IN THE SELF-CONTACT PROBLEM IN NONLINEAR ELASTICITY

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Abstract. In this paper, we consider the minimization problem of 3-dimensional nonlinear hyperelastic bodies moving in $\mathbb{R}^3$, which enables frictionless self-contact and forbids self-intersection. For this, we define a new class of admissible deformations based on a natural homotopy constraint. We study strictly orientation-preserving Sobolev maps in this new class and their global invertibility properties from a topological point of view. In this fashion, we prove that, under suitable hypotheses, such maps are actually homeomorphisms. Applying this result to the mixed displacement-traction problem, the existence of homeomorphic minimizers is shown for nonlinear stored energy functions with suitable properties.

1. Introduction

Consider an elastic body occupying the closure $\overline{\Omega}$ of a domain $\Omega \subset \mathbb{R}^3$ as its reference configuration, subjected to a boundary condition of place on a portion Γ_0 of the boundary $\partial\Omega$ and subjected to an applied body force with density $f : \Omega \to \mathbb{R}^3$ per unit mass. We assume for simplicity that the field f is independent of the deformation, i.e., that the body force is a dead load. If in a given situation, a particle occupying the point $x \in \overline{\Omega}$ in the reference configuration moves to a point represented by a vector $\varphi(x)$, the corresponding mapping $\varphi : \overline{\Omega} \to \mathbb{R}^3$, which is called a deformation, is the unknown of our problem. Let

$$\mathbb{M}^3_+ := \{F \in \mathbb{M}^3;\ \det F > 0\} \tag{1.1}$$

where $\mathbb{M}^3$ is the set of all matrices of order 3.
If the material constituting the body is hyperelastic, the unknown deformation φ undergone by the body is a stationary point of the total energy $\mathfrak{I}$ defined by

$$\mathfrak{I}(\phi) = \int_\Omega W(x, \nabla\phi(x))dx - \int_\Omega f \cdot \phi \, dx \tag{1.2}$$

where $W : (x, F) \in \overline{\Omega} \times \mathbb{M}^3_+ \to W(x, F) \in \mathbb{R}$, denotes the stored energy function of the hyperelastic material, when ϕ varies in a set of admissible deformations.

To avoid interpenetration of matter, a classical problem in Nonlinear Elasticity is to determine whether a Sobolev deformation $\phi : \Omega \to \mathbb{R}^3$ on a bounded domain $\Omega \subset \mathbb{R}^3$, is invertible in a suitable sense. If we assume the existence of a stored energy density W with suitable properties, then a stable deformed state can be found via global minimization [2]. The typical examples for such energies enforce $\phi \in W^{1,p}(\Omega; \mathbb{R}^3)$, strictly orientation-preserving in the sense that $\det \nabla\phi > 0$ almost everywhere in Ω.

However, even if $\phi \in C^1(\Omega; \mathbb{R}^3)$ and $\det \nabla\phi$ is positive everywhere, this does not suffice to guarantee global invertibility, because different ends of the body can still overlap. The variational theory is compatible with imposing global invertibility (in a weak almost everywhere sense) as a constraint, the Ciarlet and Nečas [5] condition $\int_\Omega \det(\nabla\phi)\ d\mu^3 \leq \mu^3(\mathrm{Im}\ \phi)$ is imposed on the admissible deformations then Ciarlet and Nečas [5] show that the invertibility almost everywhere is equivalent to this constraint. Furthermore, they proved that this constraint is preserved under

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weak convergence in $W^{1,p}$, $p > 3$. They were thus able to ensure the existence of minimizers for the mixed displacement-traction (Dirichlet-Neumann) problem in the class of almost everywhere invertible maps and proved that the associated minimization problem provides a mathematical model for matter to come into frictionless contact with itself but not interpenetrate. Their results have subsequently been improved by Tang Qi [19] (using ideas of Šverák (in [25]), by Giaguinta and al [9], and by Müller, Tang and Yan [17]. Some other studies of local and global invertibility in the context of elasticity can be found in [4, 6, 7, 10, 11, 13, 16, 23, 24, 18, 22].
In case of a strictly orientation-preserving map, if we also assume that its boundary values match those of a homeomorphism, the Ciarlet and Nečas condition always holds as a byproduct of a result of Ball [3]. The result also provides further topological properties of the deformation and its image. For similar purposes, this assumption also appears in [25] and other works, in particular in context of maps of finite distortion [12].

In this paper, we use a new topological condition to define a class of admissible deformations in nonlinear hyperelasticity allowing us to establish existence results in a realistic framework with mixed boundary conditions. The condition is a very natural homotopy condition that has not been used in this context before. This constraint ensure the invertibility almost everywhere (or equivalently, the injectivity almost everywhere) of every admissible deformation ϕ in the new defined class and the preservation of orientation in the sense that $\det \nabla\phi \geq 0$ almost everywhere in Ω (cf. Theorem 2.1). We begin by proving that this class of deformations is closed for the C^0 topology and that its elements don't have transversal self-intersections. We give a similar invertibility results (cf. Theorem 2.1 and Theorem 4.1) as in [3] without relying on the assumption that on $\partial\Omega$, the deformation ϕ coincides with a continuous map $\phi_0 : \overline{\Omega} \to \phi(\overline{\Omega})$ which is injective on Ω. In this fashion, we show that every strictly orientation-preserving deformation in our set of admissible deformation satisfies the conditions of Ciarlet & Nečas [5] whereas the converse is not true and then the associated minimization problem is a mathematical model of frictionless self-contact without interpenetration of matter. We also compare our constraint with the invertibility condition (INV) introduced in [18], mainly intended for settings involving cavitation in $W^{1,p}$ with $2 < p < 3$ (cf. Proposition 3.1).
Under suitable hypotheses, the admissible deformations are actually homeomorphisms (cf. Theorem 5.1 and Theorem 5.2). As a direct application, we show the existence of homeomorphic minimizers for nonlinear elastic energy functionals controlling the inner or outer distortion (cf. Theorem 6.1) without fixing the boundary values of admissible deformations as in [3] or prescribing a given Lipschitz domain as their image as in [15].

2. Set of admissible deformations

Let Ω be a bounded, open, connected subset of $\mathbb{R}^3$, whose boundary $\partial\Omega = \Gamma$ is Lipschitz continuous. Let $\triangle_{\overline{\Omega}}$ be the diagonal set of $\overline{\Omega} \times \overline{\Omega}$, that is

$$\triangle_{\overline{\Omega}} := \{(x, x) \in \overline{\Omega} \times \overline{\Omega} \; ; \; x \in \overline{\Omega}\}$$

For every mapping $\phi : \overline{\Omega} \to \mathbb{R}^3$, we define the subset $\triangle_\phi$ by

$$\triangle_\phi := \{(x, y) \in \overline{\Omega} \times \overline{\Omega} \; ; \; \phi(x) = \phi(y)\}$$

We say that $\phi \in C^0(\overline{\Omega}; \mathbb{R}^3)$ is $\mathcal{K}$-homotopic to the natural embedding $i_{\overline{\Omega}} : \overline{\Omega} \hookrightarrow \mathbb{R}^3$, if for every compact K of $\overline{\Omega} \times \overline{\Omega} \setminus \triangle_\phi$, there exists a homotopy h_K from ϕ to $i_{\overline{\Omega}}$ such that $K \subset \overline{\Omega} \times \overline{\Omega} \setminus \triangle_{h_K(\cdot,t)}$ for every $t \in [0, 1]$; in this case we write $\phi \underset{\mathcal{K}}{\simeq} i_{\overline{\Omega}}$. We define then the set of admissible deformations by

$$\Phi := \{\phi \in C^0(\overline{\Omega}; \mathbb{R}^3); \; \phi \underset{\mathcal{K}}{\simeq} i_{\overline{\Omega}}\} \tag{2.1}$$

we begin to establish that Φ is closed for the C^0 topology and it contains the embeddings isotopic to $i_{\overline{\Omega}}$.

Proposition 2.1. *Φ is closed for the C^0 topology.*

Proof. Let $\phi \in \mathrm{C}^0(\overline{\Omega};\mathbb{R}^3)$ and let ϕ_n be a sequence of elements in Φ, such that $\phi_n \to \phi$ for the C^0 topology. Let K be a compact of $\overline{\Omega}\times\overline{\Omega}\setminus\triangle_\phi$, we denote

$$\epsilon = \inf\{|\phi(x)-\phi(y)|;(x,y)\in K\} > 0. \tag{2.2}$$

There exist N such that $|\phi-\phi_N| < \epsilon/2$. Let $F_K(x,t) = (1-t)\phi(x) + t\phi_N(x)$ .For all $(x,y)\in K$ and $t\in[0,1]$

$$\begin{aligned}|F_K(x,t)-F_K(y,t)| &= |(1-t)(\phi(x)-\phi(y)) + t(\phi_N(x)-\phi_N(y))| \\ &\geq |\phi(x)-\phi(y)| - t(|\phi(x)-\phi_N(x)| - |\phi(y)-\phi_N(y)| \\ &> \epsilon - \epsilon/2 - \epsilon/2 = 0\end{aligned} \tag{2.3}$$

Then $K\subset\overline{\Omega}\times\overline{\Omega}\setminus\triangle_{F_K(\cdot,t)}$ for every $t\in[0,1]$, in particular $K\subset\overline{\Omega}\times\overline{\Omega}\setminus\triangle_{\phi_N}$. As $\phi_N \underset{\mathcal{K}}{\simeq} i_{\overline{\Omega}}$, there is a homotopy G_K from ϕ_N to $i_{\overline{\Omega}}$ such that $K\subset\overline{\Omega}\times\overline{\Omega}\setminus\triangle_{G_K(\cdot,t)}$ for every $t\in[0,1]$. The mapping $H_K : \overline{\Omega}\times[0,1]\to R^3$ defined by

$$H_K(x,t) = \begin{cases} F_K(x,2t) & \text{if } t\in[0,1/2] \\ G_K(x,2t-1) & \text{if } t\in[1/2,1]\end{cases} \tag{2.4}$$

is a homotopy from ϕ to $i_{\overline{\Omega}}$ such that $K\subset\overline{\Omega}\times\overline{\Omega}\setminus\triangle_{H_K(\cdot,t)}$ for every $t\in[0,1]$. Hence $\phi\in\Phi$. □

Remark 2.1. The C^0 closure of the embeddings isotopic to $i_{\overline{\Omega}}$ is a subset of Φ. Indeed, according to the previous proposition, it is sufficient to show that any embedding isotopic to $i_{\overline{\Omega}}$ is an element of Φ. Let ϕ be an embedding isotopic to $i_{\overline{\Omega}}$ and let denote H this isotopy. Noticing that $\triangle_{H(\cdot,t)} = \triangle_{\overline{\Omega}}$ for every $t\in[0,1]$, we deduce that ϕ is $\mathcal{K}$-homotopic to the embedding $i_{\overline{\Omega}}$.

We will show that $\phi \underset{\mathcal{K}}{\simeq} i_{\overline{\Omega}}$ contains informations to describe the self-intersections of ϕ. More precisely, we prove that any deformation with transversal self-intersection does not belong to the set Φ.

Proposition 2.2. *Let $\phi\in\mathrm{C}^0(\overline{\Omega};\mathbb{R}^3)$ and $(x,y)\in\Omega\times\Omega$ such that:*

(i) $\phi(x)=\phi(y)$;
(ii) ϕ is of class C^1 in neighborhood of x and y;
(iii) ϕ is self-transverse at (x,y), ie., $\mathrm{Im}(\nabla\phi(x)) + \mathrm{Im}(\nabla\phi(y)) = \mathbb{R}^3$.

then $\phi\notin\Phi$

Proof. Let $\tilde{\phi}$ be the mapping defined from $\overline{\Omega}\times\overline{\Omega}$ into $\mathbb{R}^3$ that associates (x,y) with $\phi(x)-\phi(y)$. Since $\mathrm{Im}(\nabla\tilde{\phi}(x,y)) = \mathbb{R}^3$, $\tilde{\phi}$ is a submersion at $(x,\ y)$. From the Implicit Function Theorem, we deduce that there exists a neighborhood V of 0 in $\mathbb{R}^6$, a neighborhood W of $(x,\ y)$ in $\Omega\times\Omega$ and a local diffeomorphism $f : V\to W$ such that $f(0) = (x,\ y)$ and

$$\tilde{\phi}\circ f = \pi \tag{2.5}$$

where $\pi : \mathbb{R}^6\to\mathbb{R}^3$ is the canonical submersion. Furthermore, one can choose f such that the unit ball $\overline{\mathbb{B}}^3\times\{(0,0,0)\}$ is included in V and such that $W\cap\triangle_{\overline{\Omega}} = \emptyset$. Suppose by contradiction that ϕ is $\mathcal{K}$-homotopic to the natural embedding $i_{\overline{\Omega}}$, and let $K = f\circ i^V_{\mathbb{S}^2}(\mathbb{S}^2)$, with $i^V_{\mathbb{S}^2}$ is the standard inclusion map of $\mathbb{S}^2$ into V. Since K is compact and $K\subset\overline{\Omega}\times\overline{\Omega}\setminus\triangle_\phi$, there exists a homotopy h_K from ϕ to $i_{\overline{\Omega}}$ such that $K\subset\overline{\Omega}\times\overline{\Omega}\setminus\triangle_{h_K(\cdot,t)}$ for every $t\in[0,1]$. Let $\tilde{i}_{\overline{\Omega}}$ be the mapping defined from $\overline{\Omega}\times\overline{\Omega}$ into $\mathbb{R}^3$ that associates (x,y) with $x-y$. Then $\tilde{\phi}\circ f\circ i^V_{\mathbb{S}^2}$ and $\tilde{i}_{\overline{\Omega}}\circ f\circ i^V_{\mathbb{S}^2}$ are homotopic in $\mathbb{R}^3\setminus 0$ and hence $\frac{\tilde{\phi}\circ f\circ i^V_{\mathbb{S}^2}}{|\tilde{\phi}\circ f\circ i^V_{\mathbb{S}^2}|} = i_{\mathbb{S}^2}$ and $\vartheta = \frac{\tilde{i}_{\overline{\Omega}}\circ f\circ i^V_{\mathbb{S}^2}}{|\tilde{i}_{\overline{\Omega}}\circ f\circ i^V_{\mathbb{S}^2}|} : \mathbb{S}^2\to\mathbb{S}^2$ are homotopic which implies that $\deg(\vartheta) = 1$. Now, notice that ϑ has the continuous extension $\frac{\tilde{i}_{\overline{\Omega}}\circ f\circ i^V_{\overline{\mathbb{B}}^3}}{|\tilde{i}_{\overline{\Omega}}\circ f\circ i^V_{\overline{\mathbb{B}}^3}|} : \overline{\mathbb{B}}^3\to\mathbb{S}^2$, then $\deg(\vartheta) = 0$. This gives us the desired contradiction. □

Proposition 2.3. *Let $\phi \in \Phi$ and let $x_0 \in \Omega$ such that ϕ is differentiable at x_0. Assume that $J_\phi(x_0) = \det(\nabla\phi(x_0)) \neq 0$. Then* $\mathrm{card}(\phi^{-1}(\phi(x_0)) = 1$ *and* $J_\phi(x_0) > 0$.

Proof. We claim that there is ρ_0 such that or every $0 < \rho \leq \rho_0$ the following assertions holds;

$$\phi(x_0 + h) \neq \phi(x_0) \text{ for every } h \in \overline{B}(0,\ \rho) \setminus \{0\} \tag{2.6}$$

$$\deg(\phi, B(x_0, \rho), \phi(x_0)) = \mathrm{sgn}(J_\phi(x_0)) \tag{2.7}$$

where $\deg(\phi, B(x_0, \rho), \phi(x_0))$ is the degree of ϕ at $\phi(x_0)$ with respect to $B(x_0, \rho)$ Since ϕ is differentiable at x_0, there exists $R_0 > 0$ such that

$$\phi(x_0 + h) = \phi(x_0) + \nabla\phi(x_0)h + |h|\epsilon(|h|) \tag{2.8}$$

for $h \in \overline{B}(0,\ R_0)$ and $\lim_{t\to 0}\epsilon(t) = 0$. Let

$$\alpha\ := \inf\{|\nabla\phi(x_0)h|\ ;\ h \in \mathbb{R}^3,\ |h| = 1\}. \tag{2.9}$$

Since $J_\phi(x_0) \neq 0$ we obtain that $\alpha > 0$ and we may find $0 < \rho_0 \leq R_0$ such that $|\epsilon(t)| \leq \dfrac{\alpha}{2}$ for every $|t| \leq \rho_0$.

(i) Claim 1. $\phi(x_0 + h) \neq \phi(x_0)$ for every $h \in \overline{B}(0, \rho_0)$ such that $h \neq 0$.
Indeed, if $0 < |h| \leq \rho_0$, then $|\epsilon(|h|)| \leq \dfrac{\alpha}{2}$ and so

$$\left|\frac{\phi(x_0 + h) - \phi(x_0)}{|h|}\right| \geq \left|\nabla\phi(x_0)\frac{h}{|h|}\right| - |\epsilon(|h|)| \geq \frac{\alpha}{2} > 0 \tag{2.10}$$

and we obtain (2.6).

(ii) Claim 2. $\deg(\phi,\ B(x_0, \rho),\ \phi(x_0)) = \mathrm{sgn}(J_\phi(x_0))$ for every $0 < \rho \leq \rho_0$.
Let us first note that from (2.6), $\deg(\phi,\ B(0, \rho),\ \phi(x_0))$ is well defined for every $0 < \rho \leq \rho_0$. Fix $0 < \rho \leq \rho_0$ and set

$$\mathrm{L}(x_0 + h)\ := \phi(x_0) + \nabla\phi(x_0)h,\ \ h \in \overline{B}(0, \rho) \tag{2.11}$$

The function L is an affine mapping defined on $\overline{B}(x_0, \rho)$. Since $J_\phi(x_0) \neq 0$ we know that L is a one-to-one mapping and so $\mathrm{L}(x_0) \neq \mathrm{L}(x)$ for every $x \in \partial B(x_0, \rho)$, i.e. $\phi(x_0) \neq \mathrm{L}(x)$ for every $x \in \partial B(x_0, \rho)$. Therefore, $\deg(\mathrm{L}, B(x_0, \rho), \phi(x_0))$ is well defined and $\deg(\mathrm{L},\ B(x_0, \rho),\ \phi(x_0)) = \mathrm{sgn}(J_\phi(x_0))$. The application H defined by

$$H(x, t)\ := t\phi(x) + (1 - t)\mathrm{L}(x),\ x \in \overline{B}(x_0, \rho),\ t \in [0, 1] \tag{2.12}$$

is a homotopy between ϕ and L. Moreover, for every $x \in \partial B(x_0, \rho)$ we have

$$|H(x, t) - \phi(x_0)| = \rho\left(\left|\nabla\phi(x_0)\frac{x - x_0}{|x - x_0|} + t\epsilon(|x - x_0|)\right|\right) \geq \frac{\rho\alpha}{2} > 0. \tag{2.13}$$

Therefore, $\phi(x_0) \notin H(\partial B(x_0, \rho), t)$ for every $t \in [0, 1]$, and we conclude that $\deg(H(\cdot, t),\ B(x_0, \rho),\ \phi(x_0))$ is well defined. By virtue of Theorem [8, Theorem 2.3], the value of $\deg(H(\cdot,\ t),\ B(x_0, \rho),\ \phi(x_0))$ is independent of t, hence, we obtain

$$\deg(\phi,\ B(x_0, \rho),\ \phi(x_0)) = \deg(\mathrm{L},\ B(x_0, \rho),\ \phi(x_0)) = \mathrm{sgn}(J_\phi(x_0)) \tag{2.14}$$

and (2.7) is proved.

Fix $0 < \rho \leq \rho_0$ and set $K = \{x_0\} \times \partial B(x_0, \rho)$. Since K is compact and $K \subset \overline{\Omega} \times \overline{\Omega} \setminus \triangle_\phi$, there exists a homotopy h_K from ϕ to $i_{\overline{\Omega}}$ such that $K \subset \overline{\Omega} \times \overline{\Omega} \setminus \triangle_{h_K(\cdot,t)}$ for every $t \in [0, 1]$ and we conclude that $\deg(h_K(\cdot, t),\ B(x_0, \rho),\ h_K(x_0, t))$ is well defined. By virtue of Theorem [8, Theorem 2.6], the value of $\deg(h_K(\cdot, t),\ B(x_0, \rho),\ h_K(x_0, t))$ is independent of t, hence, we obtain

$$\mathrm{sgn}(J_\phi(x_0)) = \deg(\phi,\ B(x_0, \rho),\ \phi(x_0)) = \deg(i_{\overline{\Omega}},\ B(x_0, \rho),\ x_0) = +1 \tag{2.15}$$

We deduce that $J_\phi(x_0) > 0$. Now, suppose by contradiction that $\mathrm{card}(\phi^{-1}(\phi(x_0)) > 1$ and let $y_0 \in \Omega \setminus \{x_0\}$ such that $\phi(y_0) = \phi(x_0)$, we can choose $0 < \delta \le \rho_0$ such that $y_0 \notin B(x_0, \delta)$. We have

$$\deg(\phi,\ B(x_0, \delta),\ \phi(y_0)) = \deg(\phi,\ B(x_0, \delta),\ \phi(x_0)) = \deg(i_{\overline{\Omega}},\ B(x_0, \delta),\ x_0) = +1 \tag{2.16}$$

But in the other hand, we set $K' = \{y_0\} \times \partial B(x_0, \delta)$. Since K' is compact and $K' \subset \overline{\Omega} \times \overline{\Omega} \setminus \triangle_\phi$, there exists a homotopy $h_{K'}$ from ϕ to $i_{\overline{\Omega}}$ such that $K' \subset \overline{\Omega} \times \overline{\Omega} \setminus \triangle_{h_{K'}(\cdot,t)}$ for every $t \in [0, 1]$ and we conclude that $\deg(h_K(\cdot, t),\ B(x_0, \rho),\ h_K(y_0, t))$ is well defined. The value of $\deg(h_{K'}(\cdot, t),\ B(x_0, \rho),\ h_{K'}(y_0, t))$ is independent of t, hence, we obtain

$$\deg(\phi,\ B(x_0, \delta),\ \phi(y_0)) = \deg(i_{\overline{\Omega}},\ B(x_0, \delta),\ y_0) = 0 \tag{2.17}$$

which is in contradiction with (2.16). □

Using the inverse function theorem for everywhere differentiable maps [21, Theorem 1] and the previous proposition, we deduce the following corollary.

Corollary 2.1. *Let $\phi \in \Phi$ such that ϕ is differentiable everywhere on Ω. Assume that $J_\phi = \det(\nabla\phi) \neq 0$ in Ω. Then $\phi_{|\Omega}$ is an embedding and ϕ^{-1} is differentiable on Ω.*

The invertibility results obtained in [3] all rely on the assumption that on $\partial\Omega$, the deformation ϕ coincides with a continuous $\phi_0 : \overline{\Omega} \to \phi(\overline{\Omega})$ which is injective on Ω. This leads to the following similar result as [3, Theorem 1].

Theorem 2.1. *Let $\phi \in \Phi \cap W^{1,p}(\Omega; \mathbb{R}^3)$ with $p > 3$. Assume that $J_\phi = \det(\nabla\phi) \neq 0$ almost everywhere in Ω. Then we have the following assertions:*

(i) $\phi(\overline{\Omega}) = \overline{\phi(\Omega) \setminus \phi(\partial\Omega)}$ and $\phi(\partial\Omega) = \partial(\phi(\Omega) \setminus \phi(\partial\Omega))$.

(ii) ϕ is one-to-one almost everywhere in $\overline{\Omega}$ and $J_\phi = \det(\nabla\phi) > 0$ almost everywhere in Ω.

(iii) Let $v \in L^\infty(\mathrm{R}^3)$. For every measurable set $E \subset \Omega$, we have :

$$\int_E v \circ \phi(x) J_\phi(x) dx = \int_{\phi(E)} v(y) dy \tag{2.18}$$

In particular, $\int_E J_\phi(x) dx = \mu^3(\phi(E))$, where μ^n is the n-dimensional Lebesgue measure on $\mathbb{R}^n$.

Proof.

(i) Since $\mathrm{int}(\partial\Omega) = 0$, we have $\phi(\overline{\Omega}) = \phi(\overline{\Omega}) \setminus \mathrm{int}(\partial\Omega) = \overline{\phi(\overline{\Omega}) \setminus \phi\partial\Omega)} = \overline{\phi(\Omega) \setminus \phi(\partial\Omega)}$.

Let $y \in \phi(\Omega) \setminus \phi(\partial\Omega)$ and let $x \in \Omega$ such that $y = \phi(x)$. We know that ϕ is $\mathcal{K}$-homotopic to the natural embedding $i_{\overline{\Omega}}$. Set $K = \{x\} \times \partial\Omega$, since K is compact and $K \subset \overline{\Omega} \times \overline{\Omega} \setminus \triangle_\phi$, there exists a homotopy h_K from ϕ to $i_{\overline{\Omega}}$ such that $K \subset \overline{\Omega} \times \overline{\Omega} \setminus \triangle_{h_K(\cdot,t)}$ for every $t \in [0, 1]$ and we conclude that $\deg(h_K(\cdot, t),\ \Omega,\ h_K(x, t))$ is well defined. By virtue of Theorem [8, Theorem 2.6], the value of $\deg(h_K(\cdot, t),\ \Omega,\ h_K(x, t))$ is independent of t, hence, we obtain $\deg(\phi,\ \Omega,\ y) = \deg(i_{\overline{\Omega}},\ \Omega,\ x) = +1$. Notice that if $y \notin \phi(\overline{\Omega})$, then $\deg(\phi,\ \Omega,\ y) = 0$. Hence $\phi(\Omega) \setminus \phi(\partial\Omega) = \{y \in \mathbb{R}^3 \setminus \phi(\partial\Omega);\ \deg(\phi; \Omega; y) \neq 0\}$ is open by the continuity of the degree and $\partial(\phi(\Omega) \setminus \phi(\partial\Omega)) = \overline{\phi(\Omega) \setminus \phi(\partial\Omega)} \setminus (\phi(\Omega) \setminus \phi(\partial\Omega)) = \phi(\overline{\Omega}) \setminus (\phi(\Omega) \setminus \phi(\partial\Omega)) = \phi(\partial\Omega)$.

(ii) Since $\phi \in W^{1,p}(\Omega; \mathbb{R}^3)$ with $p > 3$, ϕ is differentiable almost everywhere in Ω. Hence, by Proposition 2.3, ϕ is one-to-one almost everywhere in $\overline{\Omega}$ and $J_\phi = \det(\nabla\phi) > 0$ almost everywhere in Ω.

(iii) Let $v \in L^\infty(\mathrm{R}^3)$. Theorem [8, Theorem 5.30] shows that for every measurable set $E \subset \Omega$, we have :

$$\int_E v \circ \phi(x) |J_\phi(x)| dx = \int_{\mathbb{R}^3} v(y) N(\phi,\ E,\ y) dy \tag{2.19}$$

where $N(\phi,\ E,\ y) = \mathrm{card}(\phi^{-1}(y) \cap E)$. By the part (ii) we have $J_\phi = \det(\nabla\phi) > 0$ almost everywhere in Ω, $N(\phi,\ E,\ y) = 1$ almost everywhere in $\phi(E)$ and $N(\phi,\ E,\ y) = 0$ almost everywhere in $\mathbb{R}^3 \setminus \phi(E)$. Therefore

$$\int_E v \circ \phi(x) J_\phi(x) dx = \int_{\phi(E)} v(y) dy \tag{2.20}$$

In particular, taking $v = 1$, we deduce that $\int_E J_\phi(x)dx = \mu^3(\phi(E))$.

$\square$

Remark 2.2. Theorem 2.1 can be extended to the case $p = 3$ with minor modifications. In this case, continuity of ϕ up to the boundary would be an unnatural extra assumption even for smooth domains. However, even if we only have that $\phi \in \{\psi \in W^{1,3}(\Omega;\mathbb{R}^3);\ \psi \underset{\mathcal{K}}{\simeq} i_{\overline{\Omega}},\ J_\psi = \det\nabla\psi > 0$ a.e.. in $\Omega\}$, we can still follow the proof of Theorem 2.1 in subdomains compactly contained in Ω, exploiting the facts that inside Ω, deformations $\phi \in W^{1,3}(\Omega;\mathbb{R}^3)$, with $J_\phi = \det(\nabla\phi) > 0$ almost everywhere in Ω, automatically have a continuous representative [27], have a differential almost everywhere (in the classical sense) [8, Theorem 5.21] and satisfy Lusin's condition (N) [14, Corollary 3.13]. Even an explicit modulus of continuity can be obtained at any given positive distance from the boundary [22]. In particular, any such ϕ can still be approximated by smooth functions, simultaneously in $W^{1,3}$ and locally uniformly.

3. Constraints related to global invertibility

In this section, we compare our condition with two invertibility constraints often used in the literature: The Ciarlet-Nečas condition and condition (INV).

By part (ii) of Theorem 2.1, every element $\phi \in \Phi$ satisfies the Ciarlet-Nečas constraint introduced by Ciarlet and Nečas in their paper [5] :

$$\int_\Omega \det(\nabla\phi)\ d\mu^3 \leq \mu^3(\mathrm{Im}\ \phi) \tag{3.1}$$

Therefore, our model inherits all the properties of theirs. In particular, this enables us to extend their physical formal interpretation of the minimizers of the energy. If $\varphi \in C^1(\overline{\Omega};\mathbb{R}^3)$ is a solution of the minimization problem that consists to find φ such that

$$\varphi \in \Phi' = \{\phi \in \Phi;\ \det\nabla\phi \neq 0 \text{ in } \Omega,\ \phi = \varphi_0 \text{ on } \Gamma_0\} \text{ and } \mathfrak{I}(\varphi) = \inf_{\phi\in\Phi'} \mathfrak{I}(\phi) \tag{3.2}$$

and $\det(\nabla\varphi(x)) > 0$ in $\overline{\Omega}$, where the functional $\mathfrak{I}$ is defined as in (1.2), then φ fulfills the Euler-Lagrange equations. In our case, the proof is exactly word by word, as in the proof formulated by Ciarlet-Nečas in [5, Theorem 4], in particular the associated minimization problem is a mathematical model of frictionless self-contact without interpenetration of matter. Actually, our criterion is stronger: some deformations that belong to the set of the admissible deformations defined by Ciarlet-Nečas in their paper [5], do not belong to Φ. For instance, any deformation $\phi \in \{\psi \in W^{1,p}(\Omega;\mathbb{R}^3);\ J_\psi = \det\nabla\psi > 0$ a.e. in $\Omega\}$, $p > 3$, with only a transversal self-intersections and having a Lebesgue null self-intersections set fulfills criterion (3.1) but does not belong to Φ as it has a transversal self-intersection (see Proposition 2.2).

Another constraint implying both local and global invertibility properties was developed in [18], mainly intended for settings involving cavitation in $W^{1,p}$ with $2 < p < 3$. It uses the fact that the degree only depends on boundary values, and the concept of the topological image im_T, defined below.

Let $B(a, r) := \{x \in \mathbb{R}^3; |x - a| < r\} \subset \Omega$ be the ball of radius r centered at $a \in \Omega$, and suppose that $\phi \in C^0(\partial B(a, r);\mathbb{R}^3)$. We define the topological image of $B(a, r)$ under ϕ by

$$\mathrm{im}_T(\phi; B(a, r)) := \{y \in \mathbb{R}^3 \setminus \phi(\partial B(a, r)); \deg(\phi; B(a, r); y) \neq 0\} \tag{3.3}$$

Definition 3.1. (Müller-Spector condition (INV) [18]). We say that $\phi : \Omega \to \mathbb{R}^3$ satisfies condition (INV), or, shortly, $\phi \in$ INV, if the following holds: For every $a \in \Omega$, there exists a set $N_a \subset \mathbb{R}$ with $\mu^1(N_a) = 0$ such that $\phi \in C(\partial B(a, r);\mathbb{R}^3)$ for all $r \in (0, \mathrm{dist}\,(a; \partial\Omega)) \setminus N_a$,

(i) $\phi(x) \in \mathrm{im}_T(\phi; B(a, r)) \cup \phi(\partial B(a, r))$ for a.e. $x \in \overline{B(a, r)}$, and
(ii) $\phi(x) \in \mathbb{R}^3 \setminus \mathrm{im}_T(\phi; B(a, r))$ for a.e. $x \in \Omega \setminus B(a, r)$.

Condition (INV) is stable under weak convergence in $W^{1,p}$ for $p > 2$ [18, Lemma 3.3], and it implies invertibility almost everywhere when $J_\phi = \det(\nabla\phi) \neq 0$ a.e. [18, Lemma 3.4].

Proposition 3.1. *For $p \geq 3$, we have*

$$\Phi \cap \{\phi \in W^{1,p}(\Omega;\mathbb{R}^3);\ J_\phi = \det\nabla\phi > 0 \text{ a.e. in } \Omega\} \subset \mathrm{INV} \cap \{\phi \in W^{1,p}(\Omega;\mathbb{R}^3);\ J_\phi = \det\nabla\phi > 0 \text{ a.e. in } \Omega\}$$

Proof. Let $\phi \in W^{1,p}(\Omega;\mathbb{R}^3)$, $p \geq 3$, we recall that if $p = 3$ and if $J_\phi = \det\nabla\phi > 0$ a.e. in Ω or if $p > 3$, then ϕ is continuous and has a differential almost everywhere in Ω (in the classical sense). Now, let $\phi \in \Phi \cap \{\phi \in W^{1,p}(\Omega;\mathbb{R}^3);\ J_\phi = \det\nabla\phi > 0 \text{ a.e. in } \Omega\}$, $a \in \Omega$, $r \in (0, \operatorname{dist}(a;\partial\Omega))$, $y \in \phi(B(a,r)) \setminus \phi(\partial B(a,r))$ and let $x \in B(a,r)$ such that $y = \phi(x)$. We know that ϕ is $\mathcal{K}$-homotopic to the natural embedding $i_{\overline{\Omega}}$. Set $K = \{x\} \times \partial B(a,r)$, since K is compact and $K \subset \overline{\Omega} \times \overline{\Omega} \setminus \triangle_\phi$, there exists a homotopy h_K from ϕ to $i_{\overline{\Omega}}$ such that $K \subset \overline{\Omega} \times \overline{\Omega} \setminus \triangle_{h_K(\cdot,t)}$ for every $t \in [0,1]$ and we conclude that $\deg(h_K(\cdot,t),\ B(a,r),\ h_K(x,t))$ is well defined. By virtue of Theorem [8, Theorem 2.6], the value of $\deg(h_K(\cdot,t),\ B(a,r),\ h_K(x,t))$ is independent of t, hence, we obtain $\deg(\phi,\ B(a,r),\ y) = \deg(i_{\overline{\Omega}},\ B(a,r),\ x) = +1$. Notice that if $y \notin \phi(\overline{B(a,r)})$, then $\deg(\phi,\ B(a,r),\ y) = 0$. Hence $\mathrm{im}_T(\phi;B(a,r)) = \{y \in \mathbb{R}^3 \setminus \phi(\partial B(a,r));\ \deg(\phi;B(a,r);y) \neq 0\} = \phi(B(a,r)) \setminus \phi(\partial B(a,r))$. We deduce easily that $\phi \in \mathrm{INV} \cap \{\phi \in W^{1,p}(\Omega;\mathbb{R}^3);\ J_\phi = \det\nabla\phi > 0 \text{ a.e. in } \Omega\}$ □

4. The topological properties of the maps $\mathcal{K}$-homotopic to $i_{\overline{\Omega}}$

We begin to prove the following lemma.

Lemma 4.1. *Let $\phi \in C^0(\Omega;\mathbb{R}^3)$ and let $y \in \phi(\Omega)$. Then for all connected component C_y of $\phi^{-1}(y)$ with $C_y \subset\subset \Omega$ and for every $\epsilon > 0$ there exists an open $G \subset\subset \Omega$ such that $C_y \subset G$, $\operatorname{dist}(C_y,\ \partial G) \leq \epsilon$, and $y \notin \phi(\partial G)$.*

Proof. Since C_y is a compact subset of Ω, it has a positive distance to $\partial\Omega$. Choose an open sets $U \subset\subset \Omega$ with $C_y \subset U \subset\subset \Omega$ and $\operatorname{dist}(C_y,\ \partial U) \leq \epsilon$. The set $L := \overline{U} \cap \phi^{-1}(y)$ is compact. If $L = C_y$ then $\phi^{-1}(y) \cap \partial U = \emptyset$ and we take $G := U$. Otherwise, L is not connected and can be separated into two disjoint compact subsets M and $N = L \setminus M$ such that $C_y \subset M$ and $L \cap \partial U \subset N$. Since M and N have positive distance, there exists $G \subset\subset \Omega$, $G \subset U$ with $C_y \subset M \subset G$ and $N \cap \partial G = \emptyset$. In all cases, we conclude that $C_y \subset G$, $\operatorname{dist}(C_y,\ \partial G) \leq \epsilon$ and $y \notin \phi(\partial G)$. □

A mapping $f : \Omega \to \mathbb{R}^3$ is called quasilight if the connected components of every point-inverse $f^{-1}(y)$ are compact. If f is continuous, quasilight just means that for $y \in f(\Omega)$, there are no connected components of $f^{-1}(y)$ intersecting $\partial\Omega$. By Theorem [26, Theorem 3.1], a mapping $f : \Omega \to \mathbb{R}^3$ is quasilight if and only if every point $x \in \Omega$ has a neighborhood $U \subset\subset \Omega$ such that $f(x) \notin f(\partial U)$. This brings us to consider the set $\Gamma_f := \{x \in \Omega; \exists U \subset\subset \Omega,\ U \text{ open},\ x \in U \text{ and } f(x) \notin f(\partial U)\}$. Then, it is obvious that for every continuous mapping $f : \Omega \to \mathbb{R}^3$, Γ_f is open, by its definition, and the restriction $f|_{\Gamma_f}$ is quasilight. Moreover, since we can always isolate connected components with an arbitrarily close surrounding neighborhood unless they intersect the boundary (Lemma 4.1), we have

$$\Gamma_f = \bigcup_{y \in f(\Omega),\ C_y \in \mathfrak{C}_y(f)} C_y \tag{4.1}$$

where $\mathfrak{C}_y(f)$ denotes the family of all inner connected components of $f^{-1}(y)$, i.e.,

$$\mathfrak{C}_y(f) := \{C_y \subset\subset \Omega;\ C_y \text{ is connected component of } f^{-1}(y)\} \tag{4.2}$$

The next theorem is of particular interest from the point of view of Nonlinear Elasticity, because it provides a major step towards the invertibility of deformations. It summarizes the topological properties of the maps $\mathcal{K}$-homotopic to $i_{\overline{\Omega}}$. It asserts ((ii) below) that the only possible case where the preimage of a value can have several connected components occurs when all of these are contracted by the deformation to boundary points where the deformed configuration is in self-contact .

Theorem 4.1. *Let $\phi \in \Phi \cap W^{1,p}(\Omega;\mathbb{R}^3)$ with $p > 3$. Assume that $J_\phi = \det(\nabla\phi) \neq 0$ almost everywhere in Ω. Then, we have the following assertions:*

(i) $\Gamma_\phi = \Omega \setminus \phi^{-1}(\phi(\partial\Omega))$.
(ii) *If* $y \in \phi(\Omega) \setminus \phi(\partial\Omega)$, *then* $\phi^{-1}(y)$ *is a continuum contained in* Ω, *while if* $y \in \phi(\partial\Omega)$ *then each connected components of* $\phi^{-1}(y)$ *intersects* $\partial\Omega$.
(iii) *For all compact* $K \subset \Omega$ *such that int* $K = \emptyset$, *we have int* $\phi(K) = \emptyset$.

Proof.

(i) Let $y \in \phi(\Omega) \setminus \phi(\partial\Omega)$ and let C_y denote an arbitrary connected component of $\phi^{-1}(y)$. Since $y \notin \phi(\partial\Omega)$, C_y is compact and $\operatorname{dist}(C_y,\ \partial\Omega) > 0$. We deduce that $C_y \in \mathfrak{C}_y(\phi)$ and therefore $C_y \subset \Gamma_\phi$. Hence $\Omega \setminus \phi^{-1}(\phi(\partial\Omega)) \subset \Gamma_\phi$.
Since $\Gamma_\phi \subset \Omega$, to prove that $\Gamma_\phi \subset \Omega \setminus \phi^{-1}(\phi(\partial\Omega))$, it is sufficient to show that $\phi(\Gamma_\phi) \cap \phi(\partial\Omega) = \emptyset$. Let $x_0 \in \partial\Omega$ and suppose by contradiction that $z_0 := \phi(x_0) \in \phi(\Gamma_\phi)$. By the definition of Γ_ϕ, there exists an open $U \subset\subset \Omega$ with $z_0 \in \phi(U) \setminus \phi(\partial U)$, and $\phi(x) = z_0$ for an $x \in U$. Let C denote the connected component of $\phi^{-1}(z_0)$ containing x. Since C is compact and $C \subset U$, we have $C \subset \Gamma_\phi$. In particular, C has positive distance to $\mathbb{R}^3 \setminus \Gamma_\phi \supset \partial\Omega$. By Lemma 4.1, we can find an open $U_0 \subset\subset \Omega$ with $U_0 \subset U$ such that $\overline{U}_0 \subset \Gamma_\phi$ and $z_0 \in \phi(U_0) \setminus \phi(\partial U_0)$. Now choose a sequence $\{\tilde{x}_n\}_{n\geq 1} \subset \Omega$ with $\tilde{x}_n \longrightarrow x_0 \in \partial\Omega$ as $n \to \infty$. Let $\rho_n := \frac{1}{2}\operatorname{dist}(\tilde{x}_n; \partial\Omega) \leq \frac{1}{2}|\tilde{x}_n - x_0| \to 0$. For every $n \geq 1$, let $B_n \subset\subset \Omega$ be an open ball such that $B_n \subset B(\tilde{x}_n, \rho_n)$. Since $J_\phi = \det(\nabla\phi) > 0$ almost everywhere in Ω and $\int_{B_n} J_\phi(x)dx = \mu(\phi(B_n))$, by Theorem 2.1, we have $\mu(\phi(B_n)) > 0$. Moreover, $\phi \in W^{1,p}(\Omega; \mathbb{R}^3)$ with $p > 3$ implies that ϕ has the N-property, with $\mu(\partial B_n) = 0$, we obtain that $\mu(\phi(\partial B_n)) = 0$. Hence, $\phi(B_n) \setminus \phi(\partial B_n) \neq \emptyset$ and therefore we can choose $\tilde{z}_n \in \phi(B_n) \setminus \phi(\partial B_n)$. By the fact that $\phi(\partial B_n)$ is compact, we can find a neighborhood V_n of $\tilde{z}_n$ such that $V_n \cap \phi(\partial B_n) = \emptyset$. Let $b_n \in B_n$ such that $\tilde{z}_n = \phi(b_n)$. We know that ϕ is $\mathcal{K}$-homotopic to the natural embedding $i_{\overline{\Omega}}$. Set $K = \{b_n\} \times \partial B_n$, since K is compact and $K \subset \overline{\Omega} \times \overline{\Omega} \setminus \triangle_\phi$, there exists a homotopy h_K from ϕ to $i_{\overline{\Omega}}$ such that $K \subset \overline{\Omega} \times \overline{\Omega} \setminus \triangle_{h_K(\cdot,t)}$ for every $t \in [0, 1]$ and we conclude that $\deg(h_K(\cdot, t),\ B_n,\ h_K(b_n, t))$ is well defined. By virtue of Theorem [8, Theorem 2.6], the value of $\deg(h_K(\cdot, t),\ B_n,\ h_K(b_n, t))$ is independent of t, hence, we deduce that $\deg(\phi,\ B_n,\ \tilde{z}_n) = \deg(i_{\overline{\Omega}},\ B_n,\ b_n) = +1$. By continuity of the degree and by shrinking V_n if necessary, we can suppose that $\deg(\phi,\ B_n,\ V_n) \neq 0$. Since, $\phi(\partial\Omega)$ has empty interior while V_n is open, there exists $z_n \in V_n$ with $z_n \notin \phi(\partial\Omega)$. As $\deg(\phi;\ B_n;\ z_n) \neq 0$, we have $B_n \cap \phi^{-1}(z_n) \neq \emptyset$. Consequently, there exists at least one connected component C_n of $\phi^{-1}(z_n)$ in B_n, and therefore $C_n \subset \Gamma_\phi$. On the other hand, $\sup_{x\in B_n} |x - x_0| \leq \frac{3}{2}|\tilde{x}_n - x_0| \to 0$ by construction. Since $\overline{U}_0 \subset \Omega$ while $x_0 \in \partial\Omega$, this implies that $B_n \cap U_0 = \emptyset$ for all n sufficiently large, and we also infer that $z_n = \phi(x_n) \to \phi(x_0) = z_0$ by continuity of ϕ, for arbitrary $x_n \in C_n \subset B_n \cap \phi^{-1}(z_n)$. Since $\phi(\partial U_0)$ is compact, we have $z_n \in \phi(U_0) \setminus \phi(\partial U_0)$ for all sufficiently large n. As a consequence, besides $C_n \subset B_n \cap \Gamma_\phi$, $\phi^{-1}(z_n)$ has a second connected component $\tilde{C}_n$, now contained in $U_0 \cap \Gamma_\phi$. C_n and $\tilde{C}_n$, are both compact subsets of Ω. By Lemma 4.1, there exist disjoint open sets $G, \tilde{G} \subset\subset \Omega$ with $C_n \subset G, \tilde{C}_n \subset \tilde{G}$ and $z_n \in (\phi(G) \cap \phi(\tilde{G})) \setminus (\phi(\partial G) \cup \phi(\partial\tilde{G}))$. By additivity of the degree and since that ϕ is $\mathcal{K}$-homotopic to the natural embedding $i_{\overline{\Omega}}$, we obtain that $1 = \deg(\phi,\ G \cup \tilde{G},\ z_n) = \deg(\phi,\ G,\ z_n) + \deg(\phi, \tilde{G},\ z_n) = 2$. This gives us the desired contradiction.

(ii) Let $y \in \phi(\Omega) \setminus \phi(\partial\Omega)$ and suppose that $\phi^{-1}(y)$ is not connected. We therefore have at least two connected components, say C_y and $\tilde{C}_y$, both of which are compact subsets of Ω. By Lemma 4.1, there exist disjoint open sets $G, \tilde{G} \subset\subset \Omega$ with $C_y \subset G, \tilde{C}_y \subset \tilde{G}$ and $y \in (\phi(G) \cap \phi(\tilde{G})) \setminus (\phi(\partial G) \cup \phi(\partial\tilde{G}))$. By additivity of the degree and since that ϕ is $\mathcal{K}$-homotopic to the natural embedding $i_{\overline{\Omega}}$, we obtain that $1 = \deg(\phi,\ G \cup \tilde{G},\ y) = \deg(\phi,\ G,\ y) + \deg(\phi, \tilde{G},\ y) = 2$. This gives us the desired contradiction. Now let $y \in \phi(\partial\Omega)$ and suppose by contradiction that there exists a connected component C_y of $\phi^{-1}(y)$ such that $C_y \cap \partial\Omega = \emptyset$. Then $C_y \in \mathfrak{C}_y(\phi)$ and therefore $C_y \subset \Gamma_\phi = \Omega \setminus \phi^{-1}(\phi(\partial\Omega))$ which means that $y \notin \phi(\partial\Omega)$,

(iii) Let $K \subset \Omega$ be compact with int $K = \emptyset$ and suppose there exists a non-empty open set $U \subset \phi(K)$. In particular, $U \setminus \phi(\partial\Omega) \neq \emptyset$. Since ϕ is continuous, there exists $x_0 \in K$ and an open neighborhood G_0 of x_0 such that $\phi(G_0) \subset U$. Let $B(x_0, r) \subset G_0$ be an open ball. In the proof of Proposition 3.1, we proved that $\operatorname{im}_T(\phi; B(a, r)) = \{y \in \mathbb{R}^3 \setminus \phi(\partial B(a, r));\ \deg(\phi; B(a, r); y) \neq 0\} = \phi(B(a, r)) \setminus$

$\phi(\partial B(a,r))$. By continuity of the degree, $\phi(B(a,r)) \setminus \phi(\partial B(a,r))$ is then an open, in particular int $\phi(G_0) \neq \emptyset$. Choose a non-empty open set $G \subset \tilde{G}_0 := \Omega \cap \phi^{-1}$ (int $\phi(G_0)) \neq \emptyset$ such that $G \cap K = \emptyset$. This is possible because $\tilde{G}_0$ is open and K is closed with empty interior. Now let $x_1 \in G$ and $y := \phi(x_1)$. Since $\phi(G_0) \subset U \subset \phi(K)$, $\phi^{-1}(y)$ also contains a point $x_2 \in K$. Let C_1 and C_2 be the connected components of $\phi^{-1}(y)$ with $x_1 \in C_1$ and $x_2 \in C_2$, respectively. By (ii), we have one of the following two possibilities: $C_1 \subset \Omega$ and $C_1 = C_2$, or $C_1 \cap \partial\Omega \neq \emptyset$ and $C_2 \cap \partial\Omega \neq \emptyset$. In both cases, $\partial G \cap C_1 \neq \emptyset$, because C_1 is connected, contains $x_1 \in G$ and has to reach another point outside of G. With the same argument, we even get that $\partial\tilde{G} \cap C_1 \neq \emptyset$ for all open $\tilde{G} \subset G$ with $x_1 \in \tilde{G}$. In other words, $\phi(x_1) \in \phi(\partial\tilde{G})$ for all $x_1 \in \tilde{G} \subset G$. But as in (i), we can find an open ball $B \subset G$ such that $\phi(B) \setminus \phi(\partial B) \neq \emptyset$, which is a contradiction.

□

5. Global invertibility

Any map $\phi \in W^{1,p}_{loc}(\Omega;\mathbb{R}^3)$, $p > 3$, with $J_\phi = \det(\nabla\phi) > 0$ almost everywhere in Ω is automatically a map of (almost everywhere) finite distortion, with outer distortion :

$$K_O(x) := \frac{|\nabla\phi(x)|^3}{J_\phi(x)} \tag{5.1}$$

The inner distortion of ϕ is defined as

$$K_I(x) := \frac{|\mathrm{adj}\nabla\phi(x)|^3}{J_\phi(x)^2} \tag{5.2}$$

Here, $\mathrm{adj}\nabla\phi(x)$ denotes the adjugate matrix of $\nabla\phi$. We always have that $(K_O)^2 \geq cK_I$ with a constant $c > 0$, because $|F|^2 \geq c\,|\mathrm{adj}F|$ for all $F \in \mathbb{M}^3$. The investigation of maps with finite inner or outer distortion was initiated by [3, Theorem 2] (for $p > 3$) and strongly influenced by [25] (in particular for $p = 3$). Their theory is now well developed [12].

Now, we can state the following global invertibility result for the mappings that are $\mathcal{K}$-homotopic to the natural embedding $i_{\overline{\Omega}}$.

Theorem 5.1. *Let $\phi \in \Phi \cap W^{1,p}(\Omega;\mathbb{R}^3)$ with $p > 3$. Assume that $J_\phi = \det(\nabla\phi) \neq 0$ almost everywhere in Ω and*

$$\int_\Omega |(\nabla\phi(x))^{-1}|^3 J_\phi(x)dx < \infty \tag{5.3}$$

Then ϕ is a homeomorphism from $\Omega \setminus \phi^{-1}(\phi(\partial\Omega))$ onto $\phi(\Omega) \setminus \phi(\partial\Omega)$ and its inverse satisfies $\phi^{-1} \in W^{1,3}(\phi(\Omega) \setminus \phi(\partial\Omega);\mathbb{R}^3)$.

Proof. From $\phi \in W^{1,p}$ we obtain $J_\phi \in L^1$ and, by Theorem 2.1, $J_\phi > 0$ almost everywhere in Ω. which clearly implies that ϕ is a mapping of finite distortion. By simple linear algebra , we know that the adjugate matrix satisfies $\nabla\phi(x)\ \mathrm{adj}\nabla\phi(x) = I\ J_\phi(x)$ and hence for every x such that $J_\phi(x) > 0$ we obtain

$$K_I(x) = \frac{|\mathrm{adj}\nabla\phi(x)|^3}{J_\phi(x)^2} = \frac{|(\nabla\phi(x))^{-1}J_\phi(x)|^3}{J_\phi(x)^2} = |(\nabla\phi(x))^{-1}|^3 J_\phi(x) \tag{5.4}$$

Our assumption (5.3) thus implies that $K_I \in L^1$. Theorem 4.1 shows that $\tilde{\phi} = \phi_{|\Omega\setminus\phi^{-1}(\phi(\partial\Omega))}$ is quasilight thus by Theorem [20, Theroem 1.1], $\tilde{\phi}$ is discrete and open. Now let $y \in \phi(\Omega) \setminus \phi(\partial\Omega)$, since $\tilde{\phi}$ is discrete, all connected components of $\phi^{-1}(y)$ consist of a single point and then by Theorem 4.1, we infer that $\phi^{-1}(y)$ is a singleton. Hence $\tilde{\phi} : \Omega \setminus \phi^{-1}(\phi(\partial\Omega)) \to \phi(\Omega) \setminus \phi(\partial\Omega)$ is bijective and since it is open and continuous, it is a homeomorphism. Hence, by Theorem [12, Theorem 5.9] and Theorem [12, Theorem 7.1], $\phi^{-1} \in W^{1,3}(\phi(\Omega) \setminus \phi(\partial\Omega);\mathbb{R}^3)$ is a mapping of finite distortion and

$$\int_\Omega K_I(x)dx = \int_{\phi(\Omega)\setminus\phi(\partial\Omega)} |\nabla\phi^{-1}(y)|^3 dy \tag{5.5}$$

□

Under higher integrability of the outer distortion we can obtain the following stronger result.

Theorem 5.2. *Let $\phi \in \Phi \cap W^{1,p}(\Omega;\mathbb{R}^3)$ with $p > 3$. Assume that $J_\phi = \det(\nabla\phi) \neq 0$ almost everywhere in Ω and*

$$\int_\Omega \left(\frac{|\nabla\phi(x)|^3}{J_\phi(x)}\right)^q dx < \infty \tag{5.6}$$

for some $q > 2$. Then ϕ is a homeomorphism from Ω onto $\phi(\Omega)$ and its inverse satisfies $\phi^{-1} \in W^{1,3}(\phi(\Omega);\mathbb{R}^3)$.

Proof. From $\phi \in W^{1,p}$ we obtain $J_\phi \in L^1$ and, by Proposition 2.3, $J_\phi > 0$ almost everywhere in Ω. which clearly implies that ϕ is a mapping of finite distortion and it is not constant. Then, by [12, Theorem 3.4], ϕ is discrete and open. Now let $y \in \phi(\Omega)$, since ϕ is discrete, all connected components of $\phi^{-1}(y)$ consist of a single point, and then Theorem 4.1 implies that $\phi(\Omega) \cap \phi(\partial\Omega) = \emptyset$ and that $\phi^{-1}(y)$ is a singleton. Hence $\phi : \Omega \to \phi(\Omega)$ is bijective and since it is open and continuous, it is a homeomorphism. Hence, by Theorem [12, Theorem 5.9], $\phi^{-1} \in W^{1,3}(\phi(\Omega);\mathbb{R}^3)$ is a mapping of finite distortion. □

6. Results on Existence

Let the boundary $\Gamma = \partial\Omega$ be Lipschitz-continuous and let Γ_0 be a relatively open subsets of Γ with area $\Gamma_0 > 0$. Let a stored energy function $W : \Omega \times \mathbb{M}^3_+ \to \mathbb{R}$ be given that satisfies the following assumptions:

(i) Polyconvexity: For almost all $x \in \Omega$, there is a convex function $\mathfrak{W}(x,\cdot) : \mathbb{M}^3 \times \mathbb{M}^3 \times (0,+\infty) \to \mathbb{R}$ such that

$$W(F) = \mathfrak{W}(x, F, \operatorname{adj} F, \det F) \text{ for all } F \in \mathrm{M}^3_+ \tag{6.1}$$

(ii) Measurbility: The function $\mathfrak{W}(\cdot, F, H, \delta) : \Omega \to \mathbb{R}$ is measurable for all $(F, H, \delta) \in \mathbb{M}^3 \times \mathbb{M}^3 \times (0,+\infty)$.

(iii) Behavior as $\det F \to 0^+$:

$$W(F) \to +\infty \text{ as } \det F \to 0^+ \tag{6.2}$$

(iv) Coerciveness: There are constants $a > 0$ and $p > 3$ such that

$$W(F) \geqq a|F|^p \text{ for all } F \in \mathbb{M}^3_+ \tag{6.3}$$

Let the density f of the applied body force be such that the linear form $\phi \in W^{1,p}(\Omega) \to \displaystyle\int f\cdot\phi\,dx$ is continuous, and let $\varphi_0 \in W^{1,p}(\Omega)$ be given in such a way that the set

$$\Phi_p = \{\phi \in \Phi \cap W^{1,p}(\Omega;\mathbb{R}^3);\ \det\nabla\phi \neq 0 \text{ a.e. in } \Omega,\ \phi = \varphi_0 \text{ on } \Gamma_0\} \tag{6.4}$$

is not empty.

Theorem 6.1. *Assume that $\inf\limits_{\phi\in\Phi_p} \mathfrak{I}(\phi) < +\infty$, where the total energy $\mathfrak{I}$ is defined as in (1.2). Then the problem of finding a φ such that*

$$\varphi \in \Phi_p \ \text{ and } \ \mathfrak{I}(\varphi) = \inf_{\phi\in\Phi_p} \mathfrak{I}(\phi) \tag{6.5}$$

has at least one solution. Moreover, the mapping $\varphi : \overline{\Omega} \to \mathbb{R}^3$ is one to one almost everywhere and is strictly orientation-preserving in the sense that $\det\nabla\varphi > 0$ almost everywhere in Ω. More can be said :

(i) If we have in addition that for all $F \in \mathbb{M}^3_+$,

$$W(F) \geq c\frac{|\operatorname{adj} F|^3}{(\det F)^2} \tag{6.6}$$

with a constant $c > 0$, then φ is a homeomorphism from $\Omega \setminus \varphi^{-1}(\varphi(\partial\Omega))$ onto $\varphi(\Omega) \setminus \varphi(\partial\Omega)$ and its inverse satisfies $\varphi^{-1} \in W^{1,3}(\varphi(\Omega) \setminus \varphi(\partial\Omega);\mathbb{R}^3)$.

(ii) If we even have that for all $F \in \mathbb{M}^3_+$,

$$W(F) \geq c' \frac{|F|^{3q}}{(\det F)^q} \tag{6.7}$$

with constants $q > 2$ and $c' > 0$, then φ is a homeomorphism from Ω onto $\varphi(\Omega)$ and its inverse satisfies $\varphi^{-1} \in W^{1,3}(\varphi(\Omega); \mathbb{R}^3)$.

Proof. We only sketch those parts of the proof which are the same as in [2]. First, the total energy $\mathfrak{I}(\phi)$ is well defined for the functions $\phi \in \Phi_p$. Next, since for all $\phi \in \Phi_p$, we have $\det \nabla\phi > 0$ almost everywhere in Ω, then (6.3) requires that there be numbers $\alpha > 0$ and $\beta \in \mathbb{R}$ such that

$$\mathfrak{I}(\phi) \geqq \alpha \|\phi\|^p_{1,p,\Omega} + \beta \ \text{ for all } \phi \in \Phi_p \tag{6.8}$$

Since the mappings $\phi \in W^{1,p} \to \det \nabla\phi \in L^{p/3}$ and $\phi \in W^{1,p} \to \operatorname{adj}\nabla\phi \in L^{p/2}$ are sequentially weakly continuous for $p > 3$, $\mathfrak{W}$ is convex and by the arguments of [2], then we deduce that $\phi \to \int_\Omega W(\nabla\phi)dx$ is sequentially weakly lower semicontinuous on $W^{1,p}(\Omega; \mathbb{R}^3)$. Now let $\{\phi^l\}$ with $\phi^l \in \Phi_p$ be an infimizing sequence for the functional $\mathfrak{I}$. By (6.8), the sequence $\{\phi^l\}$ is bounded in the space $W^{1,p}(\Omega; \mathbb{R}^3)$. Thus there is a subsequence $\{\phi^k\}$ that converges weakly in this space to an element $\varphi \in W^{1,p}(\Omega; \mathbb{R}^3)$ satisfying $\phi = \varphi_0$ on Γ_0. Since $W^{1,p}(\Omega; \mathbb{R}^3)$ is compactly imbedded in the space $C^0(\overline{\Omega}; \mathbb{R}^3)$, then $\{\phi^k\}$ converge in $C^0(\overline{\Omega}; \mathbb{R}^3)$, and as Φ is closed for the $C^0(\overline{\Omega}; \mathbb{R}^2)$ topology, $\varphi \in \Phi$. Since $\phi^k \in \Phi$ and $\det \nabla\phi^k \neq 0$ almost everywhere in Ω, then $\det \nabla\phi^k > 0$ almost everywhere in Ω . The behavior of the stored energy function as $\det F \to 0^+$ then implies that $\det \nabla\varphi > 0$ almost everywhere in Ω ([1]). Hence $\varphi \in \Phi_p$. By Theorem 2.1, the mapping $\varphi : \overline{\Omega} \to \mathbb{R}^3$ is one to one almost everywhere. The remaining assertions follow from Theorem 5.1 and Theorem 5.2 respectively. The assumptions $\int_\Omega |(\nabla\varphi(x))^{-1}|^3 J_\varphi(x)dx < \infty$ and $\int_\Omega \left(\frac{|\nabla\varphi(x)|^3}{J_\varphi(x)}\right)^q dx < \infty$ are obtained from (6.6) or (6.7). $\square$

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