SCALAR AND TENSOR EIGENVALUE BOUNDS FOR CONFORMALLY KÄHLER EINSTEIN FOUR-MANIFOLDS

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Abstract. We prove scalar and tensor eigenvalue estimates for conformally Kähler Einstein four-manifolds. For the Chen–LeBrun–Weber metric, $\mathrm{Ric}_g = \Lambda g$, the first positive scalar eigenvalue satisfies $4\Lambda/3 < \lambda_1^{\mathrm{sc}}(g) < 1951\Lambda/1000$. Thus this metric admits a destabilizing conformal variation, as asserted in Hall–Murphy's Conjecture 5.3. The upper bound follows from boundary moments of an affine quotient on the moment polygon and a positive-coefficient polynomial identity. For any closed Einstein four-manifold conformal to a positive-scalar-curvature Kähler metric, put $m = b_2^- > 0$ and $\rho = \max s_k / \min s_k$, using the complex orientation. We prove $\lambda_m^{\mathrm{TT}} \le (4\Lambda/3)(1-\rho^{-3})$ for the full Lichnerowicz operator, with strict inequality for a nonconstant conformal factor. For the Chen–LeBrun–Weber metric, this gives two TT eigenvalues below $7\Lambda/6$, proves the instability of its normalized Ricci-flat cone, and, together with the scalar bound, gives at least three positive directions for the entropy Hessian modulo diffeomorphisms and scaling. Finally, under $\det W^+ \ge 0$, the stable cones over closed oriented positive Einstein four-manifolds are precisely those over the round sphere and the Fubini–Study projective plane.

Contents

1. Introduction

The Hessian of the shrinker entropy at a positive Einstein metric has distinct restrictions to conformal and transverse-traceless variations. The first is expressed by the scalar Laplacian; the second by the Lichnerowicz operator. We estimate both spectra for the Chen–LeBrun–Weber metric and deduce conformal instability of the compact metric and instability of its normalized Ricci-flat cone.

Page constructed a non-Kähler Einstein metric on $\mathbb{CP}^2\#\overline{\mathbb{CP}}^2$ [27]. Derdziński related the Einstein equation of a conformally Kähler four-metric to the scalar curvature of its Kähler representative [11]. Chen–LeBrun–Weber subsequently constructed the Einstein metric on $\mathbb{CP}^2\#2\overline{\mathbb{CP}}^2$ by finding a bilaterally symmetric critical class of the extremal Calabi functional [8]. LeBrun's classification

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shows that these are the only non-Kähler Einstein–Hermitian metrics on compact complex surfaces [23, Theorem A].

Hall–Haslhofer–Siepmann proved instability of the Page cone and posed the corresponding question for the Chen–LeBrun–Weber cone [14]. Hall–Murphy studied the scalar spectra [15, 16]. Their later numerical calculation identified a Rayleigh quotient below the conformal instability threshold in the exchange-anti-invariant subspace, and led to their Conjecture 5.3 [17]. Biquard–Ozuch proved compact instability by a harmonic-form construction with trace-free tensors [3]. The scalar estimate below is strictly less than 2Λ. The TT estimate, after normalization to $\mathrm{Ric} = 3g$, is strictly less than the cone threshold $15/4$.

The two calculations use different established identities. The tensor argument combines the divergence penalty of [25] with the conformal identities of Biquard–Ozuch. The scalar argument uses an affine quotient of moment functions and a specialization of the weighted toric integration formulas of Apostolov–Maschler [1]. Hall–Murphy had already used Donaldson's boundary formula for scalar spectral bounds and recorded Haslhofer's suggestion to apply it to the CLW spectrum [18]. Here the resulting quotient is bounded by a polynomial identity at the algebraic parameter of the conformally Einstein class.

1.1. **The cohomological estimate.** All manifolds and metrics are smooth. A closed manifold is compact and has no boundary. Let (M^n, g) be closed and connected, with $\mathrm{Ric}_g = \Lambda g$. We use the curvature convention

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, \qquad K(X,Y) = \langle R(X,Y)Y, X\rangle \tag{1.1}$$

for an orthonormal pair. The action on symmetric covariant tensors is

$$(\mathring{R}T)(X,Y) = \sum_i T(R(e_i, X)Y, e_i). \tag{1.2}$$

A local orthonormal frame is denoted by (e_i). All tensor norms are induced by the metric; a two-form has squared norm $|\alpha|^2 = \frac{1}{2}\sum_{i,j}\alpha_{ij}^2$. Integrals on an unoriented manifold use the Riemannian density. In particular, $\mathring{R}(g) = \mathrm{Ric}_g$. Put

$$\Delta = -\operatorname{div}\nabla, \qquad (\delta T)_j = -\nabla^i T_{ij}, \qquad L_g = \nabla^*\nabla + 2\Lambda I - 2\mathring{R}. \tag{1.3}$$

Thus L_g is the full Lichnerowicz operator and $\mathcal{E}_g = L_g - 2\Lambda I$ is the Einstein operator. A tensor is transverse-traceless, or TT, if $\delta T = 0$ and $\mathrm{tr}_g T = 0$. Write $0 = \lambda_0^{\mathrm{sc}}(g) < \lambda_1^{\mathrm{sc}}(g) \leq \cdots$ for the scalar spectrum, with multiplicities. The eigenvalues of L_g on the closed L^2 TT space, counted with multiplicity, are written $\lambda_j^{\mathrm{TT}}(g)$ in increasing order. The realization of this operator is given in Lemma 2.2. An inequality involving λ_j^{TT} includes the existence of at least j TT eigenvalues. For a quadratic form q on a vector space, $\mathrm{ind}^-(q)$ denotes the supremum of the dimensions of linear subspaces on which q is negative definite.

On an oriented four-manifold, b_2^+ and b_2^- are the positive and negative indices of the real intersection form. When a Kähler metric is specified, the orientation is its complex orientation.

Theorem A. *Let (M^4, g) be closed and connected, with $\mathrm{Ric}_g = 3g$. Suppose $g = f_0^2 k_0$, where $f_0 > 0$ is smooth and k_0 is Kähler with positive scalar curvature s_{k_0}. Set*

$$m = b_2^-(M) > 0, \qquad \rho = \frac{\max_M s_{k_0}}{\min_M s_{k_0}}. \tag{1.4}$$

Then

$$\lambda_m^{\mathrm{TT}}(g) \leq 4 - \frac{4}{\rho^3}. \tag{1.5}$$

If f_0 is nonconstant, the inequality is strict. If f_0 is constant, then

$$\dim\ker(L_g|_{\mathrm{TT}}) \geq m. \tag{1.6}$$

The ratio in (1.4) is unchanged by a constant rescaling of k_0. For $\mathrm{Ric}_g = \Lambda g$, $\Lambda > 0$, the corresponding bound is

$$\lambda_m^{\mathrm{TT}}(g) \leq \frac{4\Lambda}{3}(1-\rho^{-3}). \tag{1.7}$$

Indeed, $\widehat{g} = (\Lambda/3)g$ has $\mathrm{Ric}_{\widehat{g}} = 3\widehat{g}$, and the Levi–Civita connections agree. With $c = \Lambda/3$,

$$\mathrm{tr}_{\widehat{g}}\, T = c^{-1}\,\mathrm{tr}_g\, T, \qquad \delta_{\widehat{g}} T = c^{-1}\delta_g T, \qquad L_{\widehat{g}} T = c^{-1} L_g T. \tag{1.8}$$

Thus the TT subspaces agree and eigenvalues scale by c^{-1}.

1.2. **The scalar estimate and conformal instability.**

Theorem B. *Let g be the Chen–LeBrun–Weber Einstein metric with* $\mathrm{Ric}_g = \Lambda g$, $\Lambda > 0$. *Then*

$$\frac{4}{3}\Lambda < \lambda_1^{\mathrm{sc}}(g) < \frac{1951}{1000}\Lambda < 2\Lambda. \tag{1.9}$$

For a first positive scalar eigenfunction ψ, the conformal variation $g_t = (1+t\psi)g$ satisfies $D^2\nu_g(\psi g, \psi g) > 0$. In particular, g is linearly conformally unstable as a shrinking Ricci soliton.

Here ν is Perelman's shrinker entropy; its normalization and the second-variation formula used in the proof are given in Section 7. The lower inequality in (1.9) is the strict Lichnerowicz–Obata bound. The upper inequality gives the conformal-instability assertion of [17, Conjecture 5.3], first stated in 2014. Its proof uses exact moments of the Einstein metric.

Choose the moment coordinates so that the Kähler representative has polygon (4.10), and use the homothetic Einstein metric $g = s^{-2}k$, where $s = \mathrm{Scal}_k$. For $q = x_1 - x_2$, the function q/s is smooth and has zero g-mean. With the four moments in (6.17), the exact identities are

$$\Lambda = \frac{K}{2V}, \qquad \frac{\mathcal{R}_g(q/s)}{\Lambda} = \frac{2JV}{IK} - 2 < \frac{1951}{1000}. \tag{1.10}$$

The required integrals have rational primitives. The conformally Einstein parameter is isolated as the unique root in $(1,\infty)$ of an explicit degree-ten polynomial. Reduction modulo this polynomial, followed by a translation with positive coefficients, proves the strict inequality in (1.10); see Section 6 and Appendix A.

Hall–Murphy's approximate anti-invariant quotient was already below 2Λ [17, Section 5]. The estimate below uses that antisymmetric subspace but does not identify the first eigenvalue or its multiplicity. It is an upper bound for the Einstein metric itself. The scalar-to-tensor operator is that of Cao–Hamilton–Ilmanen [5], written explicitly in [15, Definition 5.2 and Theorem 5.4].

1.3. **The Page and Chen–LeBrun–Weber cones.** Let g_{P} and g_{CLW} denote the Page and Chen–LeBrun–Weber metrics, normalized to have Einstein constant 3. Their existence and conformally Kähler descriptions are due to Page [27], Derdziński [11], and Chen–LeBrun–Weber [8].

Theorem C. *The following inequalities hold:*

$$\lambda_1^{\mathrm{TT}}(g_{\mathrm{P}}) < \frac{7}{2}, \qquad \lambda_2^{\mathrm{TT}}(g_{\mathrm{CLW}}) < \frac{7}{2}. \tag{1.11}$$

Both Ricci-flat cones have infinite negative index on smooth TT tensors with compact support in the regular cone.

Hall–Haslhofer–Siepmann proved the instability of the Page cone and of the relevant Kähler–Einstein cones, and the stability of the Fubini–Study cone [14, Theorems 1.2–1.4]. In the preprint of 2011 they stated the Chen–LeBrun–Weber cone assertion as Conjecture 1.5. Conjecture 1.7 in the same paper concerns the compact metric as a Ricci shrinker. Biquard–Ozuch established the compact instability result [3, Theorem 7]; their conformal identities are used in the proof of Theorem A.

These two stability assertions require different bounds. With $\mathrm{Ric}_g = 3g$, a negative Einstein-operator eigenvalue is equivalent to $\lambda^{\mathrm{TT}}(L_g) < 6$, whereas a negative separated cone variation is obtained when $\lambda^{\mathrm{TT}}(L_g) < 15/4$. The latter value is the Hardy threshold; see [21, Theorem 1.2] and Proposition 8.1. Thus (1.11) gives $15/4 - \lambda_j^{\mathrm{TT}} > 1/4$ for each of the asserted eigenvalues. It does not assert that the full Lichnerowicz eigenvalues are negative. For the Chen–LeBrun–Weber metric it gives two Einstein-operator eigenvalues below $-5/2$.

Hall–Murphy obtained scalar Laplace estimates for these metrics [15, 16]. Their later numerical study also provided evidence for the cone inequality of [14, Section 7.2], by estimating the scalar-curvature derivative appearing in its sufficient condition [17, Section 5.1]. That computation is not used here. The estimate $\rho < 2$ follows instead from polynomial identities for the affine scalar curvature on the moment polygon. The coefficients are compared explicitly with the corrected formulas of [16] in Section 4.2. Numerical scalar and TT spectra for the Page metric have also been studied by Hennigar–Kunduri–Sievers–Wang [19]; their numerical spectral computation is distinct from the estimates proved here.

1.4. **Stable cones under a half-Weyl condition.** We use the following definition of cone stability, with the same transversality requirement as [14]. If $G = dr^2 + r^2 g$, set

$$Q_C(H) = \int_{(0,\infty)\times M} \left(|\nabla^G H|_G^2 - 2\langle \mathring{R}_G H, H\rangle_G \right) dv_G. \tag{1.12}$$

The cone is *stable* if $Q_C(H) \geq 0$ for every smooth compactly supported symmetric tensor with $\delta_G H = 0$. All supports are contained in the regular cone. No boundary condition at its vertex is imposed or required.

Theorem D. *Let (M^4, g) be closed, connected and oriented, with*

$$\mathrm{Ric}_g = 3g, \qquad \det W_g^+ \geq 0. \tag{1.13}$$

Its Ricci-flat cone is stable if and only if (M, g) is isometric, as a Riemannian manifold, to the unit-round S^4 or to $\mathbb{CP}^2$ with the Fubini–Study metric of Einstein constant 3. *For every other metric satisfying* (1.13), *$\pi_1(M)$ is either $\{1\}$ or $\mathbb{Z}/2$. Set*

$$m(M) = \begin{cases} b_2(M) - 1, & \pi_1(M) = 1, \\ b_2(M) + 1, & \pi_1(M) = \mathbb{Z}/2. \end{cases} \tag{1.14}$$

Then $m(M) \geq 1$ and

$$\lambda^{\mathrm{TT}}_{m(M)}(g) < \frac{7}{2}. \tag{1.15}$$

In the second case,

$$\dim\ker(L_g|_{\mathrm{TT}}) \geq b_2(M) + 1. \tag{1.16}$$

Every unstable cone in this theorem has infinite compactly supported TT index.

The geometric classification in Theorem D is supplied by Wu's Theorem 1.3 [29], LeBrun's Einstein–Hermitian classification [23, Theorem A], and his anti-holomorphic double-cover theorem [24, Proposition 1 and Theorem B]. The half-conformally-flat case uses [2, Theorem 13.30]. The spectral estimate for a free anti-holomorphic quotient uses anti-invariant harmonic forms on the cover: their products with the anti-invariant Kähler form are invariant tensors and descend. The dimension of this space is $b_2(M) + 1$; see Proposition 9.2.

Catino–Dameno have extended the conformally Kähler reduction under weaker conditions on the eigenvalues of W^+ [7]. Their result concerns the geometry of the link, not the numerical TT estimate (1.5). We retain the hypothesis $\det W^+ \geq 0$ in Theorem D. No classification without a half-Weyl hypothesis is asserted.

1.5. **The variational argument and its antecedents.** The orthogonal decomposition of metric variations into conformal, diffeomorphism and TT parts has a long history; see Koiso [20] and Besse [2, Chapter 4]. The argument in Section 2 evaluates the full-Lichnerowicz quadratic form on these parts. For an Einstein n-manifold with $\mathrm{Ric}_g = \Lambda g$, $\Lambda > 0$, define, on smooth trace-free tensors,

$$\mathcal{J}_g(T) = \langle L_g T, T\rangle_{L^2} - \frac{n}{n-1}\|\delta T\|_{L^2}^2. \tag{1.17}$$

For $\tau \le n\Lambda/(n-1)$, the exact identity is

$$\operatorname{ind}^-\big(\mathcal{J}_g - \tau\|\cdot\|_{L^2}^2\big) = \#\{\lambda \in \operatorname{spec}(L_g|_{\mathrm{TT}}) : \lambda < \tau\}, \tag{1.18}$$

where eigenvalues on the right are counted with multiplicity. This is the shifted divergence-penalty theorem of [25]. Its proof, including the spectral realization of the TT restriction, is reproduced in Section 2. No elliptic realization of the critically penalized operator on all trace-free tensors is assumed.

The conformal operator has separate antecedents. The Weitzenböck formula used by Biquard–Ozuch comes from Biquard–Rollin [4, (4.6)]. Delay established a conformal covariance formula on symmetric two-tensors [9]; Biquard–Ozuch give the four-dimensional formulation and the harmonic-form identities used below [3, Sections 1–2]. The distinction between conformal covariance and preservation of the TT condition is essential: their weights are different. Related constructions of TT tensors by differential operators are discussed by Delay [10, pp. 27–29, 54–59]. None of these operator constructions is claimed here as a new conformal covariance theorem.

After a constant rescaling of the Kähler representative, write $g = f^2 k$ with $s_k = 6/f$. If Ω_k is the Kähler form and ω is harmonic and anti-self-dual, the tensor $T_\omega = f(\omega \circ_k \Omega_k)$ is trace-free, and

$$\mathcal{J}_g(T_\omega) = \int_M (4 - 2f^{-3})|T_\omega|_g^2 \, dv_g. \tag{1.19}$$

The Einstein equation for the conformal factor yields

$$\Delta_k f = 2f^3 - 1, \qquad \frac{1}{\operatorname{Vol}_k(M)} \int_M f^3 \, dv_k = \frac{1}{2}. \tag{1.20}$$

It follows that $f^{-3} \ge 2/\rho^3$, with a uniform strict inequality when f is nonconstant. Combining (1.18)–(1.20) on the m-dimensional harmonic-form space proves (1.5). The pointwise composition, conformal powers and strictness argument are given in Section 3.

The affine scalar curvatures of the Page and Chen–LeBrun–Weber Kähler representatives are established toric data [14, 15, 16]. Their determination uses Donaldson's boundary integration identity [12, Section 2]. Section 4 recomputes the needed moments and proves $\rho < 2$ throughout the displayed parameter ranges. Neither the affine formulas nor the cone Hardy threshold are claimed as new results. The cone criterion and the broader tensor decomposition are treated in [21, 22]; Section 8 proves the quadratic-form identity for the compactly supported tensors used here.

The principal estimates are (1.5), with its cohomological multiplicity and strict form, and (1.9). The first gives a two-dimensional TT subspace on which $D^2\nu$ is positive definite. The second gives an orthogonal divergence-free representative of a conformal variation with positive second variation. Hence the positive entropy index modulo diffeomorphisms and scaling is at least three (Corollary 7.2). The cone consequence and the quotient dimension of the kernel lead to Theorem D. The geometric classification theorems, conformal covariance, weighted toric identities and entropy second variation are the quoted theorems specified in the corresponding proofs. The scalar identity is a specialization of existing toric formulas; its application gives the strict scalar bound for the CLW metric. For the wider deformation and stability theory, see the survey [28]. The paper concerns closed Einstein links and their regular Ricci-flat cones; it does not treat arbitrary Ricci-flat fillings or unrestricted positive Einstein four-manifolds.

2. A divergence penalty and the low TT spectrum

In this section $n \ge 3$, $\operatorname{Ric}_g = \Lambda g$, and $\Lambda > 0$. All pairings are real. The symbol $H^r(E)$ denotes the Sobolev space of sections of a metric vector bundle E, with derivatives of order at most r in L^2. Unless indicated otherwise, norms and pairings use g. Put

$$Q(T) = \langle L_g T, T\rangle_{L^2}, \qquad c_n = \frac{n}{n-1}, \qquad \mathcal{J}(T) = Q(T) - c_n\|\delta T\|_{L^2}^2. \tag{2.1}$$

For smooth functions and one-forms, define

$$Pv = \nabla^2 v + \frac{\Delta v}{n} g, \qquad \mathcal{K}\omega = \delta^*\omega + \frac{\delta\omega}{n} g, \qquad (\delta^*\omega)_{ij} = \frac{1}{2}(\nabla_i\omega_j + \nabla_j\omega_i). \tag{2.2}$$

Here $\delta\omega = -\nabla^i\omega_i$ and $\mathcal{K}(dv) = Pv$. Both operators in (2.2) take values in the trace-free tensors.

2.1. **Commutation identities.**

Lemma 2.1. *Let $\Delta_H = d\delta + \delta d$ on one-forms. Then*

$$\Delta_H = \nabla^*\nabla + \Lambda I, \qquad \nabla^*\nabla(dv) = d\Delta v - \Lambda\, dv, \tag{2.3}$$

$$2\delta\delta^*\omega = \nabla^*\nabla\omega + d\delta\omega - \Lambda\omega, \qquad \delta P v = d\left(\frac{n-1}{n}\Delta v - \Lambda v\right), \tag{2.4}$$

$$L_g(ag) = (\Delta a)g, \qquad L_g\delta^*\omega = \delta^*\Delta_H\omega, \tag{2.5}$$

$$L_g P v = P\Delta v, \qquad L_g\mathcal{K}\omega = \mathcal{K}\Delta_H\omega, \tag{2.6}$$

$$\operatorname{tr}(L_g T) = \Delta(\operatorname{tr} T), \qquad \delta L_g T = \Delta_H \delta T. \tag{2.7}$$

Proof. Write $R^p{}_{kij}$ for the components of $R(\partial_i,\partial_j)\partial_k$. The covector commutator and its rank-two consequence are

$$\begin{aligned} ([\nabla_i,\nabla_j]\omega)_k &= -R^p{}_{kij}\omega_p, \\ ([\nabla_i,\nabla_j]T)_{k\ell} &= -R^p{}_{kij}T_{p\ell} - R^p{}_{\ell ij}T_{kp}. \end{aligned} \tag{2.8}$$

The first equality follows by applying the definition of curvature to the scalar pairing of a covector and a vector. Applying it to each factor of a covariant tensor gives the second equality. Contracting the first equality gives

$$\nabla^i\nabla_j\omega_i = \nabla_j\nabla^i\omega_i + \operatorname{Ric}_j{}^p\omega_p. \tag{2.9}$$

Expansion of $d\delta\omega + \delta d\omega$ and substitution of (2.9) give the first identity in (2.3). Since $d^2 = 0$, $\Delta_H d = d\Delta$; its application to a function gives the second identity. Expansion of $\delta\delta^*$ gives

$$2(\delta\delta^*\omega)_j = -\nabla^i\nabla_i\omega_j - \nabla^i\nabla_j\omega_i = (\nabla^*\nabla\omega)_j + (d\delta\omega)_j - \Lambda\omega_j. \tag{2.10}$$

Applying this with $\omega = dv$ gives $\delta\nabla^2 v = d\Delta v - \Lambda\, dv$. Since $\delta(ag) = -da$, (2.4) follows.

For a variation $g + tT$, the derivative of the connection is

$$2\dot\Gamma^k_{ij} = \nabla_i T_j{}^k + \nabla_j T_i{}^k - \nabla^k T_{ij}. \tag{2.11}$$

Put $B^k{}_{ij} = \dot\Gamma^k_{ij}$. Differentiating

$$R^\ell{}_{kij} = \partial_i\Gamma^\ell_{jk} - \partial_j\Gamma^\ell_{ik} + \Gamma^\ell_{ip}\Gamma^p_{jk} - \Gamma^\ell_{jp}\Gamma^p_{ik}$$

and using $\Gamma^p_{ij} = \Gamma^p_{ji}$ gives

$$(DR)_g(T)^\ell{}_{kij} = \nabla_i B^\ell{}_{jk} - \nabla_j B^\ell{}_{ik}, \qquad (D\operatorname{Ric})_g(T)_{ij} = \nabla_p B^p{}_{ji} - \nabla_j B^p{}_{pi}. \tag{2.12}$$

In components,

$$\nabla_i B^\ell{}_{jk} = \partial_i B^\ell{}_{jk} + \Gamma^\ell_{ip}B^p{}_{jk} - \Gamma^p_{ij}B^\ell{}_{pk} - \Gamma^p_{ik}B^\ell{}_{jp}.$$

Subtraction cancels the terms containing Γ^p_{ij}. Substitution of (2.11) gives

$$2(D\operatorname{Ric})_g(T)_{ij} = \nabla^p\nabla_j T_{ip} + \nabla^p\nabla_i T_{jp} - \nabla^p\nabla_p T_{ij} - \nabla_j\nabla_i \operatorname{tr} T. \tag{2.13}$$

For the first contracted derivative, (2.8) gives

$$g^{pa}([\nabla_a,\nabla_j]T)_{ip} = -g^{pa}R^q{}_{iaj}T_{qp} - g^{pa}R^q{}_{paj}T_{iq}.$$

The curvature symmetries identify the two contractions as $-(\mathring{R}T)_{ij}$ and $\operatorname{Ric}_j{}^q T_{iq}$, respectively. Thus

$$\nabla^p\nabla_j T_{ip} - \nabla_j\nabla^p T_{ip} = \operatorname{Ric}_j{}^p T_{ip} - (\mathring{R}T)_{ij}. \tag{2.14}$$

Interchanging i, j gives the other contracted derivative; the Hessian of $\operatorname{tr} T$ is symmetric. Hence

$$\begin{aligned} 2(D\operatorname{Ric})_g(T)_{ij} = &- \nabla^p\nabla_p T_{ij} + \nabla_i\nabla^p T_{pj} + \nabla_j\nabla^p T_{pi} - \nabla_i\nabla_j \operatorname{tr} T \\ &+ \operatorname{Ric}_i{}^p T_{pj} + \operatorname{Ric}_j{}^p T_{pi} - 2(\mathring{R}T)_{ij}. \end{aligned} \tag{2.15}$$

Consequently

$$2(D\,\mathrm{Ric})_g(T) = L_g T - 2\delta^*\delta T - \nabla^2 \operatorname{tr} T. \tag{2.16}$$

Naturality under the flow of $\omega^\sharp$ gives $(D\,\mathrm{Ric})_g(\delta^*\omega) = \Lambda\delta^*\omega$. Also $\operatorname{tr}\delta^*\omega = -\delta\omega$. Substitution in (2.16) yields

$$\begin{aligned} L_g\delta^*\omega &= 2\Lambda\delta^*\omega + 2\delta^*\delta\delta^*\omega - \nabla^2\delta\omega \\ &= 2\Lambda\delta^*\omega + \delta^*(\nabla^*\nabla\omega + d\delta\omega - \Lambda\omega) - \nabla^2\delta\omega \\ &= \delta^*(\nabla^*\nabla + \Lambda)\omega. \end{aligned} \tag{2.17}$$

The last equality uses $\delta^* d(\delta\omega) = \nabla^2(\delta\omega)$. The identity for ag follows from $\nabla g = 0$ and $\mathring{R}(g) = \Lambda g$.

Taking $\omega = dv$ gives $L_g\nabla^2 v = \nabla^2\Delta v$. Hence

$$\begin{aligned} L_g P v &= \nabla^2\Delta v + \frac{\Delta^2 v}{n} g = P\Delta v, \\ L_g \mathcal{K}\omega &= \delta^*\Delta_H\omega + \frac{\Delta\delta\omega}{n} g = \mathcal{K}\Delta_H\omega. \end{aligned} \tag{2.18}$$

For the last equality, $\delta\Delta_H = \delta d\delta = \Delta\delta$ follows from $\delta^2 = 0$ and from the definition of the scalar Laplacian. For a test function a and a test one-form ω, formal self-adjointness and integration by parts give

$$\begin{aligned} \langle \operatorname{tr}(L_g T), a\rangle_{L^2} &= \langle L_g T, ag\rangle_{L^2} = \langle T, (\Delta a)g\rangle_{L^2} = \langle \Delta \operatorname{tr} T, a\rangle_{L^2}, \\ \langle \delta L_g T, \omega\rangle_{L^2} &= \langle L_g T, \delta^*\omega\rangle_{L^2} = \langle T, \delta^*\Delta_H\omega\rangle_{L^2} = \langle \Delta_H\delta T, \omega\rangle_{L^2}. \end{aligned} \tag{2.19}$$

These equalities prove (2.7). □

2.2. **The TT realization**. The analytic results used in this subsection are stated explicitly. For a smooth strongly elliptic differential operator A of order two on a vector bundle over a closed manifold, weak elliptic regularity and the global elliptic estimate give

$$u \in H^1, \quad Au \in H^r \quad \Longrightarrow \quad u \in H^{r+2}, \qquad \|u\|_{H^{r+2}} \le C_r(\|Au\|_{H^r} + \|u\|_{L^2}), \quad r \in \mathbb{Z}_{\ge 0}. \tag{2.20}$$

The equation is interpreted in distributions. We also use the compact embedding $H^{r+1} \hookrightarrow H^r$ on a closed manifold and the spectral theorem for compact self-adjoint operators. The compact embedding is [13, Lemma 1.3.4(a), p. 29]. For (2.20), the parametrix construction in [13, Lemma 1.3.5 and the vector-bundle extension, p. 30] gives an operator E of order -2 and a smoothing operator S such that $EA = I - S$. Hence, in distributions,

$$u = E(Au) + Su, \qquad \|E(Au)\|_{H^{r+2}} \le C_r\|Au\|_{H^r}, \qquad \|Su\|_{H^{r+2}} \le C_r\|u\|_{L^2}. \tag{2.21}$$

The mapping properties are [13, Lemma 1.3.4(c), pp. 29–30]. They prove both weak regularity and the estimate in (2.20). The compact self-adjoint spectral theorem is [13, Lemma 1.6.2, pp. 42–43]; its application to elliptic operators on vector bundles is [13, Lemma 1.6.3, pp. 43–45]. The page numbers refer to the 1984 edition.

Lemma 2.2. *The restriction of L_g to distributionally TT tensors is self-adjoint on*

$$H^2(S^2T^*M) \cap \mathcal{T}, \qquad \mathcal{T} = \{T \in L^2(S^2T^*M) : \operatorname{tr} T = 0,\ \delta T = 0 \text{ in distributions}\}. \tag{2.22}$$

It is bounded below and has compact resolvent. Its eigentensors are smooth. Its eigenvalues obey the min–max formula on smooth TT subspaces.

Proof. The trace is continuous from L^2 to L^2, and divergence is continuous from L^2 to H^{-1}. Thus $\mathcal{T}$ is closed. Write $L_g = \nabla^*\nabla + V$, where $V = 2\Lambda I - 2\mathring{R}$ is a smooth self-adjoint bundle endomorphism. Set $C = \max_M |V|_{\mathrm{op}}$ and choose $b > C + 1$. Then, on $H^1(S^2T^*M)$,

$$q_b(T) := Q(T) + b\|T\|_{L^2}^2 \ge \|\nabla T\|_{L^2}^2 + \|T\|_{L^2}^2. \tag{2.23}$$

Here Q is extended to H^1 by its integral formula, and $q_b(S,T)$ denotes the symmetric polarization of q_b. The integral formula gives $|q_b(S,T)| \le C_b\|S\|_{H^1}\|T\|_{H^1}$. Together with (2.23), this verifies the continuity and coercivity hypotheses of Lax–Milgram. That theorem gives, for every $U \in L^2$, a unique weak solution $V_U \in H^1$ of $(L_g + b)V_U = U$. The principal symbol of $L_g + b$ is $|\xi|^2 I$. Equation (2.20), applied with $r = 0$, gives $V_U \in H^2$. Pairing the weak equation with V_U first yields $\|V_U\|_{H^1} \le \|U\|_{L^2}$ by (2.23). The same regularity estimate then implies $\|V_U\|_{H^2} \le C_b\|U\|_{L^2}$. Thus $A_b : U \mapsto V_U$ is a bounded map $L^2 \to H^2$. The Rellich embedding $H^2 \hookrightarrow L^2$ makes A_b compact. For $U_1, U_2 \in L^2$, the weak equations give

$$\langle A_b U_1, U_2\rangle_{L^2} = q_b(A_b U_1, A_b U_2) = \langle U_1, A_b U_2\rangle_{L^2}.$$

Moreover $\langle A_b U, U\rangle_{L^2} = q_b(A_b U) > 0$ when $U \neq 0$. Consequently A_b is self-adjoint, positive and injective. Its range is exactly H^2: one inclusion follows from the estimate above; for $V \in H^2$, put $U = (L_g + b)V \in L^2$ and apply uniqueness of the weak solution to obtain $A_b U = V$. Moreover, $(\operatorname{ran} A_b)^\perp = \ker A_b = 0$. Let $A_b e_j = \rho_j e_j$ be its compact-operator spectral resolution, where $\rho_j > 0$. The inverse has domain

$$\operatorname{dom}(A_b^{-1}) = \left\{ \sum_j a_j e_j \;:\; \sum_j \rho_j^{-2}|a_j|^2 < \infty \right\} = \operatorname{ran} A_b = H^2. \tag{2.24}$$

In this basis A_b^{-1} is multiplication by the real numbers ρ_j^{-1}, and is therefore self-adjoint on this domain. Thus $A_b^{-1} - bI$ is the self-adjoint realization of L_g.

If $U \in \mathcal{T}$ and $V = (L_g + b)^{-1}U$, then (2.7), interpreted weakly, gives

$$(\Delta + b)\operatorname{tr} V = 0, \qquad (\nabla^*\nabla + \Lambda + b)\delta V = 0. \tag{2.25}$$

Pairing with $\operatorname{tr} V$ and δV, respectively, gives

$$\|d \operatorname{tr} V\|_{L^2}^2 + b\|\operatorname{tr} V\|_{L^2}^2 = 0, \qquad \|\nabla\delta V\|_{L^2}^2 + (\Lambda + b)\|\delta V\|_{L^2}^2 = 0. \tag{2.26}$$

The second pairing is legitimate since $V \in H^2$ and $\delta V \in H^1$. Thus $V \in \mathcal{T}$. For $U \in \mathcal{T}^\perp$ and $Z \in \mathcal{T}$, $\langle A_b U, Z\rangle_{L^2} = \langle U, A_b Z\rangle_{L^2} = 0$. Hence A_b also preserves $\mathcal{T}^\perp$. Its restriction to $\mathcal{T}$ is compact, positive and injective, and

$$\operatorname{ran}(A_b|_{\mathcal{T}}) = H^2 \cap \mathcal{T}. \tag{2.27}$$

Indeed, the inclusion from left to right has been proved. If $V \in H^2 \cap \mathcal{T}$, then $U = (L_g + b)V$ is TT by (2.7), and $A_b U = V$. The range is dense in $\mathcal{T}$, since its orthogonal complement is the kernel of $A_b|_{\mathcal{T}}$. Let $A_b|_{\mathcal{T}} e_j = \sigma_j e_j$ be its spectral resolution. Then $\sigma_j > 0$, and

$$\operatorname{dom}\big((A_b|_{\mathcal{T}})^{-1}\big) = \left\{ \sum_j a_j e_j \;:\; \sum_j \sigma_j^{-2}|a_j|^2 < \infty \right\} = H^2 \cap \mathcal{T}. \tag{2.28}$$

The inverse acts by σ_j^{-1} on e_j. Subtracting bI gives the self-adjoint TT realization of L_g and its orthonormal eigenbasis. The elliptic estimate

$$\|T\|_{H^{r+2}} \le C_r(\|L_g T\|_{H^r} + \|T\|_{L^2}) \qquad (r \ge 0) \tag{2.29}$$

iterated on an eigentensor gives smoothness. The orthogonal projection onto $\mathcal{T}$ commutes with A_b. It therefore commutes with $(L_g + b)^{1/2}$ on its domain. Since q_b has domain H^1, the restricted form has domain $H^1 \cap \mathcal{T}$. Its spectral partial sums converge in the q_b norm, which is equivalent to the H^1 norm by (2.23) and the continuity bound above. Thus, if (e_j) is an orthonormal TT eigenbasis and $H = \sum_j a_j e_j \in H^1 \cap \mathcal{T}$, then

$$\|H\|_{L^2}^2 = \sum_j a_j^2, \qquad Q(H) = \sum_j \lambda_j^{\mathrm{TT}} a_j^2. \tag{2.30}$$

The second series is interpreted by the lower-bounded quadratic form. The span of $e_1, \dots, e_m$ has maximal Rayleigh quotient λ_m^{TT}. Every m-dimensional subspace contains a nonzero vector orthogonal to $e_1, \dots, e_{m-1}$, whose Rayleigh quotient is at least λ_m^{TT}. This proves the min–max formula on smooth TT subspaces. To identify the negative index, let q be the number of TT eigenvalues strictly below τ. It is finite: the corresponding resolvent eigenvalues satisfy $\rho_j > (b + \tau)^{-1}$ when $b + \tau > 0$, and no such eigenvalues occur when $\tau \le -b$. The span of these q eigentensors is negative for $Q - \tau\|\cdot\|^2$. Any space of dimension greater than q contains a nonzero vector orthogonal to them. Equation (2.30) makes the latter vector's shifted quadratic form nonnegative. Thus this index is exactly q. $\square$

2.3. **Orthogonal decomposition.**

Proposition 2.3. *Every smooth trace-free symmetric tensor has a decomposition*

$$T = H + P\psi + V, \qquad H \in \mathrm{TT}, \qquad V = \delta^*\eta, \qquad \delta\eta = 0. \tag{2.31}$$

The three tensor summands are unique and depend linearly on T. They are pairwise orthogonal for the L^2 inner product and for the bilinear form induced by Q. Their divergences are also orthogonal. Moreover,

$$Q(P\psi) = c_n\|\delta P\psi\|_{L^2}^2 + c_n\Lambda\|P\psi\|_{L^2}^2, \tag{2.32}$$

$$Q(V) = 2\|\delta V\|_{L^2}^2 + 2\Lambda\|V\|_{L^2}^2. \tag{2.33}$$

Proof. On trace-free tensors, $\mathcal{K}^* = \delta$. By (2.4),

$$\mathcal{B} := \delta\mathcal{K} = \frac{1}{2}\Delta_H - \Lambda I + \left(\frac{1}{2} - \frac{1}{n}\right) d\delta. \tag{2.34}$$

With the convention $\sigma(\partial_j)(\xi) = i\xi_j$, its principal symbol is

$$\sigma_2(\mathcal{B})(\xi) = \frac{1}{2}|\xi|^2 I + \left(\frac{1}{2} - \frac{1}{n}\right) \xi \otimes \xi. \tag{2.35}$$

It has eigenvalue $|\xi|^2/2$ on $\xi^\perp$ and $(n-1)|\xi|^2/n$ on $\mathbb{R}\xi$. It is therefore elliptic. Furthermore,

$$\langle \mathcal{B}\omega, \omega\rangle_{L^2} = \|\mathcal{K}\omega\|_{L^2}^2, \qquad \ker\mathcal{B} = \ker\mathcal{K}. \tag{2.36}$$

For $\zeta \in \ker\mathcal{B}$, $\langle \delta T, \zeta\rangle_{L^2} = \langle T, \mathcal{K}\zeta\rangle_{L^2} = 0$. The elliptic spectral theorem for $\mathcal{B}$ on the closed manifold applies: its symbol is positive by (2.35), it is formally self-adjoint, and its form is nonnegative by (2.36). Let (ζ_j) be an orthonormal eigenbasis, $\mathcal{B}\zeta_j = \beta_j\zeta_j$, with $\beta_j \geq 0$. The orthogonality just proved permits the series

$$\omega = \sum_{\beta_j > 0} \beta_j^{-1}\langle \delta T, \zeta_j\rangle_{L^2}\zeta_j. \tag{2.37}$$

Write $u_j = \langle \delta T, \zeta_j\rangle_{L^2}$ and $\beta_* = \min\{\beta_j \,:\, \beta_j > 0\} > 0$. Then

$$\sum_{\beta_j>0} |\beta_j^{-1}u_j|^2 \leq \beta_*^{-2}\|\delta T\|_{L^2}^2, \qquad \sum_{\beta_j>0} \beta_j^2|\beta_j^{-1}u_j|^2 = \|\delta T\|_{L^2}^2. \tag{2.38}$$

Thus the series belongs to the operator domain, satisfies $\mathcal{B}\omega = \delta T$, and is orthogonal to $\ker\mathcal{B}$. The elliptic regularity in (2.20), iterated with smooth right-hand side, gives $\omega \in C^\infty$. If two such solutions existed, their difference would belong to both $\ker\mathcal{B}$ and its orthogonal complement, and would be zero. Set $H = T - \mathcal{K}\omega$. Then H is TT and is orthogonal to $\operatorname{im}\mathcal{K}$.

Since $\int_M \delta\omega\, dv_g = 0$, solve $\Delta\psi = \delta\omega$ with $\int_M \psi\, dv_g = 0$. The scalar Laplacian has kernel the constants, so this solution exists and is unique. Set $\eta = \omega - d\psi$. Then $\delta\eta = 0$, and $\mathcal{K}\omega = P\psi + \delta^*\eta$. Equation (2.34) gives

$$\mathcal{B}(dv) = d\left(\frac{n-1}{n}\Delta v - \Lambda v\right), \qquad \delta\mathcal{B}\omega = \left(\frac{n-1}{n}\Delta - \Lambda\right)\delta\omega \quad (\omega \in C^\infty(T^*M)). \tag{2.39}$$

Indeed, $\delta\Delta_H\omega = \Delta\delta\omega$ and $\delta d\delta\omega = \Delta\delta\omega$; the coefficient of $\Delta\delta\omega$ is $1/2 + (1/2 - 1/n) = (n-1)/n$. In particular, $\delta\eta = 0$ implies $\delta\mathcal{B}\eta = 0$. Thus $\mathcal{B}$ preserves exact and coclosed one-forms. These spaces are orthogonal, since $\langle du, \eta\rangle_{L^2} = \langle u, \delta\eta\rangle_{L^2} = 0$. It follows that

$$\langle P\psi, \delta^*\eta\rangle_{L^2} = \langle d\psi, \mathcal{B}\eta\rangle_{L^2} = 0, \qquad \langle \delta P\psi, \delta\delta^*\eta\rangle_{L^2} = 0. \tag{2.40}$$

The identities $L_g(Pv) = P\Delta v$, $L_g(\delta^*\eta) = \delta^*\Delta_H\eta$ and $\delta\Delta_H\eta = 0$ show that L_g preserves the last two spaces. Equation (2.7) gives preservation of the TT space. For summands T_i, T_j in distinct spaces, $\langle L_g T_i, T_j\rangle_{L^2} = 0$ because $L_g T_i$ remains in the space of T_i. This proves the asserted Q orthogonality. If two decompositions exist, subtract them and pair with each of the three differences. Each difference has zero squared norm. This proves uniqueness of the tensor summands and therefore their linear dependence on T.

Put $\theta = (n-1)/n$ and $u = \theta\Delta\psi - \Lambda\psi$. Then $\delta P\psi = du$ and

$$\|P\psi\|_{L^2}^2 = \langle du, d\psi\rangle_{L^2}, \qquad Q(P\psi) = \langle P\psi, P\Delta\psi\rangle_{L^2} = \langle du, d\Delta\psi\rangle_{L^2}. \tag{2.41}$$

Differentiation of the definition of u gives $d\Delta\psi = c_n(du + \Lambda\, d\psi)$, proving (2.32). For the coclosed part, (2.34) gives

$$\delta V = \mathcal{B}\eta = \tfrac{1}{2}\Delta_H\eta - \Lambda\eta, \qquad \|V\|_{L^2}^2 = \langle \eta, \mathcal{B}\eta\rangle_{L^2}. \tag{2.42}$$

Thus

$$Q(V) = \langle \Delta_H\eta, \mathcal{B}\eta\rangle_{L^2} = 2\|\mathcal{B}\eta\|_{L^2}^2 + 2\Lambda\langle \eta, \mathcal{B}\eta\rangle_{L^2}, \tag{2.43}$$

which proves (2.33). ☐

Theorem 2.4 (Shifted index identity). *For every $\tau \le c_n\Lambda$,*

$$\operatorname{ind}^-\big(\mathcal{J} - \tau\|\cdot\|_{L^2}^2\big) = \#\{j : \lambda_j^{\mathrm{TT}}(g) < \tau\}. \tag{2.44}$$

The index on the left is the supremum of the dimensions of subspaces of smooth trace-free tensors on which the indicated form is negative definite.

Proof. Substitute (2.31) into (2.1). Orthogonality and (2.32)–(2.33) give

$$\begin{aligned}\mathcal{J}(T) - \tau\|T\|_{L^2}^2 &= Q(H) - \tau\|H\|_{L^2}^2 + (c_n\Lambda - \tau)\|P\psi\|_{L^2}^2 \\ &\quad + (2-c_n)\|\delta V\|_{L^2}^2 + (2\Lambda - \tau)\|V\|_{L^2}^2.\end{aligned} \tag{2.45}$$

Since $n \ge 3$, $c_n < 2$. Indeed, $2 - c_n = (n-2)/(n-1) > 0$, $c_n\Lambda - \tau \ge 0$, and $2\Lambda - \tau \ge (2-c_n)\Lambda > 0$. If W is a negative subspace for the form on the left, the map $T \mapsto H$ is injective on W: a nonzero element in its kernel would have nonnegative energy. Its image is negative for $Q - \tau\|\cdot\|_{L^2}^2$. Conversely, on TT tensors the two forms agree. Lemma 2.2 identifies the latter index with the right-hand side of (2.44). ☐

Corollary 2.5. *Suppose a space $\mathcal{V}$ of smooth trace-free tensors has dimension m and, for some $\beta < c_n\Lambda$,*

$$\mathcal{J}(T) \le \beta\|T\|_{L^2}^2 \qquad (T \in \mathcal{V}). \tag{2.46}$$

Then $\lambda_m^{\mathrm{TT}}(g) \le \beta$. If the inequality in (2.46) *is strict for every nonzero T, then $\lambda_m^{\mathrm{TT}}(g) < \beta$.*

Proof. For every τ with $\beta < \tau < c_n\Lambda$, $\mathcal{V}$ is negative for $\mathcal{J} - \tau\|\cdot\|_{L^2}^2$. Theorem 2.4 gives at least m TT eigenvalues below τ. Letting $\tau \downarrow \beta$ gives the first conclusion. In the strict case, the set $\{T \in \mathcal{V} : \|T\|_{L^2} = 1\}$ is compact. Let

$$\beta_0 = \max_{\substack{T\in\mathcal{V}\\ \|T\|_{L^2}=1}} \mathcal{J}(T). \tag{2.47}$$

Continuity gives an attaining tensor T_0; the strict hypothesis at T_0 gives $\beta_0 < \beta$. Homogeneity implies $\mathcal{J}(T) \le \beta_0\|T\|_{L^2}^2$ on $\mathcal{V}$. The first conclusion, applied to β_0, yields $\lambda_m^{\mathrm{TT}} \le \beta_0 < \beta$. ☐

Remark 2.6 (The critical coefficient). The coefficient c_n cannot be increased while preserving finite negative index on the subspace $\operatorname{im} P$. Let $\Delta v = \mu v$ and $\|v\|_{L^2} = 1$. Equation (2.3) gives $\|\nabla^2 v\|^2 = \langle \nabla^*\nabla dv, dv\rangle = (\mu - \Lambda)\|dv\|^2 = \mu(\mu-\Lambda)$. Since $\operatorname{tr}\nabla^2 v = -\mu v$ and $|g|^2 = n$, expansion of Pv gives

$$\|Pv\|_{L^2}^2 = \frac{n-1}{n}\mu^2 - \Lambda\mu, \qquad \|\delta Pv\|_{L^2}^2 = \left(\frac{n-1}{n}\mu - \Lambda\right)\|Pv\|_{L^2}^2. \tag{2.48}$$

Consequently

$$Q(Pv) - \gamma\|\delta Pv\|_{L^2}^2 = \left[\left(1 - \gamma\frac{n-1}{n}\right)\mu + \gamma\Lambda\right]\|Pv\|_{L^2}^2. \tag{2.49}$$

For $\gamma > c_n$ this coefficient is negative at every sufficiently large scalar eigenvalue. If v_i, v_j are orthonormal scalar eigenfunctions with eigenvalues μ_i, μ_j, integration by parts gives

$$\begin{aligned}\langle Pv_i, Pv_j\rangle_{L^2} &= \left(\frac{n-1}{n}\mu_i - \Lambda\right)\mu_j\langle v_i, v_j\rangle_{L^2}, \\ \langle \delta Pv_i, \delta Pv_j\rangle_{L^2} &= \left(\frac{n-1}{n}\mu_i - \Lambda\right)\left(\frac{n-1}{n}\mu_j - \Lambda\right)\mu_j\langle v_i, v_j\rangle_{L^2}.\end{aligned} \tag{2.50}$$

Both pairings vanish for $i \neq j$. For all sufficiently large i, $\|Pv_i\|^2 > 0$ by (2.48). Their finite spans have arbitrarily large negative dimension. No ellipticity assertion for the critically penalized operator on all trace-free tensors is used in Theorem 2.4.

3. Conformal harmonic forms and a spectral bound

3.1. **The conformal identity**. For an oriented metric k in dimension four, write $\Lambda^2 = \Lambda^+ \oplus \Lambda^-$. Associate to a two-form α the skew-adjoint endomorphism A_α defined by $k(A_\alpha X, Y) = \alpha(X, Y)$. For $\alpha \in \Lambda^-$ and $\beta \in \Lambda^+$, put

$$(\alpha \circ_k \beta)(X, Y) = k(A_\alpha A_\beta X, Y). \tag{3.1}$$

The endomorphisms of opposite duality commute. Their product is self-adjoint and has zero trace. Use the bases

$$\begin{aligned} \Lambda^+ : &\quad e^{12} + e^{34}, \quad e^{13} - e^{24}, \quad e^{14} + e^{23}, \\ \Lambda^- : &\quad e^{12} - e^{34}, \quad e^{13} + e^{24}, \quad e^{14} - e^{23}. \end{aligned} \tag{3.2}$$

Denote the associated matrices by I_a^+ and I_b^-. Their entries are obtained from $(A_\alpha)^i{}_j = \alpha_{ji}$, and multiplication gives

$$(I_a^\pm)^\top = -I_a^\pm, \qquad I_a^+ I_b^- = I_b^- I_a^+, \qquad \operatorname{tr}(I_a^+ I_b^-) = 0 \quad (1 \le a, b \le 3). \tag{3.3}$$

Expanding A_α and A_β in these bases proves commutation and zero trace. Also $(A_\alpha A_\beta)^\top = A_\beta A_\alpha = A_\alpha A_\beta$, which proves symmetry.

If k is Kähler, its Kähler form Ω satisfies $A_\Omega = J$, $J^2 = -I$, and $\nabla^k \Omega = 0$. Therefore

$$|\omega \circ_k \Omega|_k^2 = 2|\omega|_k^2. \tag{3.4}$$

Indeed, multiplication by the orthogonal endomorphism J preserves the Hilbert–Schmidt norm, and $|A_\omega|_{\mathrm{HS}}^2 = 2|\omega|_k^2$. Thus $\omega \mapsto \omega \circ_k \Omega$ is injective.

We use the following conformal operator identity, independently of the TT projection in Section 2. Delay's conformally covariant tensor operator [9] precedes the construction used here. The conformal identities in the next proposition are those of Biquard–Ozuch. Its conformal covariance is the external input specified below.

Proposition 3.1 (Biquard–Ozuch). *There is a natural second-order operator $\mathcal{P}_k$ on trace-free symmetric tensors over any oriented Riemannian four-manifold such that, for a positive function f,*

$$\mathcal{P}_{f^2 k}(T) = f^{-1} \mathcal{P}_k(f^{-1} T). \tag{3.5}$$

If g is Einstein, then

$$\frac{1}{2}\nabla_g^* \nabla_g - \mathring{R}_g = \mathcal{P}_g + \frac{2}{3}\mathcal{K}_g \delta_g - \frac{\mathrm{Scal}_g}{12} I. \tag{3.6}$$

If k is Kähler and ω is a harmonic anti-self-dual two-form, then

$$\mathcal{P}_k(\omega \circ_k \Omega) = -\frac{s_k}{6}(\omega \circ_k \Omega), \qquad \delta_k(\omega \circ_k \Omega) = 0. \tag{3.7}$$

Equations (3.5) and (3.6) are [3, Proposition 3 and equation (12)]. Their symbol L denotes the operator on the left of (3.6), not L_g in (1.3); their δ_0^* is $\mathcal{K}_g$. Let d_-^* denote the adjoint of the covariant exterior derivative followed by anti-self-dual projection. It maps $\Lambda^- \otimes \Lambda^+$ to $T^*M \otimes \Lambda^+$. Define

$$\begin{aligned} \pi(\alpha \otimes \beta) &= \alpha \wedge \beta, & \sigma(\zeta) &= \sum_i e^i \otimes (e_i \lrcorner \zeta)_+, \\ \mathcal{D}_k &= \left(I - \frac{2}{3}\sigma\pi\right) d_-^*, & \mathcal{P}_k &= \mathcal{D}_k^* \mathcal{D}_k - W_k^+. \end{aligned} \tag{3.8}$$

Here $\pi : T^*M \otimes \Lambda^+ \to \Lambda^3 T^*M$ and $\sigma : \Lambda^3 T^*M \to T^*M \otimes \Lambda^+$. For a three-form ζ,

$$\begin{aligned} \pi\sigma\zeta &= \frac{1}{2}\sum_i e^i \wedge (e_i \lrcorner \zeta) + \frac{1}{2}\sum_i e^i \wedge * (e_i \lrcorner \zeta) = \frac{3}{2}\zeta, \\ \langle \sigma\zeta, \alpha \otimes \beta \rangle &= \langle \zeta, \alpha \wedge \beta \rangle \qquad (\alpha \in T^*M,\ \beta \in \Lambda^+). \end{aligned} \tag{3.9}$$

The first sum is 3ζ before multiplication by $1/2$. The second is zero because $*(e_i \lrcorner \zeta) = e^i \wedge * \zeta$. The second line proves $\sigma^* = \pi$. Hence $\Pi = I - \frac{2}{3}\sigma\pi$ satisfies $\Pi^* = \Pi$, $\Pi^2 = \Pi$, $\pi\Pi = 0$, and $\Pi|_{\ker \pi} = I$. It is the orthogonal projection onto $\ker \pi$. The action of W_k^+ on $\Lambda^- \otimes \Lambda^+$ is on the second factor. These are the constructions in [3, equations (7)–(13)], transported by (3.1). On $\omega \otimes \Omega$, d_-^* vanishes because $\delta\omega = 0$ and $\nabla\Omega = 0$. The Kähler curvature identity $W_k^+(\Omega) = (s_k/6)\Omega$ therefore gives the first equality in (3.7); this is [3, equation (26)]. For the divergence, the chosen composition is $(\omega \circ_k \Omega)_{ij} = \omega_{pi} J^p{}_j$. Since $\nabla J = 0$ and $\omega_{pi} = -\omega_{ip}$,

$$\big(\delta_k(\omega \circ_k \Omega)\big)_j = -(\nabla^i \omega_{pi}) J^p{}_j = (\nabla^i \omega_{ip}) J^p{}_j = 0. \tag{3.10}$$

The identity (3.5) applies to arbitrary Riemannian four-metrics, whereas (3.6) is used only for the Einstein metric g.

3.2. **Normalization and the trial space.** The conformal Kähler–Einstein normalization of Derdziński [11, Proposition 5], in the form [3, equation (22)], says that $f_0 s_{k_0}$ is a positive constant. It includes the constant-conformal-factor case. If that constant is c, set

$$a = (c/6)^{1/3}, \qquad k = a^2 k_0, \qquad f = f_0/a. \tag{3.11}$$

Since $s_{a^2 k_0} = a^{-2} s_{k_0}$, this choice gives $f s_k = a^{-3} f_0 s_{k_0} = a^{-3} c = 6$. Therefore

$$g = f^2 k, \qquad s_k = 6 f^{-1}, \qquad \frac{f_{\max}}{f_{\min}} = \frac{\max s_k}{\min s_k} = \rho. \tag{3.12}$$

The homothety in (3.11) changes neither the complex orientation nor the scalar-curvature ratio.

Let

$$\mathcal{H}_k^- = \{\omega \in C^\infty(\Lambda^2 T^* M) \,:\, d\omega = 0,\ \delta_k \omega = 0,\ *_k\, \omega = -\omega\}. \tag{3.13}$$

By the Hodge theorem, $\dim \mathcal{H}_k^- = b_2^-(M)$. Under a conformal change in dimension four the Hodge star on two-forms is unchanged: the factor for degree p is f^{4-2p}, which is one when $p = 2$. A closed anti-self-dual two-form is coclosed because $\delta = - * d *$ on two-forms. Hence the space in (3.13) is the same for k and g. For $\omega \in \mathcal{H}_k^-$ define

$$\widetilde{T}_\omega = \omega \circ_k \Omega, \qquad T_\omega = f \widetilde{T}_\omega. \tag{3.14}$$

$$\operatorname{tr}_g T_\omega = f^{-1} \operatorname{tr}_k \widetilde{T}_\omega = 0, \qquad |T_\omega|_g^2 = f^{-2} |\widetilde{T}_\omega|_k^2 = 2 f^{-2} |\omega|_k^2. \tag{3.15}$$

Thus both tensors are trace-free and the map $\omega \mapsto T_\omega$ is injective.

Lemma 3.2. *For every $\omega \in \mathcal{H}_k^-$,*

$$\mathcal{P}_g T_\omega = -f^{-3} T_\omega \tag{3.16}$$

and

$$\mathcal{J}_g(T_\omega) = \int_M (4 - 2 f^{-3}) |T_\omega|_g^2 \, dv_g. \tag{3.17}$$

Here $\mathcal{J}_g = Q_g - \frac{4}{3} \|\delta_g(\cdot)\|_{L^2}^2$.

Proof. Apply (3.5) and (3.7) to (3.14). Since $s_k/6 = f^{-1}$,

$$\mathcal{P}_g T_\omega = f^{-1} \mathcal{P}_k \widetilde{T}_\omega = -f^{-2} \widetilde{T}_\omega = -f^{-3} T_\omega. \tag{3.18}$$

For $\mathrm{Ric}_g = 3g$, $\mathrm{Scal}_g = 12$, and (3.6) becomes

$$L_g = 2\mathcal{P}_g + \frac{4}{3} \mathcal{K}_g \delta_g + 4I. \tag{3.19}$$

For trace-free tensors T, $\langle \mathcal{K}_g \delta_g T, T \rangle_{L^2} = \|\delta_g T\|_{L^2}^2$. Pair (3.19) with T_ω and use (3.16). Subtracting the divergence term gives (3.17). □

Lemma 3.3. *The function f satisfies*

$$\Delta_k f = 2f^3 - 1, \qquad \frac{1}{\mathrm{Vol}_k(M)} \int_M f^3 \, dv_k = \frac{1}{2}. \tag{3.20}$$

Consequently

$$f^{-3} \geq \frac{2}{\rho^3}. \tag{3.21}$$

If f is nonconstant, the inequality in (3.21) *is strict at every point.*

Proof. For $u = \log f$, the connection difference is

$$\nabla^{f^2 k}_X Y - \nabla^k_X Y = du(X)Y + du(Y)X - k(X,Y)\nabla^k u. \tag{3.22}$$

In local coordinates, put $C^p_{ij} = \delta^p_i u_j + \delta^p_j u_i - k_{ij} u^p$ and $D_k u = \nabla^p \nabla_p u = -\Delta_k u$. Then

$$\begin{aligned} \nabla_p C^p_{ij} &= 2\nabla_i \nabla_j u - k_{ij} D_k u, \quad \nabla_j C^p_{ip} = 4\nabla_i \nabla_j u, \\ C^p_{pq} C^q_{ij} - C^p_{jq} C^q_{ip} &= 2u_i u_j - 2|du|^2_k k_{ij}. \end{aligned} \tag{3.23}$$

The curvature difference is the difference of the first two terms plus the last term. Thus

$$\mathrm{Ric}_{f^2 k} = \mathrm{Ric}_k - 2\nabla^2_k u + 2du \otimes du + (\Delta_k u - 2|du|^2_k)k. \tag{3.24}$$

Take the $f^2 k$ trace. This gives

$$\mathrm{Scal}_{f^2 k} = f^{-2}(s_k + 6\Delta_k u - 6|du|^2_k) = f^{-3}(6\Delta_k f + s_k f). \tag{3.25}$$

The second equality uses $\Delta_k \log f = f^{-1}\Delta_k f + f^{-2}|df|^2_k$. Substitute $\mathrm{Scal}_g = 12$ and $s_k f = 6$, and integrate on the closed manifold. This proves (3.20). It follows that

$$f^3_{\min} \leq \frac{1}{2}, \qquad f^3_{\max} = \rho^3 f^3_{\min} \leq \frac{\rho^3}{2}. \tag{3.26}$$

This proves (3.21). If f is nonconstant, choose x_0 with $f(x_0) > f_{\min}$. Continuity gives an open neighborhood U of x_0 and $d > 0$ such that $f^3 - f^3_{\min} \geq d$ on U. Thus $\int_M (f^3 - f^3_{\min})\, dv_k \geq d\,\mathrm{Vol}_k(U) > 0$. The first inequality in (3.26) is then strict, as are the second and (3.21). □

Proof of Theorem A. By Lemmas 3.2 and 3.3,

$$\mathcal{J}_g(T_\omega) \leq \left(4 - \frac{4}{\rho^3}\right) \|T_\omega\|^2_{L^2}. \tag{3.27}$$

The tensors (3.14) form a space of dimension m. The coefficient in (3.27) is strictly less than $4 = c_4\Lambda$. Corollary 2.5 proves (1.5). If f is nonconstant, put

$$\epsilon = 2f^{-3}_{\max} - \frac{4}{\rho^3} > 0. \tag{3.28}$$

Equations (3.17) and (3.21) give $\mathcal{J}_g(T_\omega) \leq (4 - 4/\rho^3 - \epsilon)\|T_\omega\|^2$. Applying Corollary 2.5 proves the strict conclusion. If f is constant, (3.20) gives $f^3 = 1/2$. Then $\delta_g T_\omega = 0$, by (3.7) and constant rescaling, while (3.19) gives $L_g T_\omega = (4 - 2f^{-3})T_\omega = 0$. Injectivity proves (1.6). □

Corollary 3.4. *Under the hypotheses of Theorem A, $\rho^3 < 16$ implies that the Ricci-flat cone is unstable. If $\rho < 2$, then $\lambda^{\mathrm{TT}}_m(g) < 7/2$.*

Proof. The inequality $4 - 4/\rho^3 < 15/4$ is equivalent to $\rho^3 < 16$. The separated cone calculation is Proposition 8.1. If $\rho < 2$, then $4 - 4/\rho^3 < 7/2$. □

4. Scalar-curvature ratios from toric moments

The two calculations in this section use moment polygons only. For a toric Kähler surface with affine scalar curvature, the affine scalar function, up to a positive constant depending on the angular-period convention, is characterized by

$$\int_P \ell\, p\, dx = \int_{\partial P} p\, d\sigma \qquad \text{for every affine function } p. \tag{4.1}$$

Here $d\sigma$ is lattice boundary measure. The identity is Donaldson's integration formula, in the normalization of [14, Lemma 7.3]; its affine specialization is all that is used. A positive constant multiple of ℓ has the same ratio of extrema. Uniqueness in (4.1) follows because the difference of two solutions is affine and has zero squared integral over P.

4.1. **The Page trapezoid.** The Kähler metric conformal to the Page metric has moment polygon

$$T_a = \{(x_1, x_2) : x_1 \ge 0,\ x_2 \ge 0,\ a \le x_1 + x_2 \le 1\}, \qquad 0 < a < 1, \tag{4.2}$$

as in [14, Section 7.1]. Put $t = x_1 + x_2$. Integration over $0 \le x_1 \le t$, $a \le t \le 1$, gives

$$\int_{T_a} t^j\, dx = \frac{1 - a^{j+2}}{j+2} \quad (j = 0, 1, 2), \qquad \int_{\partial T_a} d\sigma = 3 - a, \qquad \int_{\partial T_a} t\, d\sigma = 2. \tag{4.3}$$

For the last two integrals the coordinate edges contribute $2(1-a)$ and $1 - a^2$, respectively, while the edges $t = 1$ and $t = a$ contribute $1 + a$ and $1 + a^2$. If the normalized affine function is $A_0 t + B_0$, (4.1) gives

$$A_0 \frac{1-a^3}{3} + B_0 \frac{1-a^2}{2} = 3 - a, \qquad A_0 \frac{1-a^4}{4} + B_0 \frac{1-a^3}{3} = 2. \tag{4.4}$$

Its solution is

$$A_0 = \frac{24a}{(1-a)(a^2+4a+1)}, \qquad B_0 = \frac{6(1-3a^2)}{(1-a)(a^2+4a+1)}. \tag{4.5}$$

To verify both equations, write $D_P = (1-a)(a^2+4a+1)$. After multiplication by D_P, their left sides are, respectively,

$$\begin{aligned} 8a(1-a^3) + 3(1-3a^2)(1-a^2) &= (3-a)D_P, \\ 6a(1-a^4) + 2(1-3a^2)(1-a^3) &= 2D_P. \end{aligned} \tag{4.6}$$

Each equality follows by multiplying the displayed polynomials. Since $D_P > 0$ for $0 < a < 1$, these equations verify the solution. Uniqueness follows from (4.1). The scalar curvature is therefore a positive multiple of

$$\ell_a(t) = 4at + 1 - 3a^2. \tag{4.7}$$

The slope is positive, so

$$\rho_{\mathrm{P}} = \frac{\ell_a(1)}{\ell_a(a)} = \frac{1 + 4a - 3a^2}{1 + a^2}. \tag{4.8}$$

The denominator is positive and

$$2(1+a^2) - (1 + 4a - 3a^2) = 5a^2 - 4a + 1 = 5\left(a - \frac{2}{5}\right)^2 + \frac{1}{5} > 0. \tag{4.9}$$

Therefore $\rho_{\mathrm{P}} < 2$ for every $0 < a < 1$. No value of the conformally Einstein parameter is required.

4.2. **The Chen–LeBrun–Weber pentagon.** The Kähler representative of the Chen–LeBrun–Weber metric is bilaterally symmetric. Up to homothety, its polygon is

$$P_a = \operatorname{conv}\{(0,0), (a,0), (a,1), (1,a), (0,a)\}, \qquad a > 1. \tag{4.10}$$

The existence and symmetry are part of [8]; the polygon normalization is [14, Section 7.2]. Symmetry and uniqueness in (4.1) give

$$\ell(x_1, x_2) = A(x_1 + x_2) + B. \tag{4.11}$$

Write $I_{ij} = \int_{P_a} x_1^i x_2^j\, dx$.

Lemma 4.1. *The moments needed for* (4.1) *are*

$$I_{00} = \frac{a^2+2a-1}{2}, \qquad I_{10} = I_{01} = \frac{a^3+3a^2-1}{6}, \tag{4.12}$$

$$I_{20} = I_{02} = \frac{a^4+4a^3-1}{12}, \qquad I_{11} = \frac{a^4+4a^3+6a^2-4a-1}{24}. \tag{4.13}$$

The boundary moments are

$$E_0 = \int_{\partial P_a} d\sigma = 1+3a, \qquad E_1 = \int_{\partial P_a} x_1\, d\sigma = \int_{\partial P_a} x_2\, d\sigma = a^2+a. \tag{4.14}$$

Proof. The polygon is the square $[0,a]^2$ minus the open triangle $x_1+x_2 > a+1$. For nonnegative integers i, j,

$$I_{ij} = \frac{a^{i+j+2}}{(i+1)(j+1)} - \frac{1}{j+1}\int_1^a x^i \left[a^{j+1} - (a+1-x)^{j+1}\right] dx. \tag{4.15}$$

For $j=0$, this is

$$I_{i0} = \frac{a^{i+2}}{i+1} - \frac{a^{i+2}-1}{i+2} + \frac{a^{i+1}-1}{i+1}. \tag{4.16}$$

Substitution of $i = 0, 1, 2$ gives I_{00}, I_{10}, I_{20}. For $i = j = 1$, the removed integral is

$$\frac{1}{2}\int_1^a x\,[a^2-(a+1-x)^2]dx = \frac{5a^4-4a^3-6a^2+4a+1}{24}. \tag{4.17}$$

Subtracting this from $a^4/4$ gives I_{11}. Interchange of coordinates gives I_{01} and I_{02}.

The five edge measures are $a, 1, a-1, 1, a$. On the diagonal edge $x_2 = a+1-x_1$, the primitive normal is $(1,1)$, so its lattice measure is dx_1, not its Euclidean arclength. The corresponding contributions to the first x_1 moment are

$$\frac{a^2}{2}, \quad a, \quad \frac{a^2-1}{2}, \quad \frac{1}{2}, \quad 0. \tag{4.18}$$

Their sum is a^2+a; the edge measures sum to $1+3a$. □

Proposition 4.2. *For every $a > 1$, the affine solution of* (4.1) *on P_a is positive and satisfies*

$$\frac{\max_{P_a} \ell}{\min_{P_a} \ell} < 2. \tag{4.19}$$

Proof. The tests $p = 1, x_1$ give

$$2I_{10}A + I_{00}B = E_0, \qquad (I_{20}+I_{11})A + I_{10}B = E_1. \tag{4.20}$$

Set $x = a-1 > 0$ and define

$$D(x) = x^6+12x^5+54x^4+120x^3+138x^2+72x+12, \tag{4.21}$$

$$U(x) = x^5+12x^4+44x^3+72x^2+50x+8, \tag{4.22}$$

$$V(x) = x^5+8x^4+24x^3+36x^2+26x+8. \tag{4.23}$$

Substitution of Lemma 4.1 gives the identities

$$I_{00}(I_{20}+I_{11}) - 2I_{10}^2 = \frac{D(x)}{144}, \tag{4.24}$$

$$E_1 I_{00} - E_0 I_{10} = -\frac{x(x^2+3x+3)}{6}, \tag{4.25}$$

$$E_0(I_{20}+I_{11}) - 2E_1 I_{10} = \frac{U(x)}{24}. \tag{4.26}$$

Set $J = I_{20}+I_{11}$ and $\mathfrak{d} = I_{00}J - 2I_{10}^2 = D(x)/144 > 0$. Multiplying the second equation in (4.20) by I_{00} and subtracting I_{10} times the first gives $\mathfrak{d}A = E_1 I_{00} - E_0 I_{10}$. Multiplying the first by J and subtracting $2I_{10}$ times the second gives $\mathfrak{d}B = E_0 J - 2E_1 I_{10}$. Substitution of (4.25)–(4.26) yields

$$A = -\frac{24x(x^2+3x+3)}{D(x)} < 0, \qquad B = \frac{6U(x)}{D(x)}. \tag{4.27}$$

The linear function $x_1 + x_2$ has minimum 0 and maximum $a + 1 = x + 2$ on P_a. Hence

$$\ell_{\max} = B = \frac{6U(x)}{D(x)}, \qquad \ell_{\min} = B + (x+2)A = \frac{6V(x)}{D(x)}. \tag{4.28}$$

The last equality follows from

$$U(x) - 4x(x+2)(x^2 + 3x + 3) = V(x). \tag{4.29}$$

All coefficients of V are positive, so $\ell_{\min} > 0$. Finally,

$$2V(x) - U(x) = x^5 + 4x^4 + 4x^3 + 2x + 8 > 0. \tag{4.30}$$

Dividing by $V(x) > 0$ proves (4.19). □

Remark 4.3. The scalar curvature in the normalization of Hall–Murphy [16] is twice the affine function ℓ used above. In the parameter a, set

$$D_a = a^6 + 6a^5 + 9a^4 + 4a^3 - 3a^2 - 6a + 1. \tag{4.31}$$

Expansion of $D(a-1)$ gives D_a. The coefficients above satisfy

$$\begin{aligned} 2A &= \frac{48(1-a^3)}{D_a}, \\ 2B &= \frac{12(a^5 + 7a^4 + 6a^3 + 2a^2 - 5a - 3)}{D_a}. \end{aligned} \tag{4.32}$$

The first expression is the corrected coefficient in the published erratum. The second is its unchanged constant term. Multiplication by two leaves $\max \ell / \min \ell$ unchanged. Thus the scalar-ratio calculation uses the corrected affine data, independently of a numerical approximation to the Einstein metric.

Proof of the spectral assertion in Theorem C. In the complex orientations, the intersection forms of the Page and Chen–LeBrun–Weber manifolds are $\langle 1\rangle \oplus \langle -1\rangle$ and $\langle 1\rangle \oplus 2\langle -1\rangle$. Thus their values of b_2^- are 1 and 2. Equations (4.9) and (4.19) give $\rho < 2$. Theorem A gives (1.11). □

5. Affine moment potentials and the scalar spectrum

5.1. **Toric conventions.** Let (M^4, k) be a closed connected toric Kähler surface, with moment map $\mu : M \to P \subset \mathbb{R}^2$. The torus coordinates have period 2π. On the inverse image of the polygon interior,

$$k = u_{ij}\, dx_i\, dx_j + U^{ij}\, d\theta_i\, d\theta_j, \qquad (U^{ij}) = (u_{ij})^{-1}, \qquad dv_k = dx_1\, dx_2\, d\theta_1\, d\theta_2. \tag{5.1}$$

The symplectic potential has boundary expression $u = \frac{1}{2}\sum_\alpha l_\alpha \log l_\alpha + v$, with v smooth on P. Here l_α are the primitive affine defining functions of the facets. Lattice boundary measure on each facet is Euclidean arclength divided by the length of its primitive normal. In these conventions, Donaldson's integration identity is

$$\int_P U^{ij}\Phi_{ij}\, dx = 2\int_{\partial P} \Phi\, d\sigma - \int_P s_k \Phi\, dx \qquad (\Phi \in C^\infty(\overline{P})). \tag{5.2}$$

We use [12, Section 2] in the normalization of [17, Proposition 2.1]; that proposition explicitly records the factor 2 associated with the factor $1/2$ in u. In particular, the function ℓ in (4.1) is $s_k/2$. The complement of the dense torus orbit is the union of the inverse images of the facets and vertices: the facet preimages are two-dimensional invariant submanifolds and the vertex preimages are points. There are finitely many. This complement has four-dimensional Riemannian measure zero. Thus integration of a torus-invariant function over M is $(2\pi)^2$ times its integral over P.

An affine function on P will also denote its smooth pullback by μ. Let w be a positive affine function and set $h = w^{-2}k$. Assume that h has constant scalar curvature κ. The Einstein condition is not imposed in the following proposition.

Proposition 5.1 (Weighted affine energy). *For affine functions q, r on P, put*

$$\mathsf{A}(q,r) = \int_P qrw^{-6}\,dx, \qquad \mathsf{B}(q,r) = \int_{\partial P} qrw^{-4}\,d\sigma. \tag{5.3}$$

Then

$$\frac{1}{4\pi^2}\int_M (q/w)(r/w)\,dv_h = \mathsf{A}(q,r), \tag{5.4}$$

$$\frac{1}{4\pi^2}\int_M \langle d(q/w), d(r/w)\rangle_h\,dv_h = \mathsf{B}(q,r) - \frac{\kappa}{2}\mathsf{A}(q,r). \tag{5.5}$$

Proof. Since $h = w^{-2}k$ in real dimension four,

$$dv_h = w^{-4}dv_k, \qquad |d(q/w)|_h^2\,dv_h = w^{-2}|d(q/w)|_k^2\,dv_k. \tag{5.6}$$

Multiplication of the first equality by $(q/w)(r/w)$ and integration over the angular coordinates gives (5.4). Both sides of (5.5) are symmetric bilinear forms in q, r. Their equality follows from equality on the diagonal by $4B(q,r) = B(q+r, q+r) - B(q-r, q-r)$. Define

$$\begin{aligned} A &= \int_P w^{-4}|dq|_k^2\,dx, & B &= \int_P qw^{-5}\langle dq, dw\rangle_k\,dx,\\ C &= \int_P q^2w^{-6}|dw|_k^2\,dx, & T &= \int_P s_k q^2 w^{-4}\,dx. \end{aligned} \tag{5.7}$$

The squared norms in these integrals mean contraction with U^{ij} on the polygon interior. They are integrable because the underlying functions are smooth on the compact manifold. Set $I = \mathsf{A}(q,q)$ and $J = \mathsf{B}(q,q)$. The energy divided by $4\pi^2$ is $E = A - 2B + C$. Affineness of q and w gives

$$(q^2w^{-4})_{ij} = 2w^{-4}q_iq_j - 8qw^{-5}(q_iw_j + w_iq_j) + 20q^2w^{-6}w_iw_j. \tag{5.8}$$

Apply (5.2) to q^2w^{-4}, which is smooth on $\overline{P}$ since $w > 0$. The result is

$$2A - 16B + 20C = 2J - T. \tag{5.9}$$

Write $D_k = \operatorname{div}_k \nabla = -\Delta_k$. Substituting $f = w^{-1}$ in (3.25) gives

$$\kappa = w^2 s_k + 6wD_kw - 12|dw|_k^2. \tag{5.10}$$

All factors in the vector field $q^2w^{-5}\nabla^k w$ are smooth on M. Since M has no boundary,

$$\int_M \operatorname{div}_k(q^2w^{-5}\nabla^k w)\,dv_k = 0.$$

Multiply (5.10) by q^2w^{-6} and integrate. Expanding the displayed divergence and dividing by $4\pi^2$ gives

$$\int_P q^2w^{-5}D_kw\,dx = -\int_P \langle d(q^2w^{-5}), dw\rangle_k\,dx = -2B + 5C. \tag{5.11}$$

Consequently

$$\kappa I = T - 12B + 18C. \tag{5.12}$$

The two linear identities yield

$$E = A - 2B + C = J - \tfrac{1}{2}T + 6B - 9C = J - \tfrac{\kappa}{2}I. \tag{5.13}$$

Replace q successively by $q + r$ and $q - r$, subtract the two identities, and divide by 4. This proves (5.5). □

Remark 5.2. Proposition 5.1 is a specialization of the weighted toric integration identities of Apostolov–Maschler [1, equations (22), (30)–(31)]. Their formula in complex dimension two reads

$$\int_P w^{-3}U^{ij}\Phi_{ij}\,dx = 2\int_{\partial P}\Phi w^{-3}\,d\sigma - \kappa\int_P \Phi w^{-5}\,dx. \tag{5.14}$$

For $F = q/w$ and $\Phi = q^2/w$, differentiation gives $\Phi_{ij} = 2wF_iF_j$. Thus (5.14) also proves (5.5). The proof above derives the same conclusion from the unweighted identity and the conformal scalar-curvature law.

5.2. **A two-dimensional Rayleigh–Ritz problem.** Let $\mathrm{Aff}(P)$ be the three-dimensional real vector space of affine functions on P, and put

$$\mathcal{A}_0 = \left\{ q \in \mathrm{Aff}(P) \, : \, \int_P q w^{-5} \, dx = 0 \right\}. \tag{5.15}$$

The functional defining this kernel is nonzero: its value at $q = w$ is $\int_P w^{-4} \, dx > 0$. Thus $\dim \mathcal{A}_0 = 2$. For $q \in \mathcal{A}_0$, q/w has zero h-mean. A nonzero such function is not constant, since $q/w = c$ would give $c \int_P w^{-4} \, dx = 0$ and hence $q = 0$.

Corollary 5.3. *Let $r_1 \le r_2$ be the two generalized eigenvalues of $(\mathsf{B} - \frac{\kappa}{2}\mathsf{A}, \mathsf{A})$ on $\mathcal{A}_0$. Then $r_1, r_2 > 0$ and*

$$\lambda_j^{\mathrm{sc}}(h) \le r_j \qquad (j = 1, 2). \tag{5.16}$$

Moreover, if $V_w = \int_P w^{-4} \, dx$ and $K_w = \int_{\partial P} w^{-2} \, d\sigma$, then

$$\kappa = \frac{2K_w}{V_w}. \tag{5.17}$$

Proof. The form A is positive definite on $\mathrm{Aff}(P)$: an affine function with zero squared integral on a polygon with nonempty interior is zero. On $\mathcal{A}_0$, (5.5) identifies the other form with the Dirichlet energy of q/w. This energy vanishes only for constant functions on the connected manifold, which were excluded above. Choose an A-orthonormal eigenbasis q_1, q_2 of this symmetric pair, ordered by $r_1 \le r_2$. For real a_i, not all zero,

$$\mathcal{R}_h\left(\frac{\sum_{i=1}^j a_i q_i}{w} \right) = \frac{\sum_{i=1}^j r_i a_i^2}{\sum_{i=1}^j a_i^2} \le r_j, \qquad j = 1, 2. \tag{5.18}$$

The map $q \mapsto q/w$ is injective, and its image here has zero mean. These j-dimensional spaces are admissible in scalar min–max, which gives (5.16). This principle follows from the compact scalar resolvent by the same eigenbasis argument as (2.30). Finally, insert $q = r = w$ in (5.5). Its left-hand side is zero, so $K_w = (\kappa/2)V_w$. □

6. An exact scalar bound for the Chen–LeBrun–Weber metric

6.1. **The Einstein class.** In this section the Einstein metric is represented as

$$g = s^{-2}k, \qquad s = \mathrm{Scal}_k > 0, \qquad \mathrm{Ric}_g = \Lambda g. \tag{6.1}$$

We fix the scale of k by (4.10). This fixes the homothety class representative $g = s^{-2}k$ and its constant Λ; no numerical value of Λ is imposed. If k is multiplied by $t > 0$, then s is multiplied by t^{-1} and $s^{-2}k$ by t^3. Both $\lambda_1^{\mathrm{sc}}(g)$ and Λ are then multiplied by t^{-3}. Their ratio is unchanged.

Let D_a be (4.31). By (4.32), write

$$\begin{aligned} s_a(t) &= b(a) + c(a)t, \qquad t = x_1 + x_2, \\ c(a) &= \frac{48(1 - a^3)}{D_a}, \\ b(a) &= \frac{12(a^5 + 7a^4 + 6a^3 + 2a^2 - 5a - 3)}{D_a}. \end{aligned} \tag{6.2}$$

The identities $c = 2A$ and $b = 2B$ follow from $s_k = 2\ell$ in (5.2). Proposition 4.2 implies $s_a > 0$ on P_a for all $a > 1$.

View M as the blow-up of $\mathbb{CP}^1 \times \mathbb{CP}^1$ at a torus-fixed point. Write $F_1, F_2, E \in H^2(M; \mathbb{Z})$ for the two ruling classes and the exceptional class, respectively. Their intersections are

$$F_1^2 = F_2^2 = 0, \quad F_1 \cdot F_2 = 1, \quad F_1 \cdot E = F_2 \cdot E = 0, \quad E^2 = -1. \tag{6.3}$$

In the angular-period convention of (5.1), the area of a toric divisor is 2π times its lattice edge length. This follows by integrating $dx \wedge d\theta$ along the edge, with $0 \le \theta < 2\pi$. The two uncut edges of P_a have length a, and the exceptional edge has length $a-1$. Pairing with F_1, F_2, E therefore gives

$$\frac{[\omega_k]}{2\pi} = a(F_1+F_2) - (a-1)E, \qquad x = a-1 > 0. \tag{6.4}$$

Indeed, if $[\omega_k]/(2\pi) = uF_1 + vF_2 - wE$, its three pairings are v, u, w, respectively; the edge areas give $u = v = a$ and $w = a-1$. Thus the class in (6.4) is $(1+x)(F_1+F_2) - xE$, the class used in [8, Section 3]. The constant 2π has no effect on the Calabi energy in real dimension four: $\mathrm{Scal}_{tk} = t^{-1}\,\mathrm{Scal}_k$ and $dv_{tk} = t^2\,dv_k$. The CLW construction supplies an extremal metric in a critical class of this family; see [8, Proposition 7, Corollary 8 and Theorem 27]. Since $dx/da = 1$, its parameter is a critical point of the function $C(a)$ defined below. The moments of t are

$$\begin{aligned} M_0 &= \frac{a^2+2a-1}{2}, \qquad & M_1 &= \frac{a^3+3a^2-1}{3}, \\ M_2 &= \frac{3a^4+12a^3+6a^2-4a-3}{12}. \end{aligned} \tag{6.5}$$

They follow from $M_0 = I_{00}$, $M_1 = 2I_{10}$ and $M_2 = 2I_{20} + 2I_{11}$. Define the affine Calabi function by

$$C(a) = \int_{P_a} s_a^2\,dx = b^2M_0 + 2bcM_1 + c^2M_2 = \frac{24N(a)}{D_a}, \tag{6.6}$$

where

$$N(a) = 3a^6 + 14a^5 + 17a^4 + 12a^3 - 5a^2 - 6a - 3. \tag{6.7}$$

For an extremal metric with this polygon, $4\pi^2 C(a)$ is its Calabi energy. The numerator of its derivative satisfies

$$N'(a)D_a - N(a)D_a' = 4\mathfrak{p}(a), \qquad C'(a) = \frac{96\mathfrak{p}(a)}{D_a^2}, \tag{6.8}$$

where

$$\begin{aligned} \mathfrak{p}(a) = a^{10} &+ 5a^9 + 6a^8 - 12a^7 - 34a^6 - 42a^5 \\ &+ 20a^3 + 21a^2 - 7a - 6. \end{aligned} \tag{6.9}$$

Thus the parameter of the CLW metric is a root of $\mathfrak{p}$ in $(1,\infty)$. Only this implication from the geometric construction is used.

Lemma 6.1. *There is exactly one root a_* of $\mathfrak{p}$ in $(1,\infty)$. It satisfies*

$$\frac{231}{118} < a_* < 2. \tag{6.10}$$

Proof. For $z = a-1$, the binomial formula gives

$$\begin{aligned} \mathfrak{p}(1+z) = z^{10} &+ 15z^9 + 96z^8 + 336z^7 + 680z^6 + 720z^5 \\ &+ 120z^4 - 624z^3 - 708z^2 - 300z - 48. \end{aligned} \tag{6.11}$$

For $z > 0$, divide by z^3 and put $H(z) = z^{-3}\mathfrak{p}(1+z)$. Then

$$\begin{aligned} H'(z) = 7z^6 &+ 90z^5 + 480z^4 + 1344z^3 + 2040z^2 + 1440z + 120 \\ &+ 708z^{-2} + 600z^{-3} + 144z^{-4} > 0. \end{aligned} \tag{6.12}$$

Moreover $\lim_{z\downarrow 0} H(z) = -\infty$ and $\lim_{z\to\infty} H(z) = +\infty$. Continuity and strict monotonicity give exactly one positive zero. At that zero, $(\mathfrak{p}(1+z))' = z^3H'(z) > 0$, so the zero is simple. For the rational lower bound, evaluation gives

$$\mathfrak{p}(231/118) = -\frac{262235040278610439893}{118^{10}} < 0, \qquad \mathfrak{p}(2) = 288 > 0. \tag{6.13}$$

The first numerator is also obtained without rational division: write $\mathfrak{p}(a) = \sum_{j=0}^{10} p_j a^j$, set $r_{10} = 1$, and put

$$r_j = 231 r_{j+1} + p_j 118^{10-j} \qquad (j = 9, \dots, 0). \tag{6.14}$$

Induction gives $r_j = \sum_{k=j}^{10} p_k 231^{k-j} 118^{10-k}$, so $r_0 = 118^{10}\mathfrak{p}(231/118)$. Uniqueness and (6.13) prove (6.10). The denominator $D_a = D(a-1)$ is positive for $a > 1$ by (4.21); hence the Calabi derivative has no pole on this interval. ☐

6.2. **The antisymmetric affine potential**. Henceforth $a = a_*$ and $s = s_{a_*}$. Put

$$q = x_1 - x_2, \qquad F = q/s. \tag{6.15}$$

The moment components are global smooth functions, and s is strictly positive. Thus $F \in C^\infty(M)$. The polygon and s are invariant under $(x_1, x_2) \mapsto (x_2, x_1)$, whereas q changes sign. Therefore

$$\int_M F \, dv_g = 4\pi^2 \int_{P_a} (x_1 - x_2) s^{-5} \, dx = 0. \tag{6.16}$$

It is nonzero since P_a has nonempty interior. Define

$$V = \int_{P_a} s^{-4} \, dx, \quad I = \int_{P_a} q^2 s^{-6} \, dx, \quad J = \int_{\partial P_a} q^2 s^{-4} \, d\sigma, \quad K = \int_{\partial P_a} s^{-2} \, d\sigma. \tag{6.17}$$

All four numbers are positive. Since $\mathrm{Scal}_g = 4\Lambda$, Proposition 5.1 and Corollary 5.3 give

$$\Lambda = \frac{K}{2V}, \qquad \frac{\mathcal{R}_g(F)}{\Lambda} = \frac{2JV}{IK} - 2, \qquad \lambda_1^{\mathrm{sc}}(g) \le \mathcal{R}_g(F). \tag{6.18}$$

Here $\mathcal{R}_g(F) = \int_M |dF|_g^2 \, dv_g / \int_M F^2 \, dv_g$. The identities involving Λ are asserted only at the CLW parameter, where (6.1) holds.

For any function f of $t = x_1 + x_2$, change variables from (x_1, x_2) to (x_1, t); the Jacobian is one. The slice intervals are $0 \le x_1 \le t$ for $0 \le t \le a$ and $t - a \le x_1 \le a$ for $a \le t \le a + 1$. Consequently

$$\begin{gathered} \int_{P_a} f(t) \, dx = \int_0^a t f(t) \, dt + \int_a^{a+1} (2a - t) f(t) \, dt, \\ \int_{\text{slice at } t} (x_1 - x_2)^2 \, dx_1 = \begin{cases} t^3/3, & 0 \le t \le a, \\ (2a - t)^3/3, & a \le t \le a + 1. \end{cases} \end{gathered} \tag{6.19}$$

The second line follows by integrating $(2x_1 - t)^2$ on the stated intervals. On the diagonal facet $t = a + 1$, lattice measure is dx_1 for $1 \le x_1 \le a$, and

$$\int_1^a (2x_1 - a - 1)^2 \, dx_1 = \frac{(a-1)^3}{3}. \tag{6.20}$$

The other four facets occur in two exchange-symmetric pairs. Equations (6.19)–(6.20) give

$$V = \int_0^a \frac{t}{s(t)^4} \, dt + \int_a^{a+1} \frac{2a - t}{s(t)^4} \, dt, \tag{6.21}$$

$$I = \frac{1}{3} \left(\int_0^a \frac{t^3}{s(t)^6} \, dt + \int_a^{a+1} \frac{(2a - t)^3}{s(t)^6} \, dt \right), \tag{6.22}$$

$$J = 2 \int_0^a \frac{t^2}{s(t)^4} \, dt + 2 \int_a^{a+1} \frac{(2a - t)^2}{s(t)^4} \, dt + \frac{(a-1)^3}{3s(a+1)^4}, \tag{6.23}$$

$$K = 2 \int_0^{a+1} s(t)^{-2} \, dt + \frac{a - 1}{s(a+1)^2}. \tag{6.24}$$

The measure on the diagonal facet is dx_1, not $\sqrt{2}\, dx_1$.

6.3. **The rational inequality**. The four integrals above are rational functions of a. For integers $j \ge 0$ and $p > j + 1$, put

$$\mathcal{I}_{j,p}(l, r; b, c) = c^{-j-1} \sum_{v=0}^{j} \binom{j}{v} (-b)^{j-v} \frac{(b + cr)^{v-p+1} - (b + cl)^{v-p+1}}{v - p + 1}. \tag{6.25}$$

If $c \neq 0$ and $b + ct > 0$ on $[l, r]$, the substitution $z = b + ct$ gives

$$\int_l^r t^j (b + ct)^{-p}\, dt = \mathcal{I}_{j,p}(l, r; b, c). \tag{6.26}$$

Indeed, $t^j dt = c^{-j-1}(z - b)^j dz$; expand $(z - b)^j$ and integrate z^{v-p}. The assumption $p > j + 1$ excludes the exponent -1.

Lemma 6.2. *At $a = a_*$, the moments in* (6.17) *satisfy*

$$3951IK - 2000JV > 0. \tag{6.27}$$

Consequently

$$\mathcal{R}_g(F) < \frac{1951}{1000}\Lambda. \tag{6.28}$$

Proof. Proposition A.2 proves (6.27) from $\mathfrak{p}(a_*) = 0$ and $a_* > 231/118$. Since $I, K > 0$, division by $1000IK$ gives

$$\frac{2JV}{IK} < \frac{3951}{1000}. \tag{6.29}$$

Subtracting 2 in (6.18) proves (6.28). ☐

Remark 6.3. The function F is a test function, not an asserted eigenfunction. The proof gives an upper bound for $\lambda_1^{\mathrm{sc}}(g)$; it does not identify its value or multiplicity. Nor is the space of affine quotients asserted to be preserved by Δ_g. Hall–Murphy's numerical value in the anti-invariant subspace [17, Section 5] is obtained from an approximate metric and is not used here.

7. Conformal instability and the entropy index

7.1. The scalar lower bound and its equality case. For a nonconstant eigenfunction on a closed Einstein four-manifold, $\Delta\psi = \lambda\psi$, integration by parts and (2.3) give

$$\|\nabla^2\psi\|_{L^2}^2 = \lambda(\lambda - \Lambda)\|\psi\|_{L^2}^2, \qquad \left\|\nabla^2\psi + \frac{\lambda}{4}\psi g\right\|_{L^2}^2 = \lambda\left(\frac{3}{4}\lambda - \Lambda\right)\|\psi\|_{L^2}^2. \tag{7.1}$$

In particular $\lambda \geq 4\Lambda/3$ when $\Lambda > 0$. Equality gives $\nabla^2\psi = -(\Lambda/3)\psi g$ pointwise, because a smooth tensor with zero squared integral is zero. Obata's theorem [26] then makes the manifold isometric to a round sphere. The CLW manifold has $b_2 = 3$, whereas S^4 has $b_2 = 0$, so equality is excluded. Combining this with (6.18) and Lemma 6.2 proves the eigenvalue inequalities in Theorem B.

7.2. The entropy Hessian. For a closed n-manifold, put

$$\begin{gathered} \mathcal{W}(g, v, \tau) = (4\pi\tau)^{-n/2}\int_M [\tau(\mathrm{Scal}_g + |dv|^2) + v - n]e^{-v}\, dv_g, \\ \nu(g) = \inf\left\{\mathcal{W}(g, v, \tau) \,:\, \tau > 0,\ v \in C^\infty(M),\ (4\pi\tau)^{-n/2}\int_M e^{-v}\, dv_g = 1\right\}. \end{gathered} \tag{7.2}$$

At a positive Einstein metric $\mathrm{Ric}_g = \Lambda g$, ν is critical. We use its Hessian on the space

$$\mathcal{S}_g = \left\{H \in C^\infty(S^2T^*M) \,:\, \delta H = 0,\ \int_M \mathrm{tr}_g H\, dv_g = 0\right\}. \tag{7.3}$$

The Cao–Hamilton–Ilmanen formula, with the proof and normalization in Cao–Zhu [6, Theorem 1.2], states that

$$D^2\nu_g(H, H) = \frac{1}{4\Lambda\,\mathrm{Vol}_g(M)}\langle(2\Lambda - L_g)H, H\rangle_{L^2} \qquad (H \in \mathcal{S}_g). \tag{7.4}$$

To match conventions, their shrinking parameter is $\tau = 1/(2\Lambda)$ and their rough Laplacian is $-\nabla^*\nabla$. On (7.3), the divergence and integrated-trace terms of their operator vanish. Its auxiliary potential solves $(\Lambda - \Delta)v = 0$ with zero mean. The lower bound $\lambda_1^{\mathrm{sc}} \geq n\Lambda/(n-1) > \Lambda$, obtained by the n-dimensional version of (7.1), gives $v = 0$. The remaining operator is $-\frac{1}{2}\nabla^*\nabla + \mathring{R} = \frac{1}{2}(2\Lambda - L_g)$, which gives the coefficient in (7.4). We retain the prefactor of [6, Theorem 1.2] in all the formulas below.

Both diffeomorphism and scaling directions lie in the radical of this Hessian. Indeed, $D\nu_{g'}(\mathcal{L}_X g') = 0$ for every nearby metric g', by diffeomorphism invariance. Differentiating in a direction H at the critical metric gives

$$D^2\nu_g(H, \mathcal{L}_X g) + D\nu_g(\mathcal{L}_X H) = 0, \qquad D^2\nu_g(H, \mathcal{L}_X g) = 0. \tag{7.5}$$

Differentiating $D\nu_{g'}(g') = 0$ gives $D^2\nu_g(H, g) + D\nu_g(H) = 0$, hence $D^2\nu_g(H, g) = 0$. These invariances follow from (7.2): pullback of g, v changes the integrals by the change-of-variables formula; replacing (g, τ) by $(cg, c\tau)$ leaves both the normalized measure and $\tau(\mathrm{Scal}_g + |dv|_g^2)$ unchanged.

Proposition 7.1 (Conformal eigendirections). *Let (M^4, g) be closed and connected, with $\mathrm{Ric}_g = \Lambda g$, $\Lambda > 0$. Suppose $\Delta_g \psi = \lambda\psi$, $\psi \neq 0$, and $4\Lambda/3 < \lambda < 2\Lambda$. Then*

$$D^2\nu_g(\psi g, \psi g) = \frac{(2\Lambda - \lambda)(3\lambda - 4\Lambda)}{4\Lambda(\lambda - \Lambda)\,\mathrm{Vol}_g(M)} \|\psi\|_{L^2}^2 > 0. \tag{7.6}$$

The metrics $g_t = (1 + t\psi)g$ are positive definite for $|t| < \|\psi\|_\infty^{-1}$. There exists $\epsilon \in (0, \|\psi\|_\infty^{-1})$ such that $\nu(g_t) > \nu(g)$ whenever $0 < |t| < \epsilon$.

Proof. Integration of the eigenfunction equation gives $\int_M \psi \, dv_g = 0$. Define

$$H = \nabla^2\psi + (\lambda - \Lambda)\psi g. \tag{7.7}$$

This tensor equals $-S(\psi)$ for the operator defined in [15, Definition 5.2 and Theorem 5.4], where it is attributed to Cao–Hamilton–Ilmanen. The commutation identities of Lemma 2.1 give, explicitly,

$$\begin{aligned} \delta H &= (\lambda - \Lambda)d\psi - (\lambda - \Lambda)d\psi = 0, \\ L_g H &= \lambda\nabla^2\psi + \lambda(\lambda - \Lambda)\psi g = \lambda H, \\ \mathrm{tr}_g H &= -\lambda\psi + 4(\lambda - \Lambda)\psi = (3\lambda - 4\Lambda)\psi. \end{aligned} \tag{7.8}$$

Thus $H \in \mathcal{S}_g$. Expanding its norm and using (7.1) yields

$$\begin{aligned} \|H\|_{L^2}^2 &= \left[\lambda(\lambda - \Lambda) - 2\lambda(\lambda - \Lambda) + 4(\lambda - \Lambda)^2\right]\|\psi\|_{L^2}^2 \\ &= (\lambda - \Lambda)(3\lambda - 4\Lambda)\|\psi\|_{L^2}^2 > 0. \end{aligned} \tag{7.9}$$

Equations (7.4) and (7.8) give

$$D^2\nu_g(H, H) = \frac{2\Lambda - \lambda}{4\Lambda\,\mathrm{Vol}_g(M)}\|H\|_{L^2}^2 > 0. \tag{7.10}$$

Since $\nabla^2\psi = \frac{1}{2}\mathcal{L}_{\nabla\psi} g$, (7.5) implies

$$D^2\nu_g(H, H) = (\lambda - \Lambda)^2 D^2\nu_g(\psi g, \psi g). \tag{7.11}$$

Substitution of (7.9) proves (7.6). For $|t| < \|\psi\|_\infty^{-1}$, $1 + t\psi \geq 1 - |t|\|\psi\|_\infty > 0$. If c denotes the positive quantity in (7.6), the second variation at this critical metric gives

$$\nu((1 + t\psi)g) = \nu(g) + \tfrac{1}{2}c\, t^2 + o(t^2) \qquad (t \to 0). \tag{7.12}$$

Choose $\epsilon > 0$ so that the absolute value of the remainder is at most $ct^2/4$ for $|t| < \epsilon$. The entropy then exceeds $\nu(g)$ for $0 < |t| < \min(\epsilon, \|\psi\|_\infty^{-1})$. In particular the conformal second variation is positive. □

Completion of Theorem B. The first eigenfunction satisfies the hypotheses of Proposition 7.1, by (7.1) and (6.28). That proposition supplies a pure conformal destabilizing variation. □

Corollary 7.2. *For the CLW Einstein metric, the positive index of $D^2\nu_g$ modulo infinitesimal diffeomorphisms and scaling is at least three.*

Proof. Constant rescaling of Theorem C gives two L^2-orthonormal TT eigentensors T_1, T_2 with eigenvalues $\mu_i < 7\Lambda/6 < 2\Lambda$. Let H be (7.7) for a first positive scalar eigenfunction. For every smooth TT tensor T,

$$\langle T, H\rangle_{L^2} = \langle \delta T, d\psi\rangle_{L^2} + (\lambda - \Lambda)\int_M \psi \operatorname{tr}_g T \, dv_g = 0. \tag{7.13}$$

The three tensors belong to $\mathcal{S}_g$ and are linearly independent. The polarized formula (7.4), together with $L_g T_i = \mu_i T_i$ and $L_g H = \lambda H$, gives

$$D^2 v_g(a_1T_1 + a_2T_2 + bH, a_1T_1 + a_2T_2 + bH) = \frac{(2\Lambda - \mu_1)a_1^2 + (2\Lambda - \mu_2)a_2^2 + (2\Lambda - \lambda)b^2\|H\|_{L^2}^2}{4\Lambda \operatorname{Vol}_g(M)}. \tag{7.14}$$

It is positive for every nonzero (a_1, a_2, b). By (7.5), a positive-definite subspace intersects the diffeomorphism and scaling directions only at zero. It therefore remains three-dimensional in the quotient. □

8. Compactly supported variations on the cone

Let (M^4, g) be closed, with $\operatorname{Ric}_g = 3g$, and let $G = dr^2 + r^2 g$. If X, Y are vector fields on M lifted independently of r, the connection is

$$\begin{aligned} &\nabla^G_{\partial_r}\partial_r = 0, \qquad &&\nabla^G_{\partial_r}X = \nabla^G_X\partial_r = r^{-1}X,\\ &\nabla^G_X Y = \nabla^g_X Y - rg(X,Y)\partial_r. && \end{aligned} \tag{8.1}$$

These equations follow by substituting $G_{rr} = 1$, $G_{ri} = 0$ and $G_{ij} = r^2 g_{ij}$ in the connection formula. They give

$$R_G(\partial_r, \cdot) = 0, \qquad R_G(X,Y)Z = R_g(X,Y)Z - g(Y,Z)X + g(X,Z)Y. \tag{8.2}$$

For lifted vectors X, Y,

$$\operatorname{Ric}_G(\partial_r,\partial_r) = 0, \quad \operatorname{Ric}_G(\partial_r, X) = 0, \quad \operatorname{Ric}_G(X,Y) = \operatorname{Ric}_g(X,Y) - 3g(X,Y) = 0. \tag{8.3}$$

Thus G is Ricci-flat.

Proposition 8.1. *Let k be a nonzero smooth TT eigentensor with $L_g k = \lambda k$. For $\varphi \in C_c^\infty((0,\infty))$, set*

$$H = r^2\varphi(r)k, \qquad H(\partial_r, \cdot) = 0. \tag{8.4}$$

Then H is TT and

$$Q_C(H) = \|k\|_{L^2(g)}^2 \int_0^\infty \left[r^4(\varphi')^2 + (\lambda - 6)r^2\varphi^2\right] dr. \tag{8.5}$$

If $\lambda < 15/4$, then $Q_C(H) < 0$ for some φ. In that case the cone has infinite negative index on smooth compactly supported TT tensors.

Proof. Let $(\widetilde{e}_a)_{a=1}^4$ be a local g-orthonormal frame and put $e_0 = \partial_r$, $e_a = r^{-1}\widetilde{e}_a$. At a point where $\nabla^g\widetilde{e}_a = 0$, the components of H and its derivatives are

$$\begin{aligned} H_{ab} &= \varphi k_{ab}, & H_{0a} &= H_{00} = 0,\\ (\nabla^G_0 H)_{ab} &= \varphi' k_{ab}, & (\nabla^G_c H)_{ab} &= r^{-1}\varphi(\nabla^g_c k)_{ab},\\ (\nabla^G_c H)_{0b} &= -r^{-1}\varphi k_{cb}, & (\nabla^G_c H)_{00} &= 0. \end{aligned} \tag{8.6}$$

The components with slots $b0$ equal those with slots $0b$. Consequently

$$\operatorname{tr}_G H = \varphi \operatorname{tr}_g k = 0, \qquad (\delta_G H)_0 = r^{-1}\varphi \operatorname{tr}_g k = 0, \qquad (\delta_G H)_b = r^{-1}\varphi(\delta_g k)_b = 0. \tag{8.7}$$

Squaring all components in (8.6) gives

$$|\nabla^G H|_G^2 = (\varphi')^2|k|_g^2 + r^{-2}\varphi^2\left(|\nabla^g k|_g^2 + 2|k|_g^2\right). \tag{8.8}$$

The constant-curvature-one operator satisfies

$$\sum_i k(R_1(e_i, X)Y, e_i) = g(X,Y)\operatorname{tr}_g k - k(X,Y) = -k(X,Y),$$

where $R_1(X,Y)Z = g(Y,Z)X - g(X,Z)Y$. Substitution in (8.2) gives

$$\langle \mathring{R}_G H, H\rangle_G = r^{-2}\varphi^2\big(\langle \mathring{R}_g k, k\rangle_g + |k|_g^2\big). \tag{8.9}$$

The two additional squared-norm terms cancel in (1.12). Since $dv_G = r^4 dr\, dv_g$ and

$$\int_M \big(|\nabla k|^2 - 2\langle \mathring{R}_g k, k\rangle\big)\, dv_g = (\lambda - 6)\|k\|_{L^2}^2, \tag{8.10}$$

we obtain (8.5).

For $\chi \in C_c^\infty(\mathbb{R})$, set

$$t = \log r, \qquad \varphi(r) = r^{-3/2}\chi(\log r). \tag{8.11}$$

Then $\varphi' = r^{-5/2}(\chi' - \frac{3}{2}\chi)$, so

$$\begin{aligned} \int_0^\infty r^4(\varphi')^2 dr &= \int_{\mathbb{R}} \left((\chi')^2 + \frac{9}{4}\chi^2\right) dt, \\ \int_0^\infty r^2\varphi^2 dr &= \int_{\mathbb{R}} \chi^2 dt. \end{aligned} \tag{8.12}$$

The cross term integrates to zero because χ is compactly supported. Hence

$$Q_C(H) = \|k\|_{L^2}^2 \int_{\mathbb{R}} \left[(\chi')^2 + \left(\lambda - \frac{15}{4}\right)\chi^2\right] dt. \tag{8.13}$$

Define

$$\eta(t) = \begin{cases} \exp(-1/(1-t^2)), & |t| < 1, \\ 0, & |t| \geq 1. \end{cases}$$

For $|t| < 1$, every derivative of η is a finite sum of terms $t^a(1-t^2)^{-b}\exp(-1/(1-t^2))$ with nonnegative integers a, b. This follows by induction and the product rule. Since $\lim_{s\to\infty} s^b e^{-s} = 0$ for every b, all one-sided derivatives tend to zero at $t = \pm 1$. The zero extension is therefore smooth. Also $\eta(0) = e^{-1} > 0$ and $\operatorname{supp}\eta = [-1, 1]$. Set $\chi_L(t) = \eta(t/L)$. Then

$$\frac{\int_{\mathbb{R}}(\chi_L')^2 dt}{\int_{\mathbb{R}} \chi_L^2 dt} = L^{-2}\frac{\int_{\mathbb{R}}(\eta')^2 dt}{\int_{\mathbb{R}} \eta^2 dt}. \tag{8.14}$$

If $\lambda < 15/4$, it suffices to choose

$$L^2 > \frac{\int_{\mathbb{R}}(\eta')^2 dt}{(15/4 - \lambda)\int_{\mathbb{R}} \eta^2 dt}.$$

Equations (8.13)–(8.14) then give $Q_C(H) < 0$. For $j \geq 1$, set $\chi_j(t) = \chi_L(t - 3Lj)$ and let H_j be the associated tensor. Its support is contained in the compact annulus $e^{3Lj-L} \leq r \leq e^{3Lj+L}$. These annuli are disjoint. For every finite family of real numbers (a_j),

$$Q_C\left(\sum_j a_j H_j\right) = \sum_j a_j^2 Q_C(H_j) < 0 \quad \text{if } (a_j) \neq 0. \tag{8.15}$$

Disjoint support and $H_j \neq 0$ also give linear independence. Thus negative subspaces exist in every finite dimension. □

Completion of Theorem C. The two bounds in (1.11) lie below 15/4. Apply Proposition 8.1 to either metric. □

9. Quotients and the half-Weyl classification

The quotient argument uses harmonic two-forms on a finite Riemannian covering. The classification theorems are applied after this construction.

9.1. **Harmonic forms on positive Kähler–Einstein surfaces.**

Lemma 9.1. *Let* (Y^4, g, J) *be a closed connected Kähler–Einstein surface with* $\mathrm{Ric}_g = 3g$*. Then* $b_1(Y) = 0$ *and* $b_2^+(Y) = 1$*. Moreover*

$$\dim \ker(L_g|_{\mathrm{TT}}) \geq b_2^-(Y) = b_2(Y) - 1. \tag{9.1}$$

Proof. For a harmonic one-form α,

$$0 = \langle \Delta_H \alpha, \alpha \rangle_{L^2} = \|\nabla \alpha\|_{L^2}^2 + 3\|\alpha\|_{L^2}^2. \tag{9.2}$$

Thus $\alpha = 0$, giving $b_1 = 0$ by Hodge theory. On self-dual two-forms, the Weitzenböck formula of [11, Section 2] is

$$\Delta_2 = \nabla^* \nabla + \frac{\mathrm{Scal}}{3} I - 2W^+. \tag{9.3}$$

For a Kähler surface, W^+ has eigenvalue $\mathrm{Scal}/6$ on $\mathbb{R}\Omega$ and $-\mathrm{Scal}/12$ on its orthogonal complement. These are the Kähler curvature identities of [11, Section 2]. Since Ω is parallel, the splitting $\Lambda^+ = \mathbb{R}\Omega \oplus \Omega^\perp$ is parallel. A harmonic self-dual form therefore has a harmonic scalar coefficient on $\mathbb{R}\Omega$, and its component ψ in $\Omega^\perp$ satisfies

$$0 = \|\nabla \psi\|_{L^2}^2 + \frac{\mathrm{Scal}}{2} \|\psi\|_{L^2}^2 = \|\nabla \psi\|_{L^2}^2 + 6\|\psi\|_{L^2}^2. \tag{9.4}$$

The scalar coefficient is constant and $\psi = 0$. Hence $\mathcal{H}^+ = \mathbb{R}\Omega$ and $b_2^+ = 1$. For every $\omega \in \mathcal{H}^-$, Proposition 3.1 and $\mathrm{Scal} = 12$ give

$$\delta_g(\omega \circ_g \Omega) = 0, \qquad \mathcal{P}_g(\omega \circ_g \Omega) = -2(\omega \circ_g \Omega). \tag{9.5}$$

Equation (3.19) then gives $L_g(\omega \circ_g \Omega) = 0$. Injectivity in (3.4) proves (9.1). □

Proposition 9.2. *Let* $(\widetilde{M}^4, \widetilde{g}, J)$ *be a closed connected Kähler–Einstein surface,* $\mathrm{Ric}_{\widetilde{g}} = 3\widetilde{g}$*. Suppose* σ *is a fixed-point-free anti-holomorphic isometric involution. Give* $M = \widetilde{M}/\langle \sigma \rangle$ *the quotient metric and the orientation induced by the complex orientation. Then*

$$\dim \ker(L_g|_{\mathrm{TT}}) \geq b_2(M) + 1. \tag{9.6}$$

Proof. An anti-complex linear isomorphism in complex dimension two preserves real orientation: complex conjugation has real determinant $(-1)^2 = 1$, and a complex-linear isomorphism has positive real determinant. Thus σ preserves orientation and commutes with the Hodge star. Being an isometry, it also commutes with the Hodge Laplacian.

Let $\widetilde{\Omega}$ be the Kähler form. Anti-holomorphicity and isometry give

$$\sigma^* \widetilde{\Omega} = -\widetilde{\Omega}. \tag{9.7}$$

If ω is harmonic, anti-self-dual and anti-invariant, then

$$\sigma^*(\omega \circ_{\widetilde{g}} \widetilde{\Omega}) = (-\omega) \circ_{\widetilde{g}} (-\widetilde{\Omega}) = \omega \circ_{\widetilde{g}} \widetilde{\Omega}. \tag{9.8}$$

It descends to a smooth tensor on M. The covering map is a local isometry, so trace, divergence and L commute with pullback. Lemma 9.1 makes the descended tensor TT and a zero mode.

Put $b = b_2(M)$. We compute the dimension of the anti-invariant anti-self-dual harmonic space. By (9.2), $b_1 = 0$ on both spaces; Poincaré duality gives $b_3 = 0$ as well. Euler characteristic is multiplicative under the degree-two covering, since a finite triangulation of M lifts to twice as many simplices in each degree. Thus

$$2 + b_2(\widetilde{M}) = 2(2 + b), \qquad b_2(\widetilde{M}) = 2b + 2. \tag{9.9}$$

By Lemma 9.1, $b_2^+(\widetilde{M}) = 1$, so $b_2^-(\widetilde{M}) = 2b + 1$. Let $\pi : \widetilde{M} \to M$ be the covering. A differential form on M pulls back to an invariant form. Conversely, on an evenly covered open set, an invariant form has the same expression on both sheets; these expressions define a smooth form on M. Since π is an orientation-preserving local isometry, $\pi^* \Delta_g = \Delta_{\widetilde{g}} \pi^*$ and $\pi^*(*_g) = *_{\widetilde{g}}\, \pi^*$. Pullback is injective, so these statements give an isomorphism between harmonic forms on M and invariant harmonic forms on $\widetilde{M}$. Their degree-two dimension is b. By (9.7), the invariant self-dual harmonic space is zero. The invariant anti-self-dual space therefore has dimension b. Since an involution has only eigenvalues 1 and -1, its anti-invariant anti-self-dual space has dimension

$$(2b+1)-b=b+1. \tag{9.10}$$

Let $\omega_1,\dots,\omega_{b+1}$ be a basis of that space, and let S_i be the descended tensors. If $\sum_i a_iS_i=0$, pullback gives $(\sum_i a_i\omega_i)\circ\widetilde{\Omega}=0$. Equation (3.4) gives $\sum_i a_i\omega_i=0$, so every $a_i=0$. This proves (9.6). □

9.2. **Application of the classification theorems.** We use the following three precise classification statements. First, a simply connected closed Einstein four-manifold of positive scalar curvature and $\det W^+\geq 0$ is either anti-self-dual or globally conformally Kähler [29, Theorem 1.3]. Second, a closed Einstein metric Hermitian with respect to a complex structure is Kähler–Einstein or is homothetic to the Page or Chen–LeBrun–Weber metric [23, Theorem A]. Third, a closed oriented Einstein four-manifold with $\det W^+>0$ is either simply connected, or has fundamental group $\mathbb{Z}/2$ and a del Pezzo double cover with an anti-holomorphic deck involution; the conformally rescaled metric on the cover is Kähler [24, Proposition 1]. The cover need not initially be assumed Kähler–Einstein.

The positive anti-self-dual Einstein classification states that a closed simply connected Einstein four-manifold with positive scalar curvature and $W^+=0$ is, up to homothety and isometry, S^4 or $\mathbb{CP}^2$ with its Fubini–Study metric and reversed complex orientation [2, Theorem 13.30]. The statement with $W^-=0$ is obtained by reversing orientation.

Lemma 9.3. *A closed simply connected Kähler–Einstein surface with* $\mathrm{Ric}=3g$ *and* $b_2=1$ *is isometric to the normalized Fubini–Study projective plane.*

Proof. By Lemma 9.1, $b_2^+=1$ and $b_2^-=0$. Thus $\chi=3$ and the signature is $\varsigma=1$. Use the endomorphism norm for $W^\pm:\Lambda^\pm\to\Lambda^\pm$. The Chern–Gauss–Bonnet and Hirzebruch signature formulas, in the Einstein case and the curvature normalization of [2, Chapter 13], are

$$\begin{aligned} 8\pi^2\chi &= \int_M\left(|W^+|^2+|W^-|^2+\frac{\mathrm{Scal}^2}{24}\right)dv_g,\\ 12\pi^2\varsigma &= \int_M(|W^+|^2-|W^-|^2)dv_g. \end{aligned} \tag{9.11}$$

Since the Kähler eigenvalues of W^+ are $2,-1,-1$, $|W^+|^2=6=\mathrm{Scal}^2/24$. Subtract twice the second identity in (9.11) from the first. This gives

$$0=8\pi^2\cdot 3-24\pi^2\cdot 1=\int_M\left(-|W^+|^2+3|W^-|^2+\frac{\mathrm{Scal}^2}{24}\right)dv_g=3\int_M|W^-|^2dv_g. \tag{9.12}$$

The integrand is continuous and nonnegative. If it were positive at one point, it would have positive integral on an open neighborhood. Therefore $W^-=0$. Apply the half-conformally-flat classification just stated, with reversed orientation. The sphere is excluded by $b_2=1$; the Einstein normalization fixes the homothety. □

Proof of Theorem D. Let $(\widetilde{M},\widetilde{g})$ be the universal Riemannian cover. A geodesic projects to a geodesic of the complete closed manifold M; its lift extends for all time. The cover is therefore complete. Since $\mathrm{Ric}_{\widetilde{g}}=3\widetilde{g}$, Myers' theorem gives $\mathrm{diam}(\widetilde{M})\leq\pi$. Hopf–Rinow implies compactness. Each fibre of the covering is closed and discrete in $\widetilde{M}$, hence finite. Thus the cover has finite degree. The lifted orientation and the local-isometry property preserve the condition $\det W^+\geq 0$. The cover is simply connected, so Wu's theorem applies.

Suppose first that $W^+_{\widetilde{g}}=0$. The half-conformally-flat classification gives the round sphere or the Fubini–Study projective plane as the cover. There is no nontrivial oriented quotient in either case. Indeed, $b_1(M)=0$ by (9.2), so $\chi(M)=2+b_2(M)\geq 2$. A cover of degree $d\geq 2$ by S^4 would instead give $\chi(M)=2/d\leq 1$. A cover by $\mathbb{CP}^2$ would give $\chi(M)=3/d\leq 3/2$. Thus the cover has degree one. The round cone is Euclidean away from the origin and $Q_C(H)=\int|\nabla H|^2\geq 0$. For the Fubini–Study cone, stability is [14, Theorem 1.3, proved in Section 5]; that theorem uses exactly the transverse compact-support condition in (1.12).

Suppose next that the universal cover is conformally Kähler. Its Kähler representative has positive scalar curvature by [23, Proposition 1]. The eigenvalues of W_k^+ for that representative are $s_k/6, -s_k/12, -s_k/12$, so $\det W_k^+ = s_k^3/864 > 0$. If $\widetilde{g} = f^2 k$, then $W_{\widetilde{g}}^+ = f^{-2} W_k^+$ as endomorphisms of two-forms. Consequently

$$\det W_{\widetilde{g}}^+ = f^{-6} \frac{s_k^3}{864} > 0. \tag{9.13}$$

The covering is a local orientation-preserving isometry, so this inequality descends to M. LeBrun's double-cover theorem now gives $\pi_1(M) = 1$ or $\mathbb{Z}/2$.

If $\pi_1(M) = 1$, LeBrun's Einstein–Hermitian classification gives three cases. In the Kähler–Einstein case, Lemma 9.1 supplies $b_2(M) - 1$ TT zero eigentensors. If $b_2(M) = 1$, Lemma 9.3 gives Fubini–Study, already treated. Otherwise $b_2(M) - 1 \geq 1$, and the zero eigentensors give (1.15). In the Page and Chen–LeBrun–Weber cases, $b_2(M) - 1$ is respectively 1 and 2, and Theorem C gives the same assertion.

If $\pi_1(M) = \mathbb{Z}/2$, apply the Einstein–Hermitian classification on the double cover. The cover cannot be Chen–LeBrun–Weber, since its Euler characteristic is 5, which cannot be twice an integer. Nor can it be the Page manifold: [24, Lemma 1] states that no nonsimply connected smooth oriented four-manifold has a covering space homeomorphic to $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$. All of its hypotheses hold here. The cover is consequently Kähler–Einstein for the complex structure supplied by LeBrun's double-cover theorem. If its other Kähler representative is $\widehat{k} = u^2 \widetilde{g}$, its Kähler form is $u^2 \widetilde{\Omega}$. Therefore

$$0 = d(u^2 \widetilde{\Omega}) = 2u\, du \wedge \widetilde{\Omega}. \tag{9.14}$$

In a unitary orthonormal coframe, $\widetilde{\Omega} = e^{12} + e^{34}$. For $\alpha = \sum_i a_i e^i$,

$$\alpha \wedge \widetilde{\Omega} = a_1 e^{134} + a_2 e^{234} + a_3 e^{123} + a_4 e^{124}.$$

The four three-forms are independent. Thus $du = 0$ and u is constant. The deck involution is therefore anti-holomorphic and isometric for the Kähler–Einstein metric itself. Proposition 9.2 proves (1.16), with $b_2(M) + 1 \geq 1$. It also gives (1.15).

Every case other than round or Fubini–Study has a TT eigenvalue below $7/2 < 15/4$. Proposition 8.1 gives instability and infinite compactly supported TT index. This proves both implications of the classification. □

Appendix A. An algebraic proof of the scalar estimate

The notation $\mathfrak{p}, D_a, I, J, V, K$ is that of Section 6. We prove the required strict inequality by polynomial identities at the root a_*. All polynomials and coefficients used in the proof are specified below.

A.1. **Rational moment formulas.** For $a > 1$, put

$$\begin{aligned} &\ell = a + 1, \qquad B_0 = a^5 + 7a^4 + 6a^3 + 2a^2 - 5a - 3, \\ &C = 4(1 - a^3), \quad B_1 = B_0 + aC, \qquad B_2 = B_0 + \ell C. \end{aligned} \tag{A.1}$$

Thus $s_a(t) = (12/D_a)(B_0 + Ct)$. In particular $B_0, B_1, B_2 > 0$, since $D_a > 0$ and $s_a > 0$ on $[0, a+1]$. Set

$$\mathsf{V} = (12/D_a)^4 V, \quad \mathsf{I} = (12/D_a)^6 I, \quad \mathsf{J} = (12/D_a)^4 J, \quad \mathsf{K} = (12/D_a)^2 K. \tag{A.2}$$

For every $a > 1$, define $V(a), I(a), J(a), K(a)$ by (6.21)–(6.24) with $s = s_a$. These positive rational integrals are defined independently of the existence of an Einstein metric at that parameter. The normalized moments in (A.2) replace $s_a(t)$ by $B_0 + Ct$. Their homogeneity degrees are respectively $-4, -6, -4, -2$; hence $2JV/(IK) = 2\mathsf{JV}/(\mathsf{IK})$. Only at $a = a_*$ is this quotient related to the Einstein constant by (6.18).

For $L > 0$, $b > 0$ and $b + cL > 0$, define $F_{j,p}(L; b, c) = \int_0^L t^j(b + ct)^{-p}\, dt$. For the five pairs used below one has

$$\begin{aligned} F_{0,2} &= \frac{L}{b(b+cL)}, & F_{2,4} &= \frac{L^3}{3b(b+cL)^3}, \\ F_{1,4} &= \frac{L^2(3b+cL)}{6b^2(b+cL)^3}, & F_{3,6} &= \frac{L^4(5b+cL)}{20b^2(b+cL)^5}, \\ F_{1,6} &= \frac{L^2(10b^3+10b^2cL+5bc^2L^2+c^3L^3)}{20b^4(b+cL)^5}. \end{aligned} \tag{A.3}$$

All arguments on the right are $(L; b, c)$. To prove these formulas, put $y = t/(b+ct)$ and $Y = L/(b+cL)$. The derivative is $dy/dt = b/(b + ct)^2 > 0$, so the substitution is one-to-one on $[0, L]$. Also $1 - cy = b/(b + ct) > 0$. Consequently

$$t = \frac{by}{1-cy}, \qquad dt = \frac{b\,dy}{(1-cy)^2}, \qquad F_{j,p}(L; b, c) = b^{j-p+1}\int_0^Y y^j(1-cy)^{p-j-2}\,dy. \tag{A.4}$$

The exponents $p - j - 2$ are respectively $0, 0, 1, 1, 3$. For the last formula the integral is $b^{-4}(Y^2/2 - cY^3 + 3c^2Y^4/4 - c^3Y^5/5)$. Substitution of Y proves (A.3); the other four formulas follow by integrating y^j or $y^j(1 - cy)$.

Let $T_L = \{x_1, x_2 \geq 0 : x_1 + x_2 \leq L\}$. Except for boundaries of measure zero, P_a is obtained from T_{a+1} by deleting the two translates $(a, 0) + T_1$ and $(0, a) + T_1$. They are disjoint because $a > 1$. On either deleted triangle, with translated coordinates u, v and $z = u + v$, the integral of $(x_1 - x_2)^2$ along the slice is

$$\int_0^z (a + 2u - z)^2\, du = a^2 z + \frac{z^3}{3}. \tag{A.5}$$

The sign of a is reversed on the other triangle, with the same squared integral. Thus

$$\begin{aligned} \mathsf{V} &= F_{1,4}(\ell; B_0, C) - 2F_{1,4}(1; B_1, C), \\ \mathsf{I} &= \tfrac{1}{3}F_{3,6}(\ell; B_0, C) - 2a^2F_{1,6}(1; B_1, C) - \tfrac{2}{3}F_{3,6}(1; B_1, C), \\ \mathsf{J} &= 2F_{2,4}(\ell; B_0, C) - 8aF_{1,4}(1; B_1, C) + \frac{(a-1)^3}{3B_2^4}, \\ \mathsf{K} &= 2F_{0,2}(\ell; B_0, C) + \frac{a-1}{B_2^2}. \end{aligned} \tag{A.6}$$

For the third line, subtract the integrands on the two changed pairs of coordinate facets: $t^2-(2a-t)^2 = 4a(t - a)$ for $a \leq t \leq a + 1$. The remaining facet is the diagonal, whose moment is (6.20). The fourth line is the sum of the four coordinate facets and the diagonal facet.

Define four polynomials in a by

$$\begin{aligned} v &= \ell^2(3B_0 + C\ell)B_1^2 - 2(3B_1 + C)B_0^2, \\ i &= \ell^4(5B_0 + C\ell)B_1^4 - 6a^2B_0^2(10B_1^3 + 10B_1^2C + 5B_1C^2 + C^3) \\ &\quad - 2B_0^2B_1^2(5B_1 + C), \\ j &= 2\ell^3B_1^2B_2 - 4aB_0(3B_1 + C)B_2 + (a-1)^3B_0B_1^2, \\ k &= 2\ell B_2 + (a-1)B_0. \end{aligned} \tag{A.7}$$

Substitution of (A.3) into (A.6), over a common denominator in each line, gives

$$\mathsf{V} = \frac{v}{6B_0^2B_1^2B_2^3}, \quad \mathsf{I} = \frac{i}{60B_0^2B_1^4B_2^5}, \quad \mathsf{J} = \frac{j}{3B_0B_1^2B_2^4}, \quad \mathsf{K} = \frac{k}{B_0B_2^2}. \tag{A.8}$$

The integrand defining I is positive on an open subset of the polygon, and the integrand defining K is positive on every open facet. Hence $\mathsf{I}, \mathsf{K} > 0$. All factors in their displayed denominators are positive, so $i, k > 0$. Define

$$\mathfrak{F}(a) = 11853\, i(a)k(a) - 20000\, v(a)j(a). \tag{A.9}$$

Equations (A.2) and (A.8) give the exact identity

$$3951IK - 2000JV = \left(\frac{D_a}{12}\right)^8 \frac{\mathfrak{F}(a)}{180B_0^3B_1^4B_2^7}. \tag{A.10}$$

Its prefactor is positive for every $a > 1$.

A.2. **Polynomial division and a positive translate.**

Lemma A.1. *Let* $\mathfrak{p}$ *be* (6.9). *There are polynomials* $Q, R \in \mathbb{Z}[a]$, *of degrees* 25 *and* 9, *such that*

$$\mathfrak{F} = \mathfrak{p}Q + R. \tag{A.11}$$

Write $Q = \sum_{j=0}^{25} q_j a^j$ *and* $R = \sum_{j=0}^{9} r_j a^j$. *Their coefficients are given in Table 1. Moreover*

$$118^9 R\left(\frac{231+z}{118}\right) = \sum_{j=0}^{9} c_j z^j, \qquad c_j > 0, \tag{A.12}$$

with the coefficients in Table 2.

Proof. For an integer polynomial A, denote the coefficient of a^j by $[a^j]A$, and extend every coefficient sequence by zero outside its stated range. Products are evaluated by

$$[a^j](AB) = \sum_{r=0}^{j}([a^r]A)([a^{j-r}]B). \tag{A.13}$$

Using (A.1) and (A.7) in this formula gives, for $0 \le j \le 35$,

$$[a^j]\mathfrak{F} = r_j + \sum_{r=0}^{10} p_r q_{j-r}, \qquad (p_0, \dots, p_{10}) = (-6, -7, 21, 20, 0, -42, -34, -12, 6, 5, 1). \tag{A.14}$$

Here q_j, r_j are the integers displayed in Table 1. To compute them successively from the stated factors, put $f_j = [a^j]\mathfrak{F}$ and set

$$\begin{aligned} q_j &= f_{j+10} - \sum_{r=0}^{9} p_r q_{j+10-r} \quad (j = 25, 24, \dots, 0), \\ r_j &= f_j - \sum_{r=0}^{10} p_r q_{j-r} \qquad\quad (0 \le j \le 9). \end{aligned} \tag{A.15}$$

Thus (A.14) is a coefficient identity in $\mathbb{Z}[a]$, proving (A.11). For instance, the coefficients in degrees 35 and 34 are $-2205 = q_{25}$ and $276759 = q_{24} + 5q_{25} = 287784 - 11025$.

The binomial theorem gives, for each $0 \le j \le 9$,

$$c_j = \sum_{k=j}^{9} \binom{k}{j} r_k 231^{k-j} 118^{9-k}. \tag{A.16}$$

This sum yields the ten integers of Table 2. Every displayed integer is positive. Formula (A.16) proves (A.12) and its sign assertion. □

Proposition A.2. *At the root* a_* *in Lemma 6.1,*

$$3951I(a_*)K(a_*) - 2000J(a_*)V(a_*) > 0. \tag{A.17}$$

Proof. Put $z_* = 118a_* - 231$. By (6.10), $z_* > 0$. Since $\mathfrak{p}(a_*) = 0$, Lemma A.1 gives

$$\mathfrak{F}(a_*) = R(a_*) = 118^{-9} \sum_{j=0}^{9} c_j z_*^j \ge 118^{-9} c_0 > 0. \tag{A.18}$$

Every other factor on the right of (A.10) is positive. That identity proves (A.17). □

Table 1. Coefficients for $\mathfrak{F} = \mathfrak{p}Q + R$. The remainder coefficients with index greater than 9 are zero.

j	q_j	r_j	j	q_j
0	−1 588 589 253 209 596	−9 531 535 506 952 461	13	476 232 516
1	740 732 446 020 673	−6 675 730 081 712 563	14	−148 476 658
2	−277 859 321 196 300	36 878 345 425 165 852	15	−50 976 437
3	86 595 295 758 476	14 790 960 069 851 972	16	−655 340 412
4	−21 232 530 012 668	−8 500 831 174 448 830	17	−315 842 653
5	2 961 425 110 884	−63 112 921 934 493 674	18	−1 962 508
6	707 881 307 862	−24 162 316 927 159 204	19	223 569 796
7	−810 603 807 347	−5 185 708 498 471 564	20	208 601 668
8	430 796 582 928	12 532 924 481 239 783	21	107 349 204
9	−175 037 545 095	2 221 544 641 356 161	22	32 506 234
10	58 852 065 720	0	23	5 004 455
11	−16 306 269 728	0	24	287 784
12	3 391 655 536	0	25	−2 205

Table 2. Positive coefficients of $118^9 R((231 + z)/118)$.

j	c_j
0	1 453 244 795 845 348 307 811 581 388 481 233
1	171 655 433 886 954 619 973 704 509 793 399 625
2	4 874 552 333 819 787 686 694 611 996 698 660
3	59 174 508 513 273 196 527 776 438 138 516
4	401 626 856 903 820 450 846 877 488 926
5	1 669 697 462 701 839 922 689 345 982
6	4 353 382 536 789 351 507 350 996
7	6 928 352 208 810 973 839 332
8	6 097 476 398 165 753 113
9	2 221 544 641 356 161

Appendix B. Operator normalizations

The following identities relate the conventions used in the proof. For $\mathrm{Ric}_g = 3g$ on a four-manifold,

$$\begin{aligned} L_g &= \nabla^*\nabla + 6I - 2\mathring{R}_g, \\ \mathcal{E}_g &= L_g - 6I, \\ L_{\mathrm{BO}} &= \tfrac{1}{2}\mathcal{E}_g, \\ \mathcal{P}_g &= \tfrac{1}{2}L_g - 2I - \tfrac{2}{3}\mathcal{K}_g\delta_g, \\ \mathcal{J}_g(T) &= \langle L_g T, T\rangle_{L^2} - \tfrac{4}{3}\|\delta_g T\|^2_{L^2}. \end{aligned} \tag{B.1}$$

On TT tensors, $\mathcal{P}_g = \frac{1}{2}L_g - 2I$. The three full-Lichnerowicz thresholds used in this paper are

property or variational range	threshold for L_g
negative Einstein-operator mode	$\lambda < 6$
shifted divergence-penalty identity	$\tau \le 4$
negative separated cone variation	$\lambda < 15/4$.

(B.2)

The value $7/2$ is a bound obtained from $\rho < 2$, not one of the operator thresholds. The proof of Theorem C does not compute an eigenvalue exactly, and does not require negativity of L_g itself.

The scalar and tensor conclusions use different eigenvalue notation. For $\mathrm{Ric}_g = \Lambda g$, the bounds for the CLW metric are

$$\lambda_1^{\mathrm{sc}}(g) < \frac{1951}{1000}\Lambda < 2\Lambda, \qquad \lambda_2^{\mathrm{TT}}(g) < \frac{7}{6}\Lambda < \frac{5}{4}\Lambda. \tag{B.3}$$

The final tensor threshold is the five-dimensional cone threshold under the constant rescaling to Einstein constant 3. The divergence-free conformal tensor in (7.7) is not TT: its trace is $(3\lambda - 4\Lambda)\psi \neq 0$.

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