PRESCRIBED STEKLOV SPECTRA AND MULTIPLICITIES ON SURFACES

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Abstract. Let $0 < a_1 \leq \cdots \leq a_N$ and $H > a_N$. On every compact connected orientable surface with $b \geq 1$ boundary components and genus at least an explicit integer $\Gamma(N, b)$, we construct a smooth metric whose first N positive Steklov eigenvalues are $a_1, \dots, a_N$ and whose next eigenvalue exceeds H. Repetitions are allowed. For connected boundary, $\Gamma(N, 1) = \lceil N(N-1)/4 \rceil$. This is the least genus permitted by the complete-graph band construction used in the proof; no optimality among all metrics is asserted. The metric may also have any sufficiently small prescribed boundary length, any positive prescribed area, and trivial isometry group. All these statements concern the ordinary, unweighted Steklov problem. The proof realises positive graph Laplacians by Dirichlet and boundary forms, estimates the complete finite spectral matrix, and applies degree after smoothing. For a family of invariant perforations of a closed hyperbolic surface, we identify the Steklov spectral subspace converging to the first positive Laplace eigenvalue with its Laplace eigenspace, equivariantly under the finite group used in the construction. The first Steklov multiplicity is at least the smallest dimension of a real irreducible constituent of that Laplace eigenspace; if the latter is irreducible, its multiplicity is preserved exactly. The surfaces of Fortier Bourque, Gruda-Mediavilla, Petri and Pineault yield hyperbolic examples of genera 10, 17 and 37 with first Steklov multiplicities 16, 21 and at least 24, respectively. They disprove Jammes's proposed equality between maximal first multiplicity and the relative chromatic number minus one.

Contents

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1. Introduction

Let (M, g) be a compact connected smooth Riemannian surface with nonempty boundary. We use the positive Laplacian and the outward unit normal. The Steklov problem is

$$\Delta_g u = 0 \quad \text{in } M, \qquad \partial_\nu u = \sigma u \quad \text{on } \partial M, \qquad \Delta_g = -\operatorname{div}_g \nabla. \tag{1.1}$$

Its eigenvalues, repeated according to multiplicity, are

$$0 = \sigma_0(M, g) < \sigma_1(M, g) \le \sigma_2(M, g) \le \cdots .$$

They are the variational eigenvalues of the Dirichlet energy relative to the boundary L^2 form [17]. We study two realisation problems: a prescribed finite initial spectrum with a fixed number of boundary components, and a first eigenspace of large dimension at fixed genus. The first construction varies the metric on a prescribed surface. The second removes disks from a fixed closed hyperbolic surface.

1.1. **Finite spectra and the topology of the surface.** Colin de Verdière prescribed finite Laplace spectra in dimensions at least three and developed the stability of multiple eigenvalues under finite-dimensional perturbations [5, 6, 7]. Jammes proved finite Steklov prescription, including multiplicities, within a prescribed conformal class in those dimensions [20, Theorem 1.2]. On surfaces, multiplicity bounds depend on topology and on the index [24, 21]; the genus cannot be suppressed from a theorem permitting arbitrary repetitions. For other boundary conditions, He prescribed any strictly increasing finite Dirichlet spectrum together with area on a fixed surface [18, Theorem 1.1]. He–Wang proved the analogous assertion for a fixed positive Robin parameter [19, Theorem 1.2]. Their surface statements prescribe simple eigenvalues. The theorem below permits repetitions, with an explicit sufficient genus and an arbitrary fixed boundary count.

Put

$$\beta_N = \frac{N(N-1)}{2}, \qquad \kappa_N = \begin{cases} 0, & 1 \le N \le 3, \\ \left\lceil \dfrac{(N-2)(N-3)}{12} \right\rceil, & N \ge 4. \end{cases}$$

The integer β_N is the cycle rank of K_{N+1}. The complete-graph genus theorem identifies κ_N with its minimum orientable genus [29].

Theorem A (Finite prescription). *Let $N, b \geq 1$ be integers, let*

$$0 < a_1 \leq \cdots \leq a_N, \qquad H > a_N,$$

and let M be a compact connected orientable smooth surface of genus

$$\gamma \geq \Gamma(N, b) := \max\left\{ \kappa_N, \left\lceil \frac{\beta_N + 1 - b}{2} \right\rceil \right\} \tag{1.2}$$

with exactly b boundary components. There is a smooth metric g on M for which

$$\sigma_j(M, g) = a_j \quad (1 \leq j \leq N), \qquad \sigma_{N+1}(M, g) > H. \tag{1.3}$$

The repetitions in the list are exact eigenvalue multiplicities.

Lemma 3.3 proves that (1.2) is necessary and sufficient for the topology to be obtained from $N+1$ disks, one primary band for each edge of K_{N+1}, and additional orientable boundary bands, without capping boundary components. This optimality statement concerns that construction only. In particular,

$$\Gamma(N, 1) = \left\lceil \frac{N(N-1)}{4} \right\rceil, \qquad b \geq \beta_N + 1 - 2\kappa_N \implies \Gamma(N, b) = \kappa_N.$$

No curvature or conformal class is prescribed. The constant eigenfunction remains at index zero; the strict inequality for σ_{N+1} prevents any additional occurrence of a_N.

Theorem B (Boundary length, area and isometries). *Under the hypotheses of Theorem A, there is $\ell_0 > 0$ such that, for every $0 < \ell < \ell_0$ and every $V > 0$, the metric can also be chosen to satisfy*

$$\mathrm{Length}_g(\partial M) = \ell, \qquad \mathrm{Area}_g(M) = V, \qquad \mathrm{Isom}(M, g) = \{1\}. \tag{1.4}$$

The threshold ℓ_0 may depend on $N, b, \gamma, a_1, \ldots, a_N, H$, but not on V.

The group $\mathrm{Isom}(M, g)$ includes orientation-reversing isometries. Smallness of ℓ is part of the statement, not a normalisation that can be removed while keeping the prescribed spectrum. A finite spectrum is also distinct from the entire spectrum. The latter determines the number and lengths of the boundary components [16, Theorem 1.7]. The compactness theorem of Jin–Wang concerns entire Steklov-isospectral families of smooth Euclidean planar domains [22, Theorem 1.2]; neither Euclidean flatness nor an entire isospectral family is asserted here.

Corollary 1.1. *For every $d, b \geq 1$ and every $\gamma \geq \Gamma(d, b)$, a surface of genus γ with b boundary components admits a metric with first Steklov multiplicity exactly d. After normalising $\sigma_1 = 1$, its next eigenvalue can be arbitrarily large and its isometry group can be trivial.*

Proof. In Theorems A and B, take $N = d$ and $a_1 = \cdots = a_d = 1$. The strict inequality $\sigma_{d+1} > H > 1$ gives multiplicity exactly d. ☐

Audet-Beaumont constructs first multiplicities tending to infinity while both genus and boundary count increase [1]. The construction appeared in the December 2024 preprint; the cited journal volume is dated 2026. Remark 1.3 of the published paper asks whether the boundary count can remain fixed. Corollary 1.1 gives an affirmative answer, including $b = 1$. By contrast, [4] uses graph spectral gaps to estimate $\sigma_1 \, \mathrm{Length}(\partial M)$.

1.2. **First multiplicity and the relative chromatic number.** Let $M_1(S)$ be the largest first positive multiplicity among smooth metrics on a compact surface S with nonempty boundary and positive smooth interior and boundary densities. Thus the allowed Rayleigh quotients are

$$\frac{\int_S \kappa|du|_g^2\, dv_g}{\int_{\partial S} \rho u^2\, ds_g}, \qquad \kappa \in C^\infty(S), \quad \rho \in C^\infty(\partial S), \quad \kappa, \rho > 0. \tag{1.5}$$

The multiplicity bounds in [24, 21] make this maximum finite. A graph embedding in S is proper if all vertices lie on ∂S. The relative chromatic number $\mathrm{Chr}_0(S)$ is the supremum of the chromatic numbers of finite graphs admitting such embeddings [21, Definition 1.5]. Jammes proposed

$$M_1(S) = \mathrm{Chr}_0(S) - 1 \tag{1.6}$$

and proved the lower bound. These are Conjecture 1.6 and Theorem 1.7, respectively, on page 147 of the published paper [21]. The initial formulation is [20, Conjecture 1.13]; the equality is also stated in [1, Section 1.1].

Theorem C (Hyperbolic multiplicity counterexamples). *There are compact connected orientable hyperbolic surfaces with smooth boundary of genus* 10 *and first Steklov multiplicity* 16*, and of genus* 17 *and first Steklov multiplicity* 21*. There are such surfaces of genus* 37 *with first Steklov multiplicity at least* 24*. For each of these three genera, the number of boundary components is unbounded among the examples. Consequently,* (1.6) *is false.*

Every example has finitely many boundary components and curvature -1. Its boundary components are circles bounding removed metric disks; they are not geodesics. Both densities in (1.5) equal one. Capping the boundary preserves the genus and every embedded graph. The Euler estimate of Section 11 gives $\mathrm{Chr}_0 \leq 14$, 17 and 24 in genera 10, 17 and 37, respectively. The multiplicity lower bounds 16, 21 and 24 exceed $\mathrm{Chr}_0 - 1$ in all three cases. This proof is independent of Theorems A and B.

1.3. **Approximation, stability and the two constructions.** Spectral convergence and preservation of multiplicity are separate assertions. Girouard–Henrot–Lagacé obtain limiting boundary problems by perforation and relate Steklov and Neumann spectra [13]. Girouard–Lagacé distribute holes in a closed manifold so that their boundary measures approximate a volume density and their Steklov spectra converge to a weighted Laplace spectrum [15, Theorem 1.1]. Girouard–Karpukhin–Lagacé formulate these problems as variational eigenvalues of measures. Their Proposition 4.11 gives a continuity criterion, and their Theorem 1.12 gives a homogenisation application [14]. Proposition 9.2 gives a quantitative form of this convergence for the invariant perforations used here: the boundary mass forms converge uniformly on bounded subsets of $H^1(X)$.

Topology-preserving approximation is also known. Bucur–Nahon approximate the normalised Steklov spectra of conformally equivalent planar domains [3, Theorem 1.1]. Karpukhin–Lagacé approximate weighted Steklov spectra by domains of the same topological type [23, Theorem 1.5]. The latter statement concerns convergence, for each fixed index, of the normalised eigenvalues. Neither conclusion alone makes nearby eigenvalues equal before passing to the limit.

The finite-dimensional stability method already gives exact multiplicities: see [20, Sections 3A, 4A and Theorem 4.2] and [21, Section 4, Theorem 4.1]. In particular, [20, Theorem 4.2], attributed there to [7, Section 4], prescribes an arbitrary finite spectrum on a positive complete-graph Laplacian and records its stability. The explicit matrices in Section 3 use a nested Soules basis [30, 9]. The estimate $c_{ij} \geq a_1/(N+1)$ permits perturbation by every symmetric matrix in a fixed ball. The graph prescription and its stability are classical; the estimates below realise this whole parameter family by ordinary surface Steklov problems. Boundary attachments occur in [26], and He uses graph approximation and attachments for Dirichlet prescription [18]. The graph-to-surface result of Zhan–Zhou compares graph Steklov eigenvalues with measure eigenvalues on closed surfaces [32, Theorem 1.2]; it is a comparison inequality, not the ordinary surface Steklov prescription in Theorem A.

Bucur–Henrot–Michetti [2, Theorem 1.1] show that, for a planar dumbbell with its ordinary boundary measure, the rescaled Steklov spectrum converges to a one-dimensional problem whose mass includes an integral along the connecting segment. Thus thinning a rectangle does not by itself reduce its boundary mass to endpoint masses. In the construction below, estimate (4.14) makes the mass on the long sides tend to zero on bounded scaled-energy sets; (5.3) excludes further bounded eigenvalues.

Our first construction uses two kinds of bands. Their widths are $\varepsilon c_e(A)$ for the edges of the prescribed graph and ε^2 for the additional bands needed to obtain the topology. The interior conductivity is always one. The boundary mass on the long sides has density of order ε^2. Estimate (4.14) implies that this part of the mass tends to zero for every bounded scaled-energy sequence. The disk averages identify all bounded spectral subspaces with subspaces of $\mathbb{R}^{N+1}$. For the first N positive eigenvalues the resulting matrices satisfy

$$\sup_{A\in\mathcal{B}}\|\mathcal{M}_\varepsilon(A)-(D+A)\|_F\longrightarrow 0,\qquad \inf_{A\in\mathcal{B}} s_{N+1}(\varepsilon,A)\longrightarrow+\infty.$$

Smoothing is carried out on the full parameter family. The degree argument is then applied to the smoothed matrix map and gives precisely D.

The distinction between the two densities in (1.5) is retained throughout. On a surface,

$$\int_M |du|^2_{e^{2f}g}\,dv_{e^{2f}g}=\int_M |du|^2_g\,dv_g,\qquad ds_{e^{2f}g}=e^f\,ds_g. \tag{1.7}$$

Thus a smooth positive boundary density is the boundary arclength of another smooth conformal metric. The energy identity does not eliminate a nonconstant interior conductivity. No such conductivity occurs in the construction of Theorem A.

For the second construction, the removed disks form an invariant set under a finite group G of orientation-preserving isometries. Harmonic filling and orthogonal projection give an equivariant isomorphism onto $E=\ker(\Delta_X-\lambda_1)$ from the span of the first $\dim E$ positive Steklov eigenfunctions, counted with multiplicity. Every nonzero invariant subspace of E contains an irreducible submodule. Hence the first Steklov multiplicity is bounded below by the smallest real irreducible dimension occurring in E. If E is irreducible, the eigenspaces of a commuting self-adjoint operator are invariant; hence that operator is scalar. The first $\dim E$ Steklov eigenvalues are then equal. The required closed hyperbolic representations are supplied by [12, Theorem 1.1 and Section 5.4]. The examples of genera 10 and 17 appeared in its December 2023 version; the genus-37 example was added in the June 2026 revision. The computer-assisted trace-formula inequalities are external inputs and are not recomputed here. Finite-group representations already enter the Steklov multiplicity construction of [1].

Both arguments use the polar normalisation of a comparison isomorphism of finite spectral spaces. This is the comparison of finite-dimensional quadratic forms used in spectral stability; see [19, Appendix A.1]. The same construction appears in [28, Sections 2.1 and 11.2]. The arguments below are independent of the local conformal-section theorem of [28]: the exact graph family and the equivariant comparison map are established here. Sections 2–7 prove Theorems A and B; Sections 8–11 prove Theorem C. Appendix A records the Sobolev estimates used in both limits. All spaces and representations are real unless complexification is specified. We write $\|B\|_F^2=\operatorname{tr}(B^{\mathsf T}B)$ and $\|B\|_{\mathrm{op}}=\sup_{|v|=1}|Bv|$. Constants are independent only of the parameters quantified in the corresponding statement.

2. Variational forms and finite spectral spaces

2.1. **The compact operator associated with the forms.** Let M be a compact connected smooth surface, possibly with finitely many Lipschitz boundary corners. Set $\mathcal{H}=H^1(M)$ and

$$q(u,v)=\int_M\langle du,dv\rangle\,dv_g,\qquad b_\omega(u,v)=\int_{\partial M}\omega uv\,ds_g,$$

where $\omega\in C^0(\partial M)$ is nonnegative and is positive on a nonempty open boundary arc. Appendix A proves the trace and extension estimates on fixed Lipschitz boundary charts, including the corner models used below. For a surface assembled from finitely many Lipschitz pieces, $H^1(M)$ consists of the piecewise H^1 functions whose traces agree on every identified side; the weak-derivative argument is included in Appendix A. Throughout, $q(u) = q(u, u)$ and $b_\omega(u) = b_\omega(u, u)$.

Lemma 2.1. *For $c > 0$, the form $a = cq + b_\omega$ is an inner product equivalent to the H^1 inner product. The operator K determined by*

$$a(Ku, v) = b_\omega(u, v) \qquad (u, v \in \mathcal{H}) \tag{2.1}$$

is compact, self-adjoint and nonnegative in $(\mathcal{H}, a)$. Its positive eigenvalues are $(1 + s_j)^{-1}$, where

$$cq(u, v) = s_j b_\omega(u, v) \qquad (v \in \mathcal{H}). \tag{2.2}$$

One has $0 = s_0 < s_1 \le s_2 \le \cdots$, and $s_j \to \infty$.

Proof. The trace bound gives $a(u) \le C\|u\|_{H^1}^2$. If a lower bound failed, there would be u_l such that

$$\|u_l\|_{H^1} = 1, \qquad q(u_l) + b_\omega(u_l) \longrightarrow 0.$$

Rellich compactness and compactness of the trace yield, after restriction to a subsequence,

$$u_l \rightharpoonup u \text{ in } H^1(M), \qquad u_l \longrightarrow u \text{ in } L^2(M), \qquad \operatorname{Tr} u_l \longrightarrow \operatorname{Tr} u \text{ in } L^2(\partial M).$$

Since $du_l \to 0$, the weak gradient of u vanishes. Connectedness gives $u = k$ for a constant k. The boundary convergence gives $0 = b_\omega(u) = k^2 \int_{\partial M} \omega$, hence $k = 0$. Then $\|u_l\|_{H^1} \to 0$, a contradiction.

Let $\mathcal{T} : (\mathcal{H}, a) \to L^2(\partial M)$ be the trace map and let M_ω denote multiplication by ω. The Riesz theorem and (2.1) give

$$K = \mathcal{T}^* M_\omega \mathcal{T}.$$

The trace is compact; multiplication is bounded and self-adjoint. Consequently K is compact and self-adjoint, with $0 \le a(Ku, u) = b_\omega(u) \le a(u)$. If $Ku = tu$, $t > 0$, then

$$t\{cq(u, v) + b_\omega(u, v)\} = b_\omega(u, v),$$

which is (2.2) with $s = t^{-1} - 1 \ge 0$. Conversely (2.2) gives $Ku = (1 + s)^{-1}u$. The equation $s = 0$ is equivalent to $du = 0$, hence its space is the one-dimensional space of constants.

For every integer m, choose m nonzero smooth traces with disjoint supports in an arc on which $\omega > 0$, and extend them to $H^1(M)$. Their mass Gram matrix is positive definite. Thus K has rank at least m for every m. Its positive eigenvalues tend to zero, so the corresponding s_j tend to infinity. □

For later use, the min–max formula is

$$s_j = \inf_{\substack{F \subset H^1(M) \\ \dim F = j+1}} \sup_{0 \ne u \in F} \frac{cq(u)}{b_\omega(u)}, \tag{2.3}$$

with quotient $+\infty$ if $b_\omega(u) = 0$. Indeed, choose an a-orthonormal eigenbasis (e_i) of $(\ker K)^{\perp_a}$, with eigenvalues $t_i = (1 + s_i)^{-1}$ in decreasing order. For $u = u_* + \sum x_i e_i$, where $u_* \in \ker K$,

$$b_\omega(u) = \sum t_i x_i^2, \qquad cq(u) = a(u_*) + \sum (1 - t_i) x_i^2.$$

The span of $e_0, \dots, e_j$ gives the upper bound in (2.3). Every $(j + 1)$-dimensional subspace contains a nonzero vector a-orthogonal to $e_0, \dots, e_{j-1}$; the displayed formulas give the lower bound.

Testing (2.2) on $H_0^1(M)$ gives weak harmonicity. Suppose now that M has smooth boundary, the metric is smooth up to that boundary, and ω is smooth and strictly positive there. The weak conormal equation is $c\partial_\nu u = s\omega u$. The fixed smooth-boundary estimates in Appendix A give $u \in C^\infty(M)$, including the boundary. Under these hypotheses the variational and smooth Steklov spectra agree.

2.2. Continuity of the forms and of their spectral projections.

Lemma 2.2. *Let q_t, b_t be bounded symmetric forms on one real Hilbert space $\mathcal{H}$. Suppose $q_t, b_t \geq 0$, b_t is compact, and $a_t = q_t + b_t \geq c\|\cdot\|^2_{\mathcal{H}}$ for a fixed $c > 0$. Order the positive eigenvalues of the associated compact operator in decreasing order, counting multiplicity, and fix a finite set $\mathcal{I}$ of indices. Suppose that, for every parameter t, these indices select complete eigenspaces and that the selected eigenvalues are separated from zero and from the unselected spectrum. If the forms are norm-continuous, the selected spectral subspace is continuous as a subspace of $\mathcal{H}$. On a compact parameter set with these separations, the conclusion persists under sufficiently small uniform perturbations by norm-continuous families $\widetilde{b}_t$ of nonnegative compact forms.*

Proof. Represent a_t, b_t by operators A_t, B_t in the fixed Hilbert inner product. The compact self-adjoint operator

$$C_t = A_t^{-1/2} B_t A_t^{-1/2} \tag{2.4}$$

has positive eigenvalues $(1 + s_j(t))^{-1}$; the corresponding vectors for the variational problem are obtained by multiplication by $A_t^{-1/2}$. For $A \geq cI$,

$$A^{-1/2} = \frac{1}{\pi} \int_0^\infty r^{-1/2}(A + rI)^{-1}\, dr.$$

The resolvent identity bounds the difference of the integrands for $A_t, A_{t'}$ by $r^{-1/2}(c + r)^{-2}\|A_t - A_{t'}\|$. The substitution $r = cs$ gives

$$\frac{1}{\pi} \int_0^\infty r^{-1/2}(c + r)^{-2}\, dr = \frac{1}{2c^{3/2}}.$$

Consequently, if $A, \widetilde{A} \geq cI$,

$$\|A^{-1/2} - \widetilde{A}^{-1/2}\| \leq \frac{\|A - \widetilde{A}\|}{2c^{3/2}}. \tag{2.5}$$

Writing $C = A^{-1/2}BA^{-1/2}$ and $\widetilde{C} = \widetilde{A}^{-1/2}\widetilde{B}\widetilde{A}^{-1/2}$, put $X = A^{-1/2}$ and $\widetilde{X} = \widetilde{A}^{-1/2}$. The identity

$$C - \widetilde{C} = (X - \widetilde{X})BX + \widetilde{X}(B - \widetilde{B})X + \widetilde{X}\widetilde{B}(X - \widetilde{X})$$

gives

$$\|C - \widetilde{C}\| \leq \frac{\|B - \widetilde{B}\|}{c} + \frac{\|B\| + \|\widetilde{B}\|}{2c^2}\|A - \widetilde{A}\|. \tag{2.6}$$

Indeed, the middle term contains $B - \widetilde{B}$ and two factors bounded by $c^{-1/2}$; the other two terms contain one difference of inverse square roots. These inequalities prove norm continuity of C_t.

Complexify $\mathcal{H}$. Locally in the parameter, choose a finite union $\mathcal{Z}$ of positively oriented simple closed contours, at positive distance from the spectrum, whose interiors contain precisely the selected eigenvalues and are pairwise disjoint. Define

$$P_t = \frac{1}{2\pi i} \int_{\mathcal{Z}} (z - C_t)^{-1}\, dz. \tag{2.7}$$

Let $d = \operatorname{dist}(\mathcal{Z}, \operatorname{spec} C) > 0$. If $\|\widetilde{C} - C\| < d/2$, then $\|(z - C)^{-1}\| \leq d^{-1}$ and $\|(z - \widetilde{C})^{-1}\| \leq 2d^{-1}$ on $\mathcal{Z}$. The resolvent identity in (2.7) therefore gives

$$\|\widetilde{P} - P\| \leq \frac{\operatorname{Length}(\mathcal{Z})}{\pi d^2}\|\widetilde{C} - C\|. \tag{2.8}$$

In particular, P_t and C_tP_t are norm-continuous. The projection commutes with complex conjugation when the enclosed spectral set is real; its range is the complexification of a real spectral space. If $\|P_t - P_{t'}\| < 1$ and $u \in \operatorname{ran} P_t$ with $P_{t'}u = 0$, then $\|u\| \leq \|P_t - P_{t'}\|\|u\|$, hence $u = 0$. Interchanging the projections gives equality of their finite ranks. Their ranges, and their images under $A_t^{-1/2}$, therefore vary continuously.

The variational operator on the selected space is continuous as well. Since $P_tC_t = C_tP_t$, the operator

$$D_t = P_tC_tP_t + I - P_t \tag{2.9}$$

is invertible: its restriction to $\operatorname{ran} P_t$ is bounded below by the distance of the selected eigenvalues from zero, and its restriction to $\ker P_t$ is the identity. The operator

$$Z_t = D_t^{-1}P_t - P_t$$

has eigenvalues $t_j^{-1} - 1 = s_j$ on $\operatorname{ran} P_t$ and is zero on $\ker P_t$. The identity $D_t^{-1} - D_{t'}^{-1} = D_t^{-1}(D_{t'} - D_t)D_{t'}^{-1}$ proves norm continuity. The operator

$$\mathscr{S}_t = A_t^{-1/2}Z_tA_t^{1/2}$$

restricts to the compressed variational operator on $A_t^{-1/2} \operatorname{ran} P_t$. Its displayed formula proves norm continuity on the fixed Hilbert space.

On a compact parameter set, choose finitely many neighbourhoods with such contours. The minimum of their positive spectral distances gives a uniform permitted perturbation. Uniform mass perturbations also preserve coercivity, because $|\widetilde{b}_t(u) - b_t(u)| \le (c/2)\|u\|^2$ implies $q_t + \widetilde{b}_t \ge (c/2)I$. The assumed norm-continuity of $\widetilde{b}_t$ makes $\widetilde{a}_t = q_t + \widetilde{b}_t$ norm-continuous. Equations (2.5)–(2.6), with coercivity constant $c/2$, give norm-continuity of $\widetilde{C}_t$. The same local contours and (2.8) then give norm-continuity of the perturbed finite spectral projections. Formula (2.9) gives the corresponding compressed variational operators. Finally, min–max for compact self-adjoint operators gives $|t_j(C) - t_j(\widetilde{C})| \le \|C - \widetilde{C}\|$ for each ordered nonnegative eigenvalue. This controls adjacent eigenvalues as well as isolated projections, without requiring them to be simple. □

2.3. **Polar identification and exact correction.**

Lemma 2.3. *Let F be a D-dimensional real inner product space and let $T : F \to \mathbb{R}^D$ be invertible. Then*

$$W = T^*(TT^*)^{-1/2} : \mathbb{R}^D \longrightarrow F \tag{2.10}$$

is an isometry. If a finite group acts orthogonally and T intertwines its actions, then W also intertwines them. If S is self-adjoint on F, its matrix in this identification is W^*SW. Equivalently, put $V = T^{-1}$, $G = V^*V$ and $H = V^*SV$. Then

$$W = VG^{-1/2}, \qquad W^*SW = G^{-1/2}HG^{-1/2}. \tag{2.11}$$

Proof. Writing $B = TT^* > 0$ gives $W^*W = B^{-1/2}BB^{-1/2} = I$. In equal dimensions this is an isomorphism. Equivariance of T and orthogonality imply equivariance of T^* and commutation of B with the action. Diagonalising B gives the same commutation for $B^{-1/2}$. Moreover, $G = T^{-*}T^{-1} = (TT^*)^{-1}$, so $VG^{-1/2} = T^{-1}(TT^*)^{1/2} = T^*(TT^*)^{-1/2} = W$. Substitution gives the second identity in (2.11). □

Lemma 2.4. *Let $r > 0$, let $\mathcal{B} = \{A \in \mathrm{Sym}_D(\mathbb{R}) : \|A\|_F \le r\}$, and let $D_0 \in \mathrm{Sym}_D(\mathbb{R})$. If a continuous map $\mathcal{M} : \mathcal{B} \to \mathrm{Sym}_D(\mathbb{R})$ satisfies*

$$\sup_{A \in \mathcal{B}} \|\mathcal{M}(A) - (D_0 + A)\|_F < r, \tag{2.12}$$

then $\mathcal{M}(A_*) = D_0$ for some $A_* \in \mathring{\mathcal{B}}$.

Proof. Set $F(A) = \mathcal{M}(A) - D_0$. On $\|A\|_F = r$, the homotopy $A + t(F(A) - A)$ has norm at least $r - \sup\|F(A) - A\|_F > 0$. Homotopy invariance of Brouwer degree gives $\deg(F, \mathring{\mathcal{B}}, 0) = \deg(I, \mathring{\mathcal{B}}, 0) = 1$. The existence property of degree gives the required zero [8, Chapter 1]. □

3. The finite matrix and the topology

3.1. A positive complete-graph Laplacian. Fix the data of Theorem A. Put $n = N + 1$ and $\mu_k = a_{n-k}$ for $1 \le k < n$. Thus $\mu_1 \ge \cdots \ge \mu_{n-1} > 0$. Define

$$q_0 = \frac{(1, \dots, 1)}{\sqrt{n}}, \qquad q_k = \frac{(\underbrace{1, \dots, 1}_{k}, -k, 0, \dots, 0)}{\sqrt{k(k+1)}}. \tag{3.1}$$

For $k \ge 1$ the squared numerator norm is $k + k^2 = k(k+1)$ and its coordinate sum is $k - k = 0$. For $1 \le k < l$, the scalar product of the numerators is $k - k = 0$. Also $|q_0|^2 = n/n = 1$. Thus $q_i \cdot q_j = \delta_{ij}$ for $0 \le i, j < n$.

Put $Q = (q_1 \ \cdots \ q_{n-1})$ and $D = \operatorname{diag}(\mu_1, \dots, \mu_{n-1})$.

Lemma 3.1. *The matrix $L_0 = QDQ^\mathsf{T}$ is the Laplacian of a complete graph with strictly positive conductances. For $i < j$,*

$$c_{ij} := -(L_0)_{ij} = \frac{\mu_{j-1}}{j} - \sum_{k=j}^{n-1} \frac{\mu_k}{k(k+1)} \ge \frac{a_1}{n}. \tag{3.2}$$

For $A \in \mathcal{B} = \{\|A\|_F \le r\}$, where $0 < r < a_1/(2n)$, define

$$L(A) = Q(D + A)Q^\mathsf{T}, \qquad c_{ij}(A) = -L(A)_{ij}. \tag{3.3}$$

All $c_{ij}(A)$ have uniform positive lower and upper bounds, and

$$Q^\mathsf{T} L(A) Q = D + A. \tag{3.4}$$

Proof. For $i < j$, terms indexed by $k < j - 1$ vanish. The term $k = j - 1$ is $-\mu_{j-1}/j$, and the terms $k \ge j$ are $\mu_k/[k(k+1)]$. Moreover,

$$\sum_{k=j}^{n-1} \frac{\mu_k}{k(k+1)} \le \mu_{j-1} \left(\frac{1}{j} - \frac{1}{n} \right).$$

This proves (3.2). Since $Q^\mathsf{T}\mathbf{1} = 0$, the row sums vanish. The diagonal entries are consequently the sums of incident conductances.

Each entry of QAQ^T has absolute value at most $\|A\|_{\mathrm{op}} \le \|A\|_F$. Thus $c_{ij}(A) > a_1/(2n)$ throughout $\mathcal{B}$. Also $|L(A)_{ij}| \le \|L(A)\|_{\mathrm{op}} \le a_N + r$. Thus the upper bound is $a_N + r$. Finally $Q^\mathsf{T} Q = I$ gives (3.4). □

In particular,

$$z^\mathsf{T} L(A) y = \sum_{i<j} c_{ij}(A)(z_i - z_j)(y_i - y_j). \tag{3.5}$$

The positive eigenvalues of L_0 are $a_1, \dots, a_N$. Equation (3.4) specifies the matrix also when entries of this list coincide.

3.2. Disk regions and boundary handles. Take $n = N + 1$ disjoint smooth disks D_i with fixed metrics. Choose finitely many disjoint boundary arcs on each disk, with pairwise disjoint collars isometric to Euclidean half-rectangles. Reserve an exposed arc on each disk, disjoint from every attachment, and choose $\beta_i \in C^\infty(\partial D_i)$ supported there with

$$\beta_i \ge 0, \qquad \int_{\partial D_i} \beta_i \, ds = 1. \tag{3.6}$$

Attach one orientable rectangular band for each edge of K_n. These are the primary bands. Their cyclic orders around the disks will be chosen below. A regular neighbourhood of the resulting graph is called its ribbon surface. Capping its boundary circles gives a cellular embedding of that graph in a closed surface.

The minimum genus in the next lemma is the complete-graph theorem of Ringel–Youngs [29]. The maximum genus follows from Xuong's theorem [31], and the absence of gaps is Duke's interpolation theorem [10]. We give the latter two arguments for the construction used here.

Lemma 3.2 (Genus range of the primary surface). *The genera of the orientable ribbon surfaces of K_n are exactly the integers*

$$\kappa_N \leq g_0 \leq \left\lfloor \frac{\beta_N}{2} \right\rfloor. \tag{3.7}$$

A ribbon surface of genus g_0 has $b_0 = \beta_N + 1 - 2g_0$ boundary components.

Proof. Every such surface retracts onto K_n, so its Euler characteristic is

$$n - \binom{n}{2} = 1 - \beta_N = 2 - 2g_0 - b_0.$$

Since $b_0 \geq 1$, this gives the upper bound in (3.7). The least genus is κ_N: for $n \geq 3$ this is [29], and for $n = 2$ the regular neighbourhood of its single edge is a disk.

For the upper endpoint, take the spanning star with centre vertex 1. Its regular neighbourhood is a disk. For $n \geq 3$ the remaining edges form the connected graph K_{n-1} on the other vertices, with β_N edges. They can be partitioned into pairs of incident edges, with one unpaired edge exactly when β_N is odd. To see this, root a spanning tree of this remaining graph. Orient all its non-tree edges arbitrarily. Process the nonroot vertices in decreasing distance from the root. At a vertex v, orient its edge to its parent towards v precisely when the other edges already entering v are odd in number. The indegree of every nonroot vertex is then even. The indegree of the root has the parity of β_N. Pair the incoming edges at each vertex, leaving one edge at the root when necessary. Each edge belongs to exactly one resulting pair or is that single unpaired edge.

Suppose that a partial ribbon surface has one boundary component and that a pair to be added consists of edges vu, vw. Choose attaching intervals at u and w, and two successive intervals on one available arc at v. Order those last two intervals so that the four endpoints alternate on the boundary circle. Label the attaching intervals in their boundary order by v_u, v_w, u, w, where v_u is paired with u and v_w with w. After attachment of the $v_u u$ band, the two remaining intervals v_w, w lie on different boundary components. The $v_w w$ band joins these components. The two attachments decrease the Euler characteristic by two and leave one boundary component. The identity $\chi = 2 - 2g - b$ therefore increases the genus by one. Insert all pairs in this way. If one edge remains, attach its band last; it splits the boundary circle into two and leaves the genus unchanged. The resulting genus is $\lfloor \beta_N/2 \rfloor$. At every insertion intervals may be chosen strictly inside unused disk arcs. Smaller unused intervals remain, and the reserved arcs in (3.6) need not meet an attachment.

It remains to obtain all intermediate integers. A rotation system is a choice of a cyclic order of the edge ends incident at each vertex. These systems are connected by transpositions of consecutive entries in a cyclic order: fix one entry and transpose neighbouring entries among the remaining ones. Let σ be the permutation given by the vertex rotations, and let α interchange the two ends of each edge. The number of boundary components is the number of cycles of $\sigma\alpha$. If consecutive distinct entries a, b are interchanged at a vertex of degree at least three and $c = \sigma(b)$, then the new rotation permutation is

$$\sigma' = (a\ b\ c)\sigma.$$

Multiplication by a transposition changes the number of cycles of a permutation by one in absolute value. A three-cycle is a product of two transpositions, so the boundary count changes by $-2, 0$, or 2. Euler's formula shows that the genus changes by $-1, 0$, or 1. A finite path between a minimum-genus system and the maximum-genus system constructed above must attain every integer between their genera. For $n = 2, 3$ there is only one cyclic order at each vertex and the two endpoints already agree. ☐

Lemma 3.3 (Exact topological criterion). *Starting with $n = N + 1$ disks and one primary band for each edge of K_n, an orientable surface of genus $\gamma \geq 0$ with $b \geq 1$ boundary components can be obtained by adding only orientable boundary bands if and only if*

$$\gamma \geq \kappa_N, \qquad 2\gamma + b \geq \beta_N + 1. \tag{3.8}$$

Every added band may have both endpoints on unused arcs of the original disks, and the reserved arcs can remain exposed. The number of added bands is necessarily

$$t = 2\gamma + b - \beta_N - 1. \tag{3.9}$$

Proof. Each additional boundary band decreases the Euler characteristic by one. Thus a construction with t such bands satisfies

$$2 - 2\gamma - b = 1 - \beta_N - t,$$

which proves (3.9) and the second inequality in (3.8). The primary surface embeds in the final surface. Attaching an orientable boundary band either keeps the genus unchanged or increases it by one. Lemma 3.2 gives the first inequality.

Conversely, assume (3.8) and put

$$u_N = \left\lfloor \frac{\beta_N}{2} \right\rfloor, \qquad g_0 = \min\{\gamma, u_N\}, \qquad b_0 = \beta_N + 1 - 2g_0.$$

Lemma 3.2 supplies a primary surface with these invariants. If $\gamma \le u_N$, then $g_0 = \gamma$ and the second inequality in (3.8) says $b \ge b_0$. Attach $b - b_0$ bands, each with both ends on the same boundary component. An orientable band of this type splits that component into two and does not change the genus. This gives the required surface.

Suppose instead that $\gamma > u_N$. If β_N is even, then $(g_0, b_0) = (u_N, 1)$. An orientable band with both ends on the boundary splits it; a second band joining the two resulting components restores one boundary component and increases the genus by one. Apply $\gamma - u_N$ such pairs, followed by $b - 1$ splitting bands. The number used is

$$2(\gamma - u_N) + b - 1 = 2\gamma + b - \beta_N - 1.$$

If β_N is odd, then $(g_0, b_0) = (u_N, 2)$. One band joining the two boundary components produces genus $u_N + 1$ and one boundary component. Apply $\gamma - u_N - 1$ of the preceding pairs and then $b - 1$ splitting bands. The count is

$$1 + 2(\gamma - u_N - 1) + b - 1 = 2\gamma + b - \beta_N - 1.$$

All counts are nonnegative in their respective cases. For the initial range $\gamma \le u_N$, their value is $b - b_0 = 2\gamma + b - \beta_N - 1$. Thus the count in (3.9) holds in all three cases.

Every primary boundary cycle contains an open disk-boundary arc. At a splitting attachment choose two intervals inside one such unused arc, leaving a nonempty interval between them and unused intervals outside them. Both new boundary cycles then contain unused disk arcs. At a joining attachment use proper subintervals of unused arcs on the two components, leaving unused subarcs on the joined boundary. This property is preserved inductively. The number of attachments is finite, so all intervals and their flat collars can be chosen disjoint from the reserved arcs. This proves the endpoint assertion. □

The additional bands are called auxiliary. They may join two arcs on the same disk or the same pair of disks as an existing band. The number of disk regions remains n. For integer γ, condition (3.8) is equivalent to (1.2); thus the bound is exact for this construction. The classification of compact orientable surfaces identifies the resulting smooth surface with the prescribed M. Its fixed finite number of bands is the only topological datum used in the estimates of the following sections. The cyclic orders of their endpoints do not enter the limiting graph quadratic form.

4. Two scales of bands

Fix the topology and the parameter ball $\mathcal{B}$ once and for all. Every band has length one. For small $\varepsilon > 0$ give it width

$$w_e(A) = \varepsilon c_e(A) \quad (e \text{ primary}), \qquad w_e(A) = \varepsilon^2 \quad (e \text{ auxiliary}). \tag{4.1}$$

Its short sides are attached to centred intervals of that length in the selected flat arcs. Positive lower and upper bounds for $c_e(A)$ ensure disjoint attachments for all $A \in \mathcal{B}$ and small ε. Denote the resulting metric surface by $M_{\varepsilon,A}$. Its boundary has finitely many corners of angle $3\pi/2$; across the open attaching seams the metric is Euclidean and smooth.

Extend each β_i by zero to the remaining boundary and put

$$\rho_{\varepsilon,A} = \sum_i \beta_i + \varepsilon^2, \qquad \theta_{\varepsilon,A} = \frac{n}{\int_{\partial M_{\varepsilon,A}} \rho_{\varepsilon,A}\, ds}, \tag{4.2}$$

$$q_{\varepsilon,A}(u,v) = \int_{M_{\varepsilon,A}} \langle du, dv\rangle, \qquad b_{\varepsilon,A}(u,v) = \theta_{\varepsilon,A} \int_{\partial M_{\varepsilon,A}} \rho_{\varepsilon,A} uv\, ds. \tag{4.3}$$

There are a fixed finite number of length-one bands. The boundary length of the cornered surface is bounded independently of A, ε, so $\theta_{\varepsilon,A} = 1 + O(\varepsilon^2)$. In particular

$$b_{\varepsilon,A}(1,1) = n. \tag{4.4}$$

The eigenvalues denoted below by $s_j(\varepsilon, A)$ are those of

$$\left(\varepsilon^{-1} q_{\varepsilon,A}, b_{\varepsilon,A}\right). \tag{4.5}$$

Unless ε is explicitly fixed, constants in Sections 4–5 may depend on the fixed topology, disks, arcs and parameter ball, but not on A or small ε. In statements at fixed ε, the constants may also depend on ε; their asserted uniformity is with respect to $A \in \mathcal{B}$.

4.1. **Means at the attaching intervals.** For $u \in H^1(D_i)$ write $c_i(u) = \operatorname{Area}(D_i)^{-1} \int_{D_i} u$.

Lemma 4.1. *If I_w is an interval of length w centred in one of the fixed flat arcs, then*

$$\left| w^{-1} \int_{I_w} u\, ds - c_i(u) \right|^2 \le C(1 + |\log w|) \int_{D_i} |du|^2. \tag{4.6}$$

Proof. Poincaré's inequality and the trace estimate of Appendix A bound the $H^{1/2}(\partial D_i)$ norm of $u - c_i(u)$ by $C\|du\|_{L^2(D_i)}$. Use an arclength coordinate of fixed period on the boundary. The Fourier coefficients of $w^{-1}\mathbf{1}_{I_w}$ have absolute value at most $C \min\{1, (|k|w)^{-1}\}$. Its squared $H^{-1/2}$ norm is at most

$$C\left(1 + \sum_{1 \le k \le w^{-1}} \frac{1}{k} + w^{-2} \sum_{k > w^{-1}} \frac{1}{k^3}\right) \le C(1 + |\log w|).$$

The dual pairing proves (4.6). The fixed coordinate normalisations only change C. □

4.2. **Boundary mass on a rectangle.**

Lemma 4.2. *Let $w > 0$, $R_w = [0,1] \times [0,w]$, and $u \in H^1(R_w)$. Put $\partial_l R_w = ([0,1] \times \{0\}) \cup ([0,1] \times \{w\})$ and $m(s) = w^{-1} \int_0^w u(s,y)\, dy$. There is an absolute constant C such that*

$$\int_{\partial_l R_w} u^2\, ds \le C\left(|m(0)|^2 + (w^{-1} + w) \int_{R_w} |du|^2\right), \tag{4.7}$$

$$\int_{R_w} |du|^2 \ge w|m(1) - m(0)|^2. \tag{4.8}$$

Proof. For almost every s, the fundamental theorem of calculus and Cauchy–Schwarz give

$$|u(s,0) - m(s)|^2 + |u(s,w) - m(s)|^2 \le 2w \int_0^w |u_y(s,y)|^2\, dy.$$

Also

$$\int_0^1 |m(s)|^2\, ds \le 2|m(0)|^2 + 2\int_0^1 |m'(s)|^2\, ds, \qquad \int_0^1 |m'|^2\, ds \le w^{-1} \int_{R_w} |u_s|^2.$$

Combining these inequalities proves (4.7). Applying Cauchy–Schwarz to $m(1) - m(0) = \int_0^1 m'$ proves (4.8). For $u \in H^1(R_w)$ choose $u_l \in C^\infty(\overline{R_w})$ with $u_l \to u$ in $H^1(R_w)$. The trace map is continuous into L^2 on each side, and $\|m_l - m\|_{H^1(0,1)} \le w^{-1/2}\|u_l - u\|_{H^1(R_w)}$. The one-dimensional trace at 0 and 1 is continuous on $H^1(0,1)$. Every term in the two inequalities therefore converges. □

4.3. **Comparison of the boundary form with disk averages.** Set

$$\|u\|_{\varepsilon,A}^2 = b_{\varepsilon,A}(u) + \varepsilon^{-1} q_{\varepsilon,A}(u), \qquad J_{\varepsilon,A}u = (c_1(u), \dots, c_n(u)). \tag{4.9}$$

When the domain is specified, write $Ju = J_{\varepsilon,A}u$.

Proposition 4.3. *For all sufficiently small ε, uniformly in $A \in \mathcal{B}$,*

$$|Ju| \le C\|u\|_{\varepsilon,A}, \tag{4.10}$$

$$|b_{\varepsilon,A}(u,v) - Ju \cdot Jv| \le C\sqrt{\varepsilon}\,\|u\|_{\varepsilon,A}\|v\|_{\varepsilon,A}. \tag{4.11}$$

These inequalities hold for all $u, v \in H^1(M_{\varepsilon,A})$. For every band with endpoints on D_i, D_j,

$$|m_e(0) - c_i(u)| + |m_e(1) - c_j(u)| \le C\sqrt{\varepsilon(1 + |\log\varepsilon|)}\,\|u\|_{\varepsilon,A}. \tag{4.12}$$

Proof. By homogeneity it suffices first to assume $\|u\|_{\varepsilon,A} \le 1$. The fixed disk Poincaré and trace inequalities give

$$\|u - c_i(u)\|_{H^1(D_i)} \le C\sqrt{\varepsilon}. \tag{4.13}$$

Since $\theta_{\varepsilon,A}$ has a uniform positive lower bound and $\int \beta_i = 1$,

$$|c_i(u)| \le \left(\int \beta_i u^2\right)^{1/2} + \left(\int \beta_i |u - c_i(u)|^2\right)^{1/2} \le C + C\sqrt{\varepsilon}.$$

This proves (4.10). The possible widths satisfy $c\varepsilon \le w_e \le C\varepsilon$ or $w_e = \varepsilon^2$. In both cases $1 + |\log w_e| \le C(1 + |\log\varepsilon|)$. Lemma 4.1 now proves (4.12). In particular, all endpoint averages are bounded when $\|u\|_{\varepsilon,A} \le 1$.

Lemma 4.2 gives

$$\varepsilon^2 \int_{\partial_l R_{w_e}} u^2 \le C\varepsilon^2 \left(1 + \frac{\varepsilon}{w_e} + w_e\varepsilon\right) \le \begin{cases} C\varepsilon^2, & e \text{ primary}, \\ C\varepsilon, & e \text{ auxiliary}. \end{cases} \tag{4.14}$$

On every disk, (4.13) and the bound for $c_i(u)$ give $\int_{\partial D_i} u^2 \le C$. Moreover,

$$\left|\int \beta_i u^2 - c_i(u)^2\right| \le 2|c_i(u)| \left(\int \beta_i |u - c_i(u)|^2\right)^{1/2} + \int \beta_i |u - c_i(u)|^2 \le C\sqrt{\varepsilon}.$$

There are finitely many disks and bands, and $\theta_{\varepsilon,A} = 1 + O(\varepsilon^2)$. Summing their contributions proves

$$\left|b_{\varepsilon,A}(u) - |Ju|^2\right| \le C\sqrt{\varepsilon}.$$

For arbitrary u, multiply this estimate by $\|u\|_{\varepsilon,A}^2$ after normalisation. Polarisation gives a bound by $C\sqrt{\varepsilon}(\|u\|_{\varepsilon,A}^2 + \|v\|_{\varepsilon,A}^2)$ for the bilinear error. Apply it to $(tu, t^{-1}v)$ and minimise over $t > 0$. If either norm is zero, that function is zero by Lemma 2.1; otherwise the minimum is $2\|u\|_{\varepsilon,A}\|v\|_{\varepsilon,A}$. Absorbing the factor in C proves (4.11). ☐

5. The complete graph limit

5.1. **Comparison functions and weak equations.** For $z = (z_1, \dots, z_n) \in \mathbb{R}^n$, define $R_\varepsilon z$ to equal z_i on D_i and $(1-s)z_i + sz_j$ on a band joining i, j. The traces agree at every attaching side. Thus $R_\varepsilon z \in H^1(M_{\varepsilon,A})$, and

$$b_{\varepsilon,A}(R_\varepsilon z, R_\varepsilon y) = z \cdot y + O(\varepsilon^2)|z||y|, \tag{5.1}$$

$$\varepsilon^{-1} q_{\varepsilon,A}(R_\varepsilon z, R_\varepsilon y) = z^{\mathsf{T}} L(A) y + \varepsilon \sum_{e \text{ auxiliary}} (z_{i_e} - z_{j_e})(y_{i_e} - y_{j_e}). \tag{5.2}$$

On each reserved arc $R_\varepsilon z = z_i$, so its β_i contribution is $z_i y_i$. On the remaining boundary $|R_\varepsilon z| \le |z|$ and $|R_\varepsilon y| \le |y|$. The boundary length is bounded and the remaining density is ε^2; multiplication by $\theta_{\varepsilon,A} = 1 + O(\varepsilon^2)$ gives (5.1). The derivative on a band is the constant $z_j - z_i$ in its longitudinal direction, which proves (5.2). These formulas imply $\|R_\varepsilon z\|_{\varepsilon,A} \le C|z|$. For a band with both endpoints on one disk, the energy summand is zero.

Proposition 5.1. *Uniformly for $A \in \mathcal{B}$, the eigenvalues $s_0(\varepsilon, A), \dots, s_{n-1}(\varepsilon, A)$ converge, in increasing order and with multiplicities, to the eigenvalues of $L(A)$. Moreover,*

$$\inf_{A \in \mathcal{B}} s_n(\varepsilon, A) \geq c\varepsilon^{-1/2} - 1 \longrightarrow +\infty. \tag{5.3}$$

Proof. Let $\ell_j(A)$ be the jth eigenvalue of $L(A)$, with $\ell_0(A) = 0$, and let $Z_j(A)$ be the span of $j+1$ corresponding orthonormal eigenvectors. For $z \in Z_j(A)$,

$$b_{\varepsilon,A}(R_\varepsilon z) \geq (1 - C\varepsilon^2)|z|^2, \qquad \varepsilon^{-1} q_{\varepsilon,A}(R_\varepsilon z) \leq (\ell_j(A) + C\varepsilon)|z|^2.$$

The first inequality makes $R_\varepsilon|_{Z_j(A)}$ injective when $C\varepsilon^2 < 1$. By (2.3),

$$s_j(\varepsilon, A) \leq \frac{\ell_j(A) + C\varepsilon}{1 - C\varepsilon^2} \qquad (0 \leq j < n). \tag{5.4}$$

The constants are uniform in A by (5.1)–(5.2).

For the lower limit, take any $\varepsilon_l \to 0$ and $A_l \to A_*$. Choose b_{ε_l,A_l}-orthonormal eigenfunctions for the first $j+1$ eigenvalues. Their energies are $O(\varepsilon_l)$. By Proposition 4.3, their disk means have a common subsequence converging to orthonormal vectors $c_0, \dots, c_j \in \mathbb{R}^n$. Pass to a further subsequence so that the eigenvalues converge to $t_0 \leq \cdots \leq t_j$.

If u_l is one of these eigenfunctions, test its equation with $R_\varepsilon z$. Disk contributions to the energy are zero. On a primary band from i to j the contribution is exactly

$$\varepsilon_l^{-1} \int_{R_{w_e}} \langle du_l, dR_\varepsilon z\rangle = c_{ij}(A_l)(z_j - z_i)(m_e(1) - m_e(0)). \tag{5.5}$$

For an auxiliary band the coefficient is ε_l. Equations (4.12) and (4.11) therefore yield

$$z^{\mathsf{T}} L(A_*) c_i = t_i z \cdot c_i \qquad (z \in \mathbb{R}^n).$$

The graph has $j+1$ orthonormal eigenvectors with eigenvalues at most t_j. Its jth eigenvalue is at most t_j, proving the lower limit.

To prove the quantitative assertion, let $F \subset H^1(M_{\varepsilon,A})$ have dimension $n+1$. Since J has range in $\mathbb{R}^n$, choose $0 \neq u \in F \cap \ker J$. If $b_{\varepsilon,A}(u) = 0$, its quotient in (2.3) is infinite. Otherwise (4.11) gives

$$b_{\varepsilon,A}(u) \leq C\sqrt{\varepsilon}\big(b_{\varepsilon,A}(u) + \varepsilon^{-1} q_{\varepsilon,A}(u)\big), \qquad \frac{q_{\varepsilon,A}(u)}{\varepsilon b_{\varepsilon,A}(u)} \geq \frac{1}{C\sqrt{\varepsilon}} - 1.$$

Thus $\sup_{0 \neq v \in F} q_{\varepsilon,A}(v)/(\varepsilon b_{\varepsilon,A}(v)) \geq C^{-1}\varepsilon^{-1/2} - 1$. Taking the infimum over all $(n+1)$-dimensional F proves (5.3).

If convergence of the first n eigenvalues were not uniform, there would be $\varepsilon_l \to 0$, $A_l \in \mathcal{B}$ and a fixed positive error. A subsequence has $A_l \to A_*$. The upper and lower limits just proved contradict that error, since the eigenvalues of $L(A)$ are continuous in A. □

5.2. **Identifications on a fixed parameter family.**

Lemma 5.2. *Fix $\varepsilon > 0$ small enough that the attaching intervals are disjoint for every $A \in \mathcal{B}$. There is a smooth identification of the labelled pieces of $M_{\varepsilon,A}$ with those of $M_{\varepsilon,0}$ which preserves all identifications of short sides. Near each corner it is a rigid translation in the Euclidean sector coordinates. On the complements of these corner neighbourhoods the maps and their inverses have uniformly bounded derivatives of every fixed order, and depend continuously on A. The bounds may depend on ε.*

Proof. Use the parameter path $t \mapsto tA$, $0 \leq t \leq 1$, and write $w(t) = w_e(tA)$ for one band width. On the compact parameter set there are constants $0 < w_* \leq w(t) \leq w^*$ and uniform bounds for its parameter derivatives. Give a short side the coordinate $-w(t)/2 \leq x \leq w(t)/2$. Choose an odd smooth function ζ on $[-1, 1]$ which equals 1 near 1 and -1 near -1. On this interval put

$$V_t(x) = \frac{w'(t)}{2} \zeta\left(\frac{2x}{w(t)}\right).$$

Extend V_t smoothly to the containing fixed boundary arc, constantly near each moving endpoint and with value zero near the arc's fixed endpoints. The positive separation of the arcs permits disjoint extensions. Their derivatives are bounded uniformly at this fixed ε. In the disk collar use the vector field $\chi(y)V_t(x)\partial_x$, where $\chi = 1$ near the boundary and $\chi = 0$ at the inner edge. On the entire band use $V_t(x)\partial_x$. The two expressions agree on neighbourhoods of the identified short sides. For orientation-reversing side identifications the oddness of V_t gives the same compatibility.

Denote the resulting piecewise vector field by $\mathcal{V}_t$. The endpoints $x_\pm(t) = \pm w(t)/2$ solve the flow equation because $V_t(x_\pm(t)) = x'_\pm(t)$. Hence the flow maps each attaching interval onto its interval at time t. Near an endpoint the field is spatially constant, so this flow is a translation on a sufficiently small sector. The sectors can be chosen with a common radius smaller than a fixed multiple of w_*. Elsewhere the variational equation for the flow is

$$\frac{d}{dt}D\Phi_t = (D\mathcal{V}_t)(\Phi_t)D\Phi_t, \qquad \det D\Phi_t = \exp\left(\int_0^t \operatorname{div} \mathcal{V}_s(\Phi_s)\, ds\right) > 0.$$

If $L_1 = \sup_{A,t} \|D\mathcal{V}_t\|_\infty$, Gronwall's inequality gives

$$\|D\Phi_t\|_\infty + \|(D\Phi_t)^{-1}\|_\infty \le 2e^{L_1 t}.$$

For $r \ge 2$, differentiation r times gives a linear equation for $D^r\Phi_t$ with coefficient $D\mathcal{V}_t(\Phi_t)$; its remaining terms are finite sums of products of $D^j\mathcal{V}_t$, $j \le r$, and $D^i\Phi_t$, $i < r$. Induction and Gronwall's inequality bound $D^r\Phi_t$ uniformly in A, t. The inverse flow satisfies the same estimates with reversed time. The fields vanish outside the chosen collars and bands, so the flows define global identifications. Continuous dependence in these integral equations gives continuity in A in every spatial C^k norm. □

5.3. **Convergence of the compressed operator.** Let $F_{\varepsilon,A}$ be the spectral space corresponding to $s_1(\varepsilon, A), \dots, s_N(\varepsilon, A)$, where eigenvalues are counted with multiplicity. Proposition 5.1 separates this space from constants and from the remaining spectrum. On $F_{\varepsilon,A}$ use the inner product $b_{\varepsilon,A}$ and define the self-adjoint operator $S_{\varepsilon,A}$ by

$$b_{\varepsilon,A}(S_{\varepsilon,A}u, v) = \varepsilon^{-1}q_{\varepsilon,A}(u, v).$$

This identity holds for $u \in F_{\varepsilon,A}$ and every $v \in H^1(M_{\varepsilon,A})$ by linearity from the weak equations of its Steklov eigenfunctions. Put

$$T_{\varepsilon,A} = Q^\mathsf{T} J|_{F_{\varepsilon,A}} : F_{\varepsilon,A} \longrightarrow \mathbb{R}^N. \tag{5.6}$$

Proposition 5.3. *For all sufficiently small ε, the maps $T_{\varepsilon,A}$ are invertible. With adjoints taken in the specified inner products, one has*

$$\|T^*_{\varepsilon,A}T_{\varepsilon,A} - I\|_{\mathrm{op}} \le C\sqrt{\varepsilon}, \tag{5.7}$$

$$\|T_{\varepsilon,A}S_{\varepsilon,A} - (D + A)T_{\varepsilon,A}\|_{\mathrm{op}} \le C\sqrt{\varepsilon(1 + |\log \varepsilon|)}. \tag{5.8}$$

If

$$W_{\varepsilon,A} = T^*_{\varepsilon,A}(T_{\varepsilon,A}T^*_{\varepsilon,A})^{-1/2}, \qquad \mathcal{M}_\varepsilon(A) = W^*_{\varepsilon,A}S_{\varepsilon,A}W_{\varepsilon,A}, \tag{5.9}$$

then $\mathcal{M}_\varepsilon$ is continuous on $\mathcal{B}$, and

$$\sup_{A\in\mathcal{B}}\|\mathcal{M}_\varepsilon(A) - (D + A)\|_F \le C\sqrt{\varepsilon(1 + |\log \varepsilon|)} \longrightarrow 0. \tag{5.10}$$

Proof. Write $\|u\|_b^2 = b_{\varepsilon,A}(u)$ and $\delta_\varepsilon = \sqrt{\varepsilon(1 + |\log \varepsilon|)}$. Proposition 5.1 gives a constant C_0 such that

$$\|S_{\varepsilon,A}\|_{\mathrm{op}} \le C_0, \qquad \|u\|_{\varepsilon,A} \le (1 + C_0)^{1/2}\|u\|_b \quad (u \in F_{\varepsilon,A}).$$

Constants are b-orthogonal to $F_{\varepsilon,A}$, and $\|1\|^2_{\varepsilon,A} = n$. Equation (4.11), applied to $(u, 1)$, gives

$$|Ju \cdot \mathbf{1}| \le C\sqrt{\varepsilon}\|u\|_b.$$

Since $QQ^\mathsf{T} = I - n^{-1}\mathbf{1}\mathbf{1}^\mathsf{T}$,

$$|Tu|^2 = |Ju|^2 - n^{-1}|Ju \cdot \mathbf{1}|^2.$$

A second application of (4.11) proves $\big||Tu|^2 - \|u\|_b^2\big| \le C\sqrt{\varepsilon}\|u\|_b^2$. For a self-adjoint operator the operator norm is the supremum of the absolute Rayleigh quotient; this proves (5.7). In particular $T^*T \ge \frac{1}{2}I$ for small ε. Since both spaces have dimension N, T is invertible.

For $z \in \mathbb{R}^n$, the explicit comparison function $R_\varepsilon z$ satisfies $JR_\varepsilon z = z$ and $\|R_\varepsilon z\|_{\varepsilon,A} \le C|z|$. The band integration in (5.5), followed by (4.12), gives for $u \in F_{\varepsilon,A}$

$$\left|\varepsilon^{-1} q_{\varepsilon,A}(u, R_\varepsilon z) - z^{\mathsf{T}} L(A) J u\right| \le C\delta_\varepsilon \|u\|_b |z|.$$

The auxiliary terms are bounded by $C\varepsilon\|u\|_b|z|$; this is at most the displayed bound. The weak equation and (4.11) applied to $(S_{\varepsilon,A}u, R_\varepsilon z)$ give

$$\left|\varepsilon^{-1} q_{\varepsilon,A}(u, R_\varepsilon z) - z \cdot J S_{\varepsilon,A} u\right| \le C\sqrt{\varepsilon}\|u\|_b |z|.$$

Subtracting, then taking the supremum over $|z| = 1$, proves $\|JS_{\varepsilon,A}u - L(A)Ju\| \le C\delta_\varepsilon\|u\|_b$. Multiplication by Q^{T} and $Q^{\mathsf{T}}L(A) = (D + A)Q^{\mathsf{T}}$ prove (5.8).

Set $B = TT^*$ and $E = TS_{\varepsilon,A} - (D + A)T$. The eigenvalues of TT^* and T^*T agree, so (5.7) gives $\|B - I\| \le C\sqrt{\varepsilon}$. The scalar functions $t^{1/2}$ and $t^{-1/2}$ are Lipschitz on $[1/2, 3/2]$; diagonalisation gives

$$\|B^{1/2} - I\| + \|B^{-1/2} - I\| \le C\sqrt{\varepsilon}.$$

Using $TW = B^{1/2}$ and $W^* = B^{-1/2}T$, obtain the exact identity

$$\mathcal{M}_\varepsilon(A) = B^{-1/2}(D + A)B^{1/2} + B^{-1/2}EW. \tag{5.11}$$

Put $K = D + A$. The identity

$$B^{-1/2}KB^{1/2} - K = (B^{-1/2} - I)KB^{1/2} + K(B^{1/2} - I)$$

and the preceding bounds give

$$\|\mathcal{M}_\varepsilon(A) - K\|_{\mathrm{op}} \le \|B^{-1/2} - I\|\,\|K\|\,\|B^{1/2}\| + \|K\|\,\|B^{1/2} - I\| + \|B^{-1/2}\|\,\|E\| \le C\delta_\varepsilon.$$

In dimension N, $\|Z\|_F \le \sqrt{N}\|Z\|_{\mathrm{op}}$, which gives the stated estimate.

Fix $\varepsilon > 0$ and apply Lemma 5.2. Under its identifications the energy coefficient tensors, boundary densities and Jacobian factors depend continuously on A in L^∞. They and their inverse metric tensors are uniformly bounded at this fixed ε. The energy and mass forms are consequently norm-continuous on one fixed H^1 space. The disk-average functionals are integrals against continuous bounded Jacobian factors on fixed labelled disks, so they are norm-continuous there as well. Lemma 2.2, including (2.9), proves continuity of the spectral space and its compressed variational operator. Equation (5.7) bounds the inverse square roots in (5.9). The formula in Lemma 2.3 then proves continuity of $\mathcal{M}_\varepsilon$. □

6. Conformal smoothing and exact prescription

6.1. **The form domain at a corner.**

Lemma 6.1. *Let M_0 be a compact orientable metric surface smooth away from finitely many Euclidean boundary corners of angle $3\pi/2$. Its conformal structure extends to a smooth bordered surface $\overline{M}$. There is a smooth compatible metric k such that the following assertions hold.*

The conformal identification maps $H^1(M_0)$ onto $H^1(\overline{M}, k)$ and preserves the Dirichlet form. A measure whose density relative to the original arclength is continuous, strictly positive, and smooth on each closed side, has a continuous nonnegative density relative to ds_k; this density vanishes precisely at the corner points.

For the band families of Section 4, at each fixed $\varepsilon > 0$ the smooth models can be identified with one surface. Their compatible metrics and their continuous boundary densities can be chosen continuously in $A \in \mathcal{B}$.

Proof. In a sector coordinate z, define a half-disk coordinate ζ by

$$z = \zeta^{3/2}, \qquad \operatorname{Im}\zeta \ge 0. \tag{6.1}$$

On overlaps away from the vertex the changes of coordinates are holomorphic with nonvanishing derivative. At an ordinary smooth boundary point use a local isothermal half-disk chart. These charts define a smooth bordered conformal surface. A partition of unity applied to positive compatible local metrics gives a smooth compatible metric k. In the corner chart,

$$|dz|^2 = \tfrac{9}{4}|\zeta|\,|d\zeta|^2. \tag{6.2}$$

The inverse metric and area factors cancel in the energy on the punctured half-disk.

We prove equality of the two H^1 spaces near the vertex. Suppose u has finite original L^2 norm and finite energy. Fix $0 < R_1 < R_2$ in the corner chart. Since r is bounded above and below on the annulus $\mathcal{A} = \{R_1 < |\zeta| < R_2,\ \operatorname{Im}\zeta > 0\}$, Fubini's theorem gives an $R_0 \in (R_1, R_2)$ for which

$$\int_0^\pi |u(R_0,\theta)|^2\, d\theta \le C(R_1,R_2)\int_{\mathcal{A}} |u|^2\, dv_{M_0}. \tag{6.3}$$

The constant is independent of u. For almost every θ and $0 < r < R_0$, the one-dimensional Sobolev representative on radial intervals satisfies

$$|u(r,\theta)|^2 \le 2|u(R_0,\theta)|^2 + 2\log(R_0/r)\int_r^{R_0} |u_s(s,\theta)|^2 s\, ds.$$

Since $\int_0^{R_0} r\log(R_0/r)\, dr = R_0^2/4$, Tonelli's theorem and (6.3) give

$$\|u\|^2_{L^2(d\zeta;\, r<R_1)} \le C\left(\|u\|^2_{L^2(M_0;\, r<R_2)} + \int_{r<R_2} |du|^2\right). \tag{6.4}$$

In particular, $u \in L^2(d\zeta)$ near the vertex. The converse implication for L^2 follows from the bounded density $\frac{9}{4}|\zeta|$. The energy equality already gives the same gradient integrability on the punctured charts.

It remains to remove the puncture in the weak derivative. Let $\chi_L(r)$ equal one for $r \le e^{-2L}$, zero for $r \ge e^{-L}$, and $L^{-1}\log(e^{-L}/r)$ on the intervening annulus. On a half-disk,

$$\int |d\chi_L|^2 = \frac{\pi}{L}, \qquad \int \chi_L^2 \le \frac{\pi}{2}e^{-2L}. \tag{6.5}$$

For $u_M = \max(-M, \min(u, M))$, the norm of $u_M d\chi_L$ is at most $M(\pi/L)^{1/2}$. The norms of $\chi_L du_M$ and $\chi_L u_M$ tend to zero by dominated convergence. The functions $(1-\chi_L)u_M$, extended by zero near the vertex, belong to the smooth-domain H^1 space. The displayed bounds make them Cauchy in that space, and their L^2 limit is u_M. Completeness therefore gives $u_M \in H^1$ across the vertex. The truncation identity $du_M = \mathbf{1}_{\{|u|<M\}}du$ holds almost everywhere. Since $u \in L^2(d\zeta)$ and $du \in L^2(d\zeta)$, dominated convergence gives $u_M \to u$ in H^1 as $M \to \infty$. This proves extension of u and equality of the two full form domains. Equation (6.4), the bounded reverse area density, and the energy equality give bounded maps in both directions on the full H^1 spaces.

On the real boundary,

$$|dz| = \tfrac{3}{2}|\zeta|^{1/2}|d\zeta|.$$

If $k = e^{2v}|d\zeta|^2$ and the original density is ρ, the new density is

$$\omega(\zeta) = \tfrac{3}{2}\rho(z(\zeta))e^{-v(\zeta)}|\zeta|^{1/2}. \tag{6.6}$$

It is continuous and positive for $\zeta \ne 0$, and tends to zero at $\zeta = 0$. The same formula applies on both sides of the corner.

Fix $\varepsilon > 0$ and use Lemma 5.2. Choose Euclidean sector coordinates centred at the moving corner points. The identifications are translations in these coordinates, so (6.1) makes their pullbacks the identity on fixed smooth half-disk charts. Away from these charts the identifications are smooth with uniformly bounded derivatives and inverses at this fixed ε. They therefore produce complex structures J_A on one fixed smooth surface, continuous in every spatial C^k norm. For a fixed smooth positive metric k_* define

$$k_A(u,v) = k_*(u,v) + k_*(J_A u, J_A v).$$

Then $k_A(J_A u, J_A v) = k_A(u,v)$. On an oriented real two-plane, any two positive J_A-invariant inner products are proportional: in a basis $(e, J_A e)$ their matrices are positive scalar multiples of the identity. Thus k_A is compatible with the required conformal structure. Its coefficients are continuous in A, and positivity and compactness give uniform ellipticity. Formula (6.6) gives continuous boundary densities in the fixed charts; away from the vertices their formulas are smooth. Hence the full family is continuous in $C^0(\partial M)$. □

6.2. Approximation by positive smooth boundary densities.

Proposition 6.2. *For each sufficiently small fixed $\varepsilon > 0$, the cornered family in* (4.5) *can be replaced by a continuous family of smooth ordinary Steklov metrics on the same topological surface with boundary length $n\varepsilon$. Its spectral subspace for the first N positive eigenvalues, identified by disk averages and polar normalisation, has matrices $\widetilde{\mathcal{M}}_\varepsilon(A)$ satisfying*

$$\sup_{A\in\mathcal{B}} \|\widetilde{\mathcal{M}}_\varepsilon(A) - (D + A)\|_F \longrightarrow 0. \tag{6.7}$$

The eigenvalue σ_{N+1} tends to infinity uniformly in A.

Proof. Apply Lemma 6.1 to $M_{\varepsilon,A}$, identifying its smooth conformal model with one fixed surface at this fixed ε. Denote its smooth compatible metric by $k_{\varepsilon,A}$ and the density representing $b_{\varepsilon,A}$ by $\omega_{\varepsilon,A}$. The Dirichlet form is $q_{\varepsilon,A}$, and

$$\omega_{\varepsilon,A} \in C^0(\partial M), \quad \omega_{\varepsilon,A} \geq 0, \quad \int_{\partial M} \omega_{\varepsilon,A}\, ds_{k_{\varepsilon,A}} = n.$$

This is a compact continuous family of densities. In finitely many fixed boundary coordinates, mollify with a nonnegative smooth kernel, use a nonnegative partition of unity, and add a positive constant tending to zero. Denote the positive result by $\widehat{\omega}_{\varepsilon,A,\tau}$ and put

$$m_{\varepsilon,A,\tau} = \int_{\partial M} \widehat{\omega}_{\varepsilon,A,\tau}\, ds_{k_{\varepsilon,A}}, \qquad \omega_{\varepsilon,A,\tau} = \frac{n}{m_{\varepsilon,A,\tau}} \widehat{\omega}_{\varepsilon,A,\tau}. \tag{6.8}$$

If $\delta_\tau = \sup_A \|\widehat{\omega}_{\varepsilon,A,\tau} - \omega_{\varepsilon,A}\|_{C^0}$ and $L_\varepsilon = \sup_A \mathrm{Length}_{k_{\varepsilon,A}}(\partial M)$, then $|m_{\varepsilon,A,\tau} - n| \leq L_\varepsilon \delta_\tau$. For $L_\varepsilon\delta_\tau < n/2$ the denominator is at least $n/2$; hence

$$\sup_A \|\omega_{\varepsilon,A,\tau} - \omega_{\varepsilon,A}\|_{C^0} \leq 2\delta_\tau + \frac{2L_\varepsilon \delta_\tau}{n} \sup_A \|\omega_{\varepsilon,A}\|_{C^0}.$$

Thus the densities are smooth, positive, continuous in A, have total mass n, and satisfy

$$\sup_A \|\omega_{\varepsilon,A,\tau} - \omega_{\varepsilon,A}\|_{C^0} \longrightarrow 0 \qquad (\tau \to 0). \tag{6.9}$$

Uniform convergence follows from uniform continuity of the compact family before mollification. The positive total mass makes the normalisation continuous.

Fix a smooth reference metric k_* on this fixed surface and set $j_{\varepsilon,A} = ds_{k_{\varepsilon,A}}/ds_{k_*}$ on its boundary. For fixed $\varepsilon, \tau > 0$, the function $d_{\varepsilon,A,\tau} = \omega_{\varepsilon,A,\tau} j_{\varepsilon,A}$ is continuous in A with values in $C^0(\partial M)$. If $A, A' \in \mathcal{B}$, the fixed trace inequality gives

$$\left\| b_{\varepsilon,A,\tau} - b_{\varepsilon,A',\tau} \right\| \leq C_* \left\| d_{\varepsilon,A,\tau} - d_{\varepsilon,A',\tau} \right\|_{C^0(\partial M)},$$

where $b_{\varepsilon,A,\tau}(u,v) = \int_{\partial M} d_{\varepsilon,A,\tau} uv\, ds_{k_*}$ and the form norm is taken on $H^1(M,k_*)$. Thus the perturbed mass forms satisfy the norm-continuity hypothesis of Lemma 2.2. For this fixed ε, the trace inequality bounds the mass-form error by

$$C_\varepsilon \|\omega_{\varepsilon,A,\tau} - \omega_{\varepsilon,A}\|_{C^0} \|u\|_{H^1} \|v\|_{H^1}.$$

The constants and coercivity constants are uniform in A. Lemma 2.2 consequently gives uniform convergence of the compact operators and their isolated first N positive spectral spaces. The disk-average maps are bounded on the fixed H^1 space and depend continuously on A. Their restrictions remain invertible as follows. Write $\mathscr{A}_{A,\tau}$ for the fixed-Hilbert-space operator representing $q_{\varepsilon,A}/\varepsilon + b_{\varepsilon,A,\tau}$, and $P_{A,\tau}$ for the corresponding orthogonal spectral projection of (2.4). The bounded projection onto the variational spectral space is

$$\Pi_{A,\tau} = \mathscr{A}_{A,\tau}^{-1/2} P_{A,\tau} \mathscr{A}_{A,\tau}^{1/2}.$$

All these operators converge uniformly in A as $\tau \to 0$; square roots are continuous on the uniformly bounded positive interval containing their spectra. The value $\tau = 0$ denotes the unsmoothed forms.

Let $V_{A,0} = (Q^\top J|_{F_{\varepsilon,A}})^{-1}$. Proposition 5.3 bounds these inverses, including their H^1 norms at this fixed ε. Since $\Pi_{A,0} V_{A,0} = V_{A,0}$,

$$K_{A,\tau} = Q^\top J \Pi_{A,\tau} V_{A,0} \longrightarrow I$$

uniformly. For $\sup_A \|K_{A,\tau} - I\| < 1/2$ this matrix is invertible. Define

$$V_{A,\tau} = \Pi_{A,\tau} V_{A,0} K_{A,\tau}^{-1}.$$

Then $Q^\top J V_{A,\tau} = I$, and $V_{A,\tau}$ maps onto the perturbed N-dimensional spectral space. Moreover $V_{A,\tau} \to V_{A,0}$ uniformly in $\mathcal{L}(\mathbb{R}^N, H^1)$. In the standard basis (e_i) of $\mathbb{R}^N$, put

$$(G_{A,\tau})_{ij} = b_{\varepsilon,A,\tau}(V_{A,\tau} e_i, V_{A,\tau} e_j), \qquad (H_{A,\tau})_{ij} = \varepsilon^{-1} q_{\varepsilon,A}(V_{A,\tau} e_i, V_{A,\tau} e_j).$$

Both matrices converge uniformly, and the smallest eigenvalue of $G_{A,0}$ has a positive lower bound. Formula (2.11) now gives

$$\widetilde{\mathcal{M}}_{\varepsilon,\tau}(A) = G_{A,\tau}^{-1/2} H_{A,\tau} G_{A,\tau}^{-1/2} \longrightarrow \mathcal{M}_\varepsilon(A) \tag{6.10}$$

uniformly in A. The same formulas prove continuity in A for fixed τ.

For each fixed ε, choose one $\tau = \tau(\varepsilon) > 0$, independent of $A \in \mathcal{B}$, so that

$$\sup_{A \in \mathcal{B}} \|\widetilde{\mathcal{M}}_{\varepsilon,\tau}(A) - \mathcal{M}_\varepsilon(A)\|_F \le \varepsilon.$$

Such a choice follows from uniform norm convergence at fixed ε. Here $\widetilde{\mathcal{M}}_{\varepsilon,\tau}$ is the matrix of the compressed variational operator for $(q_{\varepsilon,A}/\varepsilon, b_{\varepsilon,A,\tau})$, before the boundary conformal factor is imposed. To retain divergence of the next eigenvalue, first choose any $H_\varepsilon \to \infty$ below the uniform lower bound for the cornered family in (5.3). Write $C_{\varepsilon,A}$ and $\widetilde{C}_{\varepsilon,A,\tau}$ for the compact operators in (2.4), formed with $q_{\varepsilon,A}/\varepsilon$. Choose τ also to satisfy

$$\sup_A \|\widetilde{C}_{\varepsilon,A,\tau} - C_{\varepsilon,A}\| < \frac{1}{2}\left(\frac{1}{1 + H_\varepsilon/2} - \frac{1}{1 + H_\varepsilon}\right). \tag{6.11}$$

The right-hand side is positive. The compact eigenvalue with index $N+1$, with index zero assigned to the largest eigenvalue, is at most $(1 + H_\varepsilon)^{-1}$ before smoothing. The eigenvalue norm inequality in Lemma 2.2 and (6.11) place it strictly below $(1 + H_\varepsilon/2)^{-1}$ afterwards. The corresponding Steklov eigenvalue is therefore greater than $H_\varepsilon/2$. All these choices are uniform in A at fixed ε; no coercivity constant uniform as $\varepsilon \to 0$ is used.

Choose a smooth $f_{\varepsilon,A}$ with boundary value

$$e^{f_{\varepsilon,A}} = \varepsilon \omega_{\varepsilon,A,\tau(\varepsilon)}, \qquad g_{\varepsilon,A} = e^{2 f_{\varepsilon,A}} k_{\varepsilon,A}. \tag{6.12}$$

Fix a collar projection π and a smooth cutoff χ supported in that collar and equal to one near the boundary. The function $\chi \log(\varepsilon \omega_{\varepsilon,A,\tau(\varepsilon)} \circ \pi)$, extended by zero, has the required boundary value. At fixed $\varepsilon, \tau > 0$, mollification makes the densities continuous in A in every spatial C^k norm; their positive minimum is uniform on the compact parameter set. The extension is therefore continuous in those norms. In dimension two,

$$q_{g_{\varepsilon,A}} = q_{\varepsilon,A}, \qquad ds_{g_{\varepsilon,A}} = \varepsilon \omega_{\varepsilon,A,\tau(\varepsilon)}\, ds_{k_{\varepsilon,A}}. \tag{6.13}$$

Thus its ordinary Steklov equation has exactly the eigenvalues of the smoothed pair $(q_{\varepsilon,A}/\varepsilon, b_{\omega_{\varepsilon,A,\tau}})$. In the coordinates $V_{A,\tau}$ the ordinary boundary and energy Gram matrices are $\varepsilon G_{A,\tau}$ and $\varepsilon H_{A,\tau}$. Hence their polar-normalised matrix is

$$(\varepsilon G_{A,\tau})^{-1/2} (\varepsilon H_{A,\tau}) (\varepsilon G_{A,\tau})^{-1/2} = G_{A,\tau}^{-1/2} H_{A,\tau} G_{A,\tau}^{-1/2}.$$

Thus (6.10) is also the matrix of the ordinary Steklov problem. The total boundary length is $n\varepsilon$. $\square$

6.3. **Proof of Theorem A and the boundary-length assertion.**

Proof of Theorem A and the boundary-length assertion. Choose $\varepsilon_0 > 0$ so that, for every $0 < \varepsilon < \varepsilon_0$, the error in (6.7) is smaller than r and the following eigenvalue is greater than H. Lemma 2.4 gives $A_\varepsilon \in \mathcal{B}$ with

$$\widetilde{\mathcal{M}}_\varepsilon(A_\varepsilon) = D. \tag{6.14}$$

The eigenvalues of $\widetilde{\mathcal{M}}_\varepsilon(A_\varepsilon)$ are the first N positive Steklov eigenvalues. Since this matrix equals D and $\sigma_{N+1} > H > a_N$, every value in the prescribed list has exactly its prescribed multiplicity. The metric is smooth by Proposition 6.2. The topological construction and surface classification transfer it to the prescribed M.

Given $0 < \ell < n\varepsilon_0$, take $\varepsilon = \ell/n$. Equation (6.13) gives boundary length exactly ℓ. The choice of ε_0 has involved no area parameter. There is no assertion that A_ε can be selected continuously. □

7. Area and isometries

Lemma 7.1. *If two metrics on a smooth surface satisfy $g' = e^{2w} g$ with $w = 0$ on the boundary, their ordinary Steklov spectra, with multiplicities, are equal. Their boundary lengths are equal.*

Proof. For $u, v \in H^1(M)$,

$$q_{g'}(u,v) = \int_M e^{-2w} \langle du, dv \rangle_g e^{2w}\, dv_g = q_g(u,v), \qquad \int_{\partial M} uv\, ds_{g'} = \int_{\partial M} uv\, ds_g.$$

The second identity uses $w|_{\partial M} = 0$. Min–max gives equality of the spectra, with multiplicities; taking $u = v = 1$ gives equality of boundary lengths. □

Lemma 7.2. *Let a smooth positive boundary line element λ be fixed on a compact connected orientable bordered conformal surface. The group $\mathcal{A}_\lambda$ of conformal and anticonformal automorphisms preserving λ is compact and acts smoothly up to the boundary. In this conformal class there is an $\mathcal{A}_\lambda$-invariant smooth metric with boundary line element λ.*

Proof. If $2\gamma + b > 2$, double the surface across its conformal boundary charts. The double has genus $2\gamma + b - 1 \geq 2$. Every automorphism extends by reflection. Uniformisation gives a hyperbolic metric on the double in its conformal class [11], and every extended automorphism is an isometry of that metric. The isometry group is a compact Lie group [27]. Its Lie algebra consists of Killing fields. For such a field Y, integration of the Killing equation gives

$$\int |\nabla Y|^2 = \int \mathrm{Ric}(Y, Y),$$

The right-hand side is $-\int |Y|^2$. Hence $Y = 0$, the Lie algebra is zero, and the compact zero-dimensional Lie group is finite.

The remaining topological cases are the disk and the annulus. Their smooth bordered conformal models are, respectively, the closed unit disk and $r \leq |z| \leq 1$, with $0 < r < 1$; see [11]. For the disk write $\lambda = s(\theta)|d\theta|$, where $0 < m \leq s \leq M$. A conformal automorphism has the form

$$\phi(z) = e^{i\vartheta} \frac{z - a}{1 - \bar{a}z}, \qquad |a| < 1;$$

this follows by sending its zero to the origin and applying the Schwarz lemma to the map and its inverse. If ϕ preserves λ, then $|\phi'(e^{i\theta})| \leq M/m$. Since

$$\max_{|z|=1} |\phi'(z)| = \frac{1 + |a|}{1 - |a|}, \qquad |a| \leq \frac{M - m}{M + m} < 1,$$

the allowed parameters lie in a compact subset. The condition of preserving λ is closed. The same argument applies after composition with conjugation. These formulas also give smooth compactness up to the boundary.

For the annulus a conformal automorphism either preserves or exchanges its two boundary circles. The harmonic function $\log |\phi(z)|$ has the corresponding boundary values. Uniqueness for the Dirichlet problem gives $|\phi(z)| = |z|$ or $r/|z|$, respectively. Thus $\phi(z)/z$ or $z\phi(z)/r$ is holomorphic with constant modulus one, and

$$\phi(z) = e^{i\vartheta} z \quad \text{or} \quad e^{i\vartheta} r/z.$$

Together with conjugation these maps form a compact group. Its subgroup preserving λ is closed.

In all cases take any smooth compatible metric k whose boundary line element is λ and average its pullbacks with respect to the Haar probability measure of $\mathcal{A}_\lambda$:

$$\bar{k} = \int_{\mathcal{A}_\lambda} \phi^* k \, d\phi. \tag{7.1}$$

All integrands are positive conformal multiples of k, including for anticonformal ϕ, and their tangential boundary restrictions are λ^2. Their average is smooth, positive, compatible with the same conformal structure, and has the same boundary line element. Haar invariance proves that $\mathcal{A}_\lambda$ acts by isometries of $\bar{k}$. □

Proposition 7.3. *Let a smooth positive boundary line element be prescribed on a compact connected orientable bordered conformal surface. Within this conformal class, the ordinary Steklov spectrum determined by that line element can be realised with any prescribed area $V > 0$ and with trivial isometry group.*

Proof. Fix a smooth compatible reference metric k_0 and write the prescribed boundary line element as $e^{f_\partial} ds_{k_0}$. In a fixed collar with inward coordinate t, extend f_∂ constantly in t and choose a smooth cutoff χ_δ with $0 \le \chi_\delta \le 1$, equal to 1 near $t = 0$ and 0 for $t \ge \delta$. For $0 < c \le 1$, put

$$f_{\delta,c} = \chi_\delta f_\partial + (1 - \chi_\delta) \log c$$

in the collar and $f_{\delta,c} = \log c$ elsewhere. This function is smooth and has boundary value f_∂. If $B = \max(0, \sup f_\partial)$, then $f_{\delta,c} \le B$ and

$$\mathrm{Area}(e^{2f_{\delta,c}} k_0) \le e^{2B} \,\mathrm{Area}_{k_0}(\{t < \delta\}) + c^2 \,\mathrm{Area}_{k_0}(M) \le C\delta + c^2 \,\mathrm{Area}_{k_0}(M).$$

Choose δ and c so that the last expression is less than $V/2$, and put $k = e^{2f_{\delta,c}} k_0$. Its boundary line element is the prescribed one.

Apply Lemma 7.2 and use the averaged metric $\bar{k}$ of (7.1). In real dimension two the area density of a positive conformal multiple fk is $f\, dv_k$. Consequently the averaging preserves area:

$$\mathrm{Area}(\bar{k}) = \int_{\mathcal{A}_\lambda} \mathrm{Area}(\phi^* k)\, d\phi = \mathrm{Area}(k) < V/2.$$

Every member of $\mathcal{A}_\lambda$ is an isometry of $\bar{k}$. Choose an interior point p_0 and two points $p_i = \exp^{\bar{k}}_{p_0}(te_i)$, $i = 1, 2$, where e_1, e_2 is a tangent basis and $t > 0$ is sufficiently small that these geodesics are unique and lie in the interior. An isometry fixing all three points fixes p_0 and both tangent vectors, so its derivative at p_0 is the identity. The exponential map makes it the identity on a neighbourhood. The set of interior points fixed with identity derivative is closed by continuity and open by the exponential-map identity $\phi(\exp_p v) = \exp_{\phi(p)}(d\phi_p v)$. The interior is connected, so this set is the entire interior. Continuity gives the identity on the boundary as well.

Choose $r_i > 0$ so that the closed normal disks $\overline{B^{\bar{k}}_{r_i}(p_i)}$, $0 \le i \le 2$, are disjoint and contained in the interior. Define

$$\vartheta(t) = \begin{cases} \exp\bigl(-t/(1-t)\bigr), & 0 \le t < 1, \\ 0, & t \ge 1, \end{cases} \qquad w = \sum_{i=0}^{2} \delta_i \vartheta\bigl(d_{\bar{k}}(p_i, \cdot)^2 / r_i^2\bigr),$$

with distinct $\delta_i > 0$. Each summand is smooth at its centre and vanishes to infinite order at its support boundary. Its derivative with respect to t is $-\vartheta(t)/(1-t)^2 < 0$ on $(0, 1)$. Consequently the strict positive local maxima of w are exactly p_0, p_1, p_2, with distinct values $\delta_0, \delta_1, \delta_2$. Every element of $\mathcal{A}_\lambda$ preserving w fixes the three centres and is therefore the identity. Every isometry of $e^{2w}\bar{k}$ belongs to $\mathcal{A}_\lambda$; since $\bar{k}$ is invariant, its isometry condition is exactly

$$w \circ \phi = w. \tag{7.2}$$

Thus the isometry group is trivial. Choose the distinct positive amplitudes so that

$$\delta_* := \max_i \delta_i < \frac{1}{2}\log\frac{V}{\mathrm{Area}(\bar{k})}.$$

The supports are disjoint and $0 \le \vartheta \le 1$, so $0 \le w \le \delta_*$ and $\mathrm{Area}(e^{2w}\bar{k}) \le e^{2\delta_*}\,\mathrm{Area}(\bar{k}) < V$.

Choose $0 \ne \eta_0 \in C_c^\infty(\operatorname{int} M)$ with $\eta_0 \ge 0$, and put

$$\eta(x) = \int_{\mathcal{A}_\lambda} \eta_0(\phi(x))\,d\phi.$$

Its support is contained in the image of the compact set $\mathcal{A}_\lambda \times \operatorname{supp}\eta_0$ under $(\phi, x) \mapsto \phi^{-1}(x)$, hence is a compact subset of the interior. It is nonnegative and invariant, and it is nonzero because $\int_M \eta\, dv_{\bar{k}} = \int_M \eta_0\, dv_{\bar{k}} > 0$. Put $g_s = e^{2(w+s\eta)}\bar{k}$, $s \ge 0$. Every isometry of g_s belongs to $\mathcal{A}_\lambda$. For $\phi \in \mathcal{A}_\lambda$, invariance of $\bar{k}$ and η gives

$$\phi^* g_s = g_s \quad\Longleftrightarrow\quad w\circ\phi + s\eta\circ\phi = w + s\eta \quad\Longleftrightarrow\quad w\circ\phi = w.$$

Thus $\mathrm{Isom}(M, g_s) = \{1\}$ for every $s \ge 0$. Moreover

$$\frac{d}{ds}\mathrm{Area}(g_s) = 2\int_M \eta e^{2(w+s\eta)}\, dv_{\bar{k}} > 0.$$

Choose an open set $U \Subset \operatorname{int} M$ and $c > 0$ on which $\eta \ge c$. Then

$$\mathrm{Area}(g_s) \ge e^{2sc}\int_U e^{2w}\, dv_{\bar{k}} \longrightarrow \infty.$$

The intermediate value theorem gives area V. All metrics used have the same conformal structure and boundary line element. Lemma 7.1 therefore preserves their full Steklov spectra and boundary length. □

Apply Proposition 7.3 after (6.14). This proves Theorem B, including the genus-zero cases permitted by (1.2). Corollary 1.1 follows by setting $N = d$ and $a_1 = \cdots = a_d = 1$.

8. Invariant perforations of hyperbolic surfaces

The second construction will retain curvature -1. In this section let (X, h) be a closed connected orientable hyperbolic surface and let G be a finite group of orientation-preserving isometries. The group may have fixed points. Its action on functions is $(g\cdot u)(x) = u(g^{-1}x)$. All boundary problems below are posed on the perforated surface itself, not on its quotient.

8.1. Separated invariant sets.

Lemma 8.1. *For all sufficiently small $\varepsilon > 0$, there is a finite G-invariant set $P_\varepsilon \subset X$ such that*

$$d(p, p') \ge \varepsilon \;(p \ne p'), \qquad \sup_{x\in X} d(x, P_\varepsilon) \le C\varepsilon. \tag{8.1}$$

Its Voronoi cells V_p satisfy

$$B_{\varepsilon/2}(p) \subset V_p \subset \overline{B_{C\varepsilon}(p)}, \qquad c\varepsilon^2 \le |V_p| \le C'\varepsilon^2. \tag{8.2}$$

The cells are permuted by G, up to their measure-zero boundaries. The constants and the upper bound on ε may depend on the fixed pair (X, G).

Proof. Let $Z = \bigcup_{a\ne 1}\mathrm{Fix}(a)$. A nonidentity orientation-preserving isometry fixing p has a nonidentity rotation as derivative. An isometry with its value and derivative fixed at a point is determined on geodesics from that point and hence on the connected surface. Thus no such derivative is the identity. The fixed points are isolated. Each $\mathrm{Fix}(a)$ is also closed, so compactness makes it finite. Since G is finite, Z is finite.

For each $a \ne 1$, the derivative of $q \mapsto a(q) - q$ in local coordinates at a fixed point is invertible. Consequently $d(q, aq) \ge c_a d(q, \mathrm{Fix}(a))$ near its fixed points. On the complement of fixed small neighbourhoods, $d(q, aq)$ has a positive minimum. After taking minima over the finite group, there are $C_0, \varepsilon_0 > 0$ such that

$$d(q, aq) < \varepsilon < \varepsilon_0 \quad \Longrightarrow \quad d(q, Z) < C_0\varepsilon. \tag{8.3}$$

For an element without fixed points only the positive-minimum alternative is needed. Here and below $a \neq 1$ in (8.3), and $d(q, \emptyset) = +\infty$. If $Z = \emptyset$, choose ε_0 below all the positive displacement minima; the antecedent is then false.

Take ε smaller than all positive distances between distinct points of Z. Among the G-invariant ε-separated sets containing Z, choose one of maximal cardinality. Such cardinalities are bounded by a packing estimate for disjoint $\varepsilon/2$ balls; as bounded nonempty subsets of the integers they have a maximum. Suppose $d(q, P_\varepsilon) > \varepsilon$. Invariance of P_ε gives $d(gq, P_\varepsilon) > \varepsilon$ for every $g \in G$. If Gq were ε-separated, its union with P_ε would contradict maximality. Hence there are $g, h \in G$ with $gq \neq hq$ and $d(gq, hq) < \varepsilon$. For $a = h^{-1}g \neq 1$ this gives $d(q, aq) < \varepsilon$. Equation (8.3) implies $d(q, P_\varepsilon) \leq d(q, Z) < C_0\varepsilon$. Together with the case $d(q, P_\varepsilon) \leq \varepsilon$, this proves the covering bound with $C = \max\{1, C_0\}$. This proves (8.1). The proof also applies when Z is empty, beginning with any sufficiently separated orbit, or with the empty invariant set before maximisation.

Define $V_p = \{x : d(x, p) \leq d(x, p') \text{ for every } p' \in P_\varepsilon\}$. Separation gives the first inclusion in (8.2), and covering gives the second. Hyperbolic disk area gives the two area bounds. Distance invariance gives $aV_p = V_{ap}$. Lift P_ε to the universal cover $\mathbb{H}^2$ and fix a lift $\widetilde{p}$. Its Voronoi set is the intersection of the half-planes $d(x, \widetilde{p}) \leq d(x, \widetilde{q})$ over all other lifted centres. The lifted set is $C\varepsilon$-dense, so this intersection is contained in $\overline{B_{C\varepsilon}(\widetilde{p})}$. For $2C\varepsilon < \operatorname{inj}(X)$, projection is injective on this closed ball and identifies the intersection with V_p. Centres farther than $2C\varepsilon$ from $\widetilde{p}$ impose no active constraints there. Only finitely many lifted centres are within that distance. Thus V_p is the projection of a convex hyperbolic polygon; its boundary is contained in finitely many geodesic bisectors and has area zero. ☐

8.2. **Boundary length equal to cell area.** Set $a_p = |V_p|$ and choose $r_p > 0$ by

$$2\pi \sinh r_p = a_p. \tag{8.4}$$

Then $r_p \asymp \varepsilon^2$ uniformly, so the closed disks $\overline{B_{r_p}(p)}$ are disjoint for small ε. Here $B_r(p)$ denotes an open metric ball. Define

$$\Omega_\varepsilon = X \setminus \bigcup_{p \in P_\varepsilon} B_{r_p}(p). \tag{8.5}$$

For $g \in G$, one has $a_{gp} = a_p$ and hence $r_{gp} = r_p$; therefore $g\Omega_\varepsilon = \Omega_\varepsilon$. It is a compact connected surface with smooth boundary and the same genus as X. Connectedness follows by replacing each part of a path crossing a disk by a boundary arc; a perturbation first makes a given path transverse to the finitely many circles. Its boundary consists of the circles $\partial B_{r_p}(p)$.

The definitions give

$$\mathrm{Length}_h(\partial\Omega_\varepsilon) = \sum_p a_p = \mathrm{Area}_h(X), \tag{8.6}$$

$$|P_\varepsilon| \asymp \varepsilon^{-2}, \qquad\qquad \mathrm{Area}_h(X \setminus \Omega_\varepsilon) = O(\varepsilon^2). \tag{8.7}$$

For the last estimate, the area of a radius-r hyperbolic disk is $2\pi(\cosh r - 1) = O(r^2)$ for small r. Thus the number of boundary components is finite for each ε and tends to infinity.

9. Boundary mass and harmonic filling

9.1. The small-circle estimate.

Lemma 9.1. *For $U \in H^1(X)$ set $c_p = a_p^{-1} \int_{V_p} U$. Then*

$$\int_{V_p} |U - c_p|^2 \le C\varepsilon^2 \int_{V_p} |dU|^2, \tag{9.1}$$

$$\int_{\partial B_{r_p}(p)} |U - c_p|^2 \, ds \le C r_p \big(1 + \log(\varepsilon/r_p)\big) \int_{V_p} |dU|^2. \tag{9.2}$$

Proof. The lifted cells are convex, have diameter $O(\varepsilon)$, and contain $B_{\varepsilon/2}(p)$. In a Klein chart they are Euclidean convex, and the hyperbolic metric and area density are uniformly comparable to Euclidean ones. The convex-cell estimate in Appendix A, followed by subtraction of the hyperbolic rather than Euclidean mean, proves (9.1).

Use hyperbolic polar coordinates (t, θ) about p, and put

$$m(t) = \frac{1}{2\pi} \int_0^{2\pi} U(t, \theta) \, d\theta, \qquad R = \varepsilon/8.$$

For small ε, the disk $B_{2R}(p)$ is contained in V_p. The trace and Poincaré inequalities on the annulus $B_{2r_p} \setminus B_{r_p}$, rescaled to a fixed annulus, give

$$\int_{\partial B_{r_p}} |U - m(r_p)|^2 \, ds \le C r_p \int_{B_{2r_p} \setminus B_{r_p}} |dU|^2. \tag{9.3}$$

Indeed, subtract the annular mean and use its Poincaré bound; the boundary mean minimises the boundary squared distance among constants.

Cauchy–Schwarz yields

$$\begin{aligned} |m(r_p) - m(R)|^2 &\le \left(\int_{r_p}^{R} \frac{dt}{\sinh t} \right) \int_{r_p}^{R} \sinh t \, |m'(t)|^2 \, dt \\ &\le C \log(R/r_p) \int_{V_p} |dU|^2. \end{aligned} \tag{9.4}$$

At radius R, the rescaled trace inequality gives

$$\int_{\partial B_R} |U - c_p|^2 \, ds \le C \left(R^{-1} \int_{B_{2R} \setminus B_R} |U - c_p|^2 + R \int_{B_{2R} \setminus B_R} |dU|^2 \right).$$

Divide by Length$(\partial B_R) \asymp R$ and use (9.1); since $R \asymp \varepsilon$, this gives

$$|m(R) - c_p|^2 \le C \int_{V_p} |dU|^2. \tag{9.5}$$

On ∂B_{r_p}, write

$$U - c_p = (U - m(r_p)) + (m(r_p) - m(R)) + (m(R) - c_p).$$

The inequality $|x+y+z|^2 \le 3(|x|^2+|y|^2+|z|^2)$, integrated on that circle, and (9.3)–(9.5) give (9.2), since its length is comparable to r_p and $\log(R/r_p) \le \log(\varepsilon/r_p)$. For fixed ε, take $U_l \in C^\infty(X)$ converging to U in $H^1(X)$. Restriction to each cell is continuous in H^1, its mean is a bounded L^2 functional, and restriction to each circle is continuous into L^2 by the trace estimate on its fixed annulus. Passing to the limit proves both inequalities for $U \in H^1(X)$. The constants are those already obtained and remain independent of p and small ε. ☐

9.2. **Convergence of the mass forms.** Let μ_ε be arclength on $\partial\Omega_\varepsilon$, considered as a measure on X. Integrals of $H^1(X)$ functions with respect to μ_ε use their Sobolev traces on the boundary circles.

Proposition 9.2. *For every $U \in H^1(X)$,*

$$\left|\|U\|_{L^2(\mu_\varepsilon)} - \|U\|_{L^2(X)}\right| \le C\varepsilon\sqrt{1+|\log\varepsilon|}\,\|dU\|_{L^2(X)}. \tag{9.6}$$

In particular there are $\eta_\varepsilon \to 0$ such that

$$\left|\int_{\partial\Omega_\varepsilon} UV\,ds - \int_X UV\,dv_h\right| \le \eta_\varepsilon \|U\|_{H^1(X)}\|V\|_{H^1(X)}. \tag{9.7}$$

Proof. Let C_U equal c_p on V_p and the same constant on its small boundary circle. By (8.4),

$$\|C_U\|^2_{L^2(X)} = \sum_p a_p c_p^2 = \|C_U\|^2_{L^2(\mu_\varepsilon)}.$$

Summing (9.1) bounds $\|U - C_U\|_{L^2(X)}$ by $C\varepsilon\|dU\|_2$. Summing (9.2), using $r_p \asymp \varepsilon^2$, bounds its boundary counterpart by $C\varepsilon\sqrt{1+|\log\varepsilon|}\|dU\|_2$. The triangle inequality proves (9.6).

Put $a_\varepsilon = C\varepsilon\sqrt{1+|\log\varepsilon|}$, with C as in (9.6), and define

$$D_\varepsilon(U,V) = \int_{\partial\Omega_\varepsilon} UV\,ds - \int_X UV\,dv_h.$$

The norm comparison gives

$$|D_\varepsilon(U,U)| \le 2a_\varepsilon\|U\|_2\|dU\|_2 + a_\varepsilon^2\|dU\|_2^2 \le (a_\varepsilon + a_\varepsilon^2)\|U\|^2_{H^1(X)}.$$

By real polarisation and the parallelogram identity,

$$|D_\varepsilon(U,V)| \le \frac{a_\varepsilon + a_\varepsilon^2}{2}\left(\|U\|^2_{H^1} + \|V\|^2_{H^1}\right).$$

Apply this to $(tU, t^{-1}V)$ and minimise over $t > 0$. If $U = 0$ or $V = 0$, both sides vanish. Otherwise the minimum of the sum of squares is $2\|U\|_{H^1}\|V\|_{H^1}$. Thus (9.7) holds with $\eta_\varepsilon = a_\varepsilon + a_\varepsilon^2 \to 0$. ☐

9.3. **The extension operator.**

Lemma 9.3. *Let $J_\varepsilon u$ equal u on Ω_ε and the harmonic extension of its trace on every removed disk. For all sufficiently small ε, the map $J_\varepsilon : H^1(\Omega_\varepsilon) \to H^1(X)$ is linear and satisfies*

$$\int_X |dJ_\varepsilon u|^2 \le \frac{18}{5}\int_{\Omega_\varepsilon} |du|^2, \qquad J_\varepsilon a^* = a^* J_\varepsilon \quad (a \in G). \tag{9.8}$$

Moreover,

$$\|J_\varepsilon u\|_{H^1(X)} \le C\left(\|u\|_{L^2(\partial\Omega_\varepsilon)} + \|du\|_{L^2(\Omega_\varepsilon)}\right). \tag{9.9}$$

Proof. The annuli $B_{2r_p}(p) \setminus B_{r_p}(p)$ are disjoint and contained in Ω_ε. A conformal disk coordinate identifies such an annulus with $\{a < |z| < b\}$, where

$$a = \tanh(r_p/2), \qquad b = \tanh r_p, \qquad 3/2 \le b/a \le 5/2$$

for small ε. Dirichlet energy is unchanged by this coordinate. For the inner trace write $f(\theta) = \sum_{k\in\mathbb{Z}} f_k e^{ik\theta}$. Its harmonic disk extension has energy

$$E_D(f) = 2\pi\sum_{k\neq 0} |k|\,|f_k|^2. \tag{9.10}$$

The constant Fourier mode has zero Dirichlet energy. If $\mathcal{H}f$ denotes this extension, then

$$\int_{|z|<a} |\mathcal{H}f|^2\,dx\,dy = \pi a^2 \sum_{k\in\mathbb{Z}} \frac{|f_k|^2}{|k|+1} < \infty.$$

The conformal area factor is bounded on this disk. Thus an $H^{1/2}$ trace has an H^1 harmonic extension before any uniform extension estimate is used. For $k \neq 0$, use the logarithmic radius $t = \log(|z|/a) \in$

$[0, L]$, $L = \log(b/a)$. Among annular functions with prescribed inner coefficient f_k and unrestricted outer coefficient, the minimum is achieved by

$$v_k(t) = f_k \frac{\cosh(|k|(L - t))}{\cosh(|k|L)}.$$

It satisfies $v_k'' = k^2 v_k$, $v_k(0) = f_k$ and $v_k'(L) = 0$. For any other coefficient $v_k + w$ with $w(0) = 0$, integration by parts gives

$$\operatorname{Re} \int_0^L (v_k' \overline{w'} + k^2 v_k \bar{w})\, dt = \operatorname{Re}[v_k' \bar{w}]_0^L = 0.$$

Thus its energy is the sum of the energies of v_k and w, proving minimality without imposing a value at the outer boundary. Integration of $(\bar{v}_k v_k')' = |v_k'|^2 + k^2 |v_k|^2$ gives the minimum $2\pi |k| \tanh(|k|L) |f_k|^2$. Parseval's identity and summation over the modes therefore give

$$\int_{a<|z|<b} |du|^2 \geq 2\pi \sum_{k \neq 0} |k| \tanh(|k|L) |f_k|^2 \geq \tanh(\log(3/2)) E_D(f). \tag{9.11}$$

The trace belongs to $H^{1/2}$, so (9.10) is finite. Finite Fourier sums and their $H^{1/2}$ limits justify the formulas for traces of functions in H^1. Since

$$\tanh(\log(3/2)) = \frac{(3/2)^2 - 1}{(3/2)^2 + 1} = \frac{5}{13},$$

each disk filling has energy at most $13/5$ times the energy of its exterior annulus. Disjointness of these annuli gives

$$\int_X |dJ_\varepsilon u|^2 = \int_{\Omega_\varepsilon} |du|^2 + \sum_p E_{D_p} \leq \left(1 + \frac{13}{5}\right) \int_{\Omega_\varepsilon} |du|^2,$$

which proves (9.8).

The disk extension is unique: the difference of two harmonic extensions belongs to H_0^1 and has zero energy when tested against itself. Its trace agrees with the exterior trace. Integration by parts on the pieces therefore gives a global distributional first derivative in L^2, with cancelling interface terms; the resulting function lies in $H^1(X)$. For each fixed ε its volume L^2 norm is finite, since there are finitely many disks and every harmonic filling lies in H^1. An isometry in G permutes the disks and preserves the harmonic equation and its boundary values. Uniqueness gives the intertwining identity.

Apply (9.6) to $J_\varepsilon u$:

$$\|J_\varepsilon u\|_{L^2(X)} \leq \|u\|_{L^2(\partial\Omega_\varepsilon)} + C\varepsilon\sqrt{1 + |\log \varepsilon|}\|dJ_\varepsilon u\|_{L^2(X)}.$$

The coefficient is bounded for small ε. The energy bound now gives (9.9). ☐

10. Spectral convergence and irreducibility

10.1. **Convergence at each fixed index.** Write $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots$ for the Laplace spectrum of X, and $0 = \sigma_0^\varepsilon < \sigma_1^\varepsilon \leq \cdots$ for the ordinary Steklov spectrum of Ω_ε.

Proposition 10.1. *For every fixed $j \geq 0$,*

$$\sigma_j^\varepsilon \longrightarrow \lambda_j. \tag{10.1}$$

Boundary-orthonormal eigenfunctions with bounded indices have subsequences whose harmonic fillings converge weakly in $H^1(X)$ and strongly in $L^2(X)$ to orthonormal Laplace eigenfunctions with the corresponding limiting eigenvalues.

Proof. Choose volume-orthonormal Laplace eigenfunctions $\varphi_0, \ldots, \varphi_j$ and put $\Phi z = \sum_{i=0}^j z_i \varphi_i$. On this fixed finite-dimensional space, $\|\Phi z\|_{H^1}^2 \leq (1 + \lambda_j)|z|^2$ and $\|d\Phi z\|_\infty \leq C_j |z|$. If B_ε and Q_ε are the boundary mass and Dirichlet Gram matrices of their restrictions, then

$$\|B_\varepsilon - I\|_{\mathrm{op}} \leq (1 + \lambda_j)\eta_\varepsilon, \qquad \|Q_\varepsilon - \operatorname{diag}(\lambda_0, \ldots, \lambda_j)\|_{\mathrm{op}} \leq C_j^2 \operatorname{Area}(X \setminus \Omega_\varepsilon).$$

For $(1+\lambda_j)\eta_\varepsilon < 1$, the restrictions are linearly independent and

$$\sigma_j^\varepsilon \le \frac{\lambda_j}{1-(1+\lambda_j)\eta_\varepsilon}.$$

Here the numerator uses $q_{\Omega_\varepsilon}(\Phi z) \le q_X(\Phi z) \le \lambda_j |z|^2$. Consequently

$$\limsup_{\varepsilon\to 0} \sigma_j^\varepsilon \le \lambda_j. \tag{10.2}$$

Take boundary-orthonormal Steklov eigenfunctions $u_0^\varepsilon, \dots, u_j^\varepsilon$. Their energies equal their eigenvalues and are bounded by (10.2). Lemma 9.3 bounds $U_i^\varepsilon = J_\varepsilon u_i^\varepsilon$ in $H^1(X)$. By Rellich compactness, a subsequence converges weakly in H^1 and strongly in L^2 to functions U_i. Equation (9.7) gives $\int_X U_i U_k = \delta_{ik}$. Pass to a further subsequence with $\sigma_i^\varepsilon \to t_i$.

For $\phi \in C^\infty(X)$ the weak equation is

$$\int_{\Omega_\varepsilon} \langle du_i^\varepsilon, d\phi\rangle = \sigma_i^\varepsilon \int_{\partial\Omega_\varepsilon} u_i^\varepsilon \phi \, ds. \tag{10.3}$$

The missing-disk energy pairing is bounded by

$$\|dU_i^\varepsilon\|_{L^2(X)} \|d\phi\|_{L^2(X\setminus\Omega_\varepsilon)} \longrightarrow 0.$$

The right-hand side of (10.3) converges to $t_i \int_X U_i \phi$, by (9.7) and strong L^2 convergence. Therefore $\Delta_X U_i = t_i U_i$ weakly. The interior estimate recorded in Appendix A, iterated in nested charts with $\Delta U_i = t_i U_i$, gives $U_i \in H^r$ for every integer r. Sobolev embedding then gives $U_i \in C^\infty(X)$. The $j+1$ orthonormal limits have eigenvalues at most t_j, so Laplace min–max gives $\lambda_j \le t_j$. This and (10.2) prove (10.1) along the chosen subsequence. If the eigenvalue convergence failed for the full family, some sequence $\varepsilon_l \to 0$ would satisfy $|\sigma_j^{\varepsilon_l} - \lambda_j| \ge \delta > 0$. The preceding argument would give a subsequence converging to λ_j, a contradiction. Thus the eigenvalue convergence in (10.1) holds for the full family. The convergence of eigenfunctions asserted in the proposition remains subsequential. $\square$

10.2. **Equivariant identification of the spectral subspace.** Let $E = \ker(\Delta_X - \lambda_1)$ and $d = \dim E$. By Proposition 10.1, $\sigma_d^\varepsilon < \sigma_{d+1}^\varepsilon$ for sufficiently small ε. For such ε, let E_ε be the sum of the complete Steklov eigenspaces with indices $1, \dots, d$, let b_ε be its boundary inner product, and let Λ_ε be its Steklov operator. For a nonzero finite-dimensional real G-module V, set

$$d_{\min}(V) = \min\{\dim W : 0 \ne W \subset V \text{ is an irreducible real } G\text{-submodule}\}.$$

The set in this definition is nonempty: a nonzero invariant subspace of least dimension is irreducible.

Theorem 10.2 (Equivariant identification). *For all sufficiently small ε, the space E_ε is an isolated d-dimensional spectral subspace. The map*

$$T_\varepsilon = P_E J_\varepsilon|_{E_\varepsilon} : E_\varepsilon \longrightarrow E, \tag{10.4}$$

where P_E is volume-orthogonal projection, is an equivariant isomorphism and satisfies

$$\|T_\varepsilon^* T_\varepsilon - I\|_{\mathrm{op}} \longrightarrow 0. \tag{10.5}$$

In particular, the first Steklov multiplicity satisfies

$$m_1(\Omega_\varepsilon) \ge d_{\min}(E). \tag{10.6}$$

Proof. Since $0 < \lambda_1 = \cdots = \lambda_d < \lambda_{d+1}$, Proposition 10.1 separates $\sigma_1^\varepsilon, \dots, \sigma_d^\varepsilon$ from 0 and from σ_{d+1}^ε for small ε. Every element of G preserves the domain, the Dirichlet form and the boundary form. Thus E_ε is invariant and Λ_ε commutes with G. The maps J_ε and P_E commute with G as well; for P_E, this follows from invariance of E and orthogonality of the action on $L^2(X)$.

Put $\delta_\varepsilon = \max_{1\le i\le d} |\sigma_i^\varepsilon - \lambda_1|$. For $v \in E_\varepsilon$ and $\phi \in C^\infty(X)$, expansion in a b_ε-orthonormal eigenbasis gives

$$\left| \int_{\Omega_\varepsilon} \langle dv, d\phi\rangle - \lambda_1 \int_{\partial\Omega_\varepsilon} v\phi \, ds \right| \le \delta_\varepsilon \|v\|_{b_\varepsilon} \|\phi\|_{L^2(\partial\Omega_\varepsilon)}. \tag{10.7}$$

Indeed, the left side is $|b_\varepsilon((\Lambda_\varepsilon - \lambda_1)v, \phi)|$, and the boundary norm of $(\Lambda_\varepsilon - \lambda_1)v$ is at most $\delta_\varepsilon \|v\|_{b_\varepsilon}$. We claim that

$$\alpha_\varepsilon := \sup_{\substack{v \in E_\varepsilon \\ \|v\|_{b_\varepsilon}=1}} \|(I - P_E)J_\varepsilon v\|_{L^2(X)} \longrightarrow 0. \tag{10.8}$$

Otherwise there are $a > 0$, $\varepsilon_l \to 0$ and $v_l \in E_{\varepsilon_l}$ with $\|v_l\|_{b_{\varepsilon_l}} = 1$ and $\|(I - P_E)J_{\varepsilon_l} v_l\|_2 \geq a$. Their energies are at most $\sigma_d^{\varepsilon_l}$. Lemma 9.3 and Rellich compactness give, after passage to a subsequence,

$$J_{\varepsilon_l} v_l \rightharpoonup v \text{ in } H^1(X), \qquad J_{\varepsilon_l} v_l \longrightarrow v \text{ in } L^2(X).$$

For each fixed smooth ϕ, the missing-disk energy pairing has absolute value at most

$$\|dJ_{\varepsilon_l} v_l\|_{L^2(X)} \|d\phi\|_{L^2(X \setminus \Omega_{\varepsilon_l})} \longrightarrow 0.$$

Equation (9.7) gives convergence of the boundary pairing to $\int_X v\phi$. Also $\delta_{\varepsilon_l} \to 0$, while (9.6) bounds the boundary norm of ϕ. Passing to the limit in (10.7) therefore yields $\int_X \langle dv, d\phi\rangle = \lambda_1 \int_X v\phi$ for every $\phi \in C^\infty(X)$. Elliptic regularity gives $v \in E$. Consequently $(I - P_E)J_{\varepsilon_l} v_l \to 0$ in $L^2(X)$, a contradiction.

For $v \in E_\varepsilon$, Lemma 9.3 gives $\|J_\varepsilon v\|^2_{H^1(X)} \leq C(1 + \sigma_d^\varepsilon)\|v\|^2_{b_\varepsilon}$. Together with (9.7), this implies

$$\left| \|J_\varepsilon v\|^2_{L^2(X)} - \|v\|^2_{b_\varepsilon} \right| \leq C\eta_\varepsilon (1 + \sigma_d^\varepsilon)\|v\|^2_{b_\varepsilon}.$$

Orthogonality of P_E gives

$$\|T_\varepsilon v\|_2^2 = \|J_\varepsilon v\|_2^2 - \|(I - P_E)J_\varepsilon v\|_2^2 \geq \left(1 - C\eta_\varepsilon(1 + \sigma_d^\varepsilon) - \alpha_\varepsilon^2\right)\|v\|^2_{b_\varepsilon}. \tag{10.9}$$

The same identity bounds the absolute difference from $\|v\|^2_{b_\varepsilon}$ by $\left(C\eta_\varepsilon(1 + \sigma_d^\varepsilon) + \alpha_\varepsilon^2\right)\|v\|^2_{b_\varepsilon}$. The Rayleigh-quotient characterisation of the norm of a self-adjoint operator proves (10.5). For small ε the coefficient in (10.9) is at least $1/2$. Thus T_ε is injective and, since both dimensions are d, is an equivariant isomorphism.

The first Steklov eigenspace F_ε is a nonzero invariant subspace of E_ε. Its image $T_\varepsilon F_\varepsilon$ contains an irreducible submodule W of E. Hence $\dim F_\varepsilon = \dim T_\varepsilon F_\varepsilon \geq \dim W \geq d_{\min}(E)$, proving (10.6). □

Corollary 10.3 (Irreducible first eigenspace). *If E is irreducible as a real G-module, then, for all sufficiently small ε,*

$$\sigma_1^\varepsilon = \cdots = \sigma_d^\varepsilon < \sigma_{d+1}^\varepsilon. \tag{10.10}$$

The first Steklov eigenspace is isomorphic to E as a real G-module.

Proof. By Lemma 2.3, $W_\varepsilon = T_\varepsilon^*(T_\varepsilon T_\varepsilon^*)^{-1/2} : E \to E_\varepsilon$ is an equivariant isometry. Thus $W_\varepsilon^* \Lambda_\varepsilon W_\varepsilon$ is real self-adjoint and commutes with G. Each of its eigenspaces is invariant. Irreducibility implies that it has one eigenvalue. The separation from σ_{d+1}^ε was established in Theorem 10.2. □

The proof uses the eigenspaces of a self-adjoint operator, not the assertion that the real commutant is one-dimensional.

11. The relative-chromatic contradiction

11.1. **The closed hyperbolic reference surfaces.** We use the following consequence of [12, Theorem 1.1 and Section 5.4].

Theorem 11.1 (Fortier Bourque–Gruda-Mediavilla–Petri–Pineault). *There are closed connected orientable hyperbolic surfaces X_g, $g \in \{10, 17, 37\}$, and finite groups G_g of orientation-preserving isometries with the following properties. Put $E_g = \ker(\Delta_{X_g} - \lambda_1(X_g))$. Then* $\dim E_{10} = 16$ *and* $\dim E_{17} = 21$*, and these two real representations are irreducible. Every nonzero real G_{37}-invariant subspace of E_{37} has dimension at least* 24.

The surfaces and groups are defined in [12, Section 5.1]. Its Theorem 1.1 gives the dimensions 16 and 21. Its proof in Section 5.4 excludes every complex irreducible constituent of $E_g \otimes_{\mathbb{R}} \mathbb{C}$ of dimension less than μ_g, where

$$\mu_{10} = 16, \qquad \mu_{17} = 21, \qquad \mu_{37} = 24.$$

For $g = 10, 17$, the total complex dimension equals μ_g, so the complexified representation is irreducible. A proper nonzero real invariant subspace would complexify to a proper nonzero complex invariant subspace; hence E_g is real irreducible. For $0 \neq V \subset E_{37}$ invariant, the nonzero complex representation $V \otimes_{\mathbb{R}} \mathbb{C}$ contains an irreducible constituent W. Its inclusion in $E_{37} \otimes_{\mathbb{R}} \mathbb{C}$ gives

$$\dim_{\mathbb{R}} V = \dim_{\mathbb{C}}(V \otimes_{\mathbb{R}} \mathbb{C}) \geq \dim_{\mathbb{C}} W \geq 24.$$

These deductions use the representation exclusions proved by the computer-assisted trace-formula inequalities of [12]. Those inequalities and their interval-arithmetic verification are external inputs.

Apply Corollary 10.3 to X_{10} and X_{17}, and Theorem 10.2 to X_{37}. The resulting domains $\Omega_\varepsilon^{(g)}$ have curvature -1, the same genus as X_g, and first Steklov multiplicities

$$m_1(\Omega_\varepsilon^{(10)}) = 16, \qquad m_1(\Omega_\varepsilon^{(17)}) = 21, \qquad m_1(\Omega_\varepsilon^{(37)}) \geq 24.$$

Equations (8.6)–(8.7) give a finite number of boundary components for each domain and a number tending to infinity as $\varepsilon \to 0$. This proves the geometric assertions of Theorem C.

11.2. **A chromatic bound.**

Lemma 11.2. *If S is orientable of genus $\gamma \geq 1$ and has nonempty boundary, then*

$$\operatorname{Chr}_0(S) \leq \left\lfloor \frac{7 + \sqrt{1 + 48\gamma}}{2} \right\rfloor. \tag{11.1}$$

Proof. Cap the boundary circles. Every properly embedded graph in S embeds in the resulting closed genus-γ surface. It suffices to consider finite simple graphs, because loops are absent in a graph colouring problem and parallel edges do not affect its chromatic number.

If a graph has chromatic number $k \geq 7$, choose a subgraph minimal under vertex and edge deletion with that chromatic number. It is connected, since one of the connected components of a disconnected graph attains its chromatic number. Every vertex has degree at least $k-1$: after its deletion, minimality supplies a $(k-1)$-colouring; at most $k-2$ neighbour colours would leave a colour for the deleted vertex. This critical graph has no cut vertex or bridge. In fact, colour the components on the two sides with $k-1$ colours, and permute their colours to agree at a cut vertex, or to differ at the endpoints of a bridge; this would give a $(k-1)$-colouring.

Let v, e be its numbers of vertices and edges. A connected regular neighbourhood R of the graph has genus γ' and f boundary components. Let $C_1, \dots, C_c$ be the closures of the complementary components in the closed surface, with genera h_i and boundary counts b_i. Since $b_i \geq 1$ and $\sum_i b_i = f$, additivity of Euler characteristic gives

$$\gamma = \gamma' + \sum_{i=1}^{c} h_i + f - c \geq \gamma'.$$

A boundary walk of length one would be a loop. A walk of length two would use parallel edges or traverse one edge twice with both endpoint rotations of degree one. Neither occurs in this simple graph of minimum degree at least 6. Thus each walk has length at least three, and $3f \leq 2e$. Since R retracts onto the graph, $v - e = 2 - 2\gamma' - f$. Therefore

$$e \leq 3v - 6 + 6\gamma' \leq 3v - 6 + 6\gamma.$$

On the other hand $2e \geq (k-1)v$ and $v \geq k$. Thus

$$(k-7)v \leq 12\gamma - 12, \qquad k(k-7) \leq 12\gamma - 12.$$

Solving this quadratic inequality gives (11.1). If $k < 7$, the bound already holds for $\gamma \geq 1$. $\square$

For the three genera of Theorem 11.1, Lemma 11.2 gives

$$\operatorname{Chr}_0(\Omega_\varepsilon^{(10)}) \leq 14, \qquad \operatorname{Chr}_0(\Omega_\varepsilon^{(17)}) \leq 17, \qquad \operatorname{Chr}_0(\Omega_\varepsilon^{(37)}) \leq 24.$$

The constructed ordinary metrics are included among those allowed in the definition of M_1. Hence

$$M_1(\Omega_\varepsilon^{(10)}) \geq 16 > 13 \geq \mathrm{Chr}_0(\Omega_\varepsilon^{(10)}) - 1, \tag{11.2}$$

$$M_1(\Omega_\varepsilon^{(17)}) \geq 21 > 16 \geq \mathrm{Chr}_0(\Omega_\varepsilon^{(17)}) - 1, \tag{11.3}$$

$$M_1(\Omega_\varepsilon^{(37)}) \geq 24 > 23 \geq \mathrm{Chr}_0(\Omega_\varepsilon^{(37)}) - 1. \tag{11.4}$$

This proves the final assertion of Theorem C. No assertion about the precise value of Chr_0 is required.

Appendix A. The Sobolev estimates used above

The following arguments specify the estimates used in the degenerations. The principal external analytic inputs include the Riesz representation theorem, the compact self-adjoint spectral theorem, bounded positive-operator functional calculus, weak compactness in Hilbert spaces, Fourier–Plancherel theory, Sobolev approximation in fixed coordinate domains, and the local elliptic estimates stated at the end of this appendix. The coordinate and change-of-variable formulas are used on their stated fixed charts. No elliptic constant uniform over a degenerating family is inferred from these inputs.

A.1. **Trace, compactness and Poincaré inequalities.** For $v \in C_c^\infty(\mathbb{R} \times [0, \infty))$, take Fourier transform in the first variable. With $a = (1 + \xi^2)^{1/2}$,

$$a|\widehat{v}(\xi, 0)|^2 = -a \int_0^\infty \partial_y |\widehat{v}(\xi, y)|^2 \, dy \leq \int_0^\infty \left(a^2 |\widehat{v}|^2 + |\partial_y \widehat{v}|^2 \right) dy.$$

Integration in ξ gives the bounded trace $H^1 \to H^{1/2}$. Reflection, cutoffs and finite smooth boundary charts transfer it to compact smooth surfaces. The right inverse is given locally by $\widehat{v}(\xi, y) = e^{-ay} \widehat{f}(\xi)$ and then by a partition of unity. These constructions are bounded on a fixed disk or annulus, and on a compact family of smooth metrics with uniformly bounded coefficients and ellipticity constants.

For a fixed Lipschitz boundary chart $y = f(x)$, put

$$\Phi(x, t) = (x, t + f(x)), \qquad \Phi^{-1}(x, y) = (x, y - f(x)), \qquad L = \mathrm{Lip}(f).$$

These maps are bi-Lipschitz. Almost everywhere,

$$D\Phi = \begin{pmatrix} 1 & 0 \\ f'(x) & 1 \end{pmatrix}, \qquad D\Phi^{-1} = \begin{pmatrix} 1 & 0 \\ -f'(x) & 1 \end{pmatrix}, \qquad \det D\Phi = 1.$$

Change of variables and the weak chain rule give, on corresponding chart domains,

$$C_L^{-1} \|u\|_{H^1} \leq \|u \circ \Phi\|_{H^1} \leq C_L \|u\|_{H^1}.$$

If $\Gamma(x) = (x, f(x))$, then

$$|x - y| \leq |\Gamma(x) - \Gamma(y)| \leq \sqrt{1 + L^2}\, |x - y|, \qquad 1 \leq |\Gamma'(x)| \leq \sqrt{1 + L^2}$$

almost everywhere. These bounds, applied to the L^2 norm and to

$$[h]_{H^{1/2}(I)}^2 = \int_I \int_I \frac{|h(x) - h(y)|^2}{|x - y|^2} \, dx \, dy,$$

give equivalence of the $H^{1/2}$ norms under the boundary parametrisation. Conjugating the half-plane trace and its right inverse by Φ, then using a finite partition of unity, proves the trace and extension assertions for a fixed Lipschitz surface. The constants may depend on its charts; no uniformity over degenerating charts is asserted.

For a Euclidean corner of angle $3\pi/2$, the polar map $(r, \theta) \mapsto (r, 2\theta/3)$ from the sector to a half-disk is bi-Lipschitz, with a bi-Lipschitz inverse. Away from the origin its radial and angular differential factors are 1 and $2/3$. The ratios of chord lengths are bounded above and below because the two angular intervals have lengths strictly less than 2π. Change of variables therefore bounds composition in H^1 in both directions. On the boundary, comparison of arclength and distance in the double integral defining $H^{1/2}$ gives the corresponding bounds for traces. The half-disk trace and extension operators thus give the same conclusions at each corner. Using finitely many charts proves the assertions for each fixed polygonal surface used above; no uniform bound as band widths tend to zero is inferred from this argument.

On a compact curve the Fourier tail satisfies

$$\sum_{|k|>K} |\widehat{f}(k)|^2 \leq (1+K^2)^{-1/2}\|f\|^2_{H^{1/2}}.$$

Finite coordinate charts prove compactness of $H^{1/2} \hookrightarrow L^2$ and hence compactness of the trace $H^1(M) \to L^2(\partial M)$. For interior compactness, extend functions by reflection in boundary charts and use cutoffs on a fixed rectangle. The Fourier tail estimate with weight $1+|k|^2$ proves the compact embedding $H^1(M) \hookrightarrow L^2(M)$.

The mean-zero Poincaré inequality on a fixed connected compact domain follows by contradiction. A sequence with mean zero, unit L^2 norm and gradient tending to zero has an L^2-convergent subsequence. Its limit has zero weak gradient and hence is constant on every connected coordinate chart; connectedness makes the constants equal. Its zero mean and unit norm contradict one another. For a fixed annulus, rescaling the trace and this Poincaré inequality gives

$$\int_{\partial B_r} |v|^2 \leq C\left(r^{-1}\int_{B_{2r}\setminus B_r} |v|^2 + r\int_{B_{2r}\setminus B_r} |dv|^2\right). \tag{A.1}$$

The hyperbolic versions for small r follow after rescaling because their coefficients converge smoothly to the Euclidean coefficients. These arguments also give all fixed disk estimates in Section 4.

A.2. **Identified sides and the Sobolev space.** Let finitely many Lipschitz pieces be identified along full open sides by smooth isometries, with a surface as quotient. A function on the quotient belongs to H^1 if and only if its restrictions belong to H^1 and their traces agree on every paired side. The integral of its squared weak gradient is the sum of the piecewise energies.

To verify the implication from the pieces to the quotient, work in a chart meeting one open seam and flatten that seam. Write the two halves as U_+ and U_-, with common seam $\Sigma_0 = \overline{U}_+ \cap \overline{U}_-$ and opposite outward normals. For a compactly supported smooth test function ϕ, the weak Green formula on the halves gives

$$\sum_{\pm}\int_{U_\pm} u_\pm \partial_j\phi = -\sum_{\pm}\int_{U_\pm} (\partial_j u_\pm)\phi + \int_{\Sigma_0} (\mathrm{Tr}\, u_+ - \mathrm{Tr}\, u_-)\phi\nu_{+,j}.$$

The last term is zero. Hence the weak derivative is the piecewise L^2 derivative. A partition of unity proves the claim off the finitely many seam endpoints. At such endpoints use the cutoff (6.5), first for bounded truncations and then let the truncation level tend to infinity. In a finite union of sectors the cutoff energy is bounded by C/L. On the punctured chart,

$$\|\chi_L u_M\|_{H^1} \leq \|u_M\|_\infty\big(\|\chi_L\|_2 + \|d\chi_L\|_2\big) + \|\chi_L du_M\|_2 \longrightarrow 0.$$

The approximants $(1-\chi_L)u_M$ extend by zero and are Cauchy in the full chart H^1 space. Their L^2 limit is u_M. Completeness, followed by truncation convergence as $M \to \infty$, gives membership across the endpoints. Thus the function belongs to the full quotient H^1 space. Conversely, local smooth approximation on a Lipschitz chart, followed by trace continuity, gives identical traces on the two sides of a seam for every global H^1 function. Since seams have area zero, the energy identity follows from the piecewise weak derivative formula.

A.3. **Uniform Poincaré inequality for the cells.** Let $V \subset \mathbb{R}^2$ be convex, contain a ball B of radius $c\varepsilon$, and have diameter at most $C\varepsilon$. For smooth u and $x \in V$, $y \in B$,

$$|u(x)-u(y)|^2 \leq |x-y|^2\int_0^1 |du((1-t)y+tx)|^2\,dt.$$

Integrate over $V \times B$. For $0 \le t \le 1/2$, change variables in y; its Jacobian is $(1-t)^2 \ge 1/4$. For $1/2 \le t \le 1$, change variables in x; its Jacobian is $t^2 \ge 1/4$. Convexity keeps each image inside V. The result is

$$\int_V \int_B |u(x) - u(y)|^2 \, dy \, dx \le C\varepsilon^2(|V| + |B|) \int_V |du|^2.$$

Divide by $|B|$ and apply Jensen to the mean of u on B. Since $|V|/|B|$ is bounded, this gives $\int_V |u - u_B|^2 \le C\varepsilon^2 \int_V |du|^2$. The mean on V minimises the left-hand side among constants. Let $a \le \rho \le b$ be a positive density, and let $u_\rho = (\int_V \rho)^{-1} \int_V \rho u$. If u_V is the Euclidean mean, then minimisation over constants gives

$$\int_V \rho|u - u_\rho|^2 \le \int_V \rho|u - u_V|^2 \le bC\varepsilon^2 \int_V |du|^2.$$

Uniform comparison of the metric and its inverse with the Euclidean metric bounds the last integral by the corresponding Dirichlet energy with density ρ, with a constant depending only on these comparison bounds and a, b. This proves the version used in (9.1). Smooth approximation on the fixed Lipschitz cell, followed by convergence of its mean and energy, gives the inequality for H^1 functions.

A.4. **Elliptic regularity on a fixed smooth surface.** We use the following fixed-geometry elliptic estimates; see [25, Chapter 4]. For relatively compact smooth coordinate domains $U \Subset U'$ and an integer $s \ge 0$, a smooth uniformly elliptic scalar operator P of order two satisfies

$$\|u\|_{H^{s+2}(U)} \le C_s\big(\|Pu\|_{H^s(U')} + \|u\|_{L^2(U')}\big).$$

The estimate, initially for smooth functions, extends to distributional solutions by local regularisation. The boundary version for a smooth Neumann problem is

$$\|u\|_{H^{s+2}(M)} \le C_s\big(\|\Delta u\|_{H^s(M)} + \|\partial_\nu u\|_{H^{s+1/2}(\partial M)} + \|u\|_{L^2(M)}\big).$$

If $u \in H^1(M)$ is weakly harmonic and $\partial_\nu u = \sigma\omega u$ with smooth ω, the trace lies in $H^{1/2}$, so the boundary estimate first gives $u \in H^2$. Its new trace lies in $H^{3/2}$; multiplication by ω preserves this space. Iteration gives $u \in H^s$ for all integers s, and Sobolev embedding gives smoothness. On a closed surface the interior estimate provides the same conclusion for $\Delta u = \lambda u$. These regularity statements are applied to each fixed surface or to the fixed limiting surface, never with an unproved uniform constant in the shrinking parameter.

Appendix B. An explicit finite model

For $N = 4$ and $(a_1, a_2, a_3, a_4) = (1, 1, 3, 7)$, the matrix of Lemma 3.1 is

$$L_0 = \frac{1}{15}\begin{pmatrix} 62 & -43 & -13 & -3 & -3 \\ -43 & 62 & -13 & -3 & -3 \\ -13 & -13 & 32 & -3 & -3 \\ -3 & -3 & -3 & 12 & -3 \\ -3 & -3 & -3 & -3 & 12 \end{pmatrix}.$$

Its characteristic polynomial is $t(t-1)^2(t-3)(t-7)$, by (3.1). Theorems A and B therefore apply on every orientable surface satisfying

$$\gamma \ge \Gamma(4, b) = \max\left\{1, \left\lceil \frac{7-b}{2} \right\rceil\right\}.$$

In particular, genus 3 suffices for connected boundary, and genus 1 suffices for every $b \ge 5$. This finite matrix specifies the limit operator; the proof does not assert a numerical value of the small geometric parameter or of the degree-theoretic root.

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