla f_j \to 0$ and $\sqrt{U} f_j \to 0$ in L^2, and consequently $q(f_j) \to 0$. This proves closability. The closure of (5.32) is a closed nonnegative form. The representation theorem for closed forms defines its Friedrichs self-adjoint operator; see [24, Chapter VI]. Passing to the form closure in (1.9) retains the lower bound $2a$, which bounds the spectrum below by $2a$. Finally $q_\nu \leq b$ gives

$$\int (|\nabla f|^2 + S f^2)\, d\mu_F \geq (2a - b) \int f^2\, d\mu_F. \tag{5.33}$$

□

6. Weighted exhaustion and rigidity

Lemma 6.1. *Every complete properly embedded hypersurface satisfying* (1.7) *has finite weighted volume. Moreover* $F \circ X \in L^1(M, d\mu_F)$.

Proof. Write F for $F \circ X$. It is proper because $F(X) \geq a|X|^2/2$ and X is proper. Differentiation gives

$$\mathscr{D} F = \operatorname{tr} Q - q_\nu - H^2, \qquad |\nabla F|^2 = |QX|^2 - H^2, \tag{6.1}$$

$$\mathcal{L}_F F = \operatorname{tr} Q - q_\nu - |QX|^2 \leq C_0 - 2aF, \qquad C_0 = \operatorname{tr} Q - a. \tag{6.2}$$

For the first equation trace $\operatorname{Hess} F(v, z) = \langle Qv, z \rangle - H h(v, z)$. For the second use $\nabla F = (QX)^T$ and $\langle QX, \nu \rangle = H$. The inequality uses $q_\nu \geq a$ and $Q^2 \geq aQ$.

If M is compact, the assertions follow by smoothness. Otherwise F is unbounded and has arbitrarily large regular values by Sard's theorem. For a regular value t, the domain $\Omega_t = \{F < t\}$ has compact closure and smooth boundary. Its outward conormal is $\nabla F / |\nabla F|$. Weighted divergence gives

$$0 \leq \int_{\partial \Omega_t} |\nabla F| e^{-F}\, d\sigma = \int_{\Omega_t} \mathcal{L}_F F\, d\mu_F \leq \int_{\Omega_t} (C_0 - 2aF)\, d\mu_F. \tag{6.3}$$

Choose $R > (C_0 + 1)/(2a)$. For $t > R$ regular, move the negative part outside $\{F \leq R\}$ to the left:

$$\mu_F(\{R < F < t\}) \leq \int_{\{R<F<t\}} (2aF - C_0)\, d\mu_F \leq \int_{\{F \leq R\}} (C_0 - 2aF)_+\, d\mu_F < \infty. \tag{6.4}$$

The last bound is independent of t. An increasing sequence of regular values tends to infinity; monotone convergence proves $\mu_F(M) < \infty$. Increase R so that $R \geq C_0/a$. Then $2aF - C_0 \geq aF$ on $\{F > R\}$, and the same estimate proves the first moment assertion. □

Choose a smooth nonincreasing $\chi : [0, \infty) \to [0, 1]$, equal to one on $[0, 1]$ and zero on $[2, \infty)$. Set

$$\chi_R = \chi(|X|/R), \qquad |\nabla \chi_R| \leq \|\chi'\|_\infty / R. \tag{6.5}$$

These functions are smooth also at $X = 0$, where they are locally constant. They have compact support by properness, increase to one, and are eventually one on every compact set. Lemma 6.1 gives

$$\int_M |\nabla \chi_R|^2\, d\mu_F \leq R^{-2} \|\chi'\|_\infty^2 \mu_F(M) \longrightarrow 0. \tag{6.6}$$

Average inequality in Theorem 1.2. If $\int S\, d\mu_F = \infty$, the assertion holds because $a \leq q_\nu \leq b$ and $\mu_F(M) < \infty$. Otherwise $V = U - 2a \in L^1(d\mu_F)$. Apply Lemma 2.6 to Proposition 5.3 and (6.5). It gives the exact identity

$$\int_M (U - 2a)\, d\mu_F = \int_M |\nabla \log w|^2\, d\mu_F + \int_M w\, d\mathfrak{m} \geq 0. \tag{6.7}$$

This proves (1.10). The measure is initially only locally finite; the finiteness of $\int w\, d\mathfrak{m}$ is a conclusion of this identity. □

Lemma 6.2. *If equality holds in* (1.10)*, then* $\nabla h = 0$ *and* $U = 2a$*.*

Proof. Equality in (6.7) makes w constant and positive. Hence $\mathfrak{m} = (U - 2a)w^{-1} d\mu_F \geq 0$. Since its weighted average vanishes, $U = 2a$ everywhere. Thus

$$0 \leq S = 2a - q_\nu \leq a. \tag{6.8}$$

Equation (5.3) becomes

$$\tfrac{1}{2}\mathcal{L}_F S = \mathcal{B}, \qquad \mathcal{B} = |\nabla h|^2 + 2\,\mathrm{tr}\big(h^2(Q_T - a\mathrm{Id})\big) \geq |\nabla h|^2. \tag{6.9}$$

Choose an orthonormal eigenbasis of Q_T at the point, with eigenvalues $q_i \geq a$. Then

$$\mathrm{tr}(h^2(Q_T - a\mathrm{Id})) = \sum_i (q_i - a)|Be_i|^2 \geq 0. \tag{6.10}$$

No commutation of Q_T and B is used.

Multiply (6.9) by χ_R^2 and integrate. All integrations have compact support. Since $|\nabla S| \leq 2\sqrt{S}|\nabla h| \leq 2\sqrt{a}|\nabla h|$,

$$\begin{aligned}\int \chi_R^2 \mathcal{B}\, d\mu_F &= -\int \chi_R \langle \nabla \chi_R, \nabla S\rangle\, d\mu_F \\ &\leq \tfrac{1}{2}\int \chi_R^2 |\nabla h|^2\, d\mu_F + 2a \int |\nabla \chi_R|^2\, d\mu_F.\end{aligned} \tag{6.11}$$

Because $\mathcal{B} \geq |\nabla h|^2$, this implies $\int \chi_R^2 \mathcal{B}\, d\mu_F \leq 4a \int |\nabla \chi_R|^2\, d\mu_F \to 0$. On each compact set $\chi_R = 1$ eventually. Thus the smooth nonnegative function $\mathcal{B}$ vanishes, and $\nabla h = 0$. No global integrability of ∇h was assumed. □

Proposition 6.3. *A connected complete properly embedded Euclidean hypersurface with parallel nonzero shape operator is an entire product* $\mathbb{S}^p(r, c) \times E_0$*, where* E_0 *is a fixed linear subspace of dimension* $n - p$ *and the sphere lies in* $E_0^\perp$*. If it also satisfies* (1.7) *and* $U = 2a$*, it is precisely* (1.11)*.*

Proof. Equation (4.22), whose proof uses only $\nabla B = 0$, shows that the eigenvalues are constant and the eigendistributions are smooth and parallel. If v, z are unit eigenvectors in distinct eigendistributions, the parallel distribution containing z is preserved by $R(v, z)$. Thus $R(v, z)z$ belongs to that distribution and $g(R(v, z)z, v) = 0$. The Euclidean Gauss equation gives $\lambda\gamma = 0$ for distinct eigenvalues. Thus exactly one nonzero eigenvalue λ occurs, with multiplicity $p \geq 1$, and every other eigenvalue is zero.

Let $\mathcal{E}_0 = \ker h$. For a section Y of $\mathcal{E}_0$ and any tangent vector v, the Euclidean Gauss formula gives

$$\bar{D}_v Y = \nabla_v Y - h(v, Y)\nu = \nabla_v Y \in \mathcal{E}_0. \tag{6.12}$$

Lemma 4.3 makes its ambient span a fixed subspace E_0. Put $E_1 = E_0^\perp$ and let P_1 be its orthogonal projection. Then $\nu \in E_1$ and $d\nu = \lambda P_1 dX$, so

$$P_1 X - \nu/\lambda = c \in E_1 \tag{6.13}$$

is constant. The image lies in the product of E_0 with the sphere of radius $r = 1/|\lambda|$ and centre c in E_1. The differential has full rank in that product, hence the image is open there. Properness makes the image closed in the ambient space, hence closed in the product. Since $p \geq 1$, the sphere factor and the Euclidean factor are connected. The nonempty image is both open and closed in their product, so it equals that product.

Choose the outward spherical normal ω. Since the whole product is present, stationarity holds for every $\omega \in E_1$ with $|\omega| = 1$ and every $z \in E_0$:

$$\frac{p}{r} = \langle Q(c + r\omega + z), \omega\rangle. \tag{6.14}$$

Variation of z gives $\langle Qz, \omega\rangle = 0$ for all such vectors, so Q preserves E_0 and, by self-adjointness, E_1. The equations at ω and $-\omega$, after the z terms vanish, are $p/r = \langle Qc, \omega\rangle + r\langle Q\omega, \omega\rangle$ and $p/r =$

$-\langle Qc, \omega\rangle + r\langle Q\omega, \omega\rangle$. Subtracting gives $\langle Qc, \omega\rangle = 0$ for every unit $\omega \in E_1$. Since $Qc \in E_1$, one has $Qc = 0$; positive definiteness of Q gives $c = 0$. Adding the two equations gives $\langle Q\omega, \omega\rangle = p/r^2$ on the unit sphere of E_1. Polarization gives $Q|_{E_1} = (p/r^2)\mathrm{Id}$. On the cylinder $S = q_\nu = p/r^2$. Thus $U = 2a$ implies $p/r^2 = a$, and $E_1 \subset \ker(Q - a\mathrm{Id})$.

Conversely, the product in (1.11) has outward principal curvatures $1/r$ with multiplicity p and zero with multiplicity $n - p$. The subspaces $E, E^\perp$ are Q-invariant. Hence $H = p/r = ar = \langle QX, \nu\rangle$, and $S = q_\nu = a$. Its weighted volume is finite by Lemma 6.1: the displayed cylinder is a complete proper embedded hypersurface satisfying (1.7). It attains equality. Compactness forces $p = n$ and $E = \mathbb{R}^{n+1}$, giving $Q = a\mathrm{Id}$. Under equality in Theorem 1.2, Lemma 6.2 gives $\nabla h = 0$. If $h = 0$, then (6.12) applies to the full tangent bundle; Lemma 4.3 and the open-and-closed image argument give an affine hyperplane, which is excluded by that theorem. Thus $h \neq 0$, and the classification just proved applies. □

If $h = 0$, equation (6.12) and Lemma 4.3 give a fixed tangent subspace E_0 of dimension n. The orthogonal projection $P_0^\perp X$ has zero differential and is constant on M. Thus the image lies in an affine translate of E_0. Its differential has rank n, so the image is open in that translate; properness makes it closed. The affine hyperplane is connected, so it is the entire image. If $X(M) = x_0 + \nu^\perp$ satisfies stationarity, then

$$0 = \langle Q(x_0 + y), \nu\rangle \qquad (y \in \nu^\perp). \tag{6.15}$$

Thus $Q\nu = \beta\nu$ for some $\beta > 0$, and $\langle x_0, \nu\rangle = 0$. The hyperplane passes through the origin. On such a hyperplane the Euclidean envelopes satisfy $k = \ell = 0$. The reciprocal construction is therefore restricted to the nonhyperplane case.

7. Spectral and graphical consequences

7.1. **Graphs in eigendirections.**

Proof of Theorem 1.3. Rescale v so that $|v| = 1$ and put $\theta = \langle \nu, v\rangle > 0$. Equation (5.4) and $Qv = \beta v$ give

$$\mathcal{L}_F\theta = (\beta - U)\theta. \tag{7.1}$$

Suppose that M is not a hyperplane. Lemma 3.3 gives $w > 0$. For compactly supported smooth f, integration by parts against f^2/θ in (7.1) gives

$$\int_M \left(|\nabla f|^2 + (\beta - U)f^2\right) d\mu_F = \int_M \theta^2 |\nabla(f/\theta)|^2 \, d\mu_F. \tag{7.2}$$

There is no requirement that θ have a positive global minimum: on the support of f it has one. Add (7.2) to (5.31):

$$\begin{aligned} 2\int_M |\nabla f|^2 \, d\mu_F + (\beta - 2a)\int_M f^2 \, d\mu_F = &\int_M w^{-2}|\nabla(wf)|^2 \, d\mu_F \\ &+ \int_M \theta^2|\nabla(f/\theta)|^2 \, d\mu_F + \int_M wf^2 \, d\mathfrak{m}. \end{aligned} \tag{7.3}$$

All terms on the right are nonnegative.

Use $f = \chi_R$ from (6.6). If $\beta < 2a$, the left side tends to $(\beta - 2a)\mu_F(M) < 0$, a contradiction. If $\beta = 2a$, the left side tends to zero. For a fixed compact set K, choose R large enough that $\chi_R = 1$ on a neighbourhood of K. Equation (7.3) implies

$$\int_K \left(|\nabla \log w|^2 + |\nabla \log \theta|^2\right) d\mu_F \le 2\int_M |\nabla \chi_R|^2 \, d\mu_F \longrightarrow 0.$$

Exhaustion by compact sets gives

$$\nabla \log w = 0, \qquad \nabla \log \theta = 0 \quad \text{almost everywhere.} \tag{7.4}$$

Local Lipschitz regularity and connectedness imply that w and θ are constant. Equation (7.1) gives $U = \beta = 2a$. The weighted measure is finite, so equality holds in Theorem 1.2. Its classification places every normal in an a-eigenspace of Q. Self-adjointness makes that space orthogonal to the $2a$-eigenspace containing v. Thus $\theta = 0$, again a contradiction. The remaining case is a hyperplane, which passes through the origin by (6.15). □

Corollary 7.1. *Let v be a unit eigenvector of Q with eigenvalue $\beta \le 2a$. Every smooth entire graph*

$$X(y) = y + f(y)v, \qquad y \in v^{\perp},$$

satisfying (1.7) *is a hyperplane through the origin.*

Proof. Let g_f be the induced graph metric. In the coordinates $y \in v^{\perp}$, $g_f = \mathrm{Id} + df \otimes df$. Thus

$$|y - z| \le d_{g_f}(X(y), X(z)). \tag{7.5}$$

An intrinsically Cauchy sequence $X(y_j)$ therefore has $y_j \to y$ in $v^{\perp}$. Choose a convex Euclidean ball about y on which $|\nabla f| \le L$. For j large, the lift of the segment from y_j to y has length at most $\sqrt{1 + L^2}|y_j - y|$. Hence $d_{g_f}(X(y_j), X(y)) \to 0$, proving completeness. The inverse image of a compact ambient set is closed and has bounded y-coordinates, hence is compact. The graph is proper. Its normal can be chosen as

$$\nu = \frac{v - \nabla f}{\sqrt{1 + |\nabla f|^2}}, \qquad \langle \nu, v\rangle = \frac{1}{\sqrt{1 + |\nabla f|^2}} > 0.$$

Theorem 1.3 applies. No bound on f, its slope, or its volume growth has been imposed. □

7.2. **Compact spectral comparison.** We record the min–max consequence with its precise comparison factor. On a closed connected weighted manifold, let Δ_F denote the operator associated with $\int |\nabla f|^2\, d\mu_F$ on $H^1(M)$. Its eigenvalues, with multiplicity, are written

$$0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \cdots.$$

The kernel consists of constants: zero energy implies $df = 0$, and connectedness implies constancy. The compact inclusion $H^1(M) \hookrightarrow L^2(M, d\mu_F)$ gives discrete spectrum and the form min–max principle; the weight is smooth and has positive upper and lower bounds on M. We use this form of the spectral theorem as in [10, Chapter I] and [24, Chapter VI].

Proposition 7.2. *Let (M, g) be a closed connected smooth Riemannian manifold, let $F \in C^\infty(M)$, and put $d\mu_F = e^{-F} dv_g$ and $\Delta_F = -\operatorname{div}\nabla + \langle \nabla F, \nabla\rangle$. Let $c \in \mathbb{R}$, let $P \in L^\infty(M; \mathbb{R})$, and suppose that a positive Lipschitz function w satisfies*

$$\int_M \left(|\nabla f|^2 + (P - c)f^2\right) d\mu_F \ge \int_M w^{-2}|\nabla(wf)|^2\, d\mu_F \qquad (f \in H^1(M)). \tag{7.6}$$

Let ζ_j be the eigenvalues of $\Delta_F + P$. Then

$$\zeta_j \ge c + \vartheta\lambda_j \quad (j \ge 0), \qquad \vartheta = \left(\frac{\min_M w}{\max_M w}\right)^2. \tag{7.7}$$

Proof. Put $u = w^{-1}$ and write $f = uv$. The functions u, u^{-1} are bounded and Lipschitz; multiplication by u is therefore an invertible bounded map on $H^1(M)$. Equation (7.6) gives

$$\frac{\int (|\nabla f|^2 + (P - c)f^2)\, d\mu_F}{\int f^2\, d\mu_F} \ge \frac{\int u^2|\nabla v|^2\, d\mu_F}{\int u^2 v^2\, d\mu_F} \ge \frac{u_{\min}^2}{u_{\max}^2}\frac{\int |\nabla v|^2\, d\mu_F}{\int v^2\, d\mu_F}. \tag{7.8}$$

For any $(j + 1)$-dimensional subspace $E \subset H^1(M)$, its inverse image under multiplication by u has the same dimension. Take the supremum over nonzero $f \in E$ in (7.8), then the infimum over all such E. The form min–max principle gives (7.7). □

For Theorem 1.1, take $F = 0$, $P = S$ and $c = n$. For a closed hypersurface in Theorem 1.2, take $P = U$ and $c = 2a$. The conclusions are, respectively,

$$\zeta_j(\Delta + S) \geq n + \vartheta\lambda_j(\Delta), \qquad \zeta_j(\Delta_F + U) \geq 2a + \vartheta\lambda_j(\Delta_F). \tag{7.9}$$

The factor ϑ depends on the embedding; no dimension-only lower bound for it is asserted. On a minimal Clifford hypersurface, Lemma 4.5 gives constant w and $S = n$, so every comparison is an equality. On the compact equality model for Theorem 1.2, the truncated Euclidean envelopes are $k = 1/r$, $\ell = 0$. Thus w is constant and $U = 2a$, again giving equality for every j. On a noncompact hypersurface, Theorem 1.2 asserts a lower bound for a Friedrichs operator, not discreteness of its spectrum.

7.3. **The embedded low-index classification.** For a closed two-sided minimal hypersurface of $\mathbb{S}^{n+1}$, define its Morse index to be the number of negative eigenvalues, with multiplicity, of

$$J = \Delta - S - n. \tag{7.10}$$

We use the following external result only in this subsection: Perdomo's Theorem 3.1 in [29, p. 564], written there for hypersurface dimension $N - 1$ and index $N + 2$, states after setting $N = n + 1$ that

$$\operatorname{ind}(M) = n + 3 \quad \Longrightarrow \quad \int_M S\, dv_g \leq n \operatorname{Vol}(M) \tag{7.11}$$

for a compact oriented immersed minimal hypersurface. The theorem permits immersions; no converse from immersion to embedding is used here.

Corollary 7.3. *Let $n \geq 2$. A closed connected embedded minimal hypersurface of $\mathbb{S}^{n+1}$ has Morse index $n + 3$ if and only if it is a minimal Clifford hypersurface.*

Proof. The totally geodesic equator has index one. Indeed, $J = \Delta - n$ there, and the round eigenvalues are $j(j + n - 1)$, $j \geq 0$; only the constant eigenspace lies below n. Thus an index-$n + 3$ hypersurface is not totally geodesic. Lemma 3.1 makes it two-sided and oriented. Apply (7.11) and Theorem 1.1. The two inequalities give average equality, and Theorem 1.1, including its equality assertion, gives the Clifford conclusion.

Conversely, write the Clifford product with factor dimensions $p, q \geq 1$, $p + q = n$, and squared radii p/n, q/n. Its scalar eigenvalues are

$$\frac{n}{p} j(j + p - 1) + \frac{n}{q} l(l + q - 1), \qquad j, l \geq 0. \tag{7.12}$$

This follows by tensoring the spherical harmonic eigenbases on the two factors. For a round d-sphere of radius r, restriction of degree-j harmonic homogeneous polynomials gives $j(j + d - 1)/r^2$; these eigenspaces form a complete orthogonal basis. These spectral facts are the sphere and product calculations in [10]. Here $S = n$, so $J = \Delta - 2n$. In (7.12), the only eigenvalues below $2n$ arise from $(j, l) = (0, 0), (1, 0), (0, 1)$. Their multiplicities are $1, p + 1, q + 1$. The first mixed value is $2n$, and the degree-two value of either factor is $2n(d + 1)/d > 2n$. The index is consequently $1 + (p + 1) + (q + 1) = n + 3$. □

7.4. **Self-shrinkers and Gaussian curvature.** Set $Q = \frac{1}{2}\mathrm{Id}$, $d\gamma = e^{-|X|^2/4}\, dv_g$, and $\Delta_\gamma = -\mathscr{D} + \frac{1}{2}\langle X^\top, \nabla\rangle$. Stationarity becomes $H = \frac{1}{2}\langle X, \nu\rangle$. Theorem 1.2 gives

Corollary 7.4. *Let $M^n \subset \mathbb{R}^{n+1}$, $n \geq 2$, be a connected complete properly embedded self-shrinker without boundary, other than a hyperplane. Then*

$$\int_M \left(|\nabla f|^2 + S f^2\right) d\gamma \geq \frac{1}{2} \int_M f^2\, d\gamma \qquad (f \in C_c^\infty(M)), \tag{7.13}$$

and

$$\int_M S\, d\gamma \geq \frac{1}{2}\gamma(M). \tag{7.14}$$

Equality in (7.14) *holds precisely for $\mathbb{S}^p(\sqrt{2p}) \times \mathbb{R}^{n-p}$, $1 \leq p \leq n$.*

Proof. Here $a = q_\nu = 1/2$ and $U = S + 1/2$. Substitution in (1.9) and (1.10) gives the two inequalities. The entire ambient space is the least eigenspace of Q, so (1.11) gives exactly the stated cylinders. Compactness is equivalent to $p = n$ among these models. □

For comparison with the polynomial-volume-growth formulation, Cheng–Zhou's Theorem 1.3 in [15] says that a complete immersed self-shrinker is proper if and only if it has polynomial volume growth; it also identifies these conditions with finite Gaussian area. Thus the properness hypothesis of Corollary 7.4 is satisfied by every embedded self-shrinker in the complete formulation of [26, Conjecture 1.7]. Their closed formulation, Conjecture 1.6, is a special case. The following calculation specifies the entropy normalization; it is the Gaussian maximum calculation of [16, Section 7.2].

Lemma 7.5. *Let $X : M^n \to \mathbb{R}^{n+1}$ be a smooth complete self-shrinking immersion without boundary and of polynomial volume growth, with $n \geq 2$ and $H = \langle X, \nu\rangle/2$ in every local unit normal. Define*

$$F_{y,t}(M) = (4\pi t)^{-n/2} \int_M e^{-|X-y|^2/(4t)}\, dv_g, \qquad \Lambda(M) = \sup_{y\in\mathbb{R}^{n+1},\, t>0} F_{y,t}(M). \tag{7.15}$$

Then

$$\Lambda(M) = F_{0,1}(M) = (4\pi)^{-n/2}\gamma(M). \tag{7.16}$$

Proof. Write $X : M \to \mathbb{R}^{n+1}$ for the immersion. By Cheng–Zhou [15, Theorem 1.3], completeness and polynomial volume growth imply properness. Thus $\chi_R = \chi(|X|/R)$ has compact support in M; these are the cutoffs used below. Fix $y \in \mathbb{R}^{n+1}$ and $c > -1$. For $0 \leq s \leq 1$, put $t_s = 1 + cs^2$, $z_s = X - sy$, and

$$[\psi]_s = (4\pi t_s)^{-n/2} \int_M \psi e^{-|z_s|^2/(4t_s)}\, dv_g.$$

Set $t_- = \min\{1, 1+c\} > 0$ and $t_+ = \max\{1, 1+c\}$. The inequality $|X - sy|^2 \geq \frac{1}{2}|X|^2 - |y|^2$ gives

$$(4\pi t_s)^{-n/2} e^{-|X-sy|^2/(4t_s)} \leq (4\pi t_-)^{-n/2} e^{|y|^2/(4t_-)} e^{-|X|^2/(8t_+)}.$$

Polynomial volume growth is measured on the domain of the immersion: $\mathrm{Vol}_g(X^{-1}(B_R(0))) \leq C(1+R)^d$ for some finite C, d. For every integer $m \geq 0$ and $\varepsilon > 0$,

$$\int_M (1+|X|)^m e^{-\varepsilon|X|^2}\, dv_g \leq C \sum_{j=0}^{\infty} (j+2)^{m+d} e^{-\varepsilon j^2} < \infty. \tag{7.17}$$

The differentiated integrands are bounded by a fixed polynomial in $|X|$ times such a Gaussian, uniformly for $s \in [0,1]$, so dominated convergence permits differentiation. For the divergence identities, apply compactly supported cutoffs χ_R. The boundary terms are bounded by $CR^{-1}\int_{\{|X|\geq R\}}(1+|X|)^m e^{-\varepsilon|X|^2}\, dv_g$, which tends to zero by (7.17).

Since $H = \langle X, \nu\rangle/2$, one has

$$\operatorname{div} y^\top = -\tfrac{1}{2}\langle X^\perp, y^\perp\rangle, \qquad \operatorname{div} z_s^\top = n - \tfrac{1}{2}\langle X^\perp, z_s^\perp\rangle.$$

Weighted integration of these two identities gives

$$\begin{aligned} \left[\frac{\langle z_s^\top, y^\top\rangle}{t_s}\right]_s &= -[\langle z_s^\perp, y^\perp\rangle + s|y^\perp|^2]_s, \\ \left[\frac{|z_s^\top|^2}{t_s}\right]_s &= [2n - |z_s^\perp|^2 - s\langle z_s^\perp, y^\perp\rangle]_s. \end{aligned} \tag{7.18}$$

For $G(s) = F_{sy,t_s}(M)$, differentiation gives

$$G'(s) = \left[cs\left(\frac{|z_s|^2}{2t_s^2} - \frac{n}{t_s}\right) + \frac{\langle z_s, y\rangle}{2t_s}\right]_s.$$

Put $A_s = [|z_s^\perp|^2]_s$, $B_s = [\langle z_s^\perp, y^\perp\rangle]_s$ and $C_s = [|y^\perp|^2]_s$. Substitution of (7.18) gives

$$G'(s) = \frac{cs(1-t_s)}{2t_s^2}A_s + \left(-\frac{cs^2}{2t_s} + \frac{1-t_s}{2t_s}\right)B_s - \frac{s}{2}C_s.$$

Since $1 - t_s = -cs^2$, this equals $-\frac{s}{2t_s^2}(c^2s^2A_s + 2cst_sB_s + t_s^2C_s)$. Expanding the square gives

$$G'(s) = -\frac{s}{2t_s^2}[\,|csz_s^\perp + t_sy^\perp|^2\,]_s \le 0. \tag{7.19}$$

Thus $F_{y,1+c}(M) \le F_{0,1}(M)$. Every positive t has the form $1 + c$ with $c > -1$. Taking the supremum, and then taking $y = 0, t = 1$, proves (7.16). □

Consequently (7.14) is exactly

$$\int_M |A|^2 e^{-|X|^2/4}\, dv_g \ge \frac{1}{2}(4\pi)^{n/2}\Lambda(M), \tag{7.20}$$

with the sphere and cylinder equality conclusions in the closed and complete cases, respectively. This is the assertion of [26, Conjectures 1.6–1.7] in dimensions $n \ge 2$; their complete case contains the closed case. Equation (7.13) is an inequality for $\Delta_\gamma + S$, not for Δ_γ alone. No entropy-minimization or first-positive-scalar-eigenvalue assertion is deduced from it.

7.5. **The operator on a punctured cone.** The constants in the spherical and Gaussian statements can also be compared without an assertion at a singular vertex. Let $L^n \subset \mathbb{S}^{n+1}$ be minimal, and on its punctured cone put $g_C = dr^2 + r^2 g_L$. Then $S_C = r^{-2}S_L$ and

$$\Delta_{\gamma,C} = -\partial_r^2 - \frac{n}{r}\partial_r + r^{-2}\Delta_L + \frac{r}{2}\partial_r.$$

For a smooth function v on L, substitution of rv gives

$$(\Delta_{\gamma,C} + S_C - \tfrac{1}{2})(rv) = r^{-1}(\Delta_L + S_L - n)v \qquad (r > 0). \tag{7.21}$$

The cone may be incomplete at $r = 0$ and singular there. Equation (7.21) is an identity of differential expressions on the punctured cone; it is not an application of Theorem 1.2 across its vertex.

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