THE OPTIMAL EIGENVALUE COUNT IN AUBRY'S SPECTRAL SPHERE THEOREM FOR DIMENSIONS AT LEAST FOUR

ANASS NIFA

Abstract. For every $n \geq 4$, we determine the least number of positive Laplace eigenvalues whose pinching above the Lichnerowicz bound forces sphere topology under $\mathrm{Ric}_g \geq (n-1)g$. This number is n, for both homeomorphism and diffeomorphism, also in the simply connected class. At every sufficiently small prescribed positive volume V, a fixed simply connected nonspherical manifold with second homology $\mathbb{Z}^2$ admits a smooth family for which the first $n-1$ positive eigenvalues tend to n, while the nth remains uniformly separated from n. The geometric input is a fixed-volume degeneration of core-admissible doubles from a preceding paper. Applying it also to the sphere, and using measured spectral convergence, gives families on nonhomeomorphic manifolds, with the same fixed volume and Ricci lower bound, for which every scalar Laplace eigenvalue has the same limit. Their common metric-measure limit is $S^{n-2} * S^1_{2\pi q}$, where $q = V/\sigma_n$ and $\sigma_n = \mathrm{Vol}(S^n, g_{S^n})$. For $q = 1/m$, $m \geq 2$, we identify its canonical Laplacian with the invariant round-sphere Laplacian and compute all multiplicities:

$$\sum_{\ell \geq 0} a_\ell t^\ell = \frac{1+t^m}{(1-t)^{n-1}(1-t^m)}, \qquad \mu_\ell = \ell(\ell+n-1).$$

When $q = 1/m$ is sufficiently small and $m \geq 3$, the nth positive eigenvalue tends to $2(n+1)$. All families have constant volume density and a uniform positive lower bound on the volumes of unit balls.

Contents

1. Introduction

Let $n \geq 2$. For a closed connected Riemannian n-manifold (M, g), write

$$0 = \lambda_0(M,g) < \lambda_1(M,g) \leq \lambda_2(M,g) \leq \cdots$$

for the spectrum of $\Delta_g = -\mathrm{div}_g \nabla$, with multiplicities. Set

$$\mathcal{M}_n = \{(M^n, g) \,:\, M \text{ closed and connected, } \mathrm{Ric}_g \geq (n-1)g\}.$$

Lichnerowicz's inequality gives $\lambda_1 \geq n$ on $\mathcal{M}_n$. Aubry's spectral sphere theorem [1, Theorem 1, p. 388] gives a constant $c_n > 0$ such that

$$(M,g) \in \mathcal{M}_n, \qquad \lambda_n(M,g) < n + c_n \quad \Longrightarrow \quad M \cong_{\mathrm{diff}} S^n. \tag{1.1}$$

For each n, fix once and for all one such constant $c_n > 0$; throughout, c_n denotes this Aubry constant. Here $\cong_{\mathrm{top}}$ and $\cong_{\mathrm{diff}}$ denote homeomorphism and diffeomorphism, respectively.

Definition 1.1. For $\mathsf{C} \in \{\mathrm{top}, \mathrm{diff}\}$, let $k_{\mathsf{C}}(n)$ be the least positive integer k for which there is an $\varepsilon > 0$, depending only on n and k, such that

$$(M, g) \in \mathcal{M}_n, \qquad \lambda_k(M, g) < n + \varepsilon \quad \Longrightarrow \quad M \cong_{\mathsf{C}} S^n. \tag{1.2}$$

Let $k^{\mathrm{sc}}_{\mathsf{C}}(n)$ be defined by imposing in addition $\pi_1(M) = \{1\}$.

Equation (1.1) shows that all four integers in Definition 1.1 are at most n. Aubry left open the determination of the least index k for which pinching λ_k above the Lichnerowicz bound forces M to be diffeomorphic to S^n [1, p. 388]. His examples with $\lambda_{n-1} \to n$ have underlying manifold S^n; they establish the sharpness of the n-eigenvalue theorem for Gromov–Hausdorff convergence, but do not establish topological or differentiable sharpness.

For $n \geq 4$, put

$$Q_n = \begin{cases} \mathbb{CP}^{n/2}, & n \text{ even}, \\ S^{n-2} \times S^2, & n \text{ odd}, \end{cases} \qquad M_n = Q_n \# (-Q_n). \tag{1.3}$$

The projective spaces carry their complex orientations and the products their product orientations; $-Q_n$ denotes orientation reversal. Put $\sigma_n = \mathrm{Vol}(S^n, g_{S^n})$.

Theorem 1.2. *Let $n \geq 4$. The manifold M_n in* (1.3) *is simply connected and*

$$H_2(M_n; \mathbb{Z}) \cong \mathbb{Z}^2. \tag{1.4}$$

There is $q_n \in (0, 1)$ such that, for every $q \in (0, q_n)$, M_n carries a smooth family $(g_s)_{s \geq 0}$ satisfying

$$\mathrm{Ric}_{g_s} \geq (n-1) g_s, \qquad dv_{g_s} = dv_{g_0}, \qquad \mathrm{Vol}(M_n, g_s) = q\sigma_n. \tag{1.5}$$

For $1 \leq a \leq n-1$,

$$\lim_{s \to \infty} \lambda_a(M_n, g_s) = n, \tag{1.6}$$

and, for every $s \geq 0$,

$$\lambda_n(M_n, g_s) \geq n + c_n. \tag{1.7}$$

Moreover, for every $x \in M_n$ and $0 < r \leq \pi$,

$$\mathrm{Vol}_{g_s}(B_r(x)) \geq q\sigma_n \left(\frac{r}{\pi}\right)^n \qquad (s \geq 0). \tag{1.8}$$

Corollary 1.3. *For every $n \geq 4$,*

$$k_{\mathrm{top}}(n) = k_{\mathrm{diff}}(n) = k^{\mathrm{sc}}_{\mathrm{top}}(n) = k^{\mathrm{sc}}_{\mathrm{diff}}(n) = n. \tag{1.9}$$

Proof. Equation (1.1) gives the upper bound n for all four quantities. Fix $q = q_n/2$, $1 \leq k \leq n-1$, and $\varepsilon > 0$. By (1.6),

$$\lambda_k(M_n, g_s) \leq \lambda_{n-1}(M_n, g_s) < n + \varepsilon$$

for all sufficiently large s. Since $H_2(M_n; \mathbb{Z}) \cong \mathbb{Z}^2$ and $H_2(S^n; \mathbb{Z}) = 0$, M_n is not homeomorphic to S^n. Since $\pi_1(M_n) = \{1\}$, no $k < n$ has the defining property of any of the four quantities in Definition 1.1. $\square$

Reference [6, Theorem 8.1 and Lemma 5.1] supplies the geometric input: for every $n \geq 4$, every core-admissible Q^n, and every sufficiently small $q > 0$, it furnishes a fixed-volume family on $Q\#(-Q)$ with

$$\mathrm{Ric}_{g_s} \geq (n-1) g_s, \qquad dv_{g_s} = dv_{g_0}, \qquad \mathrm{Vol}(Q\#(-Q), g_s) = q\sigma_n,$$

$$(Q\#(-Q), g_s) \xrightarrow[s \to \infty]{\mathrm{GH}} S^{n-2} * S^1_{2\pi q}.$$

Sections 2–4 derive the sharp eigenvalue count from this family. The inclusion $S^{n-2} \hookrightarrow S^{n-2} * S^1_{2\pi q}$ is isometric, and hence the limit contains $n-1$ orthogonal antipodal pairs. Proposition 2.5, obtained by combining Theorem 2.4 with the persistence of this finite distance configuration under Gromov–Hausdorff convergence, therefore gives

$$\lambda_a(Q\#(-Q), g_s) \longrightarrow n \qquad (1 \le a \le n-1).$$

This pinching argument uses no general eigenvalue-continuity theorem for the singular limit. Together with Aubry's upper bound (1.1) and the topology of the doubles, this yields the sharp sphere-recognition thresholds. The full-spectrum conclusion below additionally uses De Philippis–Gigli's noncollapsed measure convergence [4] and Gigli–Mondino–Savaré's spectral convergence [5].

The fixed-volume families also give a common limit for every eigenvalue on two different manifolds. Write $L_{n,q} = S^{n-2} * S^1_{2\pi q}$, where $S^1_{2\pi q}$ has length $2\pi q$. The eigenvalues of $L_{n,q}$ below are those of the nonnegative Laplacian associated with its Cheeger energy and its n-dimensional Hausdorff measure; the convention is fixed in Section 5.

Theorem 1.4. *Let $n \ge 4$. There is $q_*(n) \in (0, q_n]$ such that, for every fixed $q \in (0, q_*(n))$, there are smooth families $(h_s)_{s\ge 0}$ on S^n and $(g_s)_{s\ge 0}$ on M_n satisfying*

$$\mathrm{Ric}_{h_s} \ge (n-1)h_s, \qquad \mathrm{Ric}_{g_s} \ge (n-1)g_s, \tag{1.10}$$

$$dv_{h_s} = dv_{h_0}, \qquad dv_{g_s} = dv_{g_0}, \tag{1.11}$$

$$\mathrm{Vol}(S^n, h_s) = q\sigma_n, \qquad \mathrm{Vol}(M_n, g_s) = q\sigma_n. \tag{1.12}$$

With $\mathcal{H}^n$ denoting the normalized Hausdorff measure,

$$(S^n, d_{h_s}, dv_{h_s}) \xrightarrow[s\to\infty]{\mathrm{mGH}} (L_{n,q}, d_{L_{n,q}}, \mathcal{H}^n), \qquad (M_n, d_{g_s}, dv_{g_s}) \xrightarrow[s\to\infty]{\mathrm{mGH}} (L_{n,q}, d_{L_{n,q}}, \mathcal{H}^n). \tag{1.13}$$

For every fixed integer $j \ge 0$,

$$\lim_{s\to\infty} \lambda_j(S^n, h_s) = \lim_{s\to\infty} \lambda_j(M_n, g_s) = \lambda_j(L_{n,q}, \mathcal{H}^n). \tag{1.14}$$

Both families have diameter at most π and satisfy

$$\mathrm{Vol}_{k_s}(B_r^{k_s}(x)) \ge q\sigma_n (r/\pi)^n \quad (0 < r \le \pi), \qquad (X, k_s) = (S^n, h_s) \text{ or } (M_n, g_s). \tag{1.15}$$

The two manifolds are simply connected, but $H_2(S^n;\mathbb{Z}) = 0$ and $H_2(M_n;\mathbb{Z}) \cong \mathbb{Z}^2$.

Taking $V_*(n) = q_*(n)\sigma_n$ gives the same statement at each prescribed volume $0 < V < V_*(n)$. In particular, the two topologies have no positive separation in any fixed finite list of eigenvalues, even when the Gromov–Hausdorff distance is included (Corollary 5.3). The convergence in (1.14) is coordinatewise; it asserts neither exact isospectrality at a finite parameter nor convergence in the supremum norm over all eigenvalue indices.

For every integer $m \ge 2$, the link $L_{n,1/m}$ is the quotient of the round sphere by the cyclic action that rotates one real two-plane and fixes its orthogonal complement. In Section 6 we compute its spectrum. The eigenvalue $\ell(\ell + n - 1)$ has multiplicity

$$a_\ell(n,m) = \binom{\ell+n-2}{n-2} + 2\sum_{r=1}^{\lfloor \ell/m \rfloor} \binom{\ell - mr + n - 2}{n-2}.$$

For every $m \ge 3$ with $1/m < q_*(n)$, it follows that $\lambda_n \to 2(n+1)$ on both families. In dimension four, $\lambda_1, \lambda_2, \lambda_3 \to 4$ and $\lambda_4, \dots, \lambda_9 \to 10$ (Corollary 6.5).

2. Spectral pinching from distance configurations

Manifolds and metrics are smooth. A closed manifold is compact without boundary. Balls are open: $B_r^g(x) = \{y : d_g(x,y) < r\}$. Tensor inequalities are inequalities of quadratic forms. The notation dv_g denotes the Riemannian density, or its positive volume form when an orientation is fixed. Smoothness of $(g_s)_{s\ge 0}$ means joint smoothness in (s,x), up to $s = 0$.

Distances on subsets are restricted ambient distances. For $m \ge 1$, S^m has its unit round metric and geodesic distance; the points of $S^0 = \{-1, 1\}$ are at distance π. Homology means singular homology with integral coefficients.

Lemma 2.1 (Lichnerowicz). *Let $n \geq 2$ and $(M, g) \in \mathcal{M}_n$. Then $\lambda_a(M, g) \geq n$ for every $a \geq 1$.*

Proof. Let $u \in C^\infty(M; \mathbb{R}) \setminus \{0\}$ satisfy $\Delta_g u = \lambda u$ with $\lambda > 0$. The integrated Bochner identity and integration by parts give

$$\lambda^2 \int_M u^2 \, dv_g = \int_M |\nabla^2 u|^2 \, dv_g + \int_M \mathrm{Ric}_g(\nabla u, \nabla u) \, dv_g,$$
$$\int_M |\nabla u|^2 \, dv_g = \lambda \int_M u^2 \, dv_g.$$

The trace identity $\mathrm{tr}_g \nabla^2 u = -\Delta_g u$ and the Cauchy–Schwarz inequality give $|\nabla^2 u|^2 \geq (\Delta_g u)^2/n$. Hence

$$\lambda^2 \int_M u^2 \, dv_g \geq \left(\frac{\lambda^2}{n} + (n-1)\lambda \right) \int_M u^2 \, dv_g.$$

Therefore

$$0 \leq (n-1)\lambda \left(\frac{\lambda}{n} - 1 \right) \int_M u^2 \, dv_g.$$

Since $(n-1)\lambda \int_M u^2 \, dv_g > 0$, one has $\lambda \geq n$. □

Definition 2.2. Let $k \geq 1$ and $\eta > 0$. A metric space (Z, d) has property $P_k(\eta)$ if there are points $x_1, \dots, x_k, y_1, \dots, y_k \in Z$ such that

$$d(x_a, y_a) > \pi - \eta \qquad (1 \leq a \leq k), \tag{2.1}$$
$$\left| d(x_a, x_b) - \frac{\pi}{2} \right| < \eta \qquad (1 \leq a \neq b \leq k). \tag{2.2}$$

For $(M, g) \in \mathcal{M}_n$ and $\varepsilon > 0$, put

$$E_\varepsilon(M, g) = \bigoplus_{n \leq \lambda \leq n+\varepsilon} \ker(\Delta_g - \lambda).$$

The sum is over the distinct eigenvalues in $[n, n+\varepsilon]$; it is a finite-dimensional subspace of $C^\infty(M; \mathbb{R})$.

Lemma 2.3 (Bertrand [2, Lemma 2.2, pp. 330–331]). *For every integer $n \geq 2$, there exists a function $\tau_n : (0, 1) \to (0, \infty)$, with $\lim_{t \downarrow 0} \tau_n(t) = 0$, such that the following holds. For every $(M, g) \in \mathcal{M}_n$, every $0 < \eta < \varepsilon < 1$, and every $p, q \in M$ with $d_g(p, q) > \pi - \eta$, there is $v_p \in E_\varepsilon(M, g)$ satisfying*

$$\| \cos d_g(p, \cdot) - v_p \|_{L^\infty(M)} \leq \tau_n(\eta/\varepsilon). \tag{2.3}$$

Theorem 2.4 (after Bertrand). *Let $n \geq 2$ and $1 \leq k \leq n+1$ be integers. For every $\varepsilon > 0$, there exists $\eta = \eta(n, k, \varepsilon) > 0$ such that*

$$(M, g) \in \mathcal{M}_n, \qquad (M, d_g) \text{ has property } P_k(\eta) \quad \Longrightarrow \quad \lambda_k(M, g) < n + \varepsilon.$$

Proof. Set $e = \min\{\varepsilon/2, 1/2\}$. Then $0 < e < \min\{1, \varepsilon\}$. Since $\tau_n(t) \to 0$ as $t \downarrow 0$, there is $t_0 \in (0, 1)$ such that $\tau_n(t) < 1/(4k)$ for $0 < t < t_0$. Choose

$$0 < \eta < \min\{e t_0, 1/(4k)\}.$$

Let $x_1, \dots, x_k, y_1, \dots, y_k$ realize $P_k(\eta)$. For each b, Lemma 2.3, applied with $(p, q, \varepsilon) = (x_b, y_b, e)$, gives $v_b \in E_e(M, g)$ such that

$$\|v_b - \cos d_g(x_b, \cdot)\|_\infty \leq \tau_n(\eta/e).$$

Let $B = (v_b(x_a))_{1 \leq a, b \leq k}$. On the diagonal, $|B_{aa} - 1| \leq \tau_n(\eta/e)$. For $a \neq b$,

$$|\cos d_g(x_b, x_a)| = \left| \sin\left(d_g(x_b, x_a) - \frac{\pi}{2} \right) \right| < \eta.$$

The maximum absolute row-sum norm therefore satisfies

$$\|B - I_k\|_{\infty \to \infty} \leq k \tau_n(\eta/e) + (k-1)\eta < \frac{2k-1}{4k} < \frac{1}{2}. \tag{2.4}$$

If $\sum_{b=1}^{k} c_b v_b = 0$, evaluation at the x_a gives $Bc = 0$. Hence

$$\|c\|_\infty = \|(I_k - B)c\|_\infty \le \|I_k - B\|_{\infty\to\infty}\|c\|_\infty < \tfrac{1}{2}\|c\|_\infty \quad \text{if } c \ne 0,$$

a contradiction. Thus $v_1, \dots, v_k$ are linearly independent and $\dim E_e(M, g) \ge k$. Since all positive eigenvalues are at least n by Lemma 2.1,

$$\lambda_k(M, g) \le n + e < n + \varepsilon.$$

$\square$

Let X and Z be nonempty compact metric spaces. A *correspondence* between them is a subset $\mathcal{R} \subset X \times Z$ whose two coordinate projections are surjective. Its distortion is

$$\operatorname{dis} \mathcal{R} = \sup_{(x,z),(x',z')\in\mathcal{R}} |d_X(x, x') - d_Z(z, z')|.$$

The Gromov–Hausdorff distance satisfies

$$d_{\mathrm{GH}}(X, Z) = \tfrac{1}{2} \inf_{\mathcal{R}} \operatorname{dis} \mathcal{R}. \tag{2.5}$$

The infimum is over all correspondences between X and Z.

Proposition 2.5. *Let $n \ge 2$ and $1 \le k \le n+1$ be integers. For every $\varepsilon > 0$, there exists $\delta = \delta(n, k, \varepsilon) > 0$ such that the following holds. Let Z be a compact metric space containing points $x_1, \dots, x_k, y_1, \dots, y_k$ with*

$$d_Z(x_a, y_a) = \pi \quad (1 \le a \le k), \qquad d_Z(x_a, x_b) = \frac{\pi}{2} \quad (1 \le a \ne b \le k). \tag{2.6}$$

For every $(M, g) \in \mathcal{M}_n$,

$$d_{\mathrm{GH}}((M, d_g), Z) < \delta \quad \Longrightarrow \quad n \le \lambda_a(M, g) < n + \varepsilon \quad (1 \le a \le k). \tag{2.7}$$

In particular, if $(M_j, g_j) \in \mathcal{M}_n$ and $(M_j, d_{g_j}) \to Z$ in Gromov–Hausdorff distance, then

$$\lim_{j\to\infty} \lambda_a(M_j, g_j) = n \qquad (1 \le a \le k).$$

The same conclusions hold whenever Z contains an isometric copy of S^{k-1}.

Proof. Let $\eta = \eta(n, k, \varepsilon)$ be given by Theorem 2.4, and set $\delta = \eta/4$. By (2.5), the strict bound $d_{\mathrm{GH}}((M, d_g), Z) < \delta$ gives a correspondence $\mathcal{R} \subset M \times Z$ with $\operatorname{dis} \mathcal{R} < 2\delta$. Its surjective projection to Z gives points $\widetilde{x}_a, \widetilde{y}_a \in M$ such that

$$(\widetilde{x}_a, x_a), (\widetilde{y}_a, y_a) \in \mathcal{R}.$$

For these points,

$$d_g(\widetilde{x}_a, \widetilde{y}_a) > \pi - 2\delta > \pi - \eta, \qquad \left| d_g(\widetilde{x}_a, \widetilde{x}_b) - \frac{\pi}{2} \right| < 2\delta < \eta \quad (a \ne b).$$

Hence $P_k(\eta)$ holds. Theorem 2.4 gives $\lambda_k(M, g) < n + \varepsilon$; Lemma 2.1 and the eigenvalue ordering give (2.7).

For a convergent sequence and each $\varepsilon > 0$, there is i_ε such that $n \le \lambda_a(M_i, g_i) < n + \varepsilon$ for $i \ge i_\varepsilon$ and $1 \le a \le k$. Thus $\lambda_a(M_i, g_i) \to n$. If $\iota : S^{k-1} \to Z$ is an isometric embedding, set $x_a = \iota(e_a)$ and $y_a = \iota(-e_a)$ for the standard basis $e_1, \dots, e_k$ of $\mathbb{R}^k$. Their distances are precisely (2.6). For $k = 1$ this uses the stated metric on S^0. $\square$

3. Fixed-volume degeneration of core-admissible doubles

For tangent vectors v, w to the boundary of a Riemannian manifold, set

$$\mathrm{II}(v,w) = -\langle \nabla_v \nu, w\rangle = \langle \nabla_v \nu_{\mathrm{out}}, w\rangle, \qquad \nu_{\mathrm{out}} = -\nu,$$

where ν is the inward unit normal.

Definition 3.1. Let $n \geq 4$. A closed connected oriented smooth n-manifold Q is *core-admissible* if there are a smoothly embedded closed n-disc $D_Q \subset Q$, a metric h_Q on $P_Q = Q \setminus \mathrm{int}_Q D_Q$, a diffeomorphism $\beta_Q : S^{n-1} \to \partial P_Q$, and constants $\kappa_Q, \mu_Q > 0$ such that

$$\beta_Q^*(h_Q|_{\partial P_Q}) = g_{S^{n-1}}, \qquad \beta_Q^* \mathrm{II}_{\partial P_Q, h_Q} \geq \kappa_Q g_{S^{n-1}}, \qquad \mathrm{Ric}_{h_Q} \geq \mu_Q h_Q. \tag{3.1}$$

Set $M_Q = Q\#(-Q)$.

For nonempty compact metric spaces A, B of diameter at most π, their spherical join is

$$A * B = ([0,\pi/2] \times A \times B)/\sim,$$

where the equivalence relation is generated by $(0,a,b) \sim (0,a,b')$ and $(\pi/2,a,b) \sim (\pi/2,a',b)$. Its distance, in the convention of [6, Section 5.1], is

$$\begin{aligned} d_{A*B}([t,a,b],[u,a',b']) = \arccos\big(&\cos t \cos u \cos d_A(a,a') \\ &+ \sin t \sin u \cos d_B(b,b')\big). \end{aligned} \tag{3.2}$$

For $0 < q < 1$, let $S^1_{2\pi q}$ be the circle of length $2\pi q$ with its intrinsic distance, and set

$$L_{n,q} = S^{n-2} * S^1_{2\pi q}. \tag{3.3}$$

Theorem 3.2 (Fixed-volume degeneration [6, Theorem 8.1 and Lemma 5.1]). *Let $n \geq 4$ and let Q^n be core-admissible. There is $q_0 = q_0(n,Q) \in (0,1)$ such that, for every $q \in (0,q_0)$, M_Q carries a smooth family $(g_s)_{s\geq 0}$ satisfying*

$$\mathrm{Ric}_{g_s} \geq (n-1) g_s, \qquad dv_{g_s} = dv_{g_0}, \qquad \mathrm{Vol}(M_Q, g_s) = q\sigma_n, \tag{3.4}$$

and

$$(M_Q, d_{g_s}) \xrightarrow[s\to\infty]{\mathrm{GH}} L_{n,q}. \tag{3.5}$$

This theorem is the direct combination of [6, Theorem 8.1] with the identification of the limit link in [6, Lemma 5.1]. After replacing the constant supplied by [6, Theorem 8.1] with its minimum with $1/2$, we may and do take $q_0 \in (0,1)$. The initial stationarity and Ricci-closability conclusions of that theorem are not used here.

Lemma 3.3. *Let $n \geq 4$ and $0 < q < 1$. The map*

$$\iota : S^{n-2} \longrightarrow L_{n,q}, \qquad \iota(x) = [0,x,z],$$

where $z \in S^1_{2\pi q}$ is arbitrary, is a well-defined isometric embedding. In particular, $L_{n,q}$ contains $n-1$ pairs satisfying (2.6) *with $k = n-1$.*

Proof. The join relation makes ι independent of z. Since spherical distances belong to $[0,\pi]$, (3.2) gives

$$d_{L_{n,q}}(\iota(x),\iota(x')) = \arccos\big(\cos d_{S^{n-2}}(x,x')\big) = d_{S^{n-2}}(x,x').$$

For the standard basis $e_1,\dots,e_{n-1}$ of $\mathbb{R}^{n-1}$, put $x_a = \iota(e_a)$ and $y_a = \iota(-e_a)$. Then

$$d(x_a,y_a) = \arccos(-1) = \pi, \qquad d(x_a,x_b) = \arccos(0) = \pi/2 \quad (a \neq b).$$

□

Proposition 3.4. *Let $n \geq 4$, let Q^n be core-admissible, and fix $0 < q < q_0(n, Q)$. Every smooth family $(g_s)_{s \geq 0}$ satisfying* (3.4) *and* (3.5) *satisfies*

$$\lim_{s \to \infty} \lambda_a(M_Q, g_s) = n \qquad (1 \leq a \leq n-1). \tag{3.6}$$

If $M_Q \not\cong_{\mathrm{diff}} S^n$, then also

$$\lambda_n(M_Q, g_s) \geq n + c_n \qquad (s \geq 0). \tag{3.7}$$

Proof. Fix $\varepsilon > 0$. Let $\delta = \delta(n, n-1, \varepsilon)$ be the constant in Proposition 2.5. By (3.5), some s_ε satisfies

$$d_{\mathrm{GH}}((M_Q, d_{g_s}), L_{n,q}) < \delta \qquad (s \geq s_\varepsilon).$$

By Lemma 3.3 and Proposition 2.5,

$$n \leq \lambda_a(M_Q, g_s) < n + \varepsilon \qquad (s \geq s_\varepsilon,\ 1 \leq a \leq n-1).$$

This proves (3.6). If $\lambda_n(M_Q, g_s) < n + c_n$ for any s, (1.1) would give $M_Q \cong_{\mathrm{diff}} S^n$. □

For a closed connected smooth n-manifold M and $q > 0$, define

$$\mathcal{G}_q(M) = \{g \,:\, \mathrm{Ric}_g \geq (n-1)g,\ \mathrm{Vol}(M, g) = q\sigma_n\},$$

where g ranges over the smooth Riemannian metrics on M.

Corollary 3.5. *Let $n \geq 4$, let Q^n be core-admissible, and let $0 < q < q_0(n, Q)$. Then $\mathcal{G}_q(M_Q)$ is nonempty and*

$$\inf_{g \in \mathcal{G}_q(M_Q)} \lambda_a(M_Q, g) = n \qquad (1 \leq a \leq n-1). \tag{3.8}$$

If $M_Q \not\cong_{\mathrm{diff}} S^n$, then

$$\inf_{g \in \mathcal{G}_q(M_Q)} \lambda_n(M_Q, g) \geq n + c_n. \tag{3.9}$$

Proof. The metrics of Theorem 3.2 belong to $\mathcal{G}_q(M_Q)$, so this class is nonempty. Fix $1 \leq a \leq n-1$ and $\varepsilon > 0$. By (3.6), some s satisfies $\lambda_a(M_Q, g_s) < n + \varepsilon$. By Lemma 2.1,

$$n \leq \inf_{g \in \mathcal{G}_q(M_Q)} \lambda_a(M_Q, g) \leq \lambda_a(M_Q, g_s) < n + \varepsilon.$$

Letting $\varepsilon \downarrow 0$ gives (3.8). If M_Q is not diffeomorphic to S^n, (1.1) implies $\lambda_n(M_Q, g) \geq n + c_n$ for every $g \in \mathcal{G}_q(M_Q)$, which gives (3.9). □

4. Nonspherical families and optimality

Lemma 4.1. *Let $n \geq 4$. The manifold Q_n in* (1.3) *is core-admissible and*

$$\pi_1(Q_n) = \{1\}, \qquad H_2(Q_n; \mathbb{Z}) \cong \mathbb{Z}.$$

Its double M_n satisfies

$$\pi_1(M_n) = \{1\}, \qquad H_2(M_n; \mathbb{Z}) \cong \mathbb{Z}^2.$$

Proof. If $n = 2m$ with $m \geq 2$, Burdick's theorem [3, Theorem C] supplies a Ricci-positive metric on $\mathbb{CP}^m \setminus \mathrm{int}\, D^{2m}$ whose boundary is round and has positive-definite outward second fundamental form.

If n is odd, then $n \geq 5$. The manifold $S^{n-2} \times S^2$ is the unit sphere bundle of the trivial rank-three vector bundle over S^{n-2}. The unit round metric on the base is a core metric: it has positive Ricci curvature and contains an isometrically embedded closed unit hemisphere [8, Definition 1.2]. The fibre has dimension 2 and the base is closed. Thus [8, Theorem 5.1] supplies a core metric on Q_n; [8, Lemma 2.18] supplies a Ricci-positive metric on $Q_n \setminus \mathrm{int}\, D^n$ with round strictly convex boundary.

In either case, write $P = Q_n \setminus \mathrm{int}\, D^n$ for the punctured manifold, denote the resulting metric by h, and choose $\beta \,:\, S^{n-1} \to \partial P$ with $\beta^*(h|_{\partial P}) = R^2 g_{S^{n-1}}$, where $R > 0$. Set $\widehat{h} = R^{-2} h$. Its Levi-Civita connection equals that of h, and its outward unit normal is $R\nu_{\mathrm{out},h}$. Thus, as covariant two-tensors,

$$\mathrm{Ric}_{\widehat{h}} = \mathrm{Ric}_h, \qquad \mathrm{II}_{\widehat{h}} = R^{-1}\mathrm{II}_h, \qquad \beta^*(\widehat{h}|_{\partial P}) = g_{S^{n-1}}.$$

The Ricci tensor and the boundary second fundamental form are positive definite. Compactness of the corresponding unit tangent bundles gives

$$\mu_Q = \min_{|v|_{\widehat{h}}=1} \mathrm{Ric}_{\widehat{h}}(v,v) > 0, \qquad \kappa_Q = \min_{|w|_{g_{S^{n-1}}}=1} (\beta^* \mathrm{II}_{\widehat{h}})(w,w) > 0.$$

These are the constants in (3.1).

The cell decomposition of $\mathbb{CP}^m$ has one cell in each dimension $0, 2, \dots, 2m$. Its 2-skeleton is S^2 and it has no 1-cells or 3-cells. Hence

$$\pi_1(\mathbb{CP}^m) = \{1\}, \qquad H_2(\mathbb{CP}^m; \mathbb{Z}) \cong \mathbb{Z}.$$

If n is odd, the two factors of $S^{n-2} \times S^2$ are simply connected. Since $n-2 \geq 3$, the integral Künneth formula gives

$$H_2(S^{n-2} \times S^2; \mathbb{Z}) \cong H_0(S^{n-2}; \mathbb{Z}) \otimes H_2(S^2; \mathbb{Z}) \cong \mathbb{Z}.$$

Thus in both cases Q_n is simply connected with second homology $\mathbb{Z}$.

Choose a bicollar $c : S^{n-1} \times (-2,2) \to Q_n$ of ∂P, with $t < 0$ in P and $t > 0$ in D^n, and put

$$\begin{aligned} U &= \mathrm{int}_{Q_n} P \cup c(S^{n-1} \times (-2,1)), \\ W &= \mathrm{int}_{Q_n} D^n \cup c(S^{n-1} \times (-1,2)). \end{aligned}$$

Then $U \cup W = Q_n$, $U \simeq P$, $W \simeq D^n$, and $U \cap W = c(S^{n-1} \times (-1,1)) \simeq S^{n-1}$. The deformation retraction $U \to P$ is obtained by replacing positive collar coordinates t by $(1-\tau)t$, $0 \leq \tau \leq 1$, and fixing P. In particular, (Q_n, P) is a good pair. Collapsing P gives $Q_n/P \cong D^n/\partial D^n \cong S^n$; hence

$$H_j(Q_n, P; \mathbb{Z}) \cong \widetilde{H}_j(S^n; \mathbb{Z}).$$

For $n \geq 4$, these groups vanish for $0 \leq j \leq 3$. The long exact sequence of the pair gives $H_0(P; \mathbb{Z}) \cong \mathbb{Z}$ and

$$H_2(P; \mathbb{Z}) \cong H_2(Q_n; \mathbb{Z}) \cong \mathbb{Z}. \tag{4.1}$$

The group $H_0(P; \mathbb{Z})$ is freely generated by the path components of P; hence P is path connected. Apply van Kampen to U, W, with basepoint in their intersection. Since $W \simeq D^n$ and $U \cap W \simeq S^{n-1}$ are simply connected, $\pi_1(P) \cong \pi_1(Q_n) = \{1\}$. A bicollar of the connected-sum sphere likewise gives an open cover of M_n whose members retract onto P and $-P$ and whose intersection retracts onto S^{n-1}. Van Kampen gives

$$\pi_1(M_n) \cong \pi_1(P) *_{\pi_1(S^{n-1})} \pi_1(-P) = \{1\}.$$

For this cover, the Mayer–Vietoris sequence contains

$$\begin{aligned} H_2(S^{n-1}; \mathbb{Z}) &\longrightarrow H_2(P; \mathbb{Z}) \oplus H_2(-P; \mathbb{Z}) \\ &\longrightarrow H_2(M_n; \mathbb{Z}) \longrightarrow H_1(S^{n-1}; \mathbb{Z}). \end{aligned}$$

Since $n - 1 \geq 3$, the first and last groups vanish. Equation (4.1) gives $H_2(M_n; \mathbb{Z}) \cong \mathbb{Z}^2$. □

Lemma 4.2. *Let $n \geq 2$ and $(M, g) \in \mathcal{M}_n$. Then $\mathrm{diam}(M, g) \leq \pi$. For every $x \in M$ and $0 < r \leq \pi$,*

$$\mathrm{Vol}_g(B_r(x)) \geq \mathrm{Vol}(M, g) \left(\frac{r}{\pi}\right)^n. \tag{4.2}$$

Proof. Let $\gamma : [0, L] \to M$ be a unit-speed minimizing geodesic of positive length. Let $E_1, \dots, E_{n-1}$ be parallel orthonormal normal fields along γ, and set $V_a(t) = \sin(\pi t/L) E_a(t)$. Nonnegativity of the index form gives

$$\begin{aligned} 0 &\leq \sum_{a=1}^{n-1} I(V_a, V_a) \\ &= \int_0^L \left((n-1) \frac{\pi^2}{L^2} \cos^2 \frac{\pi t}{L} - \mathrm{Ric}_g(\dot\gamma, \dot\gamma) \sin^2 \frac{\pi t}{L} \right) dt \\ &\leq \frac{n-1}{2} \left(\frac{\pi^2}{L} - L \right). \end{aligned}$$

Thus $L^2 \le \pi^2$. Compactness and Hopf–Rinow supply a minimizing geodesic between any two points, so $\operatorname{diam}(M, g) \le \pi$. For $R > \pi$, the open ball $B_R(x)$ equals M. Since $\operatorname{Ric}_g \ge 0$, the Bishop–Gromov comparison [7, Section 7.1] gives

$$\frac{\operatorname{Vol}_g(B_r(x))}{r^n} \ge \frac{\operatorname{Vol}_g(B_R(x))}{R^n} = \frac{\operatorname{Vol}(M, g)}{R^n}.$$

The limit $R \downarrow \pi$ proves (4.2). □

Proof of Theorem 1.2. Fix $n \ge 4$. By Lemma 4.1, Q_n is core-admissible, $\pi_1(M_n) = \{1\}$, and $H_2(M_n; \mathbb{Z}) \cong \mathbb{Z}^2$. In particular, M_n is not homeomorphic to S^n. Set $q_n = q_0(n, Q_n) \in (0, 1)$ as in Theorem 3.2. For every fixed $q \in (0, q_n)$, that theorem gives (1.5). Proposition 3.4 gives (1.6) and (1.7). Since $\operatorname{Vol}(M_n, g_s) = q\sigma_n$, Lemma 4.2 gives (1.8). □

Remark 4.3 (Dimension four). For $n = 4$ one may take $M_4 = \mathbb{CP}^2\#(-\mathbb{CP}^2)$. For every sufficiently small fixed $q > 0$ there are metrics g_s such that

$$\operatorname{Ric}_{g_s} \ge 3g_s, \qquad \operatorname{Vol}(M_4, g_s) = q\sigma_4,$$

$$\lambda_1(M_4, g_s), \lambda_2(M_4, g_s), \lambda_3(M_4, g_s) \longrightarrow 4, \qquad \lambda_4(M_4, g_s) \ge 4 + c_4.$$

The bound in (1.8) is uniform in s and in the center for fixed q, but not as $q \downarrow 0$.

5. Common spectral limits

We normalize $\mathcal{H}^n$ to equal Riemannian volume on smooth n-manifolds. For compact metric-measure spaces, measured Gromov–Hausdorff convergence, denoted by mGH, means Hausdorff convergence of the spaces and weak convergence of their measures under isometric embeddings in a common compact metric space.

Let (X, d, μ) be a compact metric space with a finite Borel measure of full support. For a real Lipschitz function u, put

$$\operatorname{lip} u(x) = \limsup_{\substack{y \to x \\ y \ne x}} \frac{|u(y) - u(x)|}{d(x, y)},$$

with value zero at isolated points. Define the relaxed energy by

$$\mathcal{E}_\mu(u) = \inf\left\{ \liminf_{i\to\infty} \int_X (\operatorname{lip} u_i)^2 \, d\mu \;:\; u_i \in \operatorname{Lip}(X),\; u_i \to u \text{ in } L^2(\mu) \right\}. \tag{5.1}$$

Thus $\mathcal{E}_\mu = 2\operatorname{Ch}_\mu$, and $W^{1,2}(X, d, \mu) = \{u \in L^2(\mu) : \mathcal{E}_\mu(u) < \infty\}$. When $\mathcal{E}_\mu$ is quadratic, write

$$\mathcal{E}_\mu(u, v) = \tfrac{1}{4}\big(\mathcal{E}_\mu(u + v) - \mathcal{E}_\mu(u - v)\big).$$

This is a densely defined closed symmetric form. Its nonnegative self-adjoint operator $\Delta_{X,\mu}$ is characterized by

$$\begin{aligned} u \in D(\Delta_{X,\mu}), \quad \Delta_{X,\mu} u = b \quad &\Longleftrightarrow \quad u \in W^{1,2}(X, d, \mu),\; b \in L^2(\mu), \\ &\qquad \mathcal{E}_\mu(u, v) = \langle b, v\rangle_{L^2(\mu)} \quad \text{for all } v \in W^{1,2}(X, d, \mu). \end{aligned}$$

Density makes b unique. On a smooth closed manifold the operator is $-\operatorname{div}\nabla$ and $\mathcal{E}_{dv_g}(u) = \int_X |\nabla u|^2 \, dv_g$. The operators and forms are complexified when complex-valued functions are used.

We use $\mathrm{RCD}(K, N)$ for the Riemannian curvature-dimension condition; $\mathrm{ncRCD}(K, N)$ additionally requires $\mu = \mathcal{H}^N$. The Cheeger energy of an RCD space is quadratic. A compact $\mathrm{RCD}(K, N)$ space, $N < \infty$, has compact Sobolev embedding and a self-adjoint Laplacian with compact resolvent [5, Proposition 6.7 and Section 7.3]. Its eigenvalues, starting at index zero and repeated with multiplicity, are characterized by

$$\lambda_j(X, \mu) = \min_{\substack{W \subset W^{1,2}(X,d,\mu) \\ \dim W = j+1}} \max_{0 \ne u \in W} \frac{\mathcal{E}_\mu(u)}{\|u\|^2_{L^2(\mu)}}. \tag{5.2}$$

This notation is used when $L^2(\mu)$ is infinite-dimensional, as it is for all the spaces below. For $c > 0$, the same approximating sequences in (5.1) give

$$W^{1,2}(X,d,c\mu) = W^{1,2}(X,d,\mu), \qquad \mathcal{E}_{c\mu} = c\mathcal{E}_\mu, \qquad \lambda_j(X,c\mu) = \lambda_j(X,\mu). \tag{5.3}$$

Proposition 5.1. *Let $n \geq 2$ and $V > 0$. Suppose that $(X_i,k_i) \in \mathcal{M}_n$, $\mathrm{Vol}(X_i,k_i) = V$, and $(X_i,d_{k_i}) \xrightarrow{\mathrm{GH}} (Z,d_Z)$ for a compact metric space Z. Then $(Z,d_Z,\mathcal{H}^n)$ is a compact* ncRCD$(n-1,n)$ *space, $\mathcal{H}^n(Z) = V$, and*

$$(X_i,d_{k_i},dv_{k_i}) \xrightarrow{\mathrm{mGH}} (Z,d_Z,\mathcal{H}^n). \tag{5.4}$$

For every integer $j \geq 0$,

$$\lambda_j(X_i,k_i) \longrightarrow \lambda_j(Z,\mathcal{H}^n). \tag{5.5}$$

Proof. The smooth spaces (X_i,d_{k_i},dv_{k_i}) are ncRCD$(n-1,n)$. By Lemma 4.2,

$$\mathrm{diam}(X_i,k_i) \leq \pi, \qquad \mathrm{Vol}_{k_i}(B_1(x_i)) \geq V/\pi^n \quad (x_i \in X_i).$$

Fix $z \in Z$ and choose basepoints x_i converging to z in a realization of the Gromov–Hausdorff convergence. The preceding lower bound excludes the collapsed alternative of [4, Theorem 1.2]. The other alternative gives pointed measured Gromov–Hausdorff convergence to $(Z,d_Z,\mathcal{H}^n,z)$ and the asserted ncRCD condition. The diameter bound gives $\mathrm{diam}\, Z \leq \pi$. A ball of radius greater than π about either basepoint contains the entire corresponding space. The pointed measured convergence therefore gives (5.4). Testing the weak convergence against the constant function one gives $\mathcal{H}^n(Z) = V$. Singletons have zero $\mathcal{H}^n$-measure, whereas $\mathcal{H}^n(Z) = V > 0$; hence Z is infinite. For every positive integer r, choose r distinct points and disjoint balls about them. Full support gives positive measure to each ball, so their indicator functions are linearly independent in $L^2(Z,\mathcal{H}^n)$. This space is therefore infinite-dimensional.

Put $\mu_i = V^{-1}dv_{k_i}$, $\mu_\infty = V^{-1}\mathcal{H}^n$, and $x_\infty = z$. The resulting probability spaces are RCD$(n-1,\infty)$ and converge in the pointed measured Gromov sense of [5, Proposition 3.30]. For each of them, including the limit, and every $u \in L^2(\mu_i)$,

$$\left(\int d(x,x_i)^2|u(x)|^2\, d\mu_i(x)\right)^{1/2} \leq \pi\|u\|_{L^2(\mu_i)}.$$

This is the uniform weak logarithmic-Sobolev-Talagrand inequality with constants $(\pi,0)$ in [5, Definition 6.4 and Remark 6.6]. The spaces are complete, separable, of full support, and of unit mass; they have the common RCD$(n-1,\infty)$ bound, pointed measured Gromov convergence, and the preceding uniform inequality. Thus [5, Theorem 7.8] applies. The source enumerates the zero eigenvalue at index one and uses the Rayleigh quotient $2\,\mathrm{Ch}(u)/\|u\|_2^2$. Its eigenvalue of index $j+1$ is therefore λ_j in (5.2). Equation (5.3) removes the normalization and proves (5.5). □

Lemma 5.2. *For $n \geq 4$, the standard sphere S^n is core-admissible. Its core may be chosen to be a round closed ball; the double obtained using the standard boundary identification is diffeomorphic to S^n.*

Proof. Fix $p \in S^n$ and $0 < \rho < \pi/2$. Let $P = \overline{B}_\rho(p)$ and $D = \overline{B}_{\pi-\rho}(-p)$, with balls taken in the unit round metric γ. Both are smoothly embedded closed discs, and $P = S^n \setminus \mathrm{int}\, D$. The standard polar parametrization $\beta : S^{n-1} \to \partial P$ gives

$$\beta^*(\gamma|_{\partial P}) = \sin^2\rho\, g_{S^{n-1}}, \qquad \beta^*\mathrm{II}_{\partial P,\gamma} = \cot\rho\sin^2\rho\, g_{S^{n-1}}.$$

Set $k = (\sin\rho)^{-2}\gamma|_P$. Constant scaling gives

$$\beta^*(k|_{\partial P}) = g_{S^{n-1}}, \qquad \beta^*\mathrm{II}_{\partial P,k} = \cos\rho\, g_{S^{n-1}}, \qquad \mathrm{Ric}_k = (n-1)\sin^2\rho\, k. \tag{5.6}$$

Thus (3.1) holds with $\kappa_{S^n} = \cos\rho > 0$ and $\mu_{S^n} = (n-1)\sin^2\rho > 0$.

Identify P with D^n by $rx \mapsto \exp_p(\rho r x)$, $0 \leq r \leq 1$, $x \in S^{n-1}$. On its two copies define

$$F_\pm(rx) = \big(\sin(\pi r/2)x,\ \pm\cos(\pi r/2)\big) \in S^n.$$

At $r = 0$, the function $\sin(\pi r/2)/r$ has a smooth positive even extension, as does $\cos(\pi r/2)$. At the common boundary, use the signed collar coordinate $t = 1 - r$ on the plus copy and $t = r - 1$ on the minus copy. Both maps then have the expression

$$(t, x) \longmapsto \big(\cos(\pi t/2)x, \sin(\pi t/2)\big).$$

The induced map on the double is smooth, bijective, and of full rank at the centers, on the collar, and for $0 < r < 1$. It is therefore a diffeomorphism. A product collar can be absorbed by a smooth increasing change of its normal coordinate. □

Proof of Theorem 1.4. By Lemmas 4.1 and 5.2, both Q_n and S^n are core-admissible. Let $q_0(n, S^n)$ be the constant in Theorem 3.2. Since $q_n = q_0(n, Q_n)$, put

$$q_*(n) = \min\{q_n, q_0(n, S^n), 1/2\} > 0. \tag{5.7}$$

Fix $q \in (0, q_*(n))$. Applying Theorem 3.2 with $Q = Q_n$ gives (g_s) on M_n. Applying it with $Q = S^n$, using the standard boundary identification of Lemma 5.2, gives (h_s) on S^n. Transport by the fixed double diffeomorphism preserves smoothness, the Ricci bound, and constancy of the volume density. These families satisfy (1.10)–(1.12), and

$$(M_n, d_{g_s}) \xrightarrow[s\to\infty]{\mathrm{GH}} L_{n,q}, \qquad (S^n, d_{h_s}) \xrightarrow[s\to\infty]{\mathrm{GH}} L_{n,q}.$$

Lemma 4.2 gives the diameter bound and (1.15).

Let $s_i \to \infty$ be any sequence. Apply Proposition 5.1 with $V = q\sigma_n$ to (M_n, g_{s_i}) and to (S^n, h_{s_i}). Their limiting measure is $\mathcal{H}^n$ on the same metric space $L_{n,q}$. This proves (1.13) and (1.14) along every such sequence, hence as $s \to \infty$. The sphere is simply connected and has $H_2(S^n; \mathbb{Z}) = 0$ for $n \geq 4$; Lemma 4.1 gives the asserted invariants of M_n. □

Corollary 5.3. *Fix $n \geq 4$ and $0 < V < q_*(n)\sigma_n$. For every integer $K \geq 1$ and every $\varepsilon > 0$ there are smooth metrics h on S^n and g on M_n such that*

$$(S^n, h), (M_n, g) \in \mathcal{M}_n, \qquad \mathrm{Vol}(S^n, h) = \mathrm{Vol}(M_n, g) = V,$$

$$d_{\mathrm{GH}}((S^n, h), (M_n, g)) < \varepsilon, \qquad \max_{1\leq j\leq K} |\lambda_j(S^n, h) - \lambda_j(M_n, g)| < \varepsilon. \tag{5.8}$$

Both metrics satisfy the local volume bound $V(r/\pi)^n$ for $0 < r \leq \pi$, at every center, and the two manifolds are not homeomorphic.

Proof. Put $q = V/\sigma_n$ and take the families of Theorem 1.4. The triangle inequality gives

$$d_{\mathrm{GH}}((S^n, h_s), (M_n, g_s)) \leq d_{\mathrm{GH}}((S^n, h_s), L_{n,q}) + d_{\mathrm{GH}}(L_{n,q}, (M_n, g_s)) \longrightarrow 0.$$

For each $1 \leq j \leq K$, (1.14) gives $|\lambda_j(S^n, h_s) - \lambda_j(M_n, g_s)| \to 0$. A single sufficiently large s satisfies all the finitely many strict bounds in (5.8). Set $h = h_s$, $g = g_s$. The other assertions follow from (1.12), (1.15), and $H_2(S^n; \mathbb{Z}) \not\cong H_2(M_n; \mathbb{Z})$. □

For two spaces with discrete scalar spectra, define

$$d_{\mathrm{sp}}(X, Y) = \sum_{j=0}^{\infty} 2^{-j-1} \min\{1, |\lambda_j(X) - \lambda_j(Y)|\}. \tag{5.9}$$

The measures specifying the spectra are part of the data; on a smooth manifold they are Riemannian volume measures. Formula (5.9) is a metric on ordered spectral sequences and a pseudometric on spaces. Indeed, $\min\{1, a + b\} \leq \min\{1, a\} + \min\{1, b\}$ for $a, b \geq 0$ gives the triangle inequality, and the sum vanishes exactly when every coordinate agrees.

Corollary 5.4. *The families of Theorem 1.4 satisfy*

$$d_{\mathrm{GH}}((S^n,h_s),(M_n,g_s))+d_{\mathrm{sp}}((S^n,h_s),(M_n,g_s))\longrightarrow 0. \tag{5.10}$$

Consequently, for every $0<q<q_*(n)$,

$$\inf_{\substack{h\in\mathcal{G}_q(S^n)\\ g\in\mathcal{G}_q(M_n)}}\left[d_{\mathrm{GH}}((S^n,h),(M_n,g))+d_{\mathrm{sp}}((S^n,h),(M_n,g))\right]=0. \tag{5.11}$$

Proof. For each integer $K\ge 0$, the part of (5.9) with $j>K$ is at most $\sum_{j>K}2^{-j-1}=2^{-K-1}$. Its finite initial sum tends to zero by (1.14). Hence the limsup of the full sum is at most 2^{-K-1} for every K, and is therefore zero. The Gromov–Hausdorff term tends to zero by the triangle inequality used in the proof of Corollary 5.3. This proves (5.10). Both metric classes in (5.11) contain the corresponding families; nonnegativity and (5.10) prove the infimum identity. □

6. The spectrum at reciprocal angles

Throughout this section, $n\ge 4$ and $m\ge 2$ is an integer. Realize the round sphere as

$$S^n=\{(x,z)\in\mathbb{R}^{n-1}\oplus\mathbb{C}\,:\,|x|^2+|z|^2=1\}.$$

Let Γ_m be the cyclic group generated by $\gamma(x,z)=(x,e^{2\pi i/m}z)$, and let $\pi_m\,:\,S^n\to S^n/\Gamma_m$ be the quotient map. The quotient has distance

$$d_{S^n/\Gamma_m}(\pi_m u,\pi_m v)=\min_{\gamma\in\Gamma_m}d_{S^n}(u,\gamma v).$$

Lemma 6.1. *There is an isometry* $S^n/\Gamma_m\cong L_{n,1/m}$. *Under this identification,*

$$\nu_m:=\mathcal{H}^n|_{L_{n,1/m}}=\frac{1}{m}(\pi_m)_*dv_{S^n}. \tag{6.1}$$

Proof. Write a point of S^n as $(\cos t\,x,\sin t\,e^{i\theta})$, where $0\le t\le\pi/2$ and $x\in S^{n-2}$. The unused coordinate is identified at each endpoint. The circle quotient is $\mathbb{R}/(2\pi/m)\mathbb{Z}$ with its intrinsic distance; it has length $2\pi/m$. For two such points, minimizing the round distance over Γ_m gives

$$\arccos\left(\cos t\cos u\,\langle x,x'\rangle+\sin t\sin u\cos\delta\right),\qquad \delta=\min_{r\in\mathbb{Z}}|\theta-\theta'+2\pi r/m|.$$

Here $0\le\delta\le\pi/m$. Since $\langle x,x'\rangle=\cos d_{S^{n-2}}(x,x')$, this is (3.2). The induced map from the orbit space to $L_{n,1/m}$ is surjective and preserves distances, hence is an isometry.

Let $F=\{(x,0)\,:\,|x|=1\}\cong S^{n-2}$. If $z\ne 0$ and $\gamma^j(x,z)=(x,z)$, then $e^{2\pi ij/m}=1$; hence the action is free on $S^n\setminus F$. Fix $u\notin F$ and put

$$a=\min_{\gamma\ne 1}d_{S^n}(u,\gamma u)>0.$$

Choose $0<r<\min\{a/4,d_{S^n}(u,F)/2\}$. For $v,w\in B_r(u)$ and $\gamma\ne 1$,

$$d_{S^n}(v,\gamma w)\ge a-d(u,v)-d(u,w)>a-2r>2r>d(v,w).$$

The quotient distance on this ball is therefore d_{S^n}. Its m translates are disjoint and form the full inverse image of $\pi_m(B_r(u))$. Thus $\pi_m\,:\,S^n\setminus F\to L_{n,1/m}\setminus\pi_m(F)$ is an m-sheeted Riemannian covering by local isometries. The restriction of π_m to F is isometric. Both F and $\pi_m(F)$ have zero n-dimensional Hausdorff measure. On an evenly covered open set U in the complement,

$$\mathrm{Vol}_{S^n}(\pi_m^{-1}(U))=m\mathcal{H}^n(U).$$

The identity also holds for each Borel subset of U. Choose a countable evenly covered open cover (U_i) of the complement and put $E_i=U_i\setminus\bigcup_{j<i}U_j$. The sets E_i are a Borel partition. Apply the local identity to each $B\cap E_i$ and sum, for any Borel set B. Since F and $\pi_m(F)$ are null, this proves (6.1). □

Lemma 6.2. *For $f \in L^2(L_{n,1/m}, \nu_m)$, pullback identifies the form domain with the invariant Sobolev space:*

$$f \in W^{1,2}(L_{n,1/m}, \nu_m) \quad \Longleftrightarrow \quad f \circ \pi_m \in H^1(S^n)^{\Gamma_m}. \tag{6.2}$$

For these functions,

$$\|f\|^2_{L^2(\nu_m)} = \frac{1}{m}\|f \circ \pi_m\|^2_{L^2(S^n)}, \qquad \mathcal{E}_{\nu_m}(f) = \frac{1}{m}\int_{S^n} |\nabla(f \circ \pi_m)|^2 \, dv_{S^n}. \tag{6.3}$$

The operator $\Delta_{L_{n,1/m},\nu_m}$ is unitarily equivalent to the restriction of the round Laplacian to $L^2(S^n)^{\Gamma_m}$, with operator domain $D(\Delta_{S^n}) \cap L^2(S^n)^{\Gamma_m}$. The quotient operator has compact resolvent.

Proof. For $f \in \mathrm{Lip}(L_{n,1/m})$, put $F_f = f \circ \pi_m$. The quotient map is 1-Lipschitz, so F_f is Lipschitz and invariant. Conversely, an invariant Lipschitz function v on S^n descends to a Lipschitz function: for all $u, w \in S^n$,

$$|v(u) - v(w)| = |v(u) - v(\gamma w)| \leq \mathrm{Lip}(v) d_{S^n}(u, \gamma w) \quad (\gamma \in \Gamma_m),$$

and one takes the minimum over γ. Off F, the local isometries in Lemma 6.1 give $\mathrm{lip}\, F_f = (\mathrm{lip}\, f) \circ \pi_m$. The exceptional sets are null. Rademacher's theorem on S^n and (6.1) therefore give

$$\int_{L_{n,1/m}} (\mathrm{lip}\, f)^2 \, d\nu_m = \frac{1}{m}\int_{S^n} |\nabla F_f|^2 \, dv_{S^n}. \tag{6.4}$$

The norm identity in (6.3) also follows from (6.1).

Lipschitz functions are dense in $L^2(\nu_m)$: continuous functions are dense for a finite Borel measure on a compact metric space, and Lipschitz functions uniformly approximate continuous functions. The map $Uf = m^{-1/2} F_f$ therefore extends by completion to an isometry from $L^2(\nu_m)$ into $L^2(S^n)^{\Gamma_m}$. Let $\mathsf{A}v = m^{-1} \sum_{\gamma \in \Gamma_m} v \circ \gamma$. Reindexing the finite sum gives $\mathsf{A}^2 = \mathsf{A}$, and $(v \mapsto v \circ \gamma)^* = (v \mapsto v \circ \gamma^{-1})$ gives $\mathsf{A}^* = \mathsf{A}$ on L^2. Each pullback is an isometry on L^2 and on H^1; hence

$$\|\mathsf{A}v\|_E \leq \frac{1}{m}\sum_{\gamma \in \Gamma_m} \|v \circ \gamma\|_E = \|v\|_E, \qquad E = L^2(S^n) \text{ or } H^1(S^n).$$

If v is invariant and smooth v_i converge to v in either space, then $\mathsf{A}v_i \to \mathsf{A}v = v$ in that space. Thus invariant smooth functions are dense in both invariant spaces. Each such smooth function is Lipschitz and descends to the quotient. The range of U is closed and contains a dense subspace; thus U is onto.

Let f_i be any sequence in (5.1) converging to f in $L^2(\nu_m)$. Then $F_{f_i} \to F_f$ in $L^2(S^n)$. Lower semicontinuity of the classical Dirichlet energy gives

$$\frac{1}{m}\mathcal{E}_{S^n}(F_f) \leq \liminf_{i\to\infty} \frac{1}{m}\mathcal{E}_{S^n}(F_{f_i}) = \liminf_{i\to\infty} \int (\mathrm{lip}\, f_i)^2 \, d\nu_m.$$

The energy on S^n is understood to be $+\infty$ outside H^1. Taking the infimum proves $m^{-1}\mathcal{E}_{S^n}(F_f) \leq \mathcal{E}_{\nu_m}(f)$. If $F_f \in H^1(S^n)^{\Gamma_m}$, choose invariant smooth $v_i \to F_f$ in H^1, and let f_i be their descents. Equation (6.4) gives

$$\mathcal{E}_{\nu_m}(f) \leq \lim_{i\to\infty} \frac{1}{m}\mathcal{E}_{S^n}(v_i) = \frac{1}{m}\mathcal{E}_{S^n}(F_f).$$

This proves (6.2) and (6.3).

Polarization gives $\mathcal{E}_{S^n}(Uf, U\psi) = \mathcal{E}_{\nu_m}(f, \psi)$. For invariant $v \in H^1(S^n)$ and $w \in H^1(S^n)$, invariance of the round form under each group element gives

$$\mathcal{E}_{S^n}(v, w \circ \gamma) = \mathcal{E}_{S^n}(v \circ \gamma^{-1}, w) = \mathcal{E}_{S^n}(v, w).$$

Averaging yields $\mathcal{E}_{S^n}(v, w) = \mathcal{E}_{S^n}(v, \mathsf{A}w)$. For invariant $b \in L^2$, self-adjointness of A likewise gives $\langle b, w\rangle = \langle b, \mathsf{A}w\rangle$.

Suppose $f \in D(\Delta_{L_{n,1/m},\nu_m})$. Put $v = Uf$ and $b = U\Delta_{L_{n,1/m},\nu_m} f$. For arbitrary $w \in H^1(S^n)$, (6.2) supplies $\psi \in W^{1,2}(L_{n,1/m}, \nu_m)$ with $U\psi = \mathsf{A}w$. Then

$$\begin{aligned}\mathcal{E}_{S^n}(v, w) &= \mathcal{E}_{S^n}(v, \mathsf{A}w) = \mathcal{E}_{\nu_m}(f, \psi)\\ &= \langle \Delta_{L_{n,1/m},\nu_m} f, \psi\rangle = \langle b, \mathsf{A}w\rangle = \langle b, w\rangle.\end{aligned}$$

Thus $v \in D(\Delta_{S^n})$ and $\Delta_{S^n} v = b$.

Conversely, let $v \in D(\Delta_{S^n})$ be invariant, and put $b = \Delta_{S^n} v$. For every $w \in H^1(S^n)$,

$$\langle b, w\rangle = \mathcal{E}_{S^n}(v, w) = \mathcal{E}_{S^n}(v, \mathsf{A}w) = \langle b, \mathsf{A}w\rangle = \langle \mathsf{A}b, w\rangle.$$

Density of H^1 in L^2 gives $b = \mathsf{A}b$. Set $f = U^{-1}v$ and $a = U^{-1}b$. For $\psi \in W^{1,2}(L_{n,1/m}, \nu_m)$,

$$\mathcal{E}_{\nu_m}(f, \psi) = \mathcal{E}_{S^n}(v, U\psi) = \langle b, U\psi\rangle = \langle a, \psi\rangle.$$

Hence $f \in D(\Delta_{L_{n,1/m},\nu_m})$ with $\Delta_{L_{n,1/m},\nu_m} f = a$. This proves the operator-domain identity and unitary equivalence. The compact embedding $H^1(S^n) \hookrightarrow L^2(S^n)$ restricts to the closed invariant subspaces and transfers under U to the quotient form domain. Its Laplacian therefore has compact resolvent. □

Theorem 6.3. *The spectrum of $(L_{n,1/m}, \nu_m)$ consists of the eigenvalues*

$$\mu_\ell = \ell(\ell + n - 1) \qquad (\ell = 0, 1, 2, \dots) \tag{6.5}$$

with multiplicities

$$a_\ell(n, m) = \binom{\ell + n - 2}{n - 2} + 2\sum_{r=1}^{\lfloor \ell/m\rfloor} \binom{\ell - mr + n - 2}{n - 2}. \tag{6.6}$$

Their generating series is

$$\sum_{\ell\geq 0} a_\ell(n, m)t^\ell = \frac{1 + t^m}{(1 - t)^{n-1}(1 - t^m)}. \tag{6.7}$$

Proof. Let $\mathcal{P}_\ell$ be the complex homogeneous polynomials of degree ℓ in the real coordinates $X_1, \dots, X_{n+1}$, and put $\mathcal{P}_\ell = 0$ for $\ell < 0$. Write $R^2 = \sum_{i=1}^{n+1} X_i^2$ and $D = \sum_{i=1}^{n+1} \partial_{X_i}^2$, and set $\mathcal{H}_\ell = \ker(D|_{\mathcal{P}_\ell})$. For monomials define $\langle X^\alpha, X^\beta\rangle = \alpha!\,\delta_{\alpha\beta}$, where $\alpha! = \prod_i \alpha_i!$, and extend sesquilinearly. If $\beta = \alpha + e_i$, then

$$\langle X_i X^\alpha, X^\beta\rangle = (\alpha_i + 1)\alpha! = \langle X^\alpha, \partial_{X_i} X^\beta\rangle;$$

both sides vanish otherwise. Multiplication by R^2 is therefore adjoint to D, so $(R^2\mathcal{P}_{\ell-2})^\perp = \ker(D|_{\mathcal{P}_\ell})$. Finite-dimensional orthogonal decomposition gives

$$\mathcal{P}_\ell = \mathcal{H}_\ell \oplus R^2\mathcal{P}_{\ell-2}. \tag{6.8}$$

The group preserves R^2 and commutes with D. Both summands are invariant, so (6.8) is equivariant.

If $H \in \mathcal{H}_\ell$ and $u = H|_{S^n}$, homogeneity gives $H(R\omega) = R^\ell u(\omega)$ for $R > 0$. The polar-coordinate formula for D gives

$$D(R^\ell u) = R^{\ell-2}\big(\ell(\ell + n - 1)u - \Delta_{S^n} u\big).$$

Therefore $\Delta_{S^n} u = \mu_\ell u$. Restriction is injective on $\mathcal{H}_\ell$ by homogeneity. The values μ_ℓ are strictly increasing because $\mu_{\ell+1} - \mu_\ell = 2\ell + n > 0$. Integration by parts makes the corresponding eigenspaces mutually orthogonal.

Repeated application of (6.8) shows that every polynomial restriction is a finite sum of harmonic restrictions. Polynomial restrictions contain constants, separate points, and are closed under complex conjugation. The Stone–Weierstrass theorem makes them dense in $C(S^n)$, and hence in $L^2(S^n)$. Averaging over Γ_m shows that the restrictions of $\mathcal{H}_\ell^{\Gamma_m}$, over all ℓ, have dense span in $L^2(S^n)^{\Gamma_m}$. The operator in Lemma 6.2 has compact resolvent. If an invariant eigenfunction has eigenvalue different from every μ_ℓ, self-adjointness makes it orthogonal to the preceding dense span, so it is zero. If its eigenvalue is μ_ℓ, subtract its orthogonal projection onto the restrictions of $\mathcal{H}_\ell^{\Gamma_m}$. The remainder is again orthogonal to every harmonic restriction and is zero. Thus the quotient eigenspace at μ_ℓ has dimension $a_\ell = \dim_{\mathbb{C}} \mathcal{H}_\ell^{\Gamma_m}$. These complex dimensions are the real multiplicities, since the operators and the group action are real and their complex kernels are complexifications of the real kernels.

Put $p_\ell = \dim_{\mathbb{C}} \mathcal{P}_\ell^{\Gamma_m}$. Multiplication by R^2 is injective, since the polynomial ring has no zero divisors. Equation (6.8), including invariants, gives

$$a_\ell = p_\ell - p_{\ell-2}. \tag{6.9}$$

Use the complex linear coordinates $x_1, \dots, x_{n-1}, z, \bar{z}$ for the polynomial algebra; the last two are independent algebraic variables over $\mathbb{C}$. A monomial $x^\alpha z^a \bar{z}^b$ is multiplied by $e^{2\pi i(a-b)/m}$. It is invariant if and only if $a - b \in m\mathbb{Z}$. Since these monomials form a diagonal basis for the action, they form a basis for the invariant subspace when this condition is imposed. As formal power series,

$$\sum_{\substack{a,b\geq 0\\ a-b\in m\mathbb{Z}}} t^{a+b} = \frac{1}{1-t^2}\left(1 + 2\sum_{r\geq 1} t^{mr}\right) = \frac{1+t^m}{(1-t^2)(1-t^m)}, \tag{6.10}$$

$$\sum_{\ell\geq 0} p_\ell t^\ell = \frac{1+t^m}{(1-t)^{n-1}(1-t^2)(1-t^m)}. \tag{6.11}$$

Indeed, $a = b$ gives $\sum_{b\geq 0} t^{2b}$; the alternatives $a = b + mr$ and $b = a + mr$, $r \geq 1$, give the other two equal sums. The $n - 1$ fixed variables contribute $(1 - t)^{-(n-1)}$. Multiplying (6.11) by $1 - t^2$ and using (6.9) proves (6.7). Finally,

$$(1-t)^{-(n-1)} = \sum_{d\geq 0} \binom{d+n-2}{n-2} t^d, \qquad \frac{1+t^m}{1-t^m} = 1 + 2\sum_{r\geq 1} t^{mr}.$$

Taking coefficients gives (6.6). Its first summand is positive for every $\ell \geq 0$; hence every value in (6.5) occurs. ☐

Corollary 6.4. *Let $n \geq 4$ and let $m \geq 2$ satisfy $1/m < q_*(n)$. Take the families in Theorem 1.4 with $q = 1/m$. Put $A_{-1} = 0$ and $A_\ell = \sum_{d=0}^{\ell} a_d(n, m)$ for $\ell \geq 0$. For each $\ell \geq 0$ and $A_{\ell-1} \leq j < A_\ell$,*

$$\lim_{s\to\infty} \lambda_j(S^n, h_s) = \lim_{s\to\infty} \lambda_j(M_n, g_s) = \ell(\ell + n - 1). \tag{6.12}$$

If $m \geq 3$, this gives

$$\lambda_j(S^n, h_s), \lambda_j(M_n, g_s) \longrightarrow n \qquad (1 \leq j \leq n-1), \tag{6.13}$$

$$\lambda_j(S^n, h_s), \lambda_j(M_n, g_s) \longrightarrow 2(n+1) \qquad \left(n \leq j \leq n - 1 + \frac{n(n-1)}{2}\right). \tag{6.14}$$

The volume of every metric in these two families is σ_n/m.

Proof. By Theorem 6.3, the eigenvalues μ_ℓ are strictly increasing, each with multiplicity $a_\ell > 0$. With the zero eigenvalue at index zero, μ_ℓ occupies exactly $A_{\ell-1}, \dots, A_\ell - 1$. The limiting measure in (1.14) is ν_m by (6.1); hence (6.12) follows. For $m \geq 3$ the sum in (6.6) is empty at $\ell = 0, 1, 2$, so

$$a_0 = 1, \qquad a_1 = n - 1, \qquad a_2 = \frac{n(n-1)}{2}.$$

Thus $A_0 = 1$, $A_1 = n$, and $A_2 = n + n(n-1)/2$. Since $\mu_1 = n$ and $\mu_2 = 2(n+1)$, this gives (6.13)–(6.14). The volume assertion is (1.12) with $q = 1/m$. ☐

Corollary 6.5. *Let $m \geq 3$ be an integer with $1/m < q_*(4)$. There are smooth families h_s on S^4 and g_s on $\mathbb{CP}^2\#(-\mathbb{CP}^2)$, each of volume σ_4/m, with $\mathrm{Ric}_{h_s} \geq 3h_s$ and $\mathrm{Ric}_{g_s} \geq 3g_s$, such that on both families*

$$\lambda_1, \lambda_2, \lambda_3 \longrightarrow 4, \qquad \lambda_4, \dots, \lambda_9 \longrightarrow 10. \tag{6.15}$$

In particular, $\lambda_4(\mathbb{CP}^2\#(-\mathbb{CP}^2), g_s) \to 10$.

Proof. In (1.3), $Q_4 = \mathbb{CP}^2$ and $M_4 = \mathbb{CP}^2\#(-\mathbb{CP}^2)$. In Corollary 6.4, the two positive multiplicities at degrees one and two are 3 and 6, while $\mu_1 = 4$ and $\mu_2 = 10$. Their index ranges are $1 \le j \le 3$ and $4 \le j \le 9$. □

Remark 6.6. The limits in (6.12) concern each fixed index. They do not assert isospectrality of two smooth metrics. For $m \ge 3$, their common limit has only $n - 1$ copies of the first positive eigenvalue n, whereas the round sphere has $n + 1$. Thus (6.14) is not pinching to the round-sphere spectrum. The value $2(n + 1)$ is a limit along the specified families, not a claim about the largest possible constant in (1.1).

References

[1] E. Aubry, *Pincement sur le spectre et le volume en courbure de Ricci positive*, Ann. Sci. École Norm. Sup. (4) **38** (2005), no. 3, 387–405. doi:10.1016/j.ansens.2005.01.002.

[2] J. Bertrand, *Pincement spectral en courbure de Ricci positive*, Comment. Math. Helv. **82** (2007), no. 2, 323–352. doi:10.4171/CMH/93. arXiv:math/0504173.

[3] B. L. Burdick, *Ricci-positive metrics on connected sums of projective spaces*, Differential Geom. Appl. **62** (2019), 212–233. doi:10.1016/j.difgeo.2018.11.005.

[4] G. De Philippis and N. Gigli, *Non-collapsed spaces with Ricci curvature bounded from below*, J. Éc. polytech. Math. **5** (2018), 613–650. doi:10.5802/jep.80.

[5] N. Gigli, A. Mondino, and G. Savaré, *Convergence of pointed non-compact metric measure spaces and stability of Ricci curvature bounds and heat flows*, Proc. Lond. Math. Soc. (3) **111** (2015), no. 5, 1071–1129. doi:10.1112/plms/pdv047. arXiv:1311.4907v1.

[6] A. Nifa, *Topological singularities in the codimension-two stratum of noncollapsed Ricci limits*, preprint, Zenodo, 2026, 94 pp. doi: 10.5281/zenodo.22670411.

[7] P. Petersen, *Riemannian Geometry*, 3rd ed., Graduate Texts in Mathematics, vol. 171, Springer, Cham, 2016. doi:10.1007/978-3-319-26654-1.

[8] P. Reiser, *Positive Ricci curvature on connected sums of fibre bundles*, arXiv:2406.02274v1, 2024.

Toulouse, France
Email address: nifa.anass@gmail.com