SHARP DEFECT BOUNDS FOR REGULAR-MATROID JACOBIANS

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Abstract. Let M be a finite connected regular matroid of rank $r \geq 2$ on m elements, and let e be the exponent of its Jacobian. The defect $\delta = er - 2m$ is nonnegative. It vanishes precisely when M is the cycle matroid of a complete graph with constant positive edge multiplicity. If $\delta > 0$, then

$$r \leq \delta, \qquad e + r - 1 \leq 3\delta, \qquad m \leq \delta^2.$$

Equality in either the rank bound or the size bound holds precisely when M is the cycle matroid of the cone over the graph obtained by doubling each edge of K_r. Thus bounded positive defect gives finitely many isomorphism classes. For a totally unimodular representation D of M of full row rank, the lattice $\mathbb{Z}^r$ with Gram matrix $e(DD^{\mathsf{T}})^{-1}$ is integral. In this lattice, the columns of D of squared length two generate an orthogonal sum of root lattices of type A; their matroid restriction is a direct sum of complete-graph matroids with constant multiplicities. At positive defect these columns span a proper subspace, whereas the remaining columns span $\mathbb{R}^r$. Connected regular matroids of Jacobian exponent at most five are classified. The least exponent of a connected regular matroid which is neither graphic nor cographic is six.

1. Introduction

Let M be a finite regular matroid. Its Jacobian $\mathrm{Jac}(M)$ is a finite abelian group whose order is the number of bases of M. If $D \in \mathrm{Mat}_{r,m}(\mathbb{Z})$ is a totally unimodular representation of full row rank, then

$$\mathrm{Jac}(M) \simeq \mathbb{Z}^r/(DD^{\mathsf{T}})\mathbb{Z}^r, \qquad r = \operatorname{rank} M, \quad m = |E(M)| \tag{1.1}$$

[3, Proposition 2.1.2]. Write $e(M)$ for its exponent, with $e(M) = 1$ when the group is trivial.

Bounded exponent does not imply bounded size, even for connected graphic matroids. For every $t \geq 2$, the simple biconnected graph H_t of [14, Theorem 1.2] has $4t - 3$ edges and

$$\mathrm{Jac}(H_t) \simeq (\mathbb{Z}/8\mathbb{Z})^{t-1}.$$

Its cycle matroid is connected. This family contradicts [11, Conjecture 1.2]. For a connected regular matroid of rank at least two, define its defect by

$$\delta(M) = e(M) \operatorname{rank} M - 2|E(M)|. \tag{1.2}$$

All matroids are finite. A matroid is connected when its ground set is nonempty and admits no partition into two nonempty sets such that every circuit lies in one part. Graphs may have loops and parallel edges. A loopless graph is biconnected when it has at least three vertices and remains connected after deletion of any vertex. Write $M(G)$ for the cycle matroid of G, and $U_{r,m}$ for the uniform matroid of rank r on m elements. For $q \geq 2$ and $k \geq 1$, the graph kK_q has k edges between each pair of distinct vertices. For $r \geq 2$, let Γ_r have vertex set $\{0, 1, \dots, r\}$, one edge $0i$ for each $i \geq 1$, and two edges ij for each $1 \leq i < j \leq r$.

Theorem 1.1. *Let M be connected and regular, of rank $r \geq 2$ on m elements. Put $e = e(M)$ and $\delta = er - 2m$. Exactly one of the following cases occurs.*

(i) *$\delta = 0$ and $M \simeq M(kK_{r+1})$ for some $k \geq 1$. In this case*

$$e = k(r + 1), \qquad \mathrm{Jac}(M) \simeq \mathbb{Z}/k\mathbb{Z} \oplus (\mathbb{Z}/(k(r + 1))\mathbb{Z})^{r-1}. \tag{1.3}$$

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(ii) $\delta > 0$ *and*

$$r \le \delta, \qquad e + r - 1 \le 3\delta, \qquad m \le \delta^2. \tag{1.4}$$

The conditions

$$\delta = r; \qquad m = \delta^2; \qquad M \simeq M(\Gamma_r) \tag{1.5}$$

are equivalent. For these matroids,

$$e = 2r + 1, \qquad m = r^2, \qquad \mathrm{Jac}(M) \simeq (\mathbb{Z}/(2r+1)\mathbb{Z})^{r-1}. \tag{1.6}$$

Corollary 1.2. *For every integer $d \ge 1$, there are only finitely many isomorphism classes of connected regular matroids of rank at least two with $0 < \delta(M) \le d$. They have rank at most d, at most d^2 elements, and exponent at most $3d - 1$.*

Proof. Theorem 1.1 gives $r \le d$, $m \le d^2$, and $e \le 3\delta - r + 1 \le 3d - 1$. There are finitely many matroids on a ground set of bounded cardinality. □

The restriction of squared length two. Assume that M is connected of rank $r \ge 2$, and put $e = e(M)$ and $\delta = \delta(M)$. Choose a representation $D = [I_r\ N]$ as in (1.1), and put

$$C = DD^{\mathsf{T}}, \quad P = D^{\mathsf{T}}C^{-1}D, \quad \Lambda = \sqrt{e}\, C^{-1/2}\mathbb{Z}^r, \quad v_i = \sqrt{e}\, C^{-1/2}D_i, \tag{1.7}$$

where D_i is the ith column of D and $C^{-1/2}$ is the positive definite inverse square root. By Proposition 2.3, e is the least positive integer for which eC^{-1} is integral. Thus Λ is integral and

$$\langle v_i, v_j \rangle = eP_{ij}, \qquad \sum_i v_i v_i^{\mathsf{T}} = eI_r.$$

For every basis B of M, the vectors $(v_i)_{i\in B}$ form a $\mathbb{Z}$-basis of Λ. Its minimum nonzero squared length is at least two (Theorem 3.1). Hence

$$\delta = \sum_{i \in E(M)} (eP_{ii} - 2) \ge 0. \tag{1.8}$$

The diagonal entry P_{ii} is the proportion of bases containing i [12, Theorem 5.1].

Set $\mathcal{R} = \{i : eP_{ii} = 2\}$ and $\mathcal{T} = E(M) \setminus \mathcal{R}$. By Theorem 4.1, each nonempty component of $M|\mathcal{R}$ is isomorphic to $M(k_j K_{q_j})$. Its column lattice is isometric to A_{q_j-1}, with multiplicity k_j on each root line. These lattices are mutually orthogonal; write W_j for their real spans and $W = \bigoplus_j W_j$. Assume $\mathcal{T} \ne \emptyset$. Then

$$\mathrm{rank}(M|\mathcal{R}) < r = \mathrm{rank}(M|\mathcal{T}) \tag{1.9}$$

by Lemmas 4.3 and 4.4. Put $h = |\mathcal{T}|$. The rank equality gives $r \le h$, and (1.8) gives $h \le \delta$, since $eP_{ii} - 2$ is a positive integer for $i \in \mathcal{T}$. Relative to $\mathbb{R}^r = W \oplus W^{\perp}$, (4.8) gives

$$S_{\mathcal{T}} := \sum_{i\in\mathcal{T}} v_i v_i^{\mathsf{T}} = \bigoplus_j (e - k_j q_j) I_{W_j} \ \oplus\ eI_{W^{\perp}}.$$

By (1.9), $S_{\mathcal{T}}$ is positive definite and $c := \dim W^{\perp} \ge 1$. Its eigenvalues are positive integers, with e occurring at least c times. Therefore

$$c(e-1) + r \le \mathrm{tr}\, S_{\mathcal{T}} = 2h + \delta \le 3\delta.$$

Together with $2m = er - \delta$ and $r \le \delta$, this gives $cm \le \delta^2$ (Proposition 5.1). In particular, $e + r - 1 \le 3\delta$ and $m \le \delta^2$.

Related results. Write $\mathcal{F} = \ker D \cap \mathbb{Z}^m$ and $\mathcal{B} = D^{\mathsf{T}}\mathbb{Z}^r$ for the integral cycle and cut lattices. Their discriminant groups are isomorphic for graphs [2, Proposition 3] and for regular matroids [3, Section 2.1]. Two regular matroids have isometric integral flow lattices if and only if the matroids obtained by contracting all coloops are isomorphic [19, Theorem 1]. The signed-permutation statement in Lemma 3.3 follows from [19, proof of Proposition 14, Corollary 15 and Lemma 16]. For a unimodular system with a vector of multiplicity $s \geq 2$, the Gale-dual lattice contains a root sublattice of type A_{s-1} [1, Proposition 6.9]. The lattice Λ in (1.7) is isometric to the metric dual of $\mathcal{B}$ multiplied by $\sqrt{e}$; its Gram matrix is eC^{-1}.

For a simple regular matroid, Heller's bound is $m \leq \binom{r+1}{2}$ [9, Theorem 4.2]. Hence, for a connected simple regular matroid of rank at least two with positive defect,

$$m \leq \frac{\delta(\delta+1)}{2}.$$

Theorem 1.1 allows parallel elements and counts them in m; its equality matroids $M(\Gamma_r)$ have parallel elements.

Lheem–Li–Quines–Zhang classify the connected regular matroids of exponent two [11, Section 3.2]. Theorem 6.1 gives the complete list of connected regular matroids of exponent at most five. The matroid R_{10} has exponent six [10, Section 3]; six is the least exponent of a connected regular matroid that is neither graphic nor cographic (Proposition 6.3 and Corollary 6.4). By Theorem 7.1, the sublattices generated by signed circuits and signed cocircuits with at most $e(M)$ elements in their supports contain $e(M)\mathcal{F}$ and $e(M)\mathcal{B}$, respectively. For a finite connected graph cellularly embedded in a closed orientable surface of genus g, deletion of at most $2ge(M(G))$ vertices yields a planar graph (Corollary 7.3).

Let $\mathcal{E}_4$ be $(\mathbb{Z}/4\mathbb{Z})^2$ with $\mathbb{Q}/\mathbb{Z}$-valued pairing represented by $\frac{1}{4}\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. For $k \geq 1$, Theorem A.4 states that $\mathcal{E}_4^{\perp k}$ is a paired graph Jacobian if and only if k is even. The realization for $k = 2$ is in [15, Remark C.7]; it contradicts [7, Conjecture 4.3] at exponent four.

2. Representations, lattices and the Jacobian

A matrix is *totally unimodular* if every square minor belongs to $\{0, 1, -1\}$. A regular matroid is the column matroid of a totally unimodular matrix. We take representations of full row rank, with columns indexed by the ground set E. A circuit is a nonempty minimal dependent set of columns. A loop is a zero column; a coloop belongs to no circuit. A singleton of either kind is connected.

Write I_s, J_s for the identity and all-one matrices of size s, and $\mathbf{e}_i$ for a coordinate vector. All inner products are Euclidean. The empty square matrix has determinant one and trivial cokernel; the trivial group has exponent one, and $\operatorname{lcm} \emptyset = 1$.

Lemma 2.1 ([19, Lemma 3 and Section 2.2]). *Let D_0 be a totally unimodular matrix with r rows and rank r. For every column basis B, the matrix $D_{0,B}^{-1}D_0$ is totally unimodular. Placing the basis columns first gives*

$$D = [I_r \; N]. \tag{2.1}$$

Two full-row-rank totally unimodular representations of isomorphic regular matroids differ by an integral unimodular row transformation and a signed column permutation [3, Lemma 2.1.1]. Take D as in (2.1) and put

$$C = DD^{\mathsf{T}}, \qquad P = D^{\mathsf{T}}C^{-1}D, \qquad Q = I_m - P. \tag{2.2}$$

The matrix C is positive definite, since $u^{\mathsf{T}}Cu = \|D^{\mathsf{T}}u\|^2 > 0$ for $u \neq 0$. The cut and cycle lattices are

$$\mathcal{B} = D^{\mathsf{T}}\mathbb{Z}^r, \qquad \mathcal{F} = \ker(D : \mathbb{Z}^m \longrightarrow \mathbb{Z}^r) = \{(-Nz, z) : z \in \mathbb{Z}^{m-r}\}. \tag{2.3}$$

The identity block gives $\mathcal{B} = (\operatorname{im}_{\mathbb{R}} D^{\mathsf{T}}) \cap \mathbb{Z}^m$ and $\mathcal{F} = (\ker_{\mathbb{R}} D) \cap \mathbb{Z}^m$. The Jacobian is

$$\operatorname{Jac}(M) = \mathbb{Z}^m/(\mathcal{B} + \mathcal{F}). \tag{2.4}$$

Under $D' = HDS$, with $H \in \mathrm{GL}_r(\mathbb{Z})$ and S a signed permutation, one has

$$\mathcal{B}' = S^{\mathsf{T}}\mathcal{B}, \qquad \mathcal{F}' = S^{\mathsf{T}}\mathcal{F}, \qquad P' = S^{\mathsf{T}}PS.$$

Thus S^{T} induces an isomorphism between the Jacobians. If $S\mathbf{e}_j = \pm\mathbf{e}_{\sigma(j)}$, then $P'_{jj} = P_{\sigma(j),\sigma(j)}$.

Lemma 2.2 ([18, Theorems 2.1(4) and 2.3]). *If K is a nonsingular integral square matrix of size s, then $|\mathbb{Z}^s/K\mathbb{Z}^s| = |\det K|$. If its Smith diagonal is $d_1, \dots, d_s$, with $d_i > 0$ and $d_i \mid d_{i+1}$, then*

$$\mathbb{Z}^s/K\mathbb{Z}^s \simeq \bigoplus_{i=1}^{s} \mathbb{Z}/d_i\mathbb{Z}.$$

Proposition 2.3. *For (2.1)–(2.2), one has*

$$\begin{gathered} \mathrm{Jac}(M) \simeq \mathbb{Z}^r/C\mathbb{Z}^r, \qquad |\mathrm{Jac}(M)| = \det C = \tau(M), \\ e(M) = \min\{a \in \mathbb{Z}_{>0} \,:\, aC^{-1} \in \mathrm{Mat}_r(\mathbb{Z})\} = \min\{a \in \mathbb{Z}_{>0} \,:\, aP \in \mathrm{Mat}_m(\mathbb{Z})\}, \end{gathered} \tag{2.5}$$

where $\tau(M)$ is the number of bases. The matrices P, Q are the orthogonal projections onto $\mathbb{R}\mathcal{B}, \mathbb{R}\mathcal{F}$, respectively. Duality interchanges these lattices and preserves the Jacobian. A direct sum of matroids gives a direct sum of Jacobians.

Proof. By [3, Section 2.1 and Proposition 2.1.2], the map $[x] \mapsto [Dx]$ is an isomorphism $\mathrm{Jac}(M) \longrightarrow \mathbb{Z}^r/C\mathbb{Z}^r$, and $|\mathrm{Jac}(M)| = \tau(M)$. Lemma 2.2 gives $|\mathrm{Jac}(M)| = \det C$. For every integer $a > 0$,

$$a(\mathbb{Z}^r/C\mathbb{Z}^r) = 0 \quad \Longleftrightarrow \quad a\mathbb{Z}^r \subseteq C\mathbb{Z}^r \quad \Longleftrightarrow \quad aC^{-1} \in \mathrm{Mat}_r(\mathbb{Z}).$$

Moreover, aC^{-1} integral implies aP integral, and the identity-column principal block of aP is aC^{-1}. The identities

$$P^{\mathsf{T}} = P, \qquad P^2 = P, \qquad PD^{\mathsf{T}} = D^{\mathsf{T}}$$

give $\mathrm{im}_{\mathbb{R}}\, P = \mathbb{R}\mathcal{B}$ and $\mathrm{im}_{\mathbb{R}}\, Q = \ker_{\mathbb{R}} D = \mathbb{R}\mathcal{F}$. The normalized dual representation is $D^* = [-N^{\mathsf{T}}\ I_{m-r}]$ [19, Section 2.3]. Its cut lattice is $\mathcal{F}$ and its cycle lattice is $\mathcal{B}$, by (2.3); the quotient (2.4) is unchanged. For $D = D_1 \oplus D_2$,

$$DD^{\mathsf{T}} = D_1D_1^{\mathsf{T}} \oplus D_2D_2^{\mathsf{T}}, \qquad \mathrm{coker}(DD^{\mathsf{T}}) \simeq \mathrm{coker}(D_1D_1^{\mathsf{T}}) \oplus \mathrm{coker}(D_2D_2^{\mathsf{T}}).$$

□

Lemma 2.4. *For every $i \in E$,*

$$P_{ii} = \frac{|\{B \,:\, B \text{ is a basis and } i \in B\}|}{\tau(M)}. \tag{2.6}$$

For a partition $E = E_1 \sqcup E_2$,

$$P_{E_1,E_2} = 0 \quad \Longleftrightarrow \quad \text{every circuit is contained in } E_1 \text{ or } E_2. \tag{2.7}$$

The connected components of M are therefore those of the graph on E with edges ij when $i \neq j$ and $P_{ij} \neq 0$. The dual has the same component partition. If M is connected and $m > 1$, then $0 < P_{ii} < 1$ for every i.

Proof. Let $\mathbb{P}_H$ be the determinantal probability measure associated with $H = \mathrm{im}_{\mathbb{R}}\, D^{\mathsf{T}}$. By [12, (2.1)], every basis B has probability

$$\mathbb{P}_H(\{B\}) = \frac{\det(D_B)^2}{\det C} = \frac{1}{\tau(M)}.$$

Since P is the orthogonal projection onto H, [12, Theorem 5.1] gives $\mathbb{P}_H(\{B \,:\, i \in B\}) = \langle P\mathbf{e}_i, \mathbf{e}_i\rangle = P_{ii}$.

Since $\ker_{\mathbb{R}} P = \ker_{\mathbb{R}} D$, the column matroids of P and D coincide. By [16, Proposition 4.2.1], every circuit lies in one part if and only if $r(E_1) + r(E_2) = r$. If $P_{E_1,E_2} = 0$, then $P = P_{E_1,E_1} \oplus P_{E_2,E_2}$, so this rank equality holds. Conversely, suppose $r(E_1) + r(E_2) = r$, and choose a basis B_j of each restriction to E_j. The set $B_1 \sqcup B_2$ spans M and has r elements, hence is a basis. Normalization at this basis makes

D block diagonal with column sets E_1, E_2, giving $P_{E_1,E_2} = 0$. Equation (2.7) identifies the connected components of M with those of the graph defined by $P_{ij} \neq 0$. Since the off-diagonal supports of P and the dual cut projection $Q = I - P$ coincide, the dual has the same components.

A loop or coloop is a singleton component. A connected matroid with $m > 1$ has neither, so every element belongs to a basis and is absent from another basis. Equation (2.6) gives the strict inequalities. □

For a regular matroid M, put

$$A = eP, \qquad a_i = A_{ii}, \qquad e = e(M). \tag{2.8}$$

Then

$$A \in \mathrm{Mat}_m(\mathbb{Z}), \quad A^2 = eA, \quad 0 \le a_i \le e, \quad \sum_{j \neq i} A_{ij}^2 = a_i(e - a_i). \tag{2.9}$$

If M is connected and $m > 1$, then $1 \le a_i \le e - 1$.

A *signed circuit* is a primitive integral cycle vector with circuit support, with either orientation. Its nonzero coordinates are $1, -1$; the same holds for signed cocircuits, which are the signed circuits of D^* [19, Lemmas 6–8]. A vector y is *conformal* to x if $\operatorname{supp} y \subseteq \operatorname{supp} x$ and $x_i y_i \ge 0$ for every i.

Lemma 2.5 (Tutte; [19, Proposition 11]). *Every integral cycle vector of a regular matroid is a sum of signed circuits conformal to it. Every integral cut vector is a sum of signed cocircuits conformal to it.*

Lemma 2.6. *Every signed circuit in a conformal decomposition of $eQ\mathbf{e}_i$ contains i with coefficient one and has at most $a_i + 1$ elements. Every signed cocircuit in a conformal decomposition of $eP\mathbf{e}_i$ contains i with coefficient one and has at most $e - a_i + 1$ elements.*

Proof. Take $U \in \{P, Q\}$ and put $x = eU\mathbf{e}_i$. Let c be a signed cocircuit if $U = P$, or a signed circuit if $U = Q$, conformal to x. Then $Uc = c$ and

$$0 < \sum_{j \in \operatorname{supp} c} |x_j| = \langle c, x \rangle = e\langle Uc, \mathbf{e}_i \rangle = ec_i.$$

Hence $c_i = 1$ and $x_i > 0$. Since x is integral,

$$e = \sum_{j \in \operatorname{supp} c} |x_j| \ge x_i + |\operatorname{supp} c| - 1, \qquad |\operatorname{supp} c| \le e - x_i + 1.$$

Now $x_i = a_i$ for $U = P$ and $x_i = e - a_i$ for $U = Q$. □

Lemma 2.7. *Let G be a finite connected graph, with loops and parallel edges allowed. Deleting one row from an oriented incidence matrix gives a totally unimodular full-row-rank representation D_G of $M(G)$. Its Gram matrix is the reduced Laplacian; hence $\mathrm{Jac}(M(G)) \simeq \mathrm{Jac}(G)$. If G is loopless and biconnected, then $M(G)$ is connected.*

Proof. Let T_G be the oriented incidence matrix. By [16, Lemmas 5.1.3 and 5.1.4], T_G is totally unimodular and represents $M(G)$. Its rows sum to zero, so deleting one row preserves the row space. Since G is connected, $\operatorname{rank} M(G) = |V(G)| - 1$ by [16, (1.3.6)]; hence D_G has full row rank. If L_o is the Laplacian reduced at the deleted vertex, then $D_G D_G^{\mathsf{T}} = L_o$. Thus [3, Proposition 2.1.2] and [18, Theorem 3.1], applied after deleting the loops, give

$$\mathrm{Jac}(M(G)) \simeq \operatorname{coker} L_o \simeq \mathrm{Jac}(G).$$

Here deleting loops leaves the Laplacian and its divisor relations unchanged. If G is loopless and biconnected, it has at least three vertices and no isolated vertex. Its cycle matroid is connected by [16, Proposition 4.1.7]. □

3. The scaled dual cut lattice

Throughout this section, M is a connected regular matroid on $m > 1$ elements, of rank $r \geq 1$. Choose a totally unimodular representation $D = [I_r\ N]$, and put

$$C = DD^{\mathsf{T}}, \qquad P = D^{\mathsf{T}}C^{-1}D, \qquad e = e(M), \qquad A = eP, \qquad a_i = A_{ii}.$$

Put

$$\Lambda = \sqrt{e}\, C^{-1/2}\mathbb{Z}^r, \qquad v_i = \sqrt{e}\, C^{-1/2}D_i \quad (i \in E). \tag{3.1}$$

Here $C^{-1/2}$ denotes the positive definite inverse square root of C. The lattice Λ is integral, since its displayed basis has Gram matrix $eC^{-1} \in \mathrm{Mat}_r(\mathbb{Z})$ by Proposition 2.3. For $V = (v_i)$,

$$V^{\mathsf{T}}V = A, \qquad VV^{\mathsf{T}} = eI_r, \qquad \|v_i\|^2 = a_i, \qquad \Lambda^* = e^{-1/2}C^{1/2}\mathbb{Z}^r. \tag{3.2}$$

In particular, $v_i \neq 0$ by Lemma 2.4. The identity columns of D give a basis of Λ. For every column basis B, total unimodularity gives $D_B \in \mathrm{GL}_r(\mathbb{Z})$, so $V_B\mathbb{Z}^r = \Lambda$. Moreover, Lemma 2.1 gives

$$V_B^{-1}V = D_B^{-1}D \quad \text{is totally unimodular.} \tag{3.3}$$

3.1. The minimum.

Theorem 3.1. *The lattice Λ contains a vector of squared length one if and only if $M \simeq U_{1,e}$. In particular, if $r \geq 2$, then*

$$\|\lambda\|^2 \geq 2 \qquad (0 \neq \lambda \in \Lambda). \tag{3.4}$$

Proof. Suppose $z \in \mathbb{Z}^r$ satisfies $ez^{\mathsf{T}}C^{-1}z = 1$, and set $x = eD^{\mathsf{T}}C^{-1}z$. Since $eC^{-1}z \in \mathbb{Z}^r$, one has $x \in \mathcal{B}$. Furthermore,

$$\|x\|^2 = e, \qquad Dx = ez.$$

Choose a signed cocircuit b in a conformal decomposition of x (Lemma 2.5), and write $b = D^{\mathsf{T}}u$, with $u \in \mathbb{Z}^r$. Conformality and the coordinates $b_j \in \{0, 1, -1\}$ give

$$0 < \langle b, x\rangle = e\, u^{\mathsf{T}}z \leq \sum_j x_j^2 = e. \tag{3.5}$$

The middle expression belongs to $e\mathbb{Z}_{>0}$, so it equals e. For each j, integrality and conformality give $x_j^2 - b_jx_j \geq 0$. Their sum is zero; hence $x_j^2 = b_jx_j$ for every j. If $b_j = 0$, this gives $x_j = 0$; if $b_j \neq 0$, then $x_j \neq 0$ and $x_j = b_j$. Thus $x = b$ and $|S| = \|b\|^2 = e$, where $S = \mathrm{supp}\, b$.

Let Σ be the diagonal sign matrix with $\Sigma_{jj} = b_j$ on S and $\Sigma_{jj} = 1$ elsewhere. Replacing (D, x, b) by $(D\Sigma, \Sigma x, \Sigma b)$ preserves total unimodularity and the identities $Dx = ez$, $b = D^{\mathsf{T}}u$; it makes $x = b = \mathbf{1}_S$. Retain the same notation. Since S is a cocircuit, $E \smallsetminus S$ is a hyperplane of M [16, Proposition 2.1.6(iii)]. Its columns have rank $r - 1$ and lie in $\ker u^{\mathsf{T}}$; hence

$$\mathrm{span}_{\mathbb{R}}\{D_j\ :\ j \notin S\} = \ker u^{\mathsf{T}}.$$

Choose $i_0 \in S$ and extend D_{i_0} to a column basis using columns outside S. After normalization at this basis, write $\widetilde{D} = D_B^{-1}D$ and $\widetilde{z} = D_B^{-1}z$. Order the basis with i_0 first. Since $u^{\mathsf{T}}D_B = (1, 0, \dots, 0)$, the first row of $\widetilde{D}$ is b^{T}; moreover, $\widetilde{D}_{i_0} = \mathbf{e}_1$, $\widetilde{z} \in \mathbb{Z}^r$, and $\widetilde{D}x = e\widetilde{z}$. The matrix $\widetilde{D}$ is totally unimodular by Lemma 2.1. Since its first row equals one on S, [19, proof of Proposition 14], applied to columns $1, a$ of $\widetilde{D}^{\mathsf{T}}$, gives

$$\widetilde{D}_{aj}\widetilde{D}_{ak} \geq 0 \qquad (a > 1,\ j, k \in S).$$

As $\widetilde{D}_{ai_0} = 0$ and $|S| = e$, one has

$$e|\widetilde{z}_a| = \left|\sum_{j\in S}\widetilde{D}_{aj}\right| = \sum_{j\in S}|\widetilde{D}_{aj}| \leq e - 1.$$

Since $\widetilde{z}_a \in \mathbb{Z}$, it follows that $\widetilde{z}_a = 0$ and $\widetilde{D}_{aj} = 0$ for every $j \in S$. Thus $\widetilde{D}_j = \mathbf{e}_1$ on S, and $r_M(S) + r_M(E \setminus S) = 1 + (r-1) = r$. By [16, Proposition 4.2.1], S is a separator of M. Connectedness and $S \neq \emptyset$ give $S = E$, whence $r = 1$ and $M \simeq U_{1,e}$.

Conversely, if $M \simeq U_{1,e}$, then $r = 1$, $m = e$, and every entry of D is 1 or -1. Hence $C = (e)$ and $\Lambda = \mathbb{Z}$. For $r \geq 2$, integrality gives $\|\lambda\|^2 \in \mathbb{Z}_{>0} \setminus \{1\}$ for every $0 \neq \lambda \in \Lambda$, proving (3.4). □

Corollary 3.2. *If $a_i = 1$ for some i, then $M \simeq U_{1,e}$. If $a_i = e - 1$ for some i, then $M \simeq U_{e-1,e}$. Outside these two cases,*

$$2 \leq a_i \leq e - 2 \qquad (i \in E). \tag{3.6}$$

For $r \geq 2$, the defect satisfies

$$\delta = er - 2m = \sum_{i \in E} (a_i - 2) \geq 0. \tag{3.7}$$

Proof. If $a_i = 1$, apply Theorem 3.1 to v_i. By Proposition 2.3 and Lemma 2.4, M^* is connected, has exponent e, and has cut projection $Q = I_m - P$. Its scaled diagonal is therefore $a_i^* = e - a_i$. If $a_i = e - 1$, then Theorem 3.1 gives $M^* \simeq U_{1,e}$, whence $M \simeq U_{e-1,e}$. Since $a_i \in \mathbb{Z}$ and $0 < a_i < e$, (3.6) follows. For $r \geq 2$, Theorem 3.1 gives $a_i \geq 2$ for every i, and

$$er = \operatorname{tr}(VV^{\mathsf{T}}) = \sum_i \|v_i\|^2 = \sum_i a_i.$$

Subtracting $2m$ gives (3.7). □

3.2. **Isometries and root reflections.** By a lattice isometry we mean an additive bijection preserving the inner product.

Lemma 3.3. *Every lattice isometry of $\mathcal{B}$ is the restriction of a signed permutation of $\mathbb{Z}^m$. For every $T \in \mathrm{O}(r)$ with $T\Lambda = \Lambda$, there are a permutation σ of E and signs $\varepsilon_i \in \{1, -1\}$ such that*

$$Tv_i = \varepsilon_i v_{\sigma(i)} \qquad (i \in E).$$

Proof. Put $U = D^{\mathsf{T}}$. An isometry of $\mathcal{B} = U\mathbb{Z}^r$ has the form $Ux \mapsto UHx$, where $H \in \mathrm{GL}_r(\mathbb{Z})$ and $H^{\mathsf{T}}CH = C$. For every set I of r row indices,

$$\det((UH)_I) = \det(U_I) \det H \in \{0, 1, -1\},$$
$$(UH)^{\mathsf{T}}(UH) = H^{\mathsf{T}}CH = C = U^{\mathsf{T}}U.$$

Thus UH is weakly unimodular and has the same Gram matrix as U. The matrix U is totally unimodular and contains I_r; [19, Lemma 16] therefore implies that UH is totally unimodular. By [19, Corollary 15], U and UH have the same multiset of nonzero row supports. Neither matrix has a zero row, because M has no loops and H is invertible. Choose a permutation matrix Π so that $U' = \Pi UH$ satisfies $|U'| = |U|$ entrywise.

For distinct a, b in the support of a row i, [19, proof of Proposition 14], applied to U and U', gives $C_{ab} \neq 0$ and

$$U'_{ia}U'_{ib} = \operatorname{sgn}(C_{ab}) = U_{ia}U_{ib}, \qquad \frac{U'_{ia}}{U_{ia}} = \frac{U'_{ib}}{U_{ib}}.$$

For a singleton support there is only one ratio. There is therefore a diagonal sign matrix R with $U' = RU$, and $UH = \Pi^{-1}RU$. The signed permutation $S = \Pi^{-1}R$ restricts to the given isometry of $\mathcal{B}$.

Let $J = D^{\mathsf{T}}C^{-1/2}$, so that $J^{\mathsf{T}}J = I_r$ and $\sqrt{e}\,J\Lambda^* = \mathcal{B}$. If $T \in \mathrm{O}(r)$ and $T\Lambda = \Lambda$, then $T\Lambda^* = (T\Lambda)^* = \Lambda^*$. Consequently,

$$\sqrt{e}\,Jw \longmapsto \sqrt{e}\,JTw$$

is an isometry of $\mathcal{B}$. Since Λ^* spans $\mathbb{R}^r$, the first part gives $SJ = JT$ for a signed permutation S. Orthogonality gives

$$S^{\mathsf{T}}J = JT^{\mathsf{T}}, \qquad J^{\mathsf{T}}S = TJ^{\mathsf{T}}, \qquad TV = VS,$$

since $V = \sqrt{e}\,J^{\mathsf{T}}$. Writing $S\mathbf{e}_i = \varepsilon_i \mathbf{e}_{\sigma(i)}$ gives the stated formula. □

If $\alpha \in \Lambda$ has squared length two, then

$$s_\alpha(x) = x - \langle x, \alpha\rangle\alpha \tag{3.8}$$

is the orthogonal reflection in $\alpha^\perp$. Since Λ is integral, $s_\alpha(\Lambda) \subseteq \Lambda$; the identity $s_\alpha^2 = I_r$ gives equality. By Lemma 3.3, s_α permutes the vectors v_i up to sign, with multiplicities.

4. The norm-two restriction

Assume henceforth that M is connected and regular of rank $r \geq 2$. Retain the notation of Section 3, and put

$$\mathcal{R} = \{i : a_i = 2\}, \qquad \mathcal{T} = E \setminus \mathcal{R}, \qquad L_0 = \sum_{i\in\mathcal{R}} \mathbb{Z}v_i, \qquad W = \mathbb{R}L_0. \tag{4.1}$$

If $\mathcal{R} = \emptyset$, set $L_0 = W = 0$.

A positive definite integral lattice generated by vectors of squared length two is an orthogonal sum of lattices of types A_s ($s \geq 1$), D_s ($s \geq 4$), E_6, E_7, E_8 [5, pp. 2–5]. We take

$$A_s = \{x \in \mathbb{Z}^{s+1} : \textstyle\sum_i x_i = 0\}; \qquad \{\alpha \in A_s : \|\alpha\|^2 = 2\} = \{\mathbf{e}_i - \mathbf{e}_j : i \neq j\}. \tag{4.2}$$

For the other irreducible types, the dual coset minima in [5, p. 5], together with evenness of the root lattices, give

L	D_s ($s \geq 4$)	E_6	E_7	E_8
$\min_{0\neq x\in L^*} \|x\|^2$	1	4/3	3/2	2.

(4.3)

Theorem 4.1. *Each component of $M|\mathcal{R}$ is $M(k_jK_{q_j})$, with $q_j \geq 2$ and $k_j \geq 1$. There is an orthogonal decomposition*

$$L_0 \simeq \perp_{j=1}^{s} A_{q_j-1}, \qquad W = \bigoplus_{j=1}^{s} W_j, \tag{4.4}$$

where W_j is the real span of the jth summand. Each root line in that summand occurs with multiplicity k_j among $(v_i)_{i\in\mathcal{R}}$. In particular,

$$\sum_{i\in\mathcal{R}} v_i v_i^{\mathsf{T}} = \bigoplus_{j=1}^{s} k_j q_j I_{W_j} \;\oplus\; 0_{W^\perp}. \tag{4.5}$$

Proof. If $\mathcal{R} = \emptyset$, all assertions hold with $s = 0$. Assume $\mathcal{R} \neq \emptyset$. The lattice L_0 is integral, positive definite, and generated by vectors of squared length two, so the root-lattice classification applies. Each irreducible summand is even; hence each of its nonzero vectors has squared length at least two. Every v_i, $i \in \mathcal{R}$, therefore belongs to one summand, and each summand L satisfies

$$L = \sum_{i\in\mathcal{R},\ v_i\in L} \mathbb{Z}v_i.$$

Fix a summand L and an index $i \in \mathcal{R}$ with $v_i \in L$. The vector $x = eD^{\mathsf{T}}C^{-1}D_i = eP\mathbf{e}_i$ belongs to $\mathcal{B}$. By Lemmas 2.5 and 2.6, $x = b + c$ for signed cocircuits b, c conformal to x: each summand has ith coordinate one, and $x_i = 2$. The summands may coincide. Put $s_i = |\operatorname{supp} b \cap \operatorname{supp} c|$. Since $b_i = c_i = 1$ and $b_j = c_j \in \{1, -1\}$ on the common support,

$$s_i \geq 1, \qquad \langle b, c\rangle = s_i, \qquad \langle b, x\rangle = e\langle Pb, \mathbf{e}_i\rangle = e, \qquad \|b\|^2 = e - s_i.$$

Write $b = D^{\mathsf{T}}u$, with $u \in \mathbb{Z}^r$. Then

$$w = e^{-1/2}C^{1/2}u \in \Lambda^*, \qquad \|w\|^2 = \frac{\|b\|^2}{e} < 1, \qquad \langle w, v_i\rangle = 1. \tag{4.6}$$

Let w_L be its orthogonal projection to $\mathbb{R}L$. For every $\ell \in L \subseteq \Lambda$, one has $\langle w_L, \ell\rangle = \langle w, \ell\rangle \in \mathbb{Z}$. Thus $w_L \in L^*$, while

$$\langle w_L, v_i\rangle = 1, \qquad 0 < \|w_L\|^2 \leq \|w\|^2 < 1.$$

The minima in (4.3) exclude D_s, E_6, E_7, E_8. Thus every summand is of type A.

Identify such a summand with A_{q-1}. Its Weyl group is the symmetric group on the q coordinates [5, pp. 5–6]. Each generating root reflection extends by the identity on $(\mathbb{R}L)^\perp$ to an isometry of Λ, by (3.8). By Lemma 3.3, it permutes the lines $\mathbb{R}v_i$ for $i \in \mathcal{R}$ with $v_i \in L$, with their multiplicities. The symmetric group is transitive on the unordered pairs $\{a, b\}$, so every root line has the same multiplicity k. Since these vectors generate $L \neq 0$, one has $k \geq 1$.

Deleting one coordinate gives a reduced incidence representation of kK_q, so these columns represent $M(kK_q)$ (Lemma 2.7). For $q = 2$, the restriction is a parallel class, including a single coloop when $k = 1$. For $q \geq 3$, kK_q is biconnected: deleting a vertex leaves the connected graph kK_{q-1}. Thus each such restriction is connected by Lemma 2.7. A dependence decomposes into dependencies in the orthogonal summands, so a minimal dependent set meets only one summand. These restrictions are therefore precisely the components of $M|\mathcal{R}$. Finally,

$$\sum_{a<b} (\mathbf{e}_a - \mathbf{e}_b)(\mathbf{e}_a - \mathbf{e}_b)^\mathsf{T} = qI_q - J_q.$$

On the coordinate-sum-zero space this is qI. Multiplicity k gives (4.5). □

Corollary 4.2. *The defect is zero if and only if $M \simeq M(kK_{r+1})$ for some $k \geq 1$. In that case* (1.3) *holds.*

Proof. By (3.7), zero defect is equivalent to $\mathcal{R} = E$. Theorem 4.1 and connectedness then give a single summand, of rank $q - 1 = r$, and $eI_r = kqI_r$.

Conversely, a reduced incidence representation of kK_q, $q \geq 3$, has

$$C = k(qI_{q-1} - J_{q-1}), \qquad C^{-1} = \frac{I_{q-1} + J_{q-1}}{kq}.$$

Since $q \geq 3$, an off-diagonal entry exists and equals $1/(kq)$. Thus aC^{-1} is integral only if $kq \mid a$, and $kqC^{-1} = I + J$ is integral. Proposition 2.3 gives $e = kq$. Each edge column is $\pm\mathbf{e}_j$ or $\pm(\mathbf{e}_i - \mathbf{e}_j)$, and its quadratic value for C^{-1} is $2/(kq)$. Hence every $a_i = 2$.

By [18, Section 3, following Theorem 3.1], there are $U_0, V_0 \in \mathrm{GL}_{q-1}(\mathbb{Z})$ such that

$$U_0(qI_{q-1} - J_{q-1})V_0 = \mathrm{diag}(1, q, \dots, q).$$

Multiplication by k gives $U_0 C V_0 = \mathrm{diag}(k, kq, \dots, kq)$. Thus Proposition 2.3 and Lemma 2.2 give

$$\mathrm{Jac}(M(kK_q)) \simeq \mathbb{Z}/k\mathbb{Z} \oplus (\mathbb{Z}/(kq)\mathbb{Z})^{q-2}.$$

□

Lemma 4.3. *If $\delta > 0$, then the columns indexed by $\mathcal{T}$ span $\mathbb{R}^r$ and*

$$r \leq h \leq \delta, \qquad h = |\mathcal{T}|. \tag{4.7}$$

Proof. Subtracting (4.5) from $VV^\mathsf{T} = eI_r$ gives

$$S_\mathcal{T} := \sum_{i \in \mathcal{T}} v_i v_i^\mathsf{T} = \bigoplus_{j=1}^{s} (e - k_j q_j) I_{W_j} \ \oplus \ eI_{W^\perp}. \tag{4.8}$$

The operator $S_\mathcal{T}$ is positive semidefinite, so $e - k_j q_j \geq 0$ for every j. If $e - k_j q_j = 0$, then

$$0 = \langle S_\mathcal{T} x, x \rangle = \sum_{i \in \mathcal{T}} \langle v_i, x \rangle^2 \qquad (x \in W_j).$$

Let $\mathcal{R}_j = \{i \in \mathcal{R} : v_i \in W_j\}$. The displayed identity makes every column in $\mathcal{T}$ orthogonal to W_j; the other root columns are orthogonal to W_j by (4.4). Consequently

$$P_{i\ell} = e^{-1}\langle v_i, v_\ell \rangle = 0 \qquad (i \in \mathcal{R}_j,\ \ell \notin \mathcal{R}_j).$$

Both $\mathcal{R}_j$ and its complement are nonempty, since $\delta > 0$ implies $\mathcal{T} \neq \emptyset$. This contradicts connectedness by Lemma 2.4. Thus $e - k_j q_j > 0$ for every j, and $S_\mathcal{T}$ is positive definite. The identity

$$\ker S_\mathcal{T} = \{x : \langle v_i, x \rangle = 0 \text{ for all } i \in \mathcal{T}\}$$

then gives $\mathrm{span}_\mathbb{R}\{v_i : i \in \mathcal{T}\} = \mathbb{R}^r$ and $h \geq r$. Finally, $a_i \geq 3$ on $\mathcal{T}$, whence $\delta = \sum_{i \in \mathcal{T}} (a_i - 2) \geq h$. □

Lemma 4.4. *If $\delta > 0$, then* $\dim W < r$.

Proof. Suppose $W = \mathbb{R}^r$. For each j, choose $B_j \subseteq \mathcal{R}$ so that $(v_i)_{i\in B_j}$ is a basis of W_j. Then $B = \bigsqcup_j B_j$ is a column basis, and $U = V_B^{-1}V$ is totally unimodular by (3.3). Let π_j be the orthogonal projection onto W_j, and put

$$\Psi_j = \{\pm\pi_j(v_i) \,:\, i \in E,\ \pi_j(v_i) \neq 0\}.$$

In the basis $(v_i)_{i\in B_j}$, these vectors are represented by the B_j-rows of U, after adjoining negatives and deleting zero or repeated columns. This matrix is totally unimodular and has rank $q_j - 1$. By Lemma 2.1, the coordinates of each vector of Ψ_j in any basis drawn from Ψ_j belong to $\{0, 1, -1\}$. By Theorem 4.1, Ψ_j contains every root of $L_0 \cap W_j \simeq A_{q_j-1}$. Heller's maximality theorem for the oriented edges of a simplex [9, (3.1)] therefore implies that every vector in Ψ_j is a root.

Since $\delta > 0$, choose $\beta = v_i$ with $i \in \mathcal{T}$. Every nonzero component of β in the orthogonal sum $\mathbb{R}^r = \bigoplus_j W_j$ is a root.

Suppose two components are nonzero, and denote them by α, γ. By Theorem 4.1, α, γ are columns up to sign. They are independent and extend, after column sign changes, to a new column basis contained in $\mathcal{R}$. Orthogonality gives $\langle\beta, \alpha\rangle = 2$. By Lemma 3.3, $s_\alpha(\beta) = \beta - 2\alpha$ is also a column up to sign. It is neither β nor $-\beta$, since its α component changes sign and its γ component does not. After orienting the two distinct columns as β and $\beta - 2\alpha$, their coordinates on the basis rows α, γ give the minor

$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix},$$

whose determinant is two. This contradicts (3.3). Thus β has exactly one nonzero component, so $\|\beta\|^2 = 2$, contrary to its choice. Therefore $W \neq \mathbb{R}^r$. □

5. Defect bounds and equality

Throughout this section, M is connected and regular of rank $r \geq 2$, and $\delta = er - 2m$. Retain $\mathcal{R}, \mathcal{T}, W$ from Section 4.

Proposition 5.1. *If $\delta > 0$, then*

$$r \leq h \leq \delta, \qquad e + r - 1 \leq 2h + \delta \leq 3\delta, \qquad m \leq \delta^2, \qquad h = |\mathcal{T}|. \tag{5.1}$$

More precisely, if $c = r - \dim W$, then

$$c \geq 1, \qquad c(e-1) + r \leq 2h + \delta \leq 3\delta, \qquad cm \leq \delta^2. \tag{5.2}$$

Equality in either size bound implies $r = \delta$.

Proof. By Lemmas 4.3 and 4.4, $r \leq h \leq \delta$ and $c \geq 1$. Formula (4.8) gives positive integral eigenvalues for $S_{\mathcal{T}}$: it is eI on the c-dimensional space $W^\perp$, and its remaining $r - c$ eigenvalues are at least one. Hence

$$c(e-1) + r = ce + (r - c) \leq \operatorname{tr} S_{\mathcal{T}} = \sum_{i\in\mathcal{T}} a_i = 2h + \delta \leq 3\delta.$$

Since $(c-1)(e-1) \geq 0$, one also has $e + r - 1 \leq c(e-1) + r \leq 2h + \delta$. Multiplying $ce \leq 3\delta - r + c$ by r and subtracting $c\delta$ yields

$$2cm = cer - c\delta \leq r(3\delta - r + c) - c\delta \leq 2\delta^2.$$

Indeed,

$$2\delta^2 - \big(r(3\delta - r + c) - c\delta\big) = (\delta - r)(2\delta - r + c) \geq 0. \tag{5.3}$$

Here $\delta - r \geq 0$ and $2\delta - r + c \geq \delta + c > 0$. Thus $cm \leq \delta^2$, with equality only if $\delta = r$. Since $c \geq 1$, this implies $m \leq \delta^2$; if $m = \delta^2$, then $\delta^2 \leq c\delta^2 \leq \delta^2$, so $c = 1$ and $\delta = r$. □

Proposition 5.2. *If $\delta = r$, then $M \simeq M(\Gamma_r)$ and $e = 2r + 1$.*

Proof. The inequalities $r \le h \le \delta$ give $h = r$. The r terms in $\delta = \sum_{i\in\mathcal{T}}(a_i - 2)$ are positive integers, so each $a_i = 3$ on $\mathcal{T}$. These r columns span by Lemma 4.3, so they form a column basis $\beta_1, \dots, \beta_r$. They form a basis of Λ by Section 3. In this basis, the coordinate matrix of V is totally unimodular by (3.3). The set $\mathcal{R}$ is nonempty: otherwise $m = r$, and every element would be a coloop, contrary to connectedness.

Let $\alpha = v_\ell$ with $\ell \in \mathcal{R}$. The reflection s_α preserves all columns up to sign by Lemma 3.3. It preserves squared lengths, so it permutes $\beta_1, \dots, \beta_r$ up to sign. In this basis it is an involutive signed permutation, and $\operatorname{rank}(s_\alpha - I) = 1$ by (3.8). After ordering its cycles, its blocks have the forms (1), (-1), and $\left(\begin{smallmatrix} 0 & \varepsilon \\ \varepsilon & 0 \end{smallmatrix}\right)$, with $\varepsilon \in \{1, -1\}$. Each nonidentity block contributes one to $\operatorname{rank}(s_\alpha - I)$, so there is exactly one such block. If that block were (-1) at i, its negative eigenspace would be $\mathbb{R}\beta_i = \mathbb{R}\alpha$. Since $\alpha \in \Lambda$, this would give $\alpha = t\beta_i$ with $t \in \mathbb{Z} \setminus \{0\}$ and $2 = 3t^2$, which is impossible. Thus s_α interchanges two basis vectors with the same sign ε and fixes the others. Its negative eigenspace is $\mathbb{R}(\beta_i - \varepsilon\beta_j)$, so $\alpha = t(\beta_i - \varepsilon\beta_j)$ with $t \in \mathbb{Z} \setminus \{0\}$. The coordinates of the column α belong to $\{0, 1, -1\}$ by total unimodularity; hence $t = \pm 1$. Therefore

$$\alpha = \pm(\beta_i - \varepsilon\beta_j). \tag{5.4}$$

Let G be the graph on $\{1, \dots, r\}$ whose edges are the two-element supports of the root columns, counted with multiplicity. The coordinate matrix consists of I_r and signed edge columns of G. If G were disconnected, the coordinate subspaces on its vertex components would contain all columns, with no column meeting two of them. Each subspace contains a basis column, and a minimal column dependence is contained in one subspace. This would give a nontrivial direct sum of M. Thus the graph G is connected. Choose a spanning tree rooted at 1. For a tree edge from parent i to child j, write its column as $\mathbf{e}_i - \varepsilon_{ij}\mathbf{e}_j$ up to sign. Set $\eta_1 = 1$ and $\eta_j = \eta_i\varepsilon_{ij}$. Replace β_i by $\eta_i\beta_i$, and orient the corresponding basis columns accordingly. Retain the notation β_i. The coordinate matrix remains totally unimodular and contains I_r. For every tree edge, $\eta_i\varepsilon_{ij}\eta_j = 1$; hence its column is a difference up to sign.

For each tree edge ij, the block description above shows that reflection in $\beta_i - \beta_j$ interchanges β_i, β_j and fixes the other basis vectors. The identity $(a\ c) = (a\ b)(b\ c)(a\ b)$ gives the transposition of the endpoints of a path by induction on its length. Since the tree is connected, these reflections generate every permutation of the basis. The Gram matrix G_β is therefore invariant under all such permutations. Its diagonal entries are three, and for any tree edge ij,

$$2 = \|\beta_i - \beta_j\|^2 = 6 - 2\langle\beta_i, \beta_j\rangle.$$

Permutation invariance makes all off-diagonal entries equal; the displayed identity makes their common value two. Thus

$$G_\beta = I_r + 2J_r. \tag{5.5}$$

For $i \ne j$, one has $\|\beta_i + \beta_j\|^2 = 10$. Since every root has squared length two, (5.4) now gives $\alpha = \pm(\beta_i - \beta_j)$ for every root column. By Lemma 3.3, these permutations preserve the lines spanned by the root columns, with their multiplicities. They act transitively on the unordered pairs, and $\mathcal{R} \ne \emptyset$; hence every pair ij occurs with the same multiplicity $k \ge 1$.

Let D_β be the coordinate matrix of V in this basis. Its columns are I_r and k copies of each difference, up to sign. Consequently,

$$C_\beta = D_\beta D_\beta^{\mathsf{T}} = (1 + kr)I_r - kJ_r, \qquad C_\beta^{-1} = \frac{I_r + kJ_r}{1 + kr}. \tag{5.6}$$

If F_β has columns β_i in orthonormal coordinates, then $V = F_\beta D_\beta$ and $F_\beta C_\beta F_\beta^{\mathsf{T}} = eI_r$. Multiplication by F_β^{-1} and $F_\beta^{-\mathsf{T}}$ gives $C_\beta = eG_\beta^{-1}$, or $G_\beta = eC_\beta^{-1}$. For $i \ne j$, which exist since $r \ge 2$, comparison with (5.5) gives

$$1 = (G_\beta)_{ii} - (G_\beta)_{ij} = \frac{e}{1 + kr}, \qquad 2 = (G_\beta)_{ij} = \frac{ek}{1 + kr}.$$

The first identity gives $e = 1 + kr$, and the second then gives $k = 2$. Thus $e = 2r + 1$. Up to column signs, D_β is the reduced incidence matrix of Γ_r, with the row at the apex deleted. It follows that $M \simeq M(\Gamma_r)$. □

Lemma 5.3. *For $r \geq 2$, the matroid $M(\Gamma_r)$ is connected and satisfies* (1.6)*. Its defect is r.*

Proof. For each $i \neq j$, a triangle contains the apex edges $0i, 0j$ and either copy of ij. Thus all apex edges, and then all other edges, belong to one matroid component. Hence $M(\Gamma_r)$ is connected. Delete the apex row from an oriented incidence matrix of Γ_r. After column sign changes, its columns are I_r and two copies of each $\mathbf{e}_i - \mathbf{e}_j$, $i < j$. Formula (5.6) with $k = 2$ gives

$$m = r + 2\binom{r}{2} = r^2, \qquad C = (2r+1)I_r - 2J_r, \qquad C^{-1} = \frac{I_r + 2J_r}{2r+1}. \tag{5.7}$$

Apply [4, Theorem 2.1(iii)] over $\mathbb{Z}$ with $(n, a, b) = (r, 2r+1, -2)$. Its hypothesis $ax + by = 1$ holds with $(x, y) = (1, r)$, and $a + nb = 1$. The Smith form of $C = aI_r + bJ_r$ is therefore $\operatorname{diag}(1, 2r+1, \dots, 2r+1)$. By Proposition 2.3 and Lemma 2.2,

$$\operatorname{Jac}(M(\Gamma_r)) \simeq (\mathbb{Z}/(2r+1)\mathbb{Z})^{r-1}.$$

Since $r \geq 2$, this group has exponent $e = 2r+1$, and $\delta = er - 2m = r$. The Gram matrix $eC^{-1} = I_r + 2J_r$ gives squared length three on the apex columns and two on the difference columns. Thus $\mathcal{R}$ consists exactly of the difference columns. Their coordinate vectors span the coordinate-sum-zero hyperplane; the invertible map $\sqrt{e}\, C^{-1/2}$ carries this span to W. Hence $\dim W = r-1$, $c = 1$, and $cm = r^2 = \delta^2$. □

Proof of Theorem 1.1. Equation (3.7) gives $\delta \geq 0$. Corollary 4.2 gives the zero-defect case and its Jacobian. For $\delta > 0$, Proposition 5.1 gives (1.4) and $m = \delta^2 \Rightarrow \delta = r$. Proposition 5.2 gives $\delta = r \Rightarrow M \simeq M(\Gamma_r)$. Lemma 5.3 gives the converse and $m = r^2 = \delta^2$, as well as (1.6). □

Corollary 5.4. *Either $M \simeq M(kK_{r+1})$ for some $k \geq 1$, or*

$$(e(M) - 1)r \geq 2|E(M)|. \tag{5.8}$$

Equality in (5.8) *holds precisely for $M(\Gamma_r)$. If the corank is at least two, the dual assertions hold with rank replaced by corank and each exceptional or equality matroid by its dual.*

Proof. One has

$$(e-1)r - 2m = \delta - r.$$

If $\delta = 0$, then Theorem 1.1 gives the exceptional family, and this difference is $-r < 0$. If $\delta > 0$, the theorem gives $\delta \geq r$, with equality precisely for $M(\Gamma_r)$. For the dual assertion, put $r^* = m - r$. The dual M^* is connected and regular and has exponent e by Lemma 2.4 and Proposition 2.3. If $r^* \geq 2$, the assertion just proved applies to M^*, with defect $er^* - 2m$. Dualizing its exceptional and equality matroids gives the stated alternatives. □

Corollary 5.5. *Among connected regular matroids of rank at least two, defect one does not occur. Defect two occurs only for $M(\Gamma_2)$, and defect three occurs only for $M(\Gamma_3)$.*

Proof. Positive defect gives $2 \leq r \leq \delta$, excluding defect one. For defect two, $r = 2$. For defect three, rank two is impossible since $\delta = 2(e - m)$ would be even. Thus $r = \delta$ for $\delta \in \{2, 3\}$, and Theorem 1.1 gives the classification. Conversely, Lemma 5.3 gives those defects for the stated matroids. □

Connectedness is necessary for a bound on rank in terms of positive defect. For every integer $s \geq 0$, the regular matroid

$$M(\Gamma_2) \oplus M(K_5)^{\oplus s}$$

has Jacobian $(\mathbb{Z}/5\mathbb{Z})^{1+3s}$ by Lemma 5.3, Corollary 4.2, and Proposition 2.3. Its exponent is therefore five, its rank is $2 + 4s$, and it has $4 + 10s$ elements. Its defect is $5(2 + 4s) - 2(4 + 10s) = 2$. It is disconnected for $s \geq 1$. The connected equality matroids $M(\Gamma_r)$ satisfy $\delta/r = 1$ and $|E| = r^2$ for every $r \geq 2$.

6. Small exponents and R_{10}

Let $T = M(\Gamma_2)$, the cycle matroid of a triangle with one doubled edge.

Theorem 6.1. *The connected regular matroids on more than one element whose Jacobians have exponent at most five are exactly those in the following table:*

e	M	$\mathrm{Jac}(M)$
2	$U_{1,2}$	$\mathbb{Z}/2\mathbb{Z}$
3	$U_{1,3},\ U_{2,3}$	$\mathbb{Z}/3\mathbb{Z}$
4	$U_{1,4},\ U_{3,4}$	$\mathbb{Z}/4\mathbb{Z}$
4	$M(K_4)$	$(\mathbb{Z}/4\mathbb{Z})^2$
5	$U_{1,5},\ U_{4,5},\ T$	$\mathbb{Z}/5\mathbb{Z}$
5	$M(K_5),\ M(K_5)^*$	$(\mathbb{Z}/5\mathbb{Z})^3$.

(6.1)

A connected regular matroid of exponent one is a single loop or a single coloop.

Proof. Put $m = |E(M)|$, $r = \operatorname{rank} M$, and $e = e(M) \le 5$. If $m = 1$, then M is a loop or a coloop and has trivial Jacobian. Assume $m > 1$. By Lemma 2.4 and Proposition 2.3, $eP_{ii} \in \mathbb{Z}$ and $0 < eP_{ii} < e$, so $e \ge 2$. In rank one, connectedness excludes zero columns. After column sign changes, a totally unimodular representation is $D = (1, \dots, 1)$. Thus $C = (m)$, $M = U_{1,m}$, and $\mathrm{Jac}(M) \simeq \mathbb{Z}/m\mathbb{Z}$. By Proposition 2.3, corank one gives $M \simeq U_{m-1,m}$ with the same Jacobian.

Suppose $r \ge 2$ and $c = m - r \ge 2$. Both $\delta(M)$ and $\delta(M^*)$ are nonnegative by Corollary 3.2. If $\delta(M) = 0$, then Corollary 4.2 gives $M \simeq M(kK_q)$ with $q = r + 1 \ge 3$ and $e = kq \le 5$. Hence $k = 1$ and $q \in \{3, 4, 5\}$; the corank condition excludes $q = 3$. The zero-defect dual cases are $M(K_4)^*$ and $M(K_5)^*$, with $M(K_4)^* \simeq M(K_4)$ [16, p. 640].

If both defects are positive, Corollary 5.4 gives

$$(e-1)r \ge 2m, \qquad (e-1)c \ge 2m. \tag{6.2}$$

Since $r + c = m$ and $e \le 5$,

$$0 \le \big((e-1)r - 2m\big) + \big((e-1)c - 2m\big) = (e-5)m \le 0.$$

Thus $e = 5$ and equality holds in both inequalities (6.2). By Corollary 5.4 and Lemma 5.3, $M \simeq M(\Gamma_r)$ and $5 = 2r+1$, so $r = 2$ and $M \simeq T$. The rank-one calculation, duality, and Corollary 4.2 and Lemma 5.3 give the groups in the table. Each $U_{1,m}$ is connected for $m \ge 2$ and is graphic; $U_{m-1,m} = U_{1,m}^*$ is connected and regular. The complete graphs and Γ_2 are loopless and biconnected, so their cycle matroids are connected and regular by Lemma 2.7. Duality preserves these properties. Thus every listed matroid occurs. □

For simple biconnected graphs, the classification in Theorem 6.1 is contained in [14, Theorem 1.1 and (5.10)].

Corollary 6.2. *A regular matroid has exponent at most five if and only if its components with more than one element occur in* (6.1) *and their exponents have least common multiple at most five.*

Proof. For the connected components M_j, Proposition 2.3 gives $\mathrm{Jac}(M) \simeq \bigoplus_j \mathrm{Jac}(M_j)$ and $e(M) = \operatorname{lcm}_j e(M_j)$, with $\operatorname{lcm} \emptyset = 1$. Apply Theorem 6.1; singleton components contribute trivial groups. □

Proposition 6.3. *The Bixby–Seymour matroid R_{10} is connected and regular, has rank five and ten elements, is neither graphic nor cographic, and satisfies*

$$\mathrm{Jac}(R_{10}) \simeq (\mathbb{Z}/3\mathbb{Z})^3 \oplus \mathbb{Z}/6\mathbb{Z}. \tag{6.3}$$

Its Jacobian exponent is six.

Proof. The matroid R_{10} is 3-connected and regular, has rank five and ten elements, and is neither graphic nor cographic [16, Section 13.1, p. 489]. Choose a normalized totally unimodular representation D. The sandpile group of [10, Definition 2.2] is

$$S(R_{10}) = \mathbb{Z}^{10}/\left(D^{\mathsf{T}}\mathbb{Z}^5 + \ker(D : \mathbb{Z}^{10} \longrightarrow \mathbb{Z}^5)\right) = \mathrm{Jac}(R_{10})$$

by (2.4). By [10, p. 158], $S(R_{10}) \simeq (\mathbb{Z}/3\mathbb{Z})^4 \oplus \mathbb{Z}/2\mathbb{Z}$. Since $\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \simeq \mathbb{Z}/6\mathbb{Z}$, (6.3) follows and $e(R_{10}) = 6$. □

Corollary 6.4. *The least Jacobian exponent of a connected regular matroid which is neither graphic nor cographic is six.*

Proof. Every matroid in Theorem 6.1 is graphic or cographic, including the singletons of exponent one. Proposition 6.3 gives a connected regular matroid in neither class with exponent six. □

The regular matroid $N = M(K_5) \oplus M(K_5)^*$ is neither graphic nor cographic [16, Section 13.1, p. 489]. By Corollary 4.2 and Proposition 2.3,

$$\mathrm{Jac}(N) \simeq (\mathbb{Z}/5\mathbb{Z})^6, \qquad e(N) = 5.$$

Thus connectedness is necessary in Corollary 6.4.

7. Integral sublattices generated by short circuits

Let M be a regular matroid of rank r on m elements, and put $e = e(M)$ and $a_i = eP_{ii}$. Loops, coloops and disconnected matroids are allowed. For $a \geq 1$, put

$$\mathcal{F}_{\leq a} = \langle c \in \mathcal{F} \,:\, c \text{ is a signed circuit}, \quad |\operatorname{supp} c| \leq a \rangle_{\mathbb{Z}},$$
$$\mathcal{B}_{\leq a} = \langle b \in \mathcal{B} \,:\, b \text{ is a signed cocircuit}, \quad |\operatorname{supp} b| \leq a \rangle_{\mathbb{Z}}.$$

Theorem 7.1. *One has*

$$e\mathcal{F} \subseteq \mathcal{F}_{\leq e} \subseteq \mathcal{F}, \qquad e\mathcal{B} \subseteq \mathcal{B}_{\leq e} \subseteq \mathcal{B}. \tag{7.1}$$

The quotients $\mathcal{F}/\mathcal{F}_{\leq e}$ and $\mathcal{B}/\mathcal{B}_{\leq e}$ are finite and annihilated by e. Over every field in which e is invertible, the signed circuits of size at most e span the cycle space, and the signed cocircuits of size at most e span the cut space.

If M is connected and its rank and corank are at least two, then

$$e\mathcal{F} \subseteq \mathcal{F}_{\leq e-1} \subseteq \mathcal{F}, \qquad e\mathcal{B} \subseteq \mathcal{B}_{\leq e-1} \subseteq \mathcal{B}.$$

The corresponding quotients are finite and annihilated by e. Over every field in which e is invertible, the signed circuits and cocircuits of size at most $e-1$ span the cycle and cut spaces, respectively.

Proof. Put $a = e$, or $a = e-1$ under the additional hypotheses. Take $(L, U) = (\mathcal{F}, Q)$ or $(\mathcal{B}, P)$, and let $L_{\leq a}$ denote the corresponding sublattice. By Proposition 2.3, each column $x = eU\mathbf{e}_i$ is integral. Since $Ux = x$, one has $x \in L$. If $x \neq 0$, then

$$x_i = e\|U\mathbf{e}_i\|^2 \in \mathbb{Z}_{>0}.$$

When $a = e-1$, Corollary 3.2 gives $2 \leq a_i \leq e-2$, so $x_i \geq 2$ for both choices of U. By Lemmas 2.5 and 2.6, x is a sum of signed circuits or cocircuits, according to L, each of size at most $e - x_i + 1 \leq a$. Zero columns also belong to $L_{\leq a}$. Thus, for every $z \in L$,

$$ez = eUz = \sum_i z_i(eU\mathbf{e}_i) \in L_{\leq a}.$$

The quotient $L/L_{\leq a}$ is a quotient of the finite group L/eL, and is annihilated by e.

In a normalized representation $D = [I_r \; N]$, the vectors $(-N\mathbf{e}_j, \mathbf{e}_j)$ form an integral basis of $\mathcal{F}$, and the vectors $D^{\mathsf{T}}\mathbf{e}_i$ form an integral basis of $\mathcal{B}$. Their images over any field K form bases of $\ker D_K$ and $\operatorname{im} D_K^{\mathsf{T}}$, respectively. If e is invertible in K, the relations $eL \subseteq L_{\leq a}$ express each basis vector as a K-linear combination of the specified signed circuits or cocircuits. □

Corollary 7.2. *For a connected regular matroid on more than one element, let g_i, g_i^* be the minimum sizes of a circuit and a cocircuit containing i. Put $g = \min_i g_i$ and $g^* = \min_i g_i^*$. Then*

$$\begin{gathered} g_i - 1 \le a_i \le e - g_i^* + 1, \qquad e \ge g_i + g_i^* - 2, \\ er \ge m(g-1), \qquad e(m-r) \ge m(g^*-1), \qquad e \ge g + g^* - 2. \end{gathered} \tag{7.2}$$

Proof. There are no loops or coloops, so $P\mathbf{e}_i$ and $Q\mathbf{e}_i$ are nonzero. By Lemmas 2.5 and 2.6, $g_i - 1 \le a_i$ and $g_i^* - 1 \le e - a_i$. Addition gives $g_i + g_i^* - 2 \le e$. Moreover,

$$\begin{aligned} m(g-1) &\le \sum_i (g_i - 1) \le \sum_i a_i = er, \\ m(g^*-1) &\le \sum_i (g_i^* - 1) \le \sum_i (e - a_i) = e(m-r). \end{aligned}$$

Finally, $g + g^* \le g_i + g_i^* \le e + 2$. □

Corollary 7.3. *Let G be a finite connected graph cellularly embedded in a closed orientable surface of genus g, and let $e = \exp \mathrm{Jac}(G)$. There is $S \subseteq V(G)$ such that*

$$|S| \le 2ge, \qquad G - S \text{ is planar.}$$

If $M(G)$ is connected and its rank and corank are at least two, one can replace $2ge$ by $2g(e-1)$.

Proof. For $g = 0$, take $S = \emptyset$. Suppose $g > 0$, and denote the surface by Σ. Put $a = e$, or $a = e - 1$ under the additional hypotheses. By Lemma 2.7 and Theorem 7.1, the oriented simple cycles of length at most a span $H_1(G;\mathbb{Q})$; loops and pairs of parallel edges have lengths one and two.

Cellularity makes G the one-skeleton of a CW decomposition of Σ. Thus inclusion induces a surjection

$$H_1(G;\mathbb{Q}) \twoheadrightarrow H_1(\Sigma;\mathbb{Q}) \simeq \mathbb{Q}^{2g}$$

by [8, Lemma 2.34(c) and Example 3.31]. Choose short cycles $\gamma_1, \dots, \gamma_{2g}$ whose images form a basis. Their vertex union S satisfies

$$|S| \le \sum_{j=1}^{2g} |V(\gamma_j)| \le \sum_{j=1}^{2g} |E(\gamma_j)| \le 2ga.$$

Put $Y = \bigcup_j \gamma_j$. The compact subgraphs Y and $G - S$ are disjoint. Choose a closed regular neighbourhood N of Y disjoint from $G - S$. If a component W of $\Sigma \setminus \operatorname{int} N$ had positive genus, [6, Theorem 1.1] would give oriented simple closed curves $\alpha, \beta \subset \operatorname{int} W$ with algebraic intersection number one. Write $[\alpha] = \sum_j q_j[\gamma_j]$ in $H_1(\Sigma;\mathbb{Q})$. Since β is disjoint from every γ_j, the intersection form [6, Section 6.1.2] gives

$$1 = [\alpha] \cdot [\beta] = \sum_j q_j [\gamma_j] \cdot [\beta] = 0,$$

a contradiction. Every component W therefore has genus zero. Capping its boundary circles gives a sphere containing the corresponding part of $G - S$. These finitely many planar graphs have a planar disjoint union. □

Appendix A. Hyperbolic pairings of exponent four

Put

$$H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad F = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Let $\mathcal{E}_4$ and $\mathcal{F}_4$ be the pairings on $(\mathbb{Z}/4\mathbb{Z})^2$ with matrices $H/4$ and $F/4$, respectively, taking values in $\mathbb{Q}/\mathbb{Z}$. Write $\perp$ for orthogonal direct sum.

For a connected graph G and a vertex o, let C be the positive Laplacian reduced at o; loops contribute zero to the Laplacian. The monodromy pairing is the nonsingular symmetric pairing

$$b_C([x],[y]) = x^{\mathsf{T}} C^{-1} y \pmod{\mathbb{Z}} \quad \text{on } \mathbb{Z}^{V(G)\setminus\{o\}}/C\mathbb{Z}^{V(G)\setminus\{o\}}; \tag{A.1}$$

see [17, Example 3.2, Theorem 3.4 and Proposition 3.7]. Equivalently, if L is the full Laplacian, $x, y \in \mathbb{Z}^{V(G)}$ have coordinate sum zero, and $f \in \mathbb{Q}^{V(G)}$ satisfies $Lf = y$, then

$$b_G([x],[y]) = x^{\mathsf{T}} f \pmod{\mathbb{Z}}. \tag{A.2}$$

Lemma A.1. *Let $E_1, \dots, E_s$ be the components of the cycle matroid of a connected graph G. Let G_i have edge set E_i and the vertices incident with these edges. Then*

$$(\mathrm{Jac}(G), b_G) \simeq \perp_{i=1}^{s} (\mathrm{Jac}(G_i), b_{G_i}).$$

A loop or a bridge contributes the trivial group.

Proof. Each loop is a singleton component with trivial Jacobian. Deleting loops leaves the Laplacian unchanged. If no edges remain, the Jacobian is trivial. Otherwise the remaining graph is an iterated vertex sum of its blocks, including parallel-edge blocks on two vertices and bridge blocks. Every circuit lies in one block. A block with at least two edges is loopless and nonseparable, and has no vertex of degree one; its cycle matroid is connected by [2, Proposition 4]. The block edge sets are therefore the components of the cycle matroid. The orthogonal vertex-sum formula [7, Proposition 2.3] gives the displayed isomorphism. A bridge block has reduced Laplacian (1) and trivial Jacobian. □

Lemma A.2 ([7, Section 4.2]). *The paired Jacobian of K_4 is $\mathcal{F}_4$.*

Lemma A.3 ([13, Tables 1.2(IV) and 3.2]). *For $k \geq 1$, the matrices $F^{\oplus k}$ and $H^{\oplus k}$ are congruent over $\mathbb{Z}/4\mathbb{Z}$ if and only if k is even.*

Theorem A.4. *For $k \geq 1$, the paired group $\mathcal{E}_4^{\perp k}$ is the entire paired Jacobian of a finite connected graph if and only if k is even. For every even k it has a simple planar realization. No loopless biconnected graph realizes it.*

Proof. Suppose $(A, b) := (\mathrm{Jac}(G), b_G) \simeq \mathcal{E}_4^{\perp k}$, and delete the loops of G. With G_i as in Lemma A.1, put $(A_i, b_i) = (\mathrm{Jac}(G_i), b_{G_i})$. Then $(A, b) \simeq \perp_{i=1}^{s} (A_i, b_i)$, so $\exp(A_i) \mid 4$. Each $M(G_i)$ is connected and regular. Theorem 6.1 gives

$$A_i \neq 0 \quad \Longrightarrow \quad A_i \simeq \mathbb{Z}/2\mathbb{Z}, \quad \mathbb{Z}/4\mathbb{Z}, \quad \text{or } (\mathbb{Z}/4\mathbb{Z})^2.$$

Since $A \simeq (\mathbb{Z}/4\mathbb{Z})^{2k}$, $A[2] := \{x \in A : 2x = 0\} = 2A$. If $A_i = \langle z \rangle \simeq \mathbb{Z}/2\mathbb{Z}$, then $z = 2y$ for some $y \in A$. The projection $\pi_i : A \to A_i$ gives $z = 2\pi_i(y) = 0$, a contradiction.

In the coordinates of $\mathcal{E}_4^{\perp k}$,

$$b(x,x) = \frac{1}{2} \sum_{j=1}^{k} x_{2j-1} x_{2j} \in \tfrac{1}{2}\mathbb{Z}/\mathbb{Z} \qquad \big(x \in (\mathbb{Z}/4\mathbb{Z})^{2k}\big).$$

Suppose $A_i = \langle z \rangle \simeq \mathbb{Z}/4\mathbb{Z}$, and write $b_i(z,z) = a/4 \pmod{\mathbb{Z}}$. If a were even, then $b_i(2z, A_i) = 0$ with $2z \neq 0$, contrary to nonsingularity. Thus a is odd, contradicting the displayed inclusion.

Thus $M(G_i) \simeq M(K_4)$ whenever $A_i \neq 0$. Such a G_i is connected and simple, with $|V(G_i)| - 1 = 3$ and $|E(G_i)| = 6$; hence $G_i \simeq K_4$. Put $t = \#\{i : A_i \neq 0\}$. Then $16^t = |A| = 16^k$, so $t = k$. Lemmas A.1 and A.2 give $(A, b) \simeq \mathcal{F}_4^{\perp k}$. Lemma A.3 implies that k is even.

For even k, identify one vertex in each of k disjoint copies of K_4. The resulting graph is simple; the copies have planar drawings in disjoint angular sectors meeting only at the identified vertex. Its paired Jacobian is $\mathcal{F}_4^{\perp k} \simeq \mathcal{E}_4^{\perp k}$ by Lemmas A.1 to A.3. If a loopless biconnected graph realized $\mathcal{E}_4^{\perp k}$, its cycle matroid would be connected by Lemma 2.7. There would be a single K_4 component, so $k = 1$, contrary to the parity condition. □

The realizations for even k follow by vertex sums from the realization of $\mathcal{E}_4^{\perp 2}$ by the vertex sum of two copies of K_4 in [15, Remark C.7], where an explicit congruence is given. The case $k = 2$ contradicts [7, Conjecture 4.3] at exponent four. The relation $\mathcal{F}_4 \perp \mathcal{F}_4 \simeq \mathcal{E}_4 \perp \mathcal{E}_4$ is in [13, Table 1.2(IV)].

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