NATURAL SPECTRA AND ARITHMETIC OF LENS SPACES

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Abstract. The lens spaces $L(49;1,6,15)$ and $L(49;1,6,20)$ have identical natural spectra precisely for five irreducible SO(5)-types. This proves the finiteness conjecture of Lauret, Miatello and Rossetti. The types admitting bounded or compact self-adjoint corrections are classified, and the least operator norm and essential norm are determined for every one-row type on the bounded-correction locus. The distinction is arithmetic: exact equality requires equality of each Casimir eigenvalue multiplicity, whereas bounded correction requires equality of the total multiplicities in each separated quadratic cluster. Periodic character identities give both conditions and the asymptotic number of unmatched frequencies. For $L(p^2;1,1\pm p,1\pm 3p)$, the bounded-correction classification holds for every prime $p\geq 5$. For flat bundles on $L(q;1^m,a)$, with $q\geq 3$ odd, $m\geq 1$, and a a unit modulo q, equality of connection spectra is characterized in terms of the holonomies. For effective prime-square lens orbifolds, the scalar spectrum determines the gcd classes of the rotation weights and implies isometry of the corresponding proper covers at orders 9 and 25.

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Introduction

Let $G=\mathrm{SO}(6)$, $K=\mathrm{SO}(5)$, and $S^5=G/K$, with curvature one. An irreducible representation τ of K defines the homogeneous bundle $G\times_K V_\tau$. Let $\Gamma\subset G$ be finite and act freely on S^5. On $L=\Gamma\backslash S^5$, denote by $\Delta_{\tau,L}$ the positive G-Casimir on the quotient bundle, with scalar product $-\frac{1}{2}\operatorname{Tr}(XY)$ on the Lie algebra of G. For exterior powers of the defining representation of K, this operator is the Hodge Laplacian [16, Section 5.1]. Equality of the Hodge spectra thus compares only finitely many natural types. The spectra of the other irreducible types impose further conditions on Γ.

2020 *Mathematics Subject Classification.* Primary 58J53; Secondary 22E46, 11F72, 11H06.

Five exact natural spectra. Consider the pair

$$L = L(49; 1, 6, 15), \qquad L' = L(49; 1, 6, 20).$$

Lauret, Miatello and Rossetti proved that these spaces are nonisometric and isospectral on forms of every degree [15]. They conjectured that only finitely many irreducible SO(5)-types have equal natural spectra on L and L'; see [15, Remark 8.2] and [16, Conjecture 5.15]. Write $\tau_{u,v}$ for the type of integral highest weight (u, v), where $u \geq v \geq 0$. Theorem 1.1 gives

$$\operatorname{Spec}(\Delta_{u,v,L}) = \operatorname{Spec}(\Delta_{u,v,L'}) \quad \Longleftrightarrow \quad (u, v) \in \{(0, 0), (1, 0), (2, 0), (1, 1), (2, 1)\}.$$

In particular the conjectured set has five elements. The types $(0, 0)$, $(1, 0)$ and $(1, 1)$ account for all Hodge degrees by duality. The remaining types are the trace-free symmetric square $(2, 0)$ and the type $(2, 1)$, of dimensions 14 and 35. For every other irreducible type, unequal multiplicities recur in infinitely many separated Casimir blocks.

There is a larger set of types for which a bounded self-adjoint operator C satisfies

$$\operatorname{Spec}(\Delta_{u,v,L} + C) = \operatorname{Spec}(\Delta_{u,v,L'}).$$

Theorem 3.1 gives an explicit set $\mathscr{Z} \subset (\mathbb{Z}/49\mathbb{Z})^2$ of cardinality 289: such an operator exists if and only if $(2u + 3, 2v + 1) \bmod 49 \in \mathscr{Z}$. It can be chosen to commute with every spectral projection of $\Delta_{u,v,L}$, with

$$\|C\| \leq u(u + 2) - 2v.$$

Here C ranges over all bounded self-adjoint operators on the Hilbert space of bundle sections. Compact corrections exist precisely for the five exact equality types (Corollary 3.8).

For each one-row type $\tau_{u,0}$ on the bounded-correction locus, Theorem 3.10 determines

$$\min_C \|C\| = \min_C \|C\|_{\mathrm{ess}} = \sup_j \left|\lambda_j(\Delta_{u,0,L}) - \lambda_j(\Delta_{u,0,L'})\right|.$$

The eigenvalues are ordered with multiplicity from the first index. Both minima equal 19 for $u = 9$ and 89 for $u = 48$. The equality includes the initial blocks, where distinct Casimir strings can overlap. In general the essential norm depends only on the eventual displacements, whereas the operator norm can also depend on the initial spectrum.

Multiplicity at an offset and multiplicity in a block. The branching inequalities for $G \supset K$ give the eigenvalues

$$r(r + 4) + w, \qquad r \geq u, \qquad w = k(k + 2) + c^2, \qquad v \leq k \leq u, \quad 0 \leq c \leq v.$$

The offsets w form a finite set. Equal offsets are combined. For the order-49 pair, the difference of their multiplicities is a periodic function $A(a, w)$, where $a = r + 2 \bmod 49$. The two spectral conditions become

$$\begin{aligned} \text{equal natural spectra} \quad &\Longleftrightarrow \quad A(a, w) = 0 \quad \text{for all } a, w, \\ \text{bounded ordered displacement} \quad &\Longleftrightarrow \quad \sum_w A(a, w) = 0 \quad \text{for all } a. \end{aligned}$$

The first condition retains every offset. The second follows from the gaps between consecutive blocks, which tend to infinity. If $\sum_w A(a, w) = 0$ for every a, the partial sums of $A(a, w)$ determine the ordered displacements in every sufficiently late block.

Multiplicity sequences along highest-weight strings have rational generating functions. Lauret and Miatello proved finite determination for finite unions of such strings [13, Proposition 2.2 and Theorem 1.2]. Lauret and Miatello give a strong spectral-to-representation converse for odd-dimensional spheres when the Casimir offsets are distinct [14, Theorem 4.2]. Their hypothesis concerns a fixed natural type. Theorem 1.1 determines all equality types for the fixed pair L, L'. The quadratic-cluster results of Section 2 give a quantitative comparison of the corresponding ordered spectra. If $E_\eta(R)$

is the least total number of unmatched copies of the two frequency lists at most R, with matching distance at most η, then for the natural pair

$$E_{\kappa/R}(R) = RC(\kappa) + O(1) \qquad (\kappa \geq 0).$$

The cutoffs are $R = K + \frac{5}{2}$, and the frequencies are $\sqrt{\lambda + 4}$. The function C is continuous, nonincreasing, and explicitly determined by the periodic multiplicities. Its value at zero vanishes precisely for exact equality, and its limit at infinity vanishes precisely for bounded correction. On the bounded-correction locus, twice the last point where $\kappa C(\kappa)$ changes slope equals the least essential norm; this value is zero if the function vanishes. Theorems 2.2 and 2.3 also give uniform formulas for tolerances below and at the limiting cluster spacing.

Prime-square orders and affine congruences. For a prime $p \geq 5$, put

$$L_p^{\pm} = L(p^2; 1, 1 \pm p, 1 \pm 3p).$$

Their equality of Hodge spectra belongs to the constructions of Lauret, Miatello and Rossetti [15] and to the Hodge-series identities of DeFord and Doyle [5, Theorem 1 and §8]. Theorem 4.1 determines all natural types admitting bounded correction. In the coordinates $(X, Y) = (2u + 3, 2v + 1) \bmod p^2$, the locus has cardinality

$$\begin{cases} 625, & p = 5, \\ 289, & p = 7, \\ 5p^2 - 4p + 40, & p \geq 11. \end{cases}$$

At $p = 5$ the two spaces are isometric. At $p = 7$ orientation-preserving isometries identify them with L, L'. For $p \geq 11$, the locus is the union of the four sets $X = 0$, $Y = 0$, $X = Y$, $X = -Y$, the set $p \mid X, Y$, and the signed-permutation orbits of

$$(3, 1),\ (5, 1),\ (5, 3),\ (7, 1),\ (7, 3).$$

The block multiplicity condition reduces to one Laurent polynomial with integer coefficients, independent of p. Equality of its exponents modulo p^2 gives finitely many affine congruences. For fixed integral arrangements, the counting functions modulo an integer are eventually quasipolynomials by Kamiya, Takemura and Terao [8, 9]. For polynomial coefficients and a fixed exponent in the modulus, eventual quasipolynomiality also follows from the theorem of Bogart, Goodrick and Woods on parametric Presburger families [4, Theorem 1.15]. Theorem 5.1 gives a single polynomial in p for each exponent k, outside one explicit finite set of primes independent of k.

Flat holonomy and orbifold covers. For a unitary representation ρ of the cyclic fundamental group, the connection spectrum on the associated flat bundle depends on the real part of its character. Lauret's twisted generating functions describe these multiplicities and their congruence lattices [10, Theorems 2.3, 3.1 and 4.2]. On $L(q; 1^m, a)$, with $q \geq 3$ odd, a a unit modulo q, and $m \geq 2$, Theorem 6.2 shows that equality of connection spectra is equivalent to equality of realified holonomy. For the shifted frequencies $k + m$, a deficit $o(R^m)$ at any fixed tolerance $0 \leq \eta < 1$ implies this equality. In dimension three the linear map from real parts of virtual characters to spectral multiplicities has a nontrivial kernel for suitable a. Theorem 7.1 determines it for every odd q and every unit a; its rational components are explicit multiples of fixed fields in real cyclotomic extensions. The isospectral line twists with $a = 2$ occur in [10, Example 4.10].

For effective lens orbifolds of order p^2, the scalar spectrum determines the numbers of rotation weights in each of the three gcd classes $1, p, p^2$ (Theorem 8.1). At orders 9 and 25, the corresponding order-p covers of scalar-isospectral orbifolds are isometric (Theorem 8.5). This proves these two cases, in every odd dimension, of the covering conjecture in [16, Conjecture 7.3]. Corresponding covers of scalar-isospectral lens manifolds are scalar isospectral [12, Theorem 4.3]; in dimension three, Bari and Hunsicker proved scalar spectral determination of lens orbifolds themselves [3, Theorem 3.1]. Finally, for $(2n - 1)$-dimensional effective lens orbifolds of prime-square order, with $n \geq 2$, equality of

the spectra on forms of degrees $0, \dots, n-2$ implies equality in every degree (Corollary 8.6). The gcd invariant gives the hypothesis in Lauret's reduction of this question to the coordinate-axis terms of the congruence lattice [12, Remark 4.6]; the moment identities for forms are those of [11, Corollary 2.3].

Part 1. Ordered spectra and natural operators

1. Exact natural spectra

1.1. **Natural Casimir operators.** Let $G = \mathrm{SO}(6)$, $K = \mathrm{SO}(5)$, and identify the unit round sphere with G/K. Consider the two quotients

$$L = L(49; 1, 6, 15), \qquad L' = L(49; 1, 6, 20).$$

All six displayed rotation weights are units modulo 49, so both cyclic actions are free. For integers $u \geq v \geq 0$, let $\tau_{u,v}$ have orthogonal highest weight $u\varepsilon_1 + v\varepsilon_2$. These are precisely the irreducible representations of K; spin representations are not representations of this group. The associated bundle on the sphere is $G \times_K V_{\tau_{u,v}}$. On it and on its quotient bundles, Δ_τ denotes the positive G-Casimir for the invariant inner product $-\frac{1}{2}\operatorname{Tr}(XY)$, following [16, Section 4]. This normalization fixes all natural eigenvalues and the shift by 4. The infinitesimal rotations in the three coordinate planes are orthonormal for $-\frac{1}{2}\operatorname{Tr}(XY)$, so the dual orthogonal coordinates ε_i have length one. The rotations in the $(i, 6)$-planes, $1 \leq i \leq 5$, map to an orthonormal basis of $T_{e_6}S^5$; hence the induced quotient metric has radius one. In an equivariant description of sections, the $\mathfrak{k}$-directions act in the fibres and the five complementary directions give the principal symbol $|\xi|^2 \,\mathrm{id}$. The Casimir is therefore elliptic; its positive self-adjoint realization on either compact quotient has finite-dimensional eigenspaces and eigenvalues tending to infinity.

The positive roots of $\mathrm{SO}(5)$ are $\varepsilon_1 - \varepsilon_2$, $\varepsilon_1 + \varepsilon_2$, ε_1, and ε_2, with half-sum $(3\varepsilon_1 + \varepsilon_2)/2$. Their four factors in the Weyl dimension formula give

$$\dim \tau_{u,v} = \frac{(2u+3)(2v+1)(u-v+1)(u+v+2)}{6}.$$

Thus the five types in the theorem have dimensions $1, 5, 14, 10, 35$, in the order stated there.

Lauret, Miatello and Rossetti constructed this pair and proved its isospectrality on differential forms of every degree [15]. Their finiteness question appears in [15, Remark 8.2] and [16, Conjecture 5.15]. The following theorem determines the complete set in that conjecture. The Hodge types $(0, 0)$, $(1, 0)$, and $(1, 1)$ belong to the original construction. The two quotients are nonisometric. An isometry of these quotients lifts to an isometry of S^5 and conjugates their cyclic deck groups. It would therefore give a unit a modulo 49 taking the set $\{\pm 1, \pm 6, \pm 15\}$ to $\{\pm 1, \pm 6, \pm 20\}$. Since the first set contains 1, up to replacing a by $-a$ only $a = 1, 6, 20$ need be considered. For $a = 1$ the image of 15 is absent from the second set; for $a = 6$ and $a = 20$ the image of 6 is respectively 36 and 22, again absent. This rules out all possible conjugacies. The value $T(6, 4, 0) = 2$ is already the invariant-dimension comparison $8 - 6$ of [15, Lemma 8.1].

Theorem 1.1 (Exact natural spectra). *For integers $u \geq v \geq 0$, the natural Casimir operators on $L(49; 1, 6, 15)$ and $L(49; 1, 6, 20)$ are $\tau_{u,v}$-isospectral if and only if*

$$(u, v) \in \{(0, 0), (1, 0), (2, 0), (1, 1), (2, 1)\}.$$

1.2. **Branching and Casimir eigenvalues.** Write $\pi_{r,k,c}$ for the irreducible G-representation of highest weight $r\varepsilon_1 + k\varepsilon_2 + c\varepsilon_3$, where

$$r \geq k \geq |c|, \qquad r, k, c \in \mathbb{Z}.$$

The half-sum of the positive roots of G is $\rho = (2, 1, 0)$. Taking the inner product of (r, k, c) with $(r, k, c) + 2\rho$ gives the Casimir eigenvalue

$$\lambda(r, k, c) = r(r+4) + k(k+2) + c^2. \tag{1.1}$$

The restriction from G to K has the following decomposition:

$$[\pi_{r,k,c}|_K \,:\, \tau_{u,v}] = 1 \quad \Longleftrightarrow \quad r \geq u \geq k \geq v \geq |c|. \tag{1.2}$$

Put $\ell = |c|$, $C_j(z) = z^j + z^{-j}$, $S_j(z) = z^j - z^{-j}$, and

$$\mathcal{D}(F, G) = F(z_1)G(z_2) - F(z_2)G(z_1).$$

On the K-torus the third G-torus coordinate is 1. The Weyl character formula [7, Theorem 25.4], in type D_3, then has zero minus-determinant and gives

$$\chi_{r,k,c}|_K = \frac{\mathcal{D}(C_{r+2} - C_\ell, C_{k+1} - C_\ell)}{\mathcal{D}(C_2 - C_0, C_1 - C_0)}.$$

The type-B_2 character of $\tau_{p,q}$ is

$$\chi_{\tau_{p,q}} = \frac{\mathcal{D}(S_{p+3/2}, S_{q+1/2})}{\mathcal{D}(S_{3/2}, S_{1/2})}.$$

The elementary identity

$$S_{1/2}(z)S_{j+1/2}(z) = C_{j+1}(z) - C_j(z)$$

telescopes in each variable. Consequently

$$\begin{aligned}\sum_{p=k}^{r}\sum_{q=\ell}^{k} \chi_{\tau_{p,q}} &= \frac{\mathcal{D}(C_{r+2} - C_{k+1}, C_{k+1} - C_\ell)}{\mathcal{D}(C_2 - C_1, C_1 - C_0)} \\ &= \frac{\mathcal{D}(C_{r+2} - C_\ell, C_{k+1} - C_\ell)}{\mathcal{D}(C_2 - C_0, C_1 - C_0)}.\end{aligned}$$

For the first equality multiply the two columns of each determinant by $S_{1/2}(z_1)$ and $S_{1/2}(z_2)$; the second equality follows from $\mathcal{D}(F + G, G) = \mathcal{D}(F, G)$. These identities hold on the regular torus and extend everywhere by continuity of characters. Linear independence of the irreducible K-characters proves (1.2), including multiplicity one. The L^2 space of quotient-bundle sections decomposes as

$$\widehat{\bigoplus}_{\pi \in \widehat{G}} V_\pi^\Gamma \otimes \mathrm{Hom}_K(V_\pi, V_\tau),$$

and the Casimir acts by its scalar on each summand [16, Proposition 4.1]. Thus the multiplicity at a given eigenvalue is the sum of $\dim V^\Gamma_{\pi_{r,k,c}}$ over all weights satisfying (1.2) and having that eigenvalue in (1.1). Coincident Casimir values contribute additively.

Let Γ, Γ' be the cyclic groups defining L, L'. Put

$$d(r, k, c) = \dim V^\Gamma_{\pi_{r,k,c}} - \dim V^{\Gamma'}_{\pi_{r,k,c}}.$$

Section 1.3 constructs a function $T : \mathbb{Z}^3 \to \mathbb{Z}$, even and 49-periodic in each coordinate, such that

$$d(r, k, 0) = T(r + 2, k + 1, 0), \tag{1.3}$$

$$d(r, k, c) + d(r, k, -c) = 2T(r + 2, k + 1, c) \quad (c > 0). \tag{1.4}$$

These identities imply the following criterion.

For $u \geq v \geq 0$, $a \in \mathbb{Z}/49\mathbb{Z}$, and $w \in \mathbb{Z}$, define

$$A_{u,v}(a, w) = \sum_{\substack{v \leq k \leq u,\ 0 \leq c \leq v \\ k(k+2)+c^2 = w}} (2 - \mathbf{1}_{c=0}) T(a, k + 1, c). \tag{1.5}$$

The possible offsets lie in the fixed interval

$$w_- = v(v + 2) \leq w \leq w_+ = u(u + 2) + v^2, \qquad H_{u,v} = w_+ - w_- = u(u + 2) - 2v.$$

This interval is independent of the first coordinate r.

Lemma 1.2. *The pair L, L' is $\tau_{u,v}$-isospectral if and only if $A_{u,v}(a, w) = 0$ for every $a \in \mathbb{Z}/49\mathbb{Z}$ and every $w \in \mathbb{Z}$.*

Proof. If all the stated sums vanish, then for each $r \geq u$ the multiplicity difference at every number $r(r+4) + w$ vanishes after summing the terms with that first coordinate. Summing over r proves isospectrality.

Conversely, let w_- and w_+ be the least and greatest integers attained by $k(k+2) + c^2$ in (1.5). Set $B = w_+ - w_-$. The consecutive differences of $r(r+4)$ are $2r+3$ on the left and $2r+5$ on the right. Both exceed B whenever $r \geq \max\{u, B\}$. Since $x(x+4)$ is strictly increasing for $x \geq 0$, the interval

$$[r(r+4) + w_-,\ r(r+4) + w_+]$$

is then disjoint from the corresponding intervals for all other integers $r' \geq u$. Choose r arbitrarily large with $r + 2 \equiv a \pmod{49}$. At $r(r+4) + w$, isospectrality and (1.3)–(1.4) give $A_{u,v}(a, w) = 0$. Every residue a is obtained in this way. The same separation, without assuming isospectrality, shows that each $A_{u,v}(a, w)$ is an integer: it is an actual multiplicity difference at any sufficiently large r in the prescribed residue class. □

1.3. **Periodic dimension data.** Let $\zeta = e^{2\pi i/49}$ and put

$$s = (1, 6, 15), \qquad s' = (1, 6, 20), \qquad \rho = (2, 1, 0).$$

For an integer triple $x = (a, b, c)$ define the Laurent polynomial

$$P_s(x; X) = \det\left(X^{x_i s_j} + X^{-x_i s_j}\right)_{1 \leq i,j \leq 3}, \qquad D_s(X) = P_s(\rho; X). \tag{1.6}$$

Define

$$T(a, b, c) = \frac{1}{49} \operatorname{Tr}_{\mathbb{Q}(\zeta)/\mathbb{Q}} \left(\frac{P_s((a, b, c); \zeta)}{D_s(\zeta)} - \frac{P_{s'}((a, b, c); \zeta)}{D_{s'}(\zeta)} \right). \tag{1.7}$$

Both denominators are nonzero. Indeed, the six residues $\pm s_1, \pm s_2, \pm s_3$ are distinct modulo 49, as are the six residues for s'. The positive roots of D_3 are $\varepsilon_i \pm \varepsilon_j$ with $i < j$. Every corresponding factor of the Weyl denominator is nonzero, because no two rotation weights agree up to sign modulo 49.

Lemma 1.3. *The function T is integer-valued, even and 49-periodic in each variable, and satisfies* (1.3)–(1.4).

Proof. Evenness and periodicity follow directly from (1.6). Write $y_j = \zeta^{s_j}$ and set

$$P_x(y) = \det(y_j^{x_i} + y_j^{-x_i})_{1 \leq i,j \leq 3}, \qquad Q_x(y) = \det(y_j^{x_i} - y_j^{-x_i})_{1 \leq i,j \leq 3}.$$

The type-D_3 Weyl group consists of coordinate permutations and sign changes whose product is 1. Its alternating sum is therefore $(P_x(y) + Q_x(y))/2$: expanding the two determinants cancels the sign vectors with product -1 and doubles those with product 1. For $x = \rho = (2, 1, 0)$ the last row of Q_x is zero, so the denominator alternant is $D_s/2$. Changing c to $-c$ leaves P_x unchanged and changes the sign of Q_x. The Weyl character quotient consequently gives

$$\chi_{r,k,0}(g) = \frac{P_s((r+2, k+1, 0); \zeta)}{D_s(\zeta)}$$

when g has rotation weights s, and, for $c > 0$,

$$\frac{\chi_{r,k,c}(g) + \chi_{r,k,-c}(g)}{2} = \frac{P_s((r+2, k+1, c); \zeta)}{D_s(\zeta)}.$$

Invariant dimensions are character averages over the cyclic group. The identity contributions cancel between Γ and Γ'. For each $j \in \{1, \dots, 6\}$, the elements obtained by taking the $7j$-th powers have rotation weights

$$j(1, -1, 1) \quad \text{and} \quad j(1, -1, -1) \pmod 7.$$

Reflection in one axis of the third rotation plane conjugates the first of these elements to the second; its determinant is -1. For a regular torus element this reflection sends y_3 to y_3^{-1}. It leaves P_x unchanged and changes the sign of Q_x, so the signed character formula just proved shows that it interchanges $\chi_{r,k,c}$ and $\chi_{r,k,-c}$. The identity extends to every torus element because characters are continuous and regular elements are dense. Thus the $c = 0$ character and the sum for $c > 0$ are unchanged by the reflection even at the singular order-seven elements. Their contributions cancel without evaluating a singular quotient.

All remaining elements have order 49 and are regular. Their character sum is the field trace in (1.7), since the automorphisms of $\mathbb{Q}(\zeta)$ are $\zeta \mapsto \zeta^j$ with $j \in (\mathbb{Z}/49\mathbb{Z})^\times$. Dividing the group sum by 49 proves (1.3)–(1.4).

The longest Weyl element sends (r, k, c) to $(-r, -k, c)$, so $V^*_{\pi_{r,k,c}} \simeq V_{\pi_{r,k,-c}}$. For a finite group F, averaging gives $(V^*)^F \simeq (V^F)^*$; hence $d(r, k, c) = d(r, k, -c)$. Equations (1.3)–(1.4) imply $T(r + 2, k + 1, c) = d(r, k, c) \in \mathbb{Z}$ whenever $r \geq k \geq c \geq 0$. Every residue triple modulo 49 has such a representative: choose $c \geq 0$, then $k \geq c$, then $r \geq k$ in the prescribed residue classes. Periodicity gives $T(a, b, c) \in \mathbb{Z}$ for every $(a, b, c) \in \mathbb{Z}^3$. □

1.4. **Finite character identities over the integers.** All subscripts in this section are taken modulo 49. Define the even integer functions $h_s, h_{s'} : \mathbb{Z}/49\mathbb{Z} \to \mathbb{Z}$ by the first three columns of Table 1, setting $h(49 - j) = h(j)$ for $1 \leq j \leq 24$. Let

$$H_s(X) = \sum_{j=0}^{48} h_s(j)X^j, \qquad \Phi(X) = 1 + X^7 + X^{14} + \cdots + X^{42}.$$

Work in the ring $\mathcal{R} = \mathbb{Z}[X]/(X^{49} - 1)$ and write $C_j = X^j + X^{-j}$. The identities $C_iC_j = C_{i+j} + C_{i-j}$ and $C_{-j} = C_j$ expand the two determinants as

$$D_s = 2(C_3 + C_4 + C_8 - C_{11} - 3C_{13} - C_{17} + C_{18} + C_{20} + C_{22} - C_{24}),$$
$$D_{s'} = 2(-C_3 + C_4 + 3C_8 + C_{10} - C_{11} - C_{13} - C_{15} + C_{17} - C_{18} - C_{22}).$$

If $D = 2\sum_j d_jC_j$ is either displayed expression, the coefficient of X^ℓ in DH is $2\sum_j d_j\big(h(\ell-j)+h(\ell+j)\big)$. Substituting the corresponding column of Table 1 gives, for each $0 \leq \ell \leq 48$,

$$2\sum_j d_j\big(h(\ell - j) + h(\ell + j)\big) = \begin{cases} 12, & \ell = 0, \\ -2, & \ell \in \{7, 14, 21, 28, 35, 42\}, \\ 0, & \text{otherwise.} \end{cases}$$

Equivalently, in $\mathcal{R}$,

$$D_s(X)H_s(X) = 14 - 2\Phi(X), \qquad D_{s'}(X)H_{s'}(X) = 14 - 2\Phi(X). \tag{1.8}$$

As $\Phi(\zeta) = 0$, this proves $D_s(\zeta)^{-1} = H_s(\zeta)/14$ and its counterpart for s'. The same two columns also give the seven identities

$$\sum_{j=0}^{6} h_s(a + 7j) = \sum_{j=0}^{6} h_{s'}(a + 7j) = 0 \qquad (0 \leq a \leq 6).$$

The trace of ζ^j is obtained by subtracting the sum over multiples of 7 from the sum over all residues modulo 49:

$$\mathrm{Tr}_{\mathbb{Q}(\zeta)/\mathbb{Q}}(\zeta^j) = \sum_{h=0}^{48} \zeta^{hj} - \sum_{h=0}^{6} \zeta^{7hj} = 49\mathbf{1}_{49|j} - 7\mathbf{1}_{7|j}.$$

For either h column, expansion of H consequently yields

$$\frac{1}{49}\,\mathrm{Tr}_{\mathbb{Q}(\zeta)/\mathbb{Q}}\big(\zeta^m H(\zeta)\big) = h(-m) - \frac{1}{7}\sum_{j \equiv -m\ (7)} h(j) = h(m).$$

TABLE 1. Integer coefficients and nonzero character differences.

j	$h_s(j)$	$h_{s'}(j)$	a	c	$T(a,j,c)$
0	0	0	6	4	-2
1	0	0	6	5	-1
2	-1	0	5	3	1
3	0	-1	5	2	-1
4	1	0	6	0	2
5	1	1	6	1	1
6	0	0	4	0	-2
7	-1	-1	6	4	-1
8	0	0	4	0	2
9	1	1	6	1	-1
10	1	1	6	0	-2
11	-1	0	6	0	2
12	0	-1	6	1	1
13	0	0	4	0	-2
14	1	1	6	4	2
15	0	0	4	0	2
16	-2	0	6	1	-1
17	-1	0	6	0	-2
18	0	-1	6	0	-2
19	1	1	6	1	-1
20	0	0	4	0	2
21	0	0	13	4	2
22	0	0	4	0	-2
23	0	-2	6	1	1
24	0	1	6	0	2

The seven zero sums and evenness of h give the last equality. It follows that

$$\frac{1}{49}\operatorname{Tr}_{\mathbb{Q}(\zeta)/\mathbb{Q}}\frac{\zeta^m}{D_s(\zeta)}=\frac{h_s(m)}{14},\qquad \frac{1}{49}\operatorname{Tr}_{\mathbb{Q}(\zeta)/\mathbb{Q}}\frac{\zeta^m}{D_{s'}(\zeta)}=\frac{h_{s'}(m)}{14}. \tag{1.9}$$

Expansion of the determinant in (1.7) now yields the periodic trace formula

$$14T(a,b,c)=\sum_{\sigma\in S_3}\operatorname{sgn}(\sigma)\sum_{e\in\{\pm1\}^3}\left[h_s\left(\sum_{i=1}^{3}e_ix_is_{\sigma(i)}\right)-h_{s'}\left(\sum_{i=1}^{3}e_ix_is'_{\sigma(i)}\right)\right],\quad x=(a,b,c). \tag{1.10}$$

The identities needed for the positive cases are

$$T(a,1,0)=T(a,2,0)=T(a,3,0)=T(a,2,1)=T(a,3,1)=0\qquad(a\in\mathbb{Z}/49\mathbb{Z}). \tag{1.11}$$

For $t=s,s'$, put

$$R_t(a;b,c)=\sum_{\sigma\in S_3}\operatorname{sgn}(\sigma)\sum_{e\in\{\pm1\}^3}h_t\big(e_1at_{\sigma(1)}+e_2bt_{\sigma(2)}+e_3ct_{\sigma(3)}\big).$$

Then $14T(a,b,c)=R_s(a;b,c)-R_{s'}(a;b,c)$. Table 2 gives their common values for $0\le a\le 24$ and the five pairs (b,c) in (1.11); evenness gives the remaining residues. The final three columns of Table 1 give nonzero values from the same finite sum (1.10). In particular, for every $b\in\mathbb{Z}/49\mathbb{Z}$ there are a and $0\le c\le 5$ with $T(a,b,c)\neq 0$.

TABLE 2. Common values of $R_s(a;b,c)$ and $R_{s'}(a;b,c)$.

a	$R(a;1,0)/4$	$R(a;2,0)/4$	$R(a;3,0)/4$	$R(a;2,1)/2$	$R(a;3,1)/2$
0	0	0	0	6	0
1	0	-3	0	0	0
2	3	0	-6	0	0
3	0	6	0	0	0
4	3	0	3	3	0
5	0	-3	0	0	-3
6	-3	0	0	-9	-1
7	0	0	0	-1	0
8	3	1	0	9	1
9	-1	3	2	0	3
10	-3	-2	-3	-3	0
11	-1	-6	-1	-1	0
12	-3	1	6	0	1
13	1	3	0	3	5
14	0	0	0	-1	0
15	-1	-2	0	-3	-5
16	2	-1	-4	0	-1
17	1	4	1	1	0
18	2	2	2	2	0
19	1	-2	-2	0	-2
20	-2	-1	0	-6	-3
21	0	0	0	-1	0
22	2	2	0	6	3
23	-2	2	4	0	2
24	-2	-4	-2	-2	0

The equality of the two R columns can also be written as a polynomial identity in $\mathcal{R}$. If j is a unit modulo 49, denote its inverse by j^{-1}. Changing the variable a in the finite sum gives

$$\sum_{a=0}^{48} R_t(a;b,c)X^a = 2\sum_{\sigma\in S_3} \operatorname{sgn}(\sigma) H_t(X^{t_{\sigma(1)}^{-1}}) C_{b t_{\sigma(2)} t_{\sigma(1)}^{-1}} C_{c t_{\sigma(3)} t_{\sigma(1)}^{-1}}.$$

Indeed the two signs of e_1 give the factor 2, by evenness of H_t, and summation over e_2, e_3 gives the two C factors. Inserting the two H polynomials gives the equal coefficients listed in Table 2, proving (1.11) in all 49 residue classes.

1.5. **Proof of the classification.**

Proof of Theorem 1.1. Suppose first that $v \geq 12$. Apply the table to the residue $b \equiv v+1 \pmod{49}$, replacing the least nonnegative representative b by $49-b$ when $b > 24$, since the function is even. Choose the resulting a and $c \leq 5$. Then $c \leq v$ and

$$c^2 \leq 25 < 2v+3.$$

Thus $w = v(v+2)+c^2$ has exactly one representation in (1.5): the pair $(k,c) = (v,c)$. Indeed, any $k \geq v+1$ gives $k(k+2) \geq v(v+2)+2v+3 > w$. Hence

$$A_{u,v}(a,w) = (2-\mathbf{1}_{c=0})T(a,v+1,c) \neq 0,$$

contrary to Lemma 1.2.

For $2 \le v \le 11, v \ne 6$, the following entries of the table satisfy the same strict inequality $c^2 < 2v+3$:

v	2	3	4	5	7	8	9	10	11
a	5	6	6	4	4	6	6	6	6
c	2	0	1	0	0	1	0	0	1

The preceding argument excludes every such v.

If $v = u = 6$, take $k = 6, c = 4, a = 6$. Here $k = 6$, and $T(6,7,4) = -1$, so $A_{6,6}(6,64) = -2$. If $v = 6$ and $u \ge 7$, take $k = 7, c = 0, a = 4$. The value $w = 63$ cannot arise from $k = 6$, because $63 - 6(6+2) = 15$ is not a square, and every $k \ge 8$ gives $k(k+2) \ge 80$. Therefore

$$A_{u,6}(4,63) = T(4,8,0) = 2.$$

This excludes the remaining cases with $v \ge 2$.

Suppose next that $v \in \{0,1\}$ and $u \ge 3$. For $v \le k \le u$, $0 \le c \le v$, the equation $k(k+2) + c^2 = 15$ has the unique solution $k = 3, c = 0$: if $k \le 2$ its left side is at most $8 + 1 = 9$, and if $k \ge 4$ it is at least 24. Thus

$$A_{u,v}(6,15) = T(6,4,0) = 2,$$

again excluding isospectrality.

The only remaining highest weights have $u \le 2$, $v \le 1$ and $u \ge v \ge 0$, hence are precisely the five pairs in the statement. For each of them, every summand in (1.5) vanishes by (1.11). Lemma 1.2 proves isospectrality. □

1.6. **The number of spectral discrepancies.** For a fixed $\tau = \tau_{u,v}$, let $m_{\tau,L}(\lambda)$ and $m_{\tau,L'}(\lambda)$ denote eigenvalue multiplicities, set equal to zero when λ is absent. Define

$$D_\tau(X) = \#\{\lambda \le X \,:\, m_{\tau,L}(\lambda) \ne m_{\tau,L'}(\lambda)\}.$$

This counts distinct spectral values, not their multiplicities. Let

$$W_{u,v} = \{k(k+2) + c^2 \,:\, v \le k \le u,\ 0 \le c \le v\}$$

and put

$$\eta_\tau = \frac{1}{49} \sum_{w \in W_{u,v}} \#\{a \in \mathbb{Z}/49\mathbb{Z} \,:\, A_{u,v}(a,w) \ne 0\}. \tag{1.12}$$

Corollary 1.4. *For every irreducible τ,*

$$D_\tau(X) = \eta_\tau \sqrt{X} + O_\tau(1) \qquad (X \longrightarrow \infty).$$

The coefficient η_τ is zero precisely for the five types in Theorem 1.1. For every other type, $\eta_\tau \ge 2/49$. For each such nonexceptional type, there are integers a, w and a nonzero integer δ such that

$$m_{\tau,L}(r(r+4)+w) - m_{\tau,L'}(r(r+4)+w) = \delta$$

for every sufficiently large r satisfying $r + 2 \equiv a \pmod{49}$.

Proof. Choose R so that all first-coordinate blocks with $r \ge R$ are disjoint from every other block, as in Lemma 1.2. Apart from finitely many values, every discrepancy is then represented uniquely by (r, w) with $r \ge R$, $w \in W_{u,v}$ and $A_{u,v}(r+2, w) \ne 0$. For each fixed w and residue a,

$$\#\{r \ge R \,:\, r + 2 \equiv a \pmod{49},\ r(r+4) + w \le X\} = \frac{\sqrt{X}}{49} + O_\tau(1),$$

since the upper bound on r is $\sqrt{X + 4 - w} - 2$. Summation gives $D_\tau(X) = \eta_\tau \sqrt{X} + O_\tau(1)$, with η_τ as in (1.12). Lemma 1.2 identifies the zero coefficient with full isospectrality. Every nonzero coefficient appearing in the classification has $a \in \{4, 5, 6, 13\}$; evenness gives the distinct residue $-a$ with the same nonzero value. Thus $\eta_\tau \ge 2/49$. For such a residue a and offset w, put $\delta = A_{u,v}(a,w) \ne 0$. Block separation gives the displayed multiplicity difference for every sufficiently large r with $r + 2 \equiv a \pmod{49}$. □

1.7. **A general finite obstruction criterion.** The following criterion applies to pairs of spherical quotients with periodic dimension differences.

Proposition 1.5. *Let $\Gamma, \Gamma' \subset \mathrm{SO}(6)$ be finite groups of the same order, acting freely on S^5. Suppose that the difference of their outer-symmetrized dimension data has the form* (1.3)–(1.4) *for a function T periodic with an integer period $q \geq 1$ in each coordinate. Suppose that, for an integer $C \geq 0$,*

(1) for every $b \pmod q$ there are a and $0 \leq c \leq C$ with $T(a, b, c) \neq 0$;

(2) there exist a_0, b_0 with $T(a_0, b_0, 0) \neq 0$.

Then only finitely many irreducible $\mathrm{SO}(5)$*-representations τ give isospectral associated bundle Laplacians on the two quotients.*

Proof. In Lemma 1.2, the interval separation depends only on the offsets, and periodicity gives the same coefficients at arbitrarily large r in each prescribed residue class. Both properties hold here with period q, so the criterion is $A_{u,v}(a, w) = 0$ for all a, w. If $v \geq C$ and $2v + 3 > C^2$, the bottom value $k = v$ and the nonzero value in (1) give a uniquely represented nonzero coefficient. Hence v is bounded. For each of the finitely many remaining v, choose $k \geq v$ with $k + 1 \equiv b_0 \pmod q$ and $2k + 1 > v^2$. The integer $k(k + 2)$ cannot equal $k'(k' + 2) + c^2$ for $v \leq k' < k$, $0 \leq c \leq v$, because the difference of the first terms is at least $2k + 1 > v^2$. It cannot arise from $k' > k$ either. If $u \geq k$, its coefficient is therefore $T(a_0, k + 1, 0) = T(a_0, b_0, 0) \neq 0$. Thus $u < k$, proving finiteness. □

2. Quadratic spectral clusters

For finite multisets $X, Y \subset \mathbb{R}$, a matching at tolerance $\eta \geq 0$ pairs labelled copies with distance at most η, with each copy occurring at most once. If $b_\eta(X, Y)$ is its maximum cardinality, put

$$E_\eta(X, Y) = |X| + |Y| - 2b_\eta(X, Y).$$

For discrete multisets bounded below and finite on bounded intervals, write $E_\eta(R; X, Y) = E_\eta(X \cap (-\infty, R], Y \cap (-\infty, R])$. All spectra and intersections retain their multiplicities.

2.1. **Matchings on a finite path.**

Lemma 2.1. *Let $x_1 < \cdots < x_s$, and let p_i, q_i be nonnegative integers. Put $\delta_i = p_i - q_i$ and $S = \sum_i |\delta_i|$. Join i to $i + 1$ when $x_{i+1} - x_i \leq \eta$, and denote the resulting interval components by C. If* $\min(p_i, q_i) \geq 3S$ *for every i, then*

$$E_\eta\big(\{x_i^{[p_i]}\}, \{x_i^{[q_i]}\}\big) = \sum_C \left| \sum_{i \in C} \delta_i \right|. \tag{2.1}$$

Here $x^{[m]}$ means m labelled copies of x. More generally, on any one interval component the same conclusion holds if its total discrepancy has absolute value at most L, all its proper prefix discrepancies have absolute value at most L, and every common multiplicity is at least $3L$.

Proof. No allowed pair can cross a missing adjacent edge. Each component must therefore leave at least the absolute value of its total discrepancy unmatched. Fix one component, renumbered $1, \dots, \ell$, and write $\Delta = \sum_{i=1}^{\ell} \delta_i$. If $\Delta \geq 0$, remove Δ source copies at x_ℓ; if $\Delta < 0$, remove $-\Delta$ target copies there. The residual capacities p_i', q_i' have equal totals. For $0 \leq i \leq \ell$ put

$$f_i = \sum_{j=1}^{i} (p_j' - q_j'), \qquad f_0 = f_\ell = 0.$$

The removal occurs at the last vertex, so $|f_i| \leq L$ for $i < \ell$. For $1 \leq i < \ell$ set

$$m_{i,i+1} = \max(f_i, 0), \qquad m_{i+1,i} = \max(-f_i, 0),$$

and set all other off-diagonal entries to zero. Let $m_{ii} = p_i' - \sum_{j \neq i} m_{ij}$. At each vertex the off-diagonal outflow is at most $2L$, whereas $p_i' \geq 3L - L = 2L$. Thus all entries are nonnegative integers. Since

outflow minus inflow is $f_i - f_{i-1} = p'_i - q'_i$, every column sum is q'_i. This matrix gives an allowed matching of all residual copies, with only diagonal or adjacent pairs. Exactly $|\Delta|$ copies were removed. Taking $L = S$ on every component gives (2.1). If $L = 0$, the diagonal matching suffices. □

Without a lower bound on the common multiplicities, (2.1) can fail. At the three points $0, 1, 2$, take one source at 0, one target at 2, and no other copies. For $\eta = 1$ the prescribed path is connected, but no pair is allowed. Its total discrepancy is zero, while its unmatched count is two.

2.2. **The periodic quadratic model.** Fix $\sigma > 0$, $\beta, c \in \mathbb{R}$, and distinct offsets $w_1 < \cdots < w_s$. For integers $r \geq r_0$ write

$$t_r = \sigma r + \beta, \qquad x_{r,i} = \sqrt{t_r^2 + w_i}, \qquad \lambda_{r,i} = t_r^2 + w_i - c. \tag{2.2}$$

Choose r_0 so that $t_r > 0$ and all these square roots are positive. Two infinite multisets of eigenvalues consist of finite multisets P, Q, together with the values $\lambda_{r,i}$ of multiplicities $p_{r,i}, q_{r,i}$. A fixed common shift c is understood to make the frequencies of the finite multisets real and nonnegative. All spectra are enumerated with multiplicity in nondecreasing order. The tail multiplicities satisfy

$$p_{r,i} - q_{r,i} = \delta_i(r \bmod q), \qquad \min(p_{r,i}, q_{r,i}) \longrightarrow \infty \quad (r \longrightarrow \infty) \tag{2.3}$$

for integers $\delta_i(a)$ and a fixed period $q \geq 1$. The convergence holds for every offset. In particular these multisets have finite multiplicity on bounded intervals and diverge to infinity.

Fix $0 < \theta < 1$ and take the whole-cluster cutoffs

$$R_K = \sigma K + \beta + \theta\sigma. \tag{2.4}$$

For all sufficiently large K, the frequencies at most R_K consist of the finite prefixes and exactly the blocks $r_0 \leq r \leq K$. Let $E_\eta(R_K)$ denote their unmatched count. The initial blocks may overlap one another or the finite prefixes.

Set

$$g_i = w_{i+1} - w_i \quad (1 \leq i < s), \qquad H = w_s - w_1, \qquad B_a = \sum_{i=1}^{s} \delta_i(a).$$

For $t \geq 0$, join adjacent offsets when $g_i \leq t$. If $\mathcal{C}(t)$ is the set of components, define

$$F_a(t) = \sum_{C \in \mathcal{C}(t)} \left| \sum_{i \in C} \delta_i(a) \right|. \tag{2.5}$$

This is a right-continuous nonincreasing step function. Its only possible jumps are at the distinct positive gaps g_i. Denote their set by G, and put

$$D_a(g) = F_a(g-) - F_a(g) \geq 0, \quad d_g = \sum_{a=0}^{q-1} D_a(g), \quad c_0 = \frac{1}{q} \sum_{a,i} |\delta_i(a)|, \quad c_+ = \frac{1}{q} \sum_a |B_a|.$$

Several adjacent edges may have the same gap; their simultaneous merger is included in the definition of $D_a(g)$. Thus

$$F_a(t) = |B_a| + \sum_{g \in G} D_a(g) \mathbf{1}_{\{t < g\}}. \tag{2.6}$$

Empty sums have value zero, in particular when $s = 1$.

Theorem 2.2 (Resolution of quadratic clusters). *Under* (2.2)–(2.3)*, define*

$$\begin{aligned} \mathcal{C}(0) &= c_0, \\ \mathcal{C}(\kappa) &= \frac{1}{q} \sum_a \int_0^1 F_a(2\kappa x)\, dx \\ &= c_+ + \frac{1}{q} \sum_{g \in G} d_g \min\left(1, \frac{g}{2\kappa}\right) \qquad (\kappa > 0). \end{aligned} \tag{2.7}$$

For every fixed $0 \le \eta_* < \sigma$*, uniformly for* $0 \le \eta \le \eta_*$ *and* $K \longrightarrow \infty$*,*

$$E_\eta(R_K) = \frac{R_K}{\sigma} C(\eta R_K) + O(1). \tag{2.8}$$

The error constant depends on the fixed model and η_**, but not on* η *or* K*.*

Proof. Choose a fixed tail start r_1 with the following properties. Its successive blocks are disjoint and have frequency gaps exceeding η_*. They cannot match any frequencies preceding the tail. Also $\min(p_{r,i}, q_{r,i}) \ge 3S$ for all $r \ge r_1$, where $S = \max_a \sum_i |\delta_i(a)|$. Such an r_1 exists by (2.3) and because the gap between successive blocks tends to σ. Deleting the finitely many preceding copies changes the unmatched count by a bounded quantity uniformly in η, K: deleting one vertex changes a maximum matching size by zero or one, and hence changes its unmatched count by at most one. Matching therefore takes place separately in each retained block, and Lemma 2.1 applies there.

The actual adjacent frequency gap is

$$x_{r,i+1} - x_{r,i} = \frac{g_i}{\sqrt{t_r^2 + w_i} + \sqrt{t_r^2 + w_{i+1}}}. \tag{2.9}$$

For large t there is a fixed constant A such that

$$\left| \tfrac{1}{2}\left(\sqrt{t^2 + w_i} + \sqrt{t^2 + w_{i+1}} \right) - t \right| \le A/t$$

for every i. If $\eta > 0$ and the actual edge condition in (2.9) differs from $g_i \le 2\eta t_r$, then

$$\left| t_r - \frac{g_i}{2\eta} \right| \le A/t_r \le A/t_{r_1}.$$

Since the t_r are spaced by σ, there are uniformly boundedly many such r for each edge, independently of η and K. At $\eta = 0$ neither graph has edges. The component deficits are bounded by S, so

$$E_\eta(R_K) = \sum_{r=r_1}^{K} F_{r \bmod q}(2\eta t_r) + O(1) \tag{2.10}$$

uniformly in the stated range.

An interval of integers is a union of complete periods and at most two intervals of fewer than q terms. Its sum of a fixed periodic function therefore differs from the mean times its cardinality by a bounded amount. Applied to each term of (2.6), this gives, uniformly for $L > 0$,

$$\sum_{r=r_1}^{K} D_{r \bmod q}(g) \mathbf{1}_{\{t_r < L\}} = \frac{d_g}{q\sigma} \min(R_K, L) + O(1).$$

Indeed the lower endpoint and the difference between R_K and t_K are fixed; changing a strict endpoint to a weak one costs a bounded amount. Periodic summation also gives

$$\sum_{r=r_1}^{K} |B_{r \bmod q}| = R_K c_+/\sigma + O(1).$$

Substituting $L = g/(2\eta)$ in (2.10) proves

$$E_\eta(R_K) = \frac{R_K c_+}{\sigma} + \frac{1}{q\sigma} \sum_g d_g \min\left(R_K, \frac{g}{2\eta} \right) + O(1).$$

At $\eta = 0$ interpret the minimum as R_K. This is (2.8). Integrating (2.6) proves (2.7). □

In this section, $R = R_K$. For fixed $\kappa \ge 0$, the theorem gives the exact scale

$$E_{\kappa/R}(R) = \frac{R}{\sigma} C(\kappa) + O(1).$$

More generally, every sequence $0 \le \eta_R \le \eta_* < \sigma$ is covered by the uniform formula, including oscillating sequences. In particular:

- If $\eta_R R \to \kappa < \infty$, then $E_{\eta_R}(R)/R \to C(\kappa)/\sigma$. More precisely, the error about $RC(\kappa)/\sigma$ is $O(1) + \frac{R}{\sigma}\big(C(\eta_R R) - C(\kappa)\big)$.
- If $\eta_R R \to 0$, then $E_{\eta_R}(R) = Rc_0/\sigma + O(1)$.
- If $\eta_R R \to \infty$, then $E_{\eta_R}(R)/R \to c_+/\sigma$. Once $2\eta_R R \geq \max(G \cup \{0\})$, the more precise formula is

$$E_{\eta_R}(R) = \frac{Rc_+}{\sigma} + \frac{1}{2q\sigma\eta_R} \sum_g d_g g + O(1). \tag{2.11}$$

The second term is $o(R)$ in this regime. For a fixed positive $\eta < \sigma$, it is a constant.

If G is empty or every $d_g = 0$, then (2.8) gives $E_\eta(R) = Rc_+/\sigma + O(1)$.

The profile is continuous and nonincreasing on $[0, \infty)$, piecewise of the form $A + B/\kappa$, and tends from c_0 to c_+. Define the finite positive measure

$$\nu = \sum_{g \in G} (d_g/q)\, \delta_g.$$

Away from the finitely many thresholds $2\kappa \in G$,

$$\frac{d}{d\kappa}\big[\kappa C(\kappa)\big] = \frac{1}{q} \sum_a F_a(2\kappa). \tag{2.12}$$

Thus the profile determines the residue-averaged component deficit and the measure ν, but need not determine the individual $\delta_i(a)$: replacing every $\delta_i(a)$ by $-\delta_i(a)$ leaves every F_a unchanged.

2.3. **The boundary at the cluster spacing.** The distance between successive clusters tends to σ. For tolerances tending to σ from below, the first-order term in this distance determines a second asymptotic formula.

Theorem 2.3. *Assume the periodic quadratic model, $H = w_s - w_1 > 0$, and*

$$\sum_{a=0}^{q-1} B_a = 0. \tag{2.13}$$

For any fixed $0 < \eta_0 < \sigma$, uniformly for $\eta_0 \leq \eta < \sigma$,

$$E_\eta(R) = \frac{c_+}{\sigma} \max\left(0, R - \frac{H}{2(\sigma - \eta)}\right) + O(1). \tag{2.14}$$

Uniformly for $\eta \geq \sigma$, $E_\eta(R) = O(1)$. Together with Theorem 2.2, taking for example $\eta_* = \eta_0 = \sigma/2$, these statements cover arbitrary sequences of nonnegative tolerances. In particular, for fixed $\kappa > 0$,

$$E_{\sigma - \kappa/R}(R) = \frac{Rc_+}{\sigma} \max\left(0, 1 - \frac{H}{2\kappa}\right) + O(1). \tag{2.15}$$

Proof. Delete a fixed finite initial region. Every retained block has diameter less than η_0, the blocks are ordered, and the distance between nonconsecutive blocks exceeds σ. For $\eta < \sigma$, components of the allowed support graph are therefore consecutive runs of blocks; a run is separated precisely where the adjacent support gap

$$h_r = x_{r+1,1} - x_{r,s} = \sigma - \frac{H}{2t_r} + O(t_r^{-2})$$

exceeds η. Within a block all adjacent edges are allowed. Condition (2.13) implies that every sum of consecutive B_r is uniformly bounded. Since the within-block discrepancies also belong to a finite set, every discrepancy sum over an arbitrary interval of the full ordered support is bounded by a fixed L. In particular all prefix sums and the total of any component have absolute value at most L. After deleting a further fixed initial region the common multiplicities are at least $3L$. Lemma 2.1 then applies to components of arbitrary length, with a bound independent of that length.

Put $\varepsilon = \sigma - \eta > 0$ and $A_\varepsilon = H/(2\varepsilon)$. The ideal interblock graph joins successive blocks when $t_r \le A_\varepsilon$. Its edge decisions differ from those of the actual graph at only boundedly many indices, uniformly in ε. A difference implies

$$|\varepsilon t_r - H/2| \le A/t_r$$

for a fixed A. Increase the tail start so that $A/t_r \le H/4$. It follows that $\varepsilon t_r \ge H/4$, and hence

$$|t_r - A_\varepsilon| \le A/(\varepsilon t_r) \le 4A/H.$$

The bound follows from the spacing σ of the t_r. Changing one such edge changes the component-deficit sum by a bounded amount: each of the two component discrepancies has absolute value at most L. Thus the true and ideal deficits differ by $O(1)$ uniformly.

The ideal graph consists of one initial connected run, possibly empty, followed by single blocks. Its initial run has bounded discrepancy. The sum of $|B_r|$ on the remaining blocks, by periodic summation, is

$$\frac{c_+}{\sigma}\max(0, R - A_\varepsilon) + O(1).$$

This proves (2.14), including cases in which the threshold lies outside the retained range. Since $H > 0$, $h_r < \sigma$ eventually. At $\eta = \sigma$ all retained support is connected and Lemma 2.1 gives $O(1)$ unmatched copies. Monotonicity of the unmatched count proves the uniform statement for all $\eta \ge \sigma$. □

For a single positive offset w, the adjacent-block gap equals $\sigma - \sigma w/(2t_r^2) + O(t_r^{-3})$; its boundary layer has scale R^{-2}, not R^{-1}. For a single zero offset the gap is exactly σ. For a negative single offset the gap is greater than σ. For example, take $\sigma = 1$, $\beta = 0$, $w_1 = -1$, $r \ge 2$, $p_{r,1} = r + \max\{(-1)^r, 0\}$, and $q_{r,1} = r + \max\{-(-1)^r, 0\}$. The block discrepancies have zero mean, but $E_1(R_K) = K + O(1)$, since no two distinct support points can be paired. If every multiplicity difference is zero, matching equal copies in each block gives $E_\eta(R) \le |P| + |Q|$ for every $\eta \ge 0$.

2.4. **Complete ordered spectra and the initial prefix.**

Theorem 2.4. *In the periodic quadratic model, let $d_{\mathrm{pre}} = |P| - |Q|$. The following are equivalent:*

(1) $\sup_j |\lambda_j - \lambda'_j| < \infty$ *for the complete ordered spectra, without an index shift;*

(2) $B_a = 0$ *for every residue a and $d_{\mathrm{pre}} = 0$;*

(3) $\sqrt{\lambda_j + c} - \sqrt{\lambda'_j + c} \longrightarrow 0$.

The equivalence does not require common multiplicities tending to infinity, provided the displayed ordered sequences are infinite. If condition 2 holds, a bound in condition 1 is

$$\max\left\{H, \max_{1\le j\le |P|} |P_j - Q_j|\right\},$$

where P_j, Q_j are the sorted prefix eigenvalues and the second maximum is zero for empty prefixes.

Proof. The eigenvalue gaps between consecutive tail blocks tend to infinity. If condition 1 holds with bound C, choose a point in each sufficiently late gap at distance greater than C from the spectra on both sides. The two counting functions at that point must agree: otherwise the first index at which one list has crossed the point and the other has not would have displacement greater than C. Subtract the equalities at consecutive gaps. The total discrepancy B_r of each sufficiently late block is zero. Periodicity then gives $B_a = 0$ for every residue, and the equality of cumulative counts forces $d_{\mathrm{pre}} = 0$.

Conversely, condition 2 allows an arbitrary sorted matching of the prefixes and an arbitrary matching inside each equal-cardinality block. The latter displacements are at most H. These matchings give a bijection with the stated uniform bound. A bounded-displacement bijection of two ordered discrete multisets implies the same bound for their ordered matching: if $\lambda_j > \lambda'_j + C$, the first j target copies require j source copies strictly below λ_j, where there are at most $j - 1$ copies. Reversing the two lists proves the other inequality. This argument also permits coincidences and overlaps among

the initial blocks. Condition 1 implies condition 3 by dividing the eigenvalue difference by the sum of the two square roots, which tends to infinity.

Finally, consecutive frequency blocks have separating gaps tending to $\sigma > 0$. If condition 3 holds, the cumulative counts at every sufficiently late gap agree: unequal counts would force an ordered pair across a gap of width at least $\sigma/2$, at indices tending to infinity. Subtracting the equalities at consecutive gaps gives $B_r = 0$ eventually. Periodicity gives $B_a = 0$ for all a, after which either cumulative equality gives $d_{\mathrm{pre}} = 0$. □

The equality $c_+ = 0$ is equivalent to $B_a = 0$ for every a and imposes no condition on d_{pre}. For example, put $w_1 = 0$, $p_{r,1} = q_{r,1} = r^4$, $r \geq 1$, and insert one additional low source eigenvalue. Every positive-tolerance deficit is bounded, but the complete ordered spectra cross infinitely many widening gaps and have unbounded eigenvalue displacement.

Theorem 2.5. *Assume common multiplicity growth, $B_a = 0$ for every a, and $d_{\mathrm{pre}} = 0$. Define*

$$s_i(a) = \sum_{j=1}^{i} \delta_j(a) \quad (1 \leq i < s), \qquad D_* = \max\{g_i \,:\, s_i(a) \neq 0 \text{ for some } a\},$$

with an empty maximum equal to zero. Then

$$\limsup_{j\to\infty} |\lambda_j - \lambda'_j| = D_*. \tag{2.16}$$

Indeed all sufficiently late ordered pairs have displacement at most D_*. Within every sufficiently late r-block there are exactly $|s_i(r \bmod q)|$ ordered pairs of displacement g_i for each boundary i; all other displacements are zero. If different boundaries have identical g_i, their counts are added. The supremum over all indices may also depend on the finite initial spectra.

Proof. After a fixed initial region, the blocks are separated, and the equal prefix and block cardinalities align their complete ordered indices. Fix one block and write $P_i = \sum_{j\leq i} p_{r,j}$ and $Q_i = \sum_{j\leq i} q_{r,j}$, so $P_i - Q_i = s_i$. The integer interval

$$I_i = (\min(P_i, Q_i), \max(P_i, Q_i)] \cap \mathbb{Z}$$

contains exactly $|s_i|$ ranks. Put $M = \max_{a,\, 1\leq i<s} |s_i(a)|$, with $M = 0$ if $s = 1$. Once every common multiplicity exceeds M,

$$\min(P_{i+1}, Q_{i+1}) \geq \min(P_i, Q_i) + \min(p_{r,i+1}, q_{r,i+1}) > \max(P_i, Q_i).$$

The intervals I_i are therefore disjoint and separated. At ranks in I_i, one list has just crossed boundary i and the other has not; the two values are exactly $\lambda_{r,i}$ and $\lambda_{r,i+1}$. At all other ranks both lists occupy the same offset. This proves the histogram. Every residue occurs infinitely often, so every active gap occurs infinitely often, proving (2.16). □

Proposition 2.6. *Assume common multiplicity growth and* (2.13)*. Then*

$$\limsup_{j\to\infty} \left| \sqrt{\lambda_j + c} - \sqrt{\lambda'_j + c} \right| = \begin{cases} 0, & B_a = 0 \text{ for every } a \text{ and } d_{\mathrm{pre}} = 0, \\ \sigma, & \text{otherwise.} \end{cases}$$

Proof. After a fixed initial region the blocks are disjoint. Let P_r, Q_r be the complete cumulative counts through block r. Then

$$P_r - Q_r = d_{\mathrm{pre}} + \sum_{h=r_0}^{r} B_{h \bmod q}$$

is bounded and periodic in r. The two cardinalities of block $r + 1$ tend to infinity; hence $\min(P_{r+1}, Q_{r+1}) > \max(P_r, Q_r)$ eventually. The integer intervals between P_r and Q_r are disjoint. Their ranks pair successive blocks, and all remaining ranks pair values in the same block. Since

$$\max_i x_{r,i} - \min_i x_{r,i} = O(r^{-1}), \qquad x_{r+1,1} - x_{r,s} = \sigma + O(r^{-1}),$$

the limsup is at most σ. If every cumulative discrepancy vanishes, subtracting consecutive equalities gives $B_a = 0$ for every a and then $d_{\mathrm{pre}} = 0$; Theorem 2.4 gives the limit zero. Otherwise a nonzero cumulative discrepancy recurs periodically. Its rank interval is nonempty at infinitely many boundaries, and the corresponding frequency differences tend to σ. □

2.5. **Bounded and compact spectral corrections.** For self-adjoint matrices, the distance between unitary orbits is the maximum ordered eigenvalue displacement, and commuting representatives attain this distance [2, Theorems 1.1–1.2].

Proposition 2.7. *Let A be self-adjoint, bounded below, and have compact resolvent on an infinite-dimensional separable complex Hilbert space. Write its ordered eigenvalues as a_j, and let $b_j \to \infty$ be another ordered sequence, bounded below and with finite multiplicities. There exists a bounded self-adjoint C with $\operatorname{Spec}(A + C) = \{b_j\}$ if and only if $D = \sup_j |a_j - b_j| < \infty$. In that case*

$$\inf_C \|C\| = D,$$
$$\inf_C \|C\|_{\mathrm{ess}} = \limsup_{j\to\infty} |a_j - b_j|.$$

Both infima are attained by the same operator diagonal in an eigenbasis of A. A compact self-adjoint correction exists if and only if $a_j - b_j \to 0$.

Proof. For a bounded self-adjoint D_0 and $|\operatorname{Im} z| > \|D_0\|$,

$$(A + D_0 - z)^{-1} = (A - z)^{-1}\big(I + D_0(A - z)^{-1}\big)^{-1}.$$

The inverse on the right exists by its norm-convergent geometric series. Put $T = A + D_0$ on $\mathcal{D}(A)$. It is symmetric, and both $T \pm it$ are onto for $t > \|D_0\|$. If $x \in \mathcal{D}(T^*)$, choose $y \in \mathcal{D}(T)$ with $(T - it)y = (T^* - it)x$. Then $x - y \in \ker(T^* - it) = \operatorname{Ran}(T + it)^\perp = 0$. Thus $T = T^*$; it is bounded below by $a_1 - \|D_0\|$, and the displayed resolvent is compact. The variational formula [18, Theorem 4.14 and Corollary 4.13]

$$\lambda_j(A + D_0) = \inf_{\substack{V\subset\mathcal{D}(A)\\ \dim V = j}} \sup_{\substack{x\in V\\ \|x\|=1}} \langle (A + D_0)x, x\rangle$$

follows by taking V to be the span of the first j eigenvectors of $A + D_0$ for one inequality; every j-dimensional subspace meets the orthogonal complement of the first $j - 1$ eigenvectors for the other. Comparing its Rayleigh quotients gives $|\lambda_j(A + D_0) - a_j| \le \|D_0\|$. In particular $D \le \|C\|$ for every correction C.

Conversely, in an orthonormal eigenbasis e_j of A, set $Ce_j = (b_j - a_j)e_j$. If $D < \infty$, this defines a bounded self-adjoint operator of norm D, and $(A + C)e_j = b_j e_j$. For P_N the projection onto $\operatorname{span}\{e_1, \dots, e_N\}$,

$$\|C - P_N C\| = \sup_{j>N} |b_j - a_j|.$$

Every compact K satisfies $Ke_j \to 0$: every norm-convergent subsequence has limit zero, since $\langle Ke_j, x\rangle = \langle e_j, K^*x\rangle \to 0$. Compactness then excludes a subsequence bounded away from zero. Thus

$$\|C - K\| \ge \limsup_j \|(C - K)e_j\| = \limsup_j |b_j - a_j|.$$

Taking the infimum over K and then letting $N \to \infty$ proves the essential-norm formula for this diagonal correction.

For every compact self-adjoint K,

$$\lambda_j(A + K) - a_j \longrightarrow 0. \tag{2.17}$$

Indeed, $P_N \to I$ strongly and this convergence is uniform on the compact closure of the image under K of the unit ball: a finite ε-net, the bound $\|I - P_N\| \le 1$, and the triangle inequality give uniform convergence on that compact set. Since $K = K^*$,

$$\|K - P_N K P_N\| \le 2\|(I - P_N)K\| \longrightarrow 0.$$

The operator $A + P_N K P_N$ changes only the finite P_N block. Beyond a finite spectral level its ordered eigenvalues are exactly those of A, since the changed finite block still has exactly N eigenvalues. Min–max and approximation prove (2.17). For any such K, min–max applied to $A + C$ and $A + K$ now gives

$$\limsup_j |b_j - a_j| \le \|C - K\|.$$

Taking the infimum over compact self-adjoint K gives the essential norm; restricting to self-adjoint K does not change that infimum, since symmetrization cannot increase the distance from self-adjoint C. Finally, if $a_j - b_j \to 0$, the diagonal correction is compact. Conversely, every compact correction satisfies (2.17). □

In the balanced quadratic model, Theorem 2.5 therefore computes the optimal essential norm as D_*. The infima range over all bounded self-adjoint operators on the given Hilbert space.

2.6. **A common-multiplicity estimate on odd spherical quotients.**

Lemma 2.8. *Let $\Gamma \subset \mathrm{SO}(6)$ be a finite group of odd order q acting freely on S^5. Fix integers $k \ge |c|$, and let $\pi_{r,k,c}$ be the irreducible representation of highest weight (r, k, c), $r \ge k$. Then, for every nonidentity $\gamma \in \Gamma$,*

$$|\chi_{r,k,c}(\gamma)| = O_{\gamma,k,c}((r+1)^2), \tag{2.18}$$

and

$$\dim V^{\Gamma}_{\pi_{r,k,c}} = \frac{1}{q} \dim \pi_{r,k,c} + O_{\Gamma,k,c}((r+1)^2), \tag{2.19}$$

$$\dim \pi_{r,k,c} = \frac{\big((r+2)^2 - (k+1)^2\big)\big((r+2)^2 - c^2\big)\big((k+1)^2 - c^2\big)}{12}. \tag{2.20}$$

In particular

$$\dim V^{\Gamma}_{\pi_{r,k,c}} = \frac{(k+1)^2 - c^2}{12q}\, r^4 + O_{\Gamma,k,c}(r^3),$$

with a strictly positive leading coefficient. Each nonidentity character, as a function of r, is a quasipolynomial of period dividing q and degree at most two.

Proof. Write the eigenvalues of γ as $e^{\pm i\theta_j}$, $j = 1, 2, 3$. Freeness excludes eigenvalue 1 for $\gamma \ne 1$, and odd order excludes eigenvalue -1. Thus the cosine nodes $x_j = \cos\theta_j$ lie in $(-1, 1)$ and $\sin\theta_j \ne 0$. Put $\ell = (r+2, k+1, c)$ and $\rho = (2, 1, 0)$. The Weyl character formula [7, Theorem 25.4], whose type-D_3 alternating sum ranges over permutations and even sign changes, gives

$$\chi_{r,k,c}(\gamma) = \frac{\det\big(2\cos(\ell_i\theta_j)\big) + \det\big(2i\sin(\ell_i\theta_j)\big)}{\det\big(2\cos(\rho_i\theta_j)\big)}, \tag{2.21}$$

initially at distinct cosine nodes, with its continuous extension at coincident nodes. Changing the choice of orientations of the rotation planes changes the sine factors consistently; it does not affect the following bound.

In the cosine determinant write $\cos(m\theta) = T_m(x)$; in the sine determinant factor $\prod_j \sin\theta_j$ from its columns and substitute $\sin(m\theta)/\sin\theta = U_{m-1}(x)$. For $m = 0$ the latter expression is zero, and negative m is treated by its minus sign. The denominator is a fixed nonzero constant times the Vandermonde in x_1, x_2, x_3, since its rows are $4x^2 - 2, 2x, 2$. Each numerator after the indicated factorization is an alternating polynomial in these nodes. For a node of multiplicity m, replace its columns in both numerator and denominator by derivatives of orders $0, \dots, m-1$, divided by their factorials. Taylor expansion and cancellation give the quotient at coincident nodes. The resulting denominator is a nonzero constant times $\prod_{\alpha<\beta}(\xi_\beta - \xi_\alpha)^{m_\alpha m_\beta}$, where the ξ_α are the distinct nodes and m_α their multiplicities. There are three columns, so the derivative orders are at most two. Only the first row depends on r.

At any fixed $x = \cos\theta \in (-1, 1)$ and for $0 \le d \le 2$,

$$T_{r+2}^{(d)}(x) = O_x((r+1)^d), \qquad U_{r+1}^{(d)}(x) = O_x((r+1)^d).$$

Indeed $d/dx = -(\sin\theta)^{-1} d/d\theta$, applied to $\cos((r+2)\theta)$ or $\sin((r+2)\theta)/\sin\theta$, gives these estimates. All other rows and nodes are fixed. The confluent determinant formula therefore proves (2.18): distinct nodes give $O(1)$, a double node gives $O(r+1)$, and a triple node gives $O((r+1)^2)$. The factored product is

$$\prod_j \sin\theta_j = \left(\prod_j \varepsilon_j\right) \prod_j \sqrt{1 - x_j^2}, \qquad \varepsilon_j = \operatorname{sgn}(\sin\theta_j).$$

The signs are locally constant, and this product is nonzero at coincident cosine nodes.

Each differentiated term is a polynomial in r of degree at most two multiplied by $e^{\pm i(r+2)\theta_j}$. Since $e^{i\theta_j}$ has order dividing q, restriction to each residue class modulo q is polynomial. Averaging the character over Γ proves (2.19). Finally the Weyl dimension formula [7, Proposition 28.1] is the product over the positive type-D_3 roots $e_i \pm e_j$:

$$\prod_{1 \le i < j \le 3} \frac{\ell_i^2 - \ell_j^2}{\rho_i^2 - \rho_j^2}.$$

The denominator is $(4-1)(4-0)(1-0) = 12$, which gives (2.20) and the stated positive leading coefficient. □

For fixed q, k, c, the constants can be chosen uniformly over all such subgroups of order q: the possible eigenangles belong to the finite set of qth roots of unity, and the preceding confluent formulas give a finite list of constants. These constants may depend on q, k, c.

For any fixed natural SO(5) type, each offset is contributed by a nonempty finite collection of these fixed (k, c) strings. Summing their positive leading coefficients proves the common-multiplicity hypothesis (2.3) for every offset on any two fixed odd-order free spherical quotients. In particular, an offset with zero signed discrepancy must still be retained: it provides the common copies required by the finite-flow matching.

2.7. **Quasipolynomial discrepancies.** Retain the quadratic support, finite prefixes, and cutoffs of (2.2) and (2.4), and suppose both complete ordered lists are infinite. Replace (2.3) by

$$p_{r,i} - q_{r,i} = P_{a,i}(r), \qquad a = r \bmod q, \tag{2.22}$$

where $D \in \mathbb{Z}_{\ge 1}$ and each $P_{a,i}$ is a real polynomial of degree at most D with integer values on its residue class. Assume, for all sufficiently large r,

$$\min(p_{r,i}, q_{r,i}) \ge 3 \sum_j |P_{r \bmod q, j}(r)| \quad \text{for every } i. \tag{2.23}$$

Write $b_i(a)$ for the coefficient of r^D in $P_{a,i}$, and define $F_a^{(D)}(t)$ by (2.5) with $\delta_i(a)$ replaced by $b_i(a)$. Define

$$C_D(\kappa) = \frac{1}{q} \sum_a \int_0^1 x^D F_a^{(D)}(2\kappa x)\, dx.$$

Theorem 2.9. *Under* (2.22)–(2.23)*, for any fixed* $0 \le \eta_* < \sigma$*,*

$$E_\eta(R) = \frac{R^{D+1}}{\sigma^{D+1}} C_D(\eta R) + O(R^D) \tag{2.24}$$

uniformly for $0 \le \eta \le \eta_*$*. The complete ordered eigenvalue displacement is bounded if and only if* $d_{\mathrm{pre}} = 0$ *and each polynomial* $\sum_i P_{a,i}$ *is identically zero. In this balanced case its sharp eventual bound is*

$$\max\left\{ g_i \,:\, \sum_{j \le i} P_{a,j} \text{ is not the zero polynomial for some } a \right\}.$$

An empty maximum equals zero.

Proof. Choose a fixed $r_1 \geq \max\{r_0, 1\}$ beyond which the blocks are separated by more than η_* and (2.23) holds. Deleting preceding copies costs $O(1)$. Lemma 2.1 applies separately in each retained block. There are boundedly many indices where the true and ideal adjacent edge conditions differ; each such block has deficit $O(R^D)$, so replacing the graph costs $O(R^D)$. For every one of the finitely many possible components C,

$$\left\|\sum_{i\in C} P_{a,i}(r)\right| - r^D \left|\sum_{i\in C} b_i(a)\right\| = O(r^{D-1}).$$

Summing this error to K gives $O(R^D)$. Weighted periodic summation on each threshold interval gives

$$\sum_{\substack{r_1\leq r\leq K\\ r\equiv a\ (q)}} r^D \mathbf{1}_{\{t_r<L\}} = \frac{1}{q\sigma^{D+1}(D+1)} \min(R, L)^{D+1} + O(R^D)$$

uniformly in $L > 0$. Indeed, comparison of the increasing function x^D with its integral on intervals of length q gives an endpoint error $O(K^D)$; the fixed lower endpoint and $t_r = \sigma r + \beta$ change this by $O(R^D)$. Expanding the finite step function $F_a^{(D)}$ proves (2.24).

For the ordered criterion, equality of all sufficiently late cumulative counts again forces the block-total polynomial on each residue to vanish at infinitely many integers, hence identically. It then forces the prefix cardinalities to agree. Conversely, matching within the equal-cardinality blocks and between the equal-cardinality prefixes gives the uniform displacement bound. In the balanced case put $S_r = \sum_i |P_{r \bmod q,i}(r)|$. Every prefix discrepancy has absolute value at most S_r, while every common multiplicity is at least $3S_r$. If $S_r > 0$, the rank intervals in the proof of Theorem 2.5 are therefore disjoint and pair only adjacent offsets. If $S_r = 0$, the block lists agree. A nonzero prefix polynomial has only finitely many zeros on its residue class; hence every active gap recurs infinitely often, and no other nonzero displacement occurs eventually. □

When all leading component coefficients vanish, the leading profile in (2.24) is zero; this says $E_\eta(R) = O(R^D)$, not that the deficit is bounded. Lower-degree terms must then be examined. The bounded periodic theorem is the degree-zero case with its stronger $O(1)$ remainder.

Corollary 2.10. *Fix two free odd-order subgroups of* SO(6) *of the same order q and a fixed natural* SO(5) *type. Their multiplicity difference at each quadratic offset is an exact quasipolynomial in r of period dividing q and degree at most two. The hypotheses of Theorem 2.9 hold, after a fixed initial region, with $D = 2$, or with $D = 1$ when the discrepancies have degree at most one. If their nonidentity singular character contributions cancel, their multiplicity differences are periodic and Theorem 2.2 applies.*

Proof. The identical identity terms $q^{-1} \dim \pi_{r,k,c}$ cancel between the two character averages. Lemma 2.8 makes the remaining differences quasipolynomials of degree at most two. Their coefficients on each residue class are real, since the imaginary part is a polynomial vanishing at infinitely many integers. There are finitely many strings at each offset. Their common multiplicities grow with positive coefficient as r^4, whereas the total absolute discrepancy is $O(r^2)$; hence (2.23) holds eventually. At regular nonidentity elements the character has degree zero. If all singular contributions cancel, only these periodic terms remain. □

3. Bounded and compact perturbations of natural spectra

Retain the lens spaces and the Casimir normalization of Section 1. Eigenvalues are listed in nondecreasing order, with multiplicity.

For $X, Y \in \mathbb{Z}/49\mathbb{Z}$ define $\mathscr{Z}$ by the following alternatives:

(1) $X = 0$, $Y = 0$, $X = Y$ or $X = -Y$;

(2) $7 \mid X$ and $7 \mid Y$;

(3) one coordinate is divisible by 7, and the other belongs to $\{\pm 1, \pm 3\}$;

(4) both coordinates belong to $\{\pm 1, \pm 3, \pm 5\}$.

Divisibility in this definition is well defined on residues.

Theorem 3.1. *For every integral $u \geq v \geq 0$, the following conditions are equivalent.*

(1) $(2u+3, 2v+1) \bmod 49 \in \mathscr{Z}$.

(2) $\sup_j |\lambda_j(\Delta_{u,v,L}) - \lambda_j(\Delta_{u,v,L'})| < \infty$.

(3) $|\sqrt{\lambda_j(\Delta_{u,v,L}) + 4} - \sqrt{\lambda_j(\Delta_{u,v,L'}) + 4}| \longrightarrow 0$.

(4) There is a bounded self-adjoint operator C on the bundle space over L such that $\mathrm{Spec}(\Delta_{u,v,L} + C) = \mathrm{Spec}(\Delta_{u,v,L'})$.

(5) Such a C can be chosen to commute with every spectral projection of $\Delta_{u,v,L}$.

If these conditions hold, the difference in (2) is at most

$$H_{u,v} = u(u+2) - 2v$$

at every index, and $\|C\| \leq H_{u,v}$ is possible. The same locus results if bounded self-adjoint corrections are permitted on both sides.

The locus $\mathscr{Z}$ contains exactly 289 *of the* 2401 *residue pairs. The two uncorrected spectra are equal precisely for*

$$(u, v) = (0,0), (1,0), (2,0), (1,1), (2,1).$$

Every other dominant type in this residue locus has unequal exact spectra and admits a bounded commuting correction.

For every bounded self-adjoint C on the Hilbert space of sections over L, the sum $\Delta_{u,v,L} + C$ has domain $\mathcal{D}(\Delta_{u,v,L})$.

Lemma 3.2. *The function T is alternating in its three variables and its sum over any complete period in any variable is zero.*

Proof. The alternating determinant in (1.6) also shows that T is alternating in its three variables. For a unit j modulo 49,

$$\sum_{b=0}^{48} (\zeta^{jb} + \zeta^{-jb}) = 0.$$

Summing the relevant row of the determinant proves that the sum of T over a complete period in any variable is zero. □

3.1. **The alternating rectangle.** For fixed (u, v) put

$$W = \{k(k+2) + c^2 \,:\, v \leq k \leq u,\ 0 \leq c \leq v\},$$

$$A(a, w) = \sum_{\substack{v \leq k \leq u,\ 0 \leq c \leq v \\ k(k+2)+c^2 = w}} (2 - \mathbf{1}_{c=0}) T(a, k+1, c), \qquad B(a) = \sum_{w \in W} A(a, w).$$

All representation collisions at an offset are aggregated in A. The complete contribution with first coordinate r lies in

$$I_r = [r(r+4) + v(v+2),\ r(r+4) + u(u+2) + v^2],$$

an interval of width $H_{u,v}$. Its total multiplicity difference is $B(r+2)$ for every $r \geq u$, not only for large r.

Lemma 3.3. *The total $B_{u,v}(a)$ is periodic separately in u and v, of period* 49. *It has the signed-rectangle expression*

$$2B_{u,v}(a) = \sum_{b=-u-1}^{u+1} \sum_{c=-v}^{v} T(a, b, c). \tag{3.1}$$

In the coordinates $X = 2u+3$, $Y = 2v+1$, its zero set is invariant under independent sign changes and interchange of X, Y. It vanishes on the four walls $X = 0$, $Y = 0$, $X = Y$, $X = -Y$ modulo 49.

Proof. Evenness in c changes the definition of B into $\sum_{b=v+1}^{u+1}\sum_{c=-v}^{v} T(a,b,c)$. The square with $-v \le b, c \le v$ has sum zero by alternation in b, c; evenness in b proves (3.1). Its right-hand side provides an extension off the dominant cone, so its two coordinates can be reduced to residues independently. Write $J_N = \sum_{j=-N}^{N} e_{j \bmod 49}$ in the group algebra and let $\mathbf{e} = \sum_{j=0}^{48} e_j$. Then $J_{N+49} = J_N + 2\mathbf{e}$ and, for $0 \le N \le 48$, $J_{48-N} = 2\mathbf{e} - J_N$. Pairing either argument with $\mathbf{e}$ gives zero by Lemma 3.2. This proves periodicity and sign change under $2N+1 \mapsto -(2N+1)$. Interchanging the arguments negates the pairing. Finally $J_{24} = \mathbf{e}$, while coincident arguments have zero alternating pairing. These give the four walls. □

Lemma 3.4. *Outside the four walls, the zero vectors $(B_{u,v}(a))_{a \bmod 49}$ are exactly the signed-permutation orbits of the following unordered pairs of least positive absolute representatives of X, Y:*

$$\{1,3\},\{1,5\},\{3,5\},\{1,7\},\{1,14\},\{1,21\},\{3,7\},\{3,14\},\{3,21\},\{7,14\},\{7,21\},\{14,21\}.$$

Proof. The sign and permutation symmetries reduce the complement of the walls to $1 \le Y < X \le 24$. For a residue Z, put

$$N_Z = 25(Z-1) \pmod{49}, \qquad 0 \le N_Z \le 48,$$
$$j_Z(b) = \#\{j : -N_Z \le j \le N_Z,\ j \equiv b \pmod{49}\}.$$

For $X = 2u+3$ and $Y = 2v+1$, formula (3.1) gives

$$B_{u,v}(a) = \frac{1}{2} \sum_{b,c \bmod 49} j_X(b) j_Y(c) T(a,b,c).$$

Every term is specified by Table 1 and (1.10). Define

$$Q(X,Y) = \frac{1}{4}\sum_{a=0}^{24} B_{u,v}(a)^2.$$

Evenness in a gives $B = 0$ if and only if $Q(X,Y) = 0$. Expansion of the displayed finite sums gives Table 3. Each row lists $Q(X,1),\dots,Q(X,X-1)$ in that order. Its zero positions are exactly the twelve pairs in the statement.

□

The four walls have $1 + 4 \cdot 48 = 193$ points: every pairwise intersection is the origin because 2 is a unit modulo 49. Each nonwall orbit has eight points. Lemma 3.4 therefore gives $193 + 12 \cdot 8 = 289$. It also gives exactly the alternatives defining $\mathscr{Z}$.

3.2. **Complete ordered eigenvalue lists.**

Lemma 3.5. *For this pair and any fixed (u,v), $B(a) = 0$ for every residue if and only if the complete ordered eigenvalue differences are bounded, and if and only if the complete ordered shifted-square-root differences tend to zero. If $B = 0$, the bound $H_{u,v}$ holds at every index.*

Proof. Apply Theorem 2.4 with $\sigma = 1$, $t_r = r+2$, $c = 4$, $r_0 = u$, and offset set W. The two finite prefixes are empty. The multiplicity discrepancies are $A(r+2, w)$ for every $r \ge u$, by (1.3)–(1.4); their block sums are $B(r+2)$. Lemma 1.3 gives periodicity. Thus the three conditions in that theorem give the stated equivalences and the bound $H_{u,v}$ at every index. The allowance for overlapping initial blocks in Theorem 2.4 retains all initial representations and all Casimir collisions. □

Proof of Theorem 3.1. The rectangle lemmas identify condition 1 with $B = 0$, and Lemma 3.5 identifies this with conditions 2 and 3. Proposition 2.7 identifies condition 2 with condition 4 and gives a diagonal correction of norm at most $H_{u,v}$. This correction commutes with every spectral projection, giving condition 5; condition 5 implies condition 4. If $\Delta_{u,v,L} e_j = \lambda_j e_j$ and $C e_j = c_j e_j$, then

$$\sum_j \lambda_j^2 |c_j x_j|^2 \le \|C\|^2 \sum_j \lambda_j^2 |x_j|^2;$$

TABLE 3. Values of $Q(X, Y)$ for $1 \le Y < X \le 24$.

X	$Q(X,1), \dots, Q(X,X-1)$
2	12
3	0, 12
4	12, 24, 48
5	0, 12, 0, 36
6	24, 83, 72, 59, 72
7	0, 48, 0, 84, 6, 84
8	24, 89, 72, 71, 72, 12, 84
9	6, 12, 6, 36, 6, 81, 6, 79
10	12, 30, 48, 24, 36, 71, 84, 59, 34
11	6, 24, 24, 48, 18, 75, 28, 71, 12, 40
12	12, 6, 12, 30, 12, 89, 34, 83, 10, 24, 22
13	12, 51, 36, 45, 36, 24, 28, 20, 39, 41, 27, 49
14	0, 24, 0, 28, 10, 28, 0, 28, 10, 28, 28, 10, 28
15	12, 49, 36, 41, 36, 20, 28, 24, 49, 45, 47, 51, 20, 28
16	10, 6, 10, 30, 10, 88, 34, 92, 12, 34, 24, 10, 51, 10, 45
17	6, 22, 24, 40, 18, 71, 28, 75, 22, 48, 40, 24, 47, 28, 27, 18
18	10, 26, 40, 24, 30, 72, 56, 80, 32, 40, 48, 30, 47, 28, 35, 20, 24
19	6, 10, 6, 34, 6, 79, 20, 81, 10, 36, 22, 12, 49, 10, 39, 6, 12, 26
20	20, 78, 60, 62, 60, 12, 56, 20, 71, 70, 67, 82, 24, 28, 12, 64, 55, 44, 65
21	0, 40, 0, 56, 12, 56, 0, 56, 12, 56, 28, 26, 28, 0, 28, 26, 28, 56, 12, 56
22	20, 82, 60, 70, 60, 20, 56, 12, 65, 62, 55, 78, 12, 28, 24, 76, 67, 68, 71, 24, 56
23	10, 10, 10, 34, 10, 92, 20, 88, 6, 30, 18, 6, 45, 10, 51, 12, 24, 32, 12, 76, 12, 64
24	10, 30, 40, 40, 30, 80, 56, 72, 26, 24, 24, 26, 35, 28, 47, 32, 48, 48, 32, 68, 56, 44, 20

hence $C\mathcal{D}(\Delta_{u,v,L}) \subset \mathcal{D}(\Delta_{u,v,L})$. If corrections C, C' on both sides give a common ordered spectrum μ_j, the variational estimate gives

$$|\lambda_j(L) - \lambda_j(L')| \le |\lambda_j(L) - \mu_j| + |\mu_j - \lambda_j(L')| \le \|C\| + \|C'\|.$$

Thus the residue locus is unchanged. Equality of the uncorrected spectra is characterized by Theorem 1.1. □

3.3. **The eventual displacement.** The positive multiplicity estimate is Lemma 2.8. In this order-49 case, for every fixed k, c it gives

$$\dim V^{\Gamma}_{r,k,c} = \frac{(k+1)^2 - c^2}{588} r^4 + O_{k,c}(r^3),$$

and the same formula for Γ'. Thus the common multiplicity at every offset tends to infinity, including offsets with zero signed discrepancy.

Proposition 3.6. *Suppose $B = 0$ and write $W = \{w_1 < \cdots < w_s\}$. Put*

$$D_* = \max\{w_{i+1} - w_i \,:\, \sum_{j \le i} A(a, w_j) \neq 0 \text{ for some } a \bmod 49\}, \tag{3.2}$$

with empty maximum zero. Then $\limsup_j |\lambda_j(L) - \lambda_j(L')| = D_*$, and all sufficiently late differences are at most D_*.

Proof. Apply Theorem 2.5 with $t_r = r + 2$, $c = 4$ and discrepancies $A(r+2, w_i)$. The total multiplicities agree for every $r \ge u$; hence the initial blocks have equal cardinalities. The exact histogram in that theorem also proves recurrence of every active adjacent gap, giving the limsup equality. □

3.4. **Resolution at every tolerance.** Let W contain every offset, including those with zero discrepancy. For $a \in \mathbb{Z}/49\mathbb{Z}$ and $t \geq 0$, join consecutive offsets whose gap is at most t, and define

$$F_a(t) = \sum_C \left| \sum_{w \in C} A(a, w) \right|, \qquad d_g = \sum_{a=0}^{48} \big(F_a(g-) - F_a(g) \big).$$

Here C ranges over the resulting interval components; all edges of a given gap g are joined simultaneously. Put

$$c_0 = \frac{1}{49} \sum_{a,w} |A(a, w)|, \qquad c_+ = \frac{1}{49} \sum_a |B(a)|, \qquad R_K = K + \frac{5}{2}.$$

Theorem 3.7. *Fix $u \geq v \geq 0$. For frequencies $\sqrt{\lambda + 4}$ of the two natural $\tau_{u,v}$ spectra and their specified cutoffs, define*

$$C(0) = c_0, \qquad C(\kappa) = c_+ + \frac{1}{49} \sum_g d_g \min\{1, g/(2\kappa)\} \quad (\kappa > 0).$$

The following estimates have error constants independent of both the tolerance and K within each displayed range:

$$E_\eta(R_K) = R_K C(\eta R_K) + O_{u,v}(1) \qquad (0 \leq \eta \leq \tfrac{1}{2}), \tag{3.3}$$

$$E_\eta(R_K) = c_+ \max\{0, R_K - H_{u,v}/[2(1-\eta)]\} + O_{u,v}(1) \qquad (\tfrac{1}{2} \leq \eta < 1), \tag{3.4}$$

$$E_\eta(R_K) = O_{u,v}(1) \qquad (\eta \geq 1). \tag{3.5}$$

For the scalar type $(0, 0)$ the spectra agree and every deficit is zero; the second display is interpreted directly in that case. Consequently, for every fixed $\kappa \geq 0$,

$$E_{\kappa/R_K}(R_K) = R_K C(\kappa) + O_{u,v}(1),$$

and for every fixed $\kappa > 0$,

$$E_{1-\kappa/R_K}(R_K) = R_K c_+ \max\{0, 1 - H_{u,v}/(2\kappa)\} + O_{u,v}(1).$$

In addition,

$$\limsup_{j \to \infty} \left| \sqrt{\lambda_j(L) + 4} - \sqrt{\lambda_j(L') + 4} \right| = \begin{cases} 0, & (2u+3, 2v+1) \bmod 49 \in \mathscr{Z}, \\ 1, & \text{otherwise.} \end{cases}$$

Proof. The quadratic model has $\sigma = 1$, $t_r = r + 2$, $c = 4$, and $\delta_i(r) = A(r + 2, w_i)$. Lemma 1.3 gives the exact periodic discrepancies, and Lemma 2.8 gives common multiplicities tending to infinity at every offset. Summing the first-coordinate period of T gives

$$\sum_{a=0}^{48} B(a) = 0.$$

The width $H_{u,v} = u(u+2) - 2v$ is positive for every nontrivial dominant type. Indeed $u \geq 1$ implies $H_{u,v} \geq u^2 > 0$. There is no separate finite prefix: both complete decompositions start at $r = u$. Therefore Theorems 2.2 and 2.3 give the three uniform estimates. The fixed-scale formulas follow by substitution. Proposition 2.6 and Theorem 3.1 give the final dichotomy. □

Corollary 3.8. *A compact self-adjoint correction giving equality of the two natural spectra exists exactly for*

$$(u, v) \in \{(0,0), (1,0), (2,0), (1,1), (2,1)\}.$$

For every bounded-correctable type, write $W = \{w_1 < \cdots < w_s\}$ and put

$$D_{u,v} = \max \left\{ w_{i+1} - w_i \,:\, \sum_{j \leq i} A(a, w_j) \neq 0 \text{ for some } a \bmod 49 \right\},$$

with empty maximum zero. Then

$$\inf_C \|C\|_{\mathrm{ess}} = \limsup_j |\lambda_j(L) - \lambda_j(L')| = D_{u,v}, \qquad \inf_C \|C\| = \sup_j |\lambda_j(L) - \lambda_j(L')| \le H_{u,v}.$$

Both infima are attained by one correction diagonal in an eigenbasis of $\Delta_{u,v,L}$. In every sufficiently late block of residue a, exactly $\left|\sum_{j\le i} A(a, w_j)\right|$ ordered pairs cross the ith adjacent gap, and all other ordered displacements are zero.

Proof. Theorem 2.5 gives the histogram and the limsup; Proposition 2.7 gives both optimal norms and the compact criterion. On the bounded locus $B = 0$, the identity $D_{u,v} = 0$ is equivalent to the vanishing of every prefix sum and hence every $A(a, w)$. The exact classification in Theorem 3.1 then leaves precisely the five displayed types. Conversely their uncorrected spectra agree, so the zero correction suffices. □

Corollary 3.9. *For the natural pair, a bounded self-adjoint correction giving equality of the two spectra exists if and only if $\lim_{\kappa\to\infty} C(\kappa) = 0$. Such a correction can be chosen compact if and only if $C(0) = 0$, which here is also equivalent to exact equality of the complete spectra. On the bounded-correction locus,*

$$\inf_C \|C\|_{\mathrm{ess}} = D_{u,v} = \max\{g \,:\, d_g > 0\},$$

with empty maximum zero. Equivalently, when $C \neq 0$, this norm is twice the last point at which $\kappa C(\kappa)$ changes slope; when $C = 0$, the norm is zero.

Proof. Theorem 3.1 and Corollary 3.8 give the bounded and compact criteria, since

$$\lim_{\kappa\to\infty} C(\kappa) = \frac{1}{49}\sum_a |B(a)|, \qquad C(0) = \frac{1}{49}\sum_{a,w} |A(a, w)|.$$

Suppose $B = 0$. For a threshold t, all component discrepancies vanish exactly when every prefix sum at a missing edge vanishes. If $t \ge D_{u,v}$, this holds by the definition of $D_{u,v}$. If $0 \le t < D_{u,v}$, an active edge of gap $D_{u,v}$ remains missing, so some component discrepancy is nonzero. Hence $\sum_a F_a(t)$ is positive precisely for $0 \le t < D_{u,v}$, unless $D_{u,v} = 0$. Its last nonzero drop is therefore at $D_{u,v}$. The profile formula and Corollary 3.8 give the result. □

The operator norm is determined by the complete ordered lists; the essential norm is determined by their eventual displacements.

For example, a nonzero c_+ produces a linear deficit at each fixed tolerance $0 < \eta < 1$, but only a bounded deficit at tolerance 1. The approach to 1 is determined by (3.4). On the bounded-correction locus $c_+ = 0$, the lower transition still detects the nonzero exact discrepancies unless the type is one of the five exact exceptions. The upper transition depends on the block sums $B(a)$; the lower transition depends on the component sums defining F_a.

3.5. **Optimal correction norms for one-row types.** The weight multiplicities of the type-D_3 representations of highest weight $r\varepsilon_1 + k\varepsilon_2$ are given by Lauret and Rossi Bertone [17, Theorem 3.3]. Their Remark 3.17 describes the corresponding natural spectra for one-row types. Theorem 3.10 determines the complete correction locus and both optimal norms for L, L'.

For the natural type $\tau_{u,0}$, write $A_u = \Delta_{u,0,L}$ and $A'_u = \Delta_{u,0,L'}$, with the groups, bundles, and Casimir normalization of Section 1.

Write $V_{r,k,0}$ for the irreducible G-representation with highest weight $(r, k, 0)$, and put

$$p_{r,k} = \dim V_{r,k,0}^{\Gamma}, \qquad q_{r,k} = \dim V_{r,k,0}^{\Gamma'}.$$

The branching rule and the Casimir eigenvalue give the complete multiset decompositions

$$\begin{aligned} \mathrm{Spec}(A_u) &= \coprod_{r=u}^{\infty}\coprod_{k=0}^{u}\{r(r+4) + k(k+2)\}^{[p_{r,k}]}, \\ \mathrm{Spec}(A'_u) &= \coprod_{r=u}^{\infty}\coprod_{k=0}^{u}\{r(r+4) + k(k+2)\}^{[q_{r,k}]}. \end{aligned} \tag{3.6}$$

This is the specialization of (1.2) and (1.1) to $v = 0$: the only strings have $c = 0$ and $r \geq u \geq k \geq 0$, each with branching multiplicity one. Both lists begin at $r = u$.

Theorem 3.10 (Optimal norms for one-row types). *Let $u \geq 0$ be an integer. A bounded self-adjoint operator C such that $\mathrm{Spec}(A_u + C) = \mathrm{Spec}(A'_u)$ exists if and only if*

$$u \bmod 49 \in \mathcal{B}, \qquad \mathcal{B} = \{0, 1, 2, 9, 16, 23, 30, 37, 44, 45, 46, 47, 48\}.$$

For $u = 0, 1, 2$ the spectra are equal and the zero operator suffices. In every other allowed case, write $u = 49t + b$ with $0 \leq b < 49$. Then

$$\inf_C \|C\| = \inf_C \|C\|_{\mathrm{ess}} = \sup_j |\lambda_j(A_u) - \lambda_j(A'_u)| = \limsup_j |\lambda_j(A_u) - \lambda_j(A'_u)| = d_u,$$

where

$$d_u = \begin{cases} 98t - 9, & b \in \{0, 1, 2\}, \quad t \geq 1, \\ 98t + 2b + 1, & b \in \{9, 16, 23, 30, 37\}, \\ 98t + 89, & b \in \{44, 45, 46, 47, 48\}. \end{cases} \tag{3.7}$$

Both infima are attained by one correction diagonal in an eigenbasis of A_u. A compact self-adjoint correction exists precisely for $u = 0, 1, 2$.

3.5.1. *Two invariant tensors.*

Lemma 3.11. *Let λ, μ be highest weights of representations of G. For $\Lambda = \Gamma$ or Γ' and a nonzero $\eta \in (V_\mu^*)^\Lambda$, Cartan multiplication by η gives an injection*

$$(V_\lambda^*)^\Lambda \longrightarrow (V_{\lambda+\mu}^*)^\Lambda.$$

In particular, for $r \geq k \geq 0$,

$$\dim V^\Lambda_{r+2,k,0} \geq \dim V^\Lambda_{r,k,0}, \qquad \dim V^\Lambda_{r+1,k+1,0} \geq \dim V^\Lambda_{r,k,0}. \tag{3.8}$$

Proof. Complexify G and choose a highest-weight vector v_λ in V_λ. The map

$$\ell \longmapsto f_\ell, \qquad f_\ell(g) = \ell(g v_\lambda),$$

embeds V_λ^* in $\mathbb{C}[G_\mathbb{C}]$: if $f_\ell = 0$, then ℓ vanishes on the span of $G_\mathbb{C} v_\lambda$, which is V_λ. The tensor $v_\lambda \otimes v_\mu$ is a highest-weight vector of weight $\lambda + \mu$. This weight space in the tensor product is one-dimensional. Complete reducibility [7, Theorem 16.4] therefore places this vector in the unique irreducible summand of highest weight $\lambda + \mu$, which it generates. Moreover,

$$f_\ell(g) f_\eta(g) = (\ell \otimes \eta)\big(g(v_\lambda \otimes v_\mu)\big)$$

is the corresponding matrix coefficient. Thus multiplication maps the two embedded components to the component of highest weight $\lambda + \mu$.

If $f_\eta f_\ell = 0$ and $f_\eta \neq 0$, then f_ℓ vanishes on the nonempty open set $\{g : f_\eta(g) \neq 0\}$. Regular functions are holomorphic, and $G_\mathbb{C}$ is connected; the identity theorem gives $f_\ell = 0$. Multiplication by f_η is therefore injective. If ℓ and η are Λ-invariant, then f_ℓ and f_η are invariant under left translation by Λ, and so is their product. Multiplication by f_η therefore induces the displayed injection on invariant vectors.

Let $e_1, \dots, e_6$ be the real coordinate basis. Both groups preserve each oriented coordinate two-plane. Hence

$$e_1 \wedge e_2 \in \bigwedge^2 \mathbb{C}^6 = V_{1,1,0}$$

is nonzero and invariant. The equality here is the irreducible adjoint representation: the equivariant map $v \wedge w \mapsto [x \mapsto \langle w, x\rangle v - \langle v, x\rangle w]$ identifies $\bigwedge^2 \mathbb{C}^6$ with $\mathfrak{so}(6, \mathbb{C})$, whose highest weight is $\epsilon_1 + \epsilon_2$. The quadratic polynomial

$$x_1^2 + x_2^2 - x_3^2 - x_4^2$$

is also nonzero and invariant. Its Euclidean Laplacian is $2+2-2-2=0$. Harmonic quadratics have dimension $21-1=20$ and contain the square of a highest-weight linear coordinate, of weight $2\varepsilon_1$. Formula (2.20) gives $\dim V_{2,0,0}=20$, so the harmonic quadratics equal this representation. The dual of the highest weight (r,k,c) is $(r,k,-c)$; thus these two components and all $V_{r,k,0}$ are self-dual. Taking $\mu=(2,0,0)$ and $\mu=(1,1,0)$ gives (3.8). □

Lemma 3.12. *For every $r\ge 9$ and every $0\le k\le r$,*

$$p_{r,k}\ge 28, \qquad q_{r,k}\ge 28.$$

Proof. Under $\mathrm{Spin}(6)\simeq \mathrm{SU}(4)$ [7, Example 32.10], the order-49 lifts have weights

$$(11,39,44,4), \qquad (38,12,17,31).$$

Their sums are zero modulo 49, and their six pair sums are the six weights $\pm1,\pm6,\pm15$, respectively $\pm1,\pm6,\pm20$. Thus their exterior squares give the specified orthogonal generators. For either tuple $t=(t_1,t_2,t_3,t_4)$ define, in $\mathbb{Z}[Z]/(Z^{49}-1)$,

$$\sum_{n\ge0} h_n^{(t)}T^n=\prod_{i=1}^{4}(1-TZ^{t_i})^{-1}, \qquad h_n^{(t)}=0 \quad (n<0). \tag{3.9}$$

The A_3 Dynkin labels corresponding to $(r,k,0)$ are $(k,r-k,k)$; the associated partition is therefore $(r+k,r,k,0)$. For independent variables $z_1,\dots,z_4$ put $D_j=\prod_{i\ne j}(z_j-z_i)$. The generating function and Lagrange interpolation give

$$h_n(z_1,\dots,z_4)=\sum_{j=1}^{4}\frac{z_j^{n+3}}{D_j} \quad (n\ge-3), \qquad h_{-3}=h_{-2}=h_{-1}=0.$$

Taking determinants in $h_{\lambda_i-i+j}=\sum_\ell z_\ell^{\lambda_i+4-i}z_\ell^{j-1}/D_\ell$ gives the Weyl quotient

$$\det(h_{\lambda_i-i+j})_{i,j=1}^4=\frac{\det(z_j^{\lambda_i+4-i})_{i,j=1}^4}{\det(z_j^{4-i})_{i,j=1}^4}.$$

Both sides are polynomials. Since $\lambda_4=0$, the last row on the left is $(0,0,0,1)$. Specializing $z_i=Z^{t_i}$ therefore gives the invariant dimension as

$$[Z^0]\det\bigl(h^{(t)}_{\lambda_i-i+j}\bigr)_{i,j=1}^{3}, \qquad (\lambda_1,\lambda_2,\lambda_3)=(r+k,r,k), \tag{3.10}$$

where $[Z^0]$ denotes the coefficient of the residue class zero. Indeed averaging Z^m over $Z=\zeta^j$, $0\le j<49$, gives $\mathbf{1}_{49|m}$, so this coefficient counts invariant vectors. For the following eleven pairs, (3.9) and (3.10) give

(r,k)	$p_{r,k}$	$q_{r,k}$	(r,k)	$p_{r,k}$	$q_{r,k}$
$(9,0)$	28	28	$(9,6)$	660	660
$(9,1)$	91	91	$(9,7)$	844	842
$(9,2)$	226	226	$(9,8)$	594	594
$(9,3)$	334	334	$(9,9)$	527	527
$(9,4)$	472	472	$(10,0)$	30	30
$(9,5)$	655	657			

The coefficients in (3.9) are determined by the integer recurrences

$$H_0^{(0)}(a)=\mathbf{1}_{a=0}, \quad H_n^{(0)}(a)=0\ (n>0), \qquad H_n^{(j)}(a)=H_n^{(j-1)}(a)+H_{n-1}^{(j)}(a-t_j), \tag{3.11}$$

with $H_{-1}^{(j)}=0$, $1\le j\le 4$ and $a\in\mathbb{Z}/49$. Here $h_n^{(t)}=\sum_a H_n^{(4)}(a)Z^a$.

Put $d=r-k$. If $d\ge9$, choose $d_0\in\{9,10\}$ with $d\equiv d_0 \pmod 2$. Starting at $(d_0,0)$, apply the first injection in (3.8) $(d-d_0)/2$ times and the second k times. This gives the desired lower bound at (r,k). If $0\le d\le 8$, start at $(9,9-d)$ and apply the second injection $r-9$ times. Its final weight is $(r,r-d)=(r,k)$. Every starting dimension in the table is at least 28. □

3.5.2. *The cumulative multiplicity differences.* Let $T(a,b,c)$ be the periodic integer-valued function in (1.10). Put $T_0(a,b) = T(a,b,0)$. Equation (1.3) gives

$$p_{r,k} - q_{r,k} = T_0(r+2, k+1). \tag{3.12}$$

By (1.10), T_0 is 49-periodic and even in both variables; Lemma 3.2 gives $\sum_{b=0}^{48} T_0(a,b) = 0$.

For $a \in \mathbb{Z}/49$ and $k \geq 0$, put

$$S_a(k) = \sum_{j=0}^{k} T_0(a, j+1), \qquad S_a(-1) = 0.$$

Thus $S_a(k+49) = S_a(k)$. For $0 \leq j \leq 5$ define the integer intervals

$$I_j = \{7j+3, \dots, 7j+8\}, \qquad J_j = \{7j+5, 7j+6\},$$

and the vectors

$$c^{(1)} = (1,1,-1,1,-1,-1), \quad c^{(2)} = (1,0,2,-2,0,-1), \quad c^{(3)} = (1,-2,1,-1,2,-1).$$

For $K = I$ or J and $0 \leq \bar{k} < 49$, write

$$F_{K,c}(\bar{k}) = 2 \sum_{j=0}^{5} c_j \mathbf{1}_{K_j}(\bar{k}).$$

Summing (1.10) for $b = 1, \dots, k+1$ gives the complete list below. A row means $S_{a_+} = S_{-a_+} = F_{K,c}$ and $S_{a_-} = S_{-a_-} = -F_{K,c}$; the remaining residues have $S_a = 0$.

a_+	a_-	K	c
6	8	I	$c^{(1)}$
10	4	J	$c^{(1)}$
13	15	I	$c^{(2)}$
17	11	J	$c^{(2)}$
22	20	I	$c^{(3)}$
18	24	J	$c^{(3)}$

(3.13)

Every entry of this table is an identity of the finite integer sums (1.10), with the coefficient arrays $h_s, h_{s'}$ in Table 1. It follows that

$$|S_a(k)| \leq 4, \qquad S_a(k) S_a(k-1) \geq 0, \tag{3.14}$$

because consecutive intervals in either family are separated by a zero entry. Also

$$\begin{gathered} S_a(k) = 0 \text{ for every } a \quad \Longleftrightarrow \quad k \bmod 49 \in \mathcal{B}, \\ \mathcal{B} = \{0,1,2,9,16,23,30,37,44,45,46,47,48\}. \end{gathered} \tag{3.15}$$

Indeed the union of the supports in (3.13) is $\bigcup_{j=0}^{5} I_j$; already the row $a = 6$ is nonzero everywhere on this union.

3.5.3. *A matching in every block.* Suppose $u \geq 9$ and $u \bmod 49 \in \mathcal{B}$. For fixed $r \geq u$ put

$$a = r+2 \pmod{49}, \quad p_k = p_{r,k}, \quad q_k = q_{r,k}, \quad w_k = k(k+2),$$

$$f_k = S_a(k), \qquad f_{-1} = f_u = 0.$$

For $0 \leq k < u$ define

$$\begin{gathered} m_{k,k+1} = \max(f_k, 0), \qquad m_{k+1,k} = \max(-f_k, 0), \\ m_{k,k} = p_k - \max(f_k, 0) - \max(-f_{k-1}, 0), \end{gathered} \tag{3.16}$$

with the diagonal formula also at $k = u$, and all other entries zero. By (3.14), the two nonnegative quantities subtracted in the diagonal formula cannot both be positive. Their sum is at most 4.

Lemma 3.12 therefore gives $m_{k,k} \geq 24$. The incoming off-diagonal multiplicity is $\max(f_{k-1}, 0) + \max(-f_k, 0)$, which is also at most 4. All entries are nonnegative integers, and the row sums are p_k. The column sum at k is

$$p_k - f_k + f_{k-1} = p_k - T_0(a, k+1) = q_k.$$

Thus (3.16) is a bijection between all labelled copies in the two r-blocks. It moves a copy only across an adjacent gap

$$w_{k+1} - w_k = 2k + 3 \quad \text{with } S_a(k) \neq 0.$$

Consequently, with an empty maximum equal to zero, every block has a matching of displacement at most

$$D(u) = \max\{2k + 3 \,:\, 0 \leq k < u,\ S_a(k) \neq 0 \text{ for some } a\}. \tag{3.17}$$

The union of these block matchings remains a bijection when eigenvalues from different blocks coincide. If N_u, N'_u are the two counting functions, this bijection gives, for every real E,

$$N_u(E) \leq N'_u(E + D(u)), \qquad N'_u(E) \leq N_u(E + D(u)).$$

Both counts are finite. Substituting the jth eigenvalue in the two inequalities yields

$$|\lambda_j(A_u) - \lambda_j(A'_u)| \leq D(u) \qquad (j \geq 1). \tag{3.18}$$

No disjointness of the initial blocks is needed for this implication.

For the converse inequality, put $H = u(u+2)$. The r-block lies in $[r(r+4), r(r+4) + H]$. Its distance from the end of the preceding block is $2r + 3 - H$, which is positive for

$$r \geq R_u := \max\left\{u, \left\lceil \frac{H}{2} \right\rceil - 1\right\}. \tag{3.19}$$

For such r, all preceding complete blocks lie below this one, and their total multiplicities agree by (3.15). Write

$$P_k = \sum_{j=0}^{k} p_j, \qquad Q_k = \sum_{j=0}^{k} q_j.$$

Then $P_k - Q_k = f_k$. Since each $p_k, q_k \geq 28$ and $|f_k| \leq 4$,

$$\min(P_{k+1}, Q_{k+1}) > \max(P_k, Q_k).$$

The $|f_k|$ integer ranks in $(\min(P_k, Q_k), \max(P_k, Q_k)]$ therefore pair the two adjacent values $r(r+4)+w_k$ and $r(r+4) + w_{k+1}$ exactly. All other ranks pair equal values. Choose k attaining (3.17). The row $a = 6$ of (3.13) has $S_6(k) \neq 0$ at this k. There are infinitely many $r \equiv 4 \pmod{49}$ beyond R_u. Each of these blocks contains an ordered displacement $D(u)$. Together with (3.18), this proves

$$\sup_j |\lambda_j(A_u) - \lambda_j(A'_u)| = \limsup_{j\to\infty} |\lambda_j(A_u) - \lambda_j(A'_u)| = D(u). \tag{3.20}$$

3.5.4. *The complete formula.*

Proof of Theorem 3.10. By Lemma 3.5 and Proposition 2.7, a bounded correction exists exactly when every block total vanishes. Here the block total is $S_{r+2}(u)$, by (3.12). Equation (3.15) therefore gives exactly $u \bmod 49 \in \mathcal{B}$.

For $u = 0, 1, 2$, the table (3.13) has $S_a(k) = 0$ at every $0 \leq k \leq u$. Thus each individual discrepancy is zero, and (3.6) gives equality of the complete spectra. Every other u with residue in $\mathcal{B}$ is at least 9. The matching (3.16) and (3.20) therefore apply.

The active boundary residues are exactly

$$\bigcup_{j=0}^{5} \{7j + 3, \ldots, 7j + 8\} \pmod{49}.$$

If $b \in \{0,1,2\}$ and $t \geq 1$, the last active boundary below u is $49t-6$. If $b \in \{9,16,23,30,37\}$, it is $u-1$. If $44 \leq b \leq 48$, it is $49t+43$. Substitution in $2k+3$ gives (3.7). Proposition 2.7 and (3.20) give both norm infima and their simultaneous attainment by the diagonal correction

$$C_u e_j = (\lambda_j(A'_u) - \lambda_j(A_u))e_j.$$

In every nontrivial case $d_u > 0$, so the same proposition excludes compact corrections. □

Corollary 3.13. *For $\tau_{9,0}$, the least full and essential correction norms are* 19. *For $\tau_{48,0}$, both norms are* 89.

Proof. Substitute $u = 9$ and $u = 48$ in (3.7). For $u = 48$, the last active boundary is $k = 43$, between 1935 and 2024, with $S_6(43) = -2$. The separated tail begins at $r = 1199$ by (3.19); the block matching above also includes every earlier r. □

3.5.5. *Two resolution profiles.* For $\tau_{9,0}$ the offsets are $0, 3, 8, 15, 24, 35, 48, 63, 80, 99$. The two cumulative-discrepancy arrays in (3.13) give

$$C_{9,0}(\kappa) = \frac{48}{49}\left(\min\{1, 15/(2\kappa)\} + \min\{1, 19/(2\kappa)\}\right).$$

The value at $\kappa = 0$ is the right limit. For $\tau_{48,0}$, which has dimension 40425, put

$$\mathcal{G} = (15, 19, 29, 33, 43, 47, 57, 61, 71, 75, 85, 89).$$

Give the first four and last four entries weight $d_g = 48$ and the middle four entries weight $d_g = 64$. The same table gives

$$C_{48,0}(\kappa) = \frac{1}{49}\sum_{g\in\mathcal{G}} d_g \min\{1, g/(2\kappa)\}. \tag{3.21}$$

The value at 0 is $640/49$. For $\kappa \geq 89/2$ it is $16640/(49\kappa)$. These are the profiles in the eigenvalue-offset normalization of Section 3.4.

Part 2. Prime-square orders and congruences

4. Natural spectra at prime-square orders

For $p \geq 5$ prime, put $q = p^2$ and

$$L_p^{\pm} = L(q; 1, 1 \pm p, 1 \pm 3p).$$

These are members of the Lauret–Miatello–Rossetti family [15]. Their Hodge spectra agree in every degree; in [5, Theorem 1 and Sections 8–9], take $r = p$, $t = 1$, and $a = (0, 1, 3)$. The entries of a are distinct modulo every prime $p \geq 5$, so the hereditary condition there holds.

For a pair $(X, Y) \in (\mathbb{Z}/q\mathbb{Z})^2$, its *signed-permutation orbit* consists of $(\epsilon X, \delta Y)$ and $(\epsilon Y, \delta X)$, where $\epsilon, \delta \in \{1, -1\}$. Define

$$\mathcal{W}_q = \{X = 0\} \cup \{Y = 0\} \cup \{X = Y\} \cup \{X = -Y\}, \qquad \mathcal{G}_p = \{(X, Y) : p \mid X,\ p \mid Y\},$$

and let $\mathcal{O}_q$ be the union of the five signed-permutation orbits represented by

$$(3,1), \quad (5,1), \quad (5,3), \quad (7,1), \quad (7,3).$$

Theorem 4.1 (Bounded corrections at prime-square orders). *Let $\tau_{u,v}$ be an integral dominant* SO(5)*-type, with integers $u \geq v \geq 0$, and let $\Delta_{u,v}^{\pm}$ be the positive curvature-one* SO(6) *Casimirs on the associated natural bundles over $L_p^{\pm}$. Set*

$$X = 2u + 3 \pmod{p^2}, \qquad Y = 2v + 1 \pmod{p^2}, \qquad H = u(u+2) - 2v.$$

Then a bounded self-adjoint correction K with

$$\mathrm{Spec}(\Delta_{u,v}^{+} + K) = \mathrm{Spec}(\Delta_{u,v}^{-}),$$

including all multiplicities, exists precisely in the following cases.

(1) If $p = 5$, every pair (u, v) is allowed.
(2) If $p = 7$, (X, Y) belongs to $\mathcal{W}_{49} \cup \mathcal{G}_7 \cup \mathcal{O}_{49}$ or to one of the four additional signed-permutation orbits represented by

$$(14, 1), \quad (21, 1), \quad (14, 3), \quad (21, 3).$$

(3) If $p \geq 11$, (X, Y) belongs to $\mathcal{W}_{p^2} \cup \mathcal{G}_p \cup \mathcal{O}_{p^2}$.

Whenever a correction exists, it can be chosen to commute with $\Delta^+_{u,v}$ and to satisfy $\|K\| \leq H$. At $p = 5$, the two spaces are isometric and the zero correction is available after the induced unitary identification. Allowing bounded self-adjoint corrections on both sides gives the same classification.

The cardinalities of these residue sets are respectively

$$625 \quad (p = 5), \qquad 289 \quad (p = 7), \qquad \boxed{5p^2 - 4p + 40} \quad (p \geq 11).$$

4.1. **The fixed Laurent polynomial.** Let the four independent indeterminates be t, w, x, y. Define

$$E_d = 1 - t^2 w^{2d}, \qquad F_d = 1 - t^4 w^{2d}, \qquad A(w) = (w^2 - 1)(w^4 - 1)(w^6 - 1),$$

$$\Lambda_t = E_3 E_{-3} \prod_{1 \leq |j| \leq 5} F_j, \qquad Q_{d,e} = \frac{\Lambda_t}{F_d F_e F_{d+e} E_d E_e}.$$

Only the following six ordered pairs occur:

$$\mathcal{P} = \{(1, 3), (-1, -3), (-1, 2), (1, -2), (-3, -2), (3, 2)\}.$$

For each pair in $\mathcal{P}$, $Q_{d,e}$ belongs to $\mathbb{Z}[t, w, w^{-1}]$. Indeed, after removing F_d, F_e, F_{d+e} from Λ_t, cancel E_f directly if $|f| = 3$; if $|f| \leq 2$, cancel it from F_{2f} by the identity $F_{2f} = E_f(1 + t^2 w^{2f})$. The required factors F_{2f} have not already been removed. This also gives $\deg_t Q_{d,e} = 28$.

Define the six signed monomials $m_{d,e}(w)$ by

(d, e)	$(1, 3)$	$(-1, -3)$	$(-1, 2)$	$(1, -2)$	$(-3, -2)$	$(3, 2)$
$m_{d,e}$	w^{18}	w^{-6}	$-w^9$	$-w^3$	w^{-9}	w^{21}.

The fixed Laurent polynomial is

$$J(t, w, x, y) = \sum_{(d,e) \in \mathcal{P}} m_{d,e}(w) Q_{d,e}(t, w)(x^d y^e - x^e y^d). \tag{4.1}$$

Write its nonzero monomials as

$$J = \sum_{\nu \in I} j_\nu \, t^{a_\nu} w^{b_\nu} x^{c_\nu} y^{d_\nu}.$$

The polynomial J has 1816 monomials, $a_\nu \in \{0, 2, \ldots, 28\}$, $|j_\nu| \leq 6$, and at most 216 monomials at any one t-degree. For $s \in \mathbb{Z}/q\mathbb{Z}$, define the integer

$$C_s(p; X, Y) = \sum_{\substack{\nu \in I, \ \epsilon, \delta \in \{\pm 1\} \\ a_\nu + p b_\nu + \epsilon(1 + p c_\nu) X + \delta(1 + p d_\nu) Y \equiv s \pmod q}} \epsilon \delta \, j_\nu. \tag{4.2}$$

There are only 7264 summands before equal residues are combined.

Theorem 4.2. *For every prime $p \geq 5$ and every integral dominant type (u, v), the following conditions are equivalent:*

(1) For every $0 \leq r < p$, the p integers $C_r, C_{r+p}, \ldots, C_{r+(p-1)p}$ are equal.
(2) $\sup_j |\lambda_j(\Delta^+_{u,v}) - \lambda_j(\Delta^-_{u,v})| < \infty$.
(3) There is a bounded self-adjoint operator K such that $\mathrm{Spec}(\Delta^+_{u,v} + K) = \mathrm{Spec}(\Delta^-_{u,v})$.
(4) Such an operator K can be chosen to commute with $\Delta^+_{u,v}$.

If these conditions hold, the ordered eigenvalue difference is at most H at every index, and $\|K\| \leq H$ is possible. Allowing bounded self-adjoint corrections on both sides gives the same criterion.

4.2. **Reduction from spectra to one Fourier coefficient.** For $z = (z_1, z_2, z_3)$ put

$$D(z) = \det(z_j^{\rho_i} + z_j^{-\rho_i})_{i,j=1}^3, \qquad \rho = (2, 1, 0).$$

The positive Casimir eigenvalue in the string (r, k, c) is $r(r+4) + k(k+2) + c^2$, and the branching condition is $r \geq u \geq k \geq v \geq |c|$. The Weyl character formula and this branching rule have the normalization of Section 1.

All p-th powers of the two generators have identical rotation coordinates, and the identity contributions agree. The other powers are regular: $1 + pa_i \not\equiv \pm(1 + pa_j) \pmod{p^2}$ for distinct $a_i, a_j \in \{0, 1, 3\}$. Consequently the difference of the outer-symmetrized characters is q-periodic in the three shifted highest-weight coordinates, even in each coordinate, alternating under interchange of coordinates, and has zero sum over a full period in each coordinate. Indeed, every row frequency is a unit $\pm 2h(1 \pm pa_j)$ modulo q, so its geometric sum over a full period is zero. Only regular elements occur in the Weyl quotients.

Choose a primitive q-th root t, put $(a_1, a_2, a_3) = (0, 1, 3)$, and set

$$z_{h,j}^{\pm} = t^{2h(1 \pm pa_j)}, \qquad C_\ell(z) = \det(z_j^{\ell_i} + z_j^{-\ell_i})_{i,j=1}^3.$$

For $\ell = (a, b, c)$ define the normalized cosine-character discrepancy by

$$T_p(a, b, c) = \frac{1}{q} \sum_{h \in (\mathbb{Z}/q\mathbb{Z})^\times} \left(\frac{C_\ell(z_h^+)}{D(z_h^+)} - \frac{C_\ell(z_h^-)}{D(z_h^-)} \right).$$

If $\delta_p(r, k, c)$ is the full invariant-dimension discrepancy in the string (r, k, c), the type-D_3 character formula gives

$$\delta_p(r, k, 0) = T_p(r + 2, k + 1, 0), \qquad \delta_p(r, k, c) + \delta_p(r, k, -c) = 2T_p(r + 2, k + 1, c) \quad (c > 0).$$

Indeed the sine determinants cancel between c and $-c$, and all nonunit powers cancel between the two groups before any Weyl quotient is evaluated. Thus the total discrepancy in a fixed first-coordinate block is

$$B_{u,v}(a) = \sum_{b=v+1}^{u+1} \sum_{c=-v}^{v} T_p(a, b, c).$$

The square $[-v, v]^2$ contributes zero by alternation in b, c. The two remaining outer strips in b agree by evenness. Consequently

$$2B_{u,v}(a) = \sum_{b=-u-1}^{u+1} \sum_{c=-v}^{v} T_p(a, b, c). \tag{4.3}$$

The rectangle sum is periodic in u, v: increasing either variable by q adds two strips whose sums vanish over full periods. Consequently dominant types with the same residues of u, v give the same $B_{u,v}$. Both complete spectral decompositions begin at $r = u$.

Put $w = t^p$. The generator rotation coordinates in the half-angle parameter t are $t^{2(1+pa_i)}$. Define

$$S_d = tw^d - t^{-1}w^{-d}, \quad z_d = t^2w^{2d}, \quad c_d = z_d + z_d^{-1}, \quad R_X(d) = \frac{t^X w^{dX} - t^{-X} w^{-dX}}{S_d}.$$

The Dirichlet identity

$$\sum_{b=-(X-1)/2}^{(X-1)/2} (t^{2b(1+pd)} + t^{-2b(1+pd)}) = 2R_X(d)$$

holds for the odd positive lengths arising from u, v and extends by periodicity. At the specified roots, $R_{X+q}(d) = R_X(d)$ and $R_{-X}(d) = -R_X(d)$. Interchanging X, Y interchanges the last two determinant rows. Thus the condition $B_{u,v} = 0$ is preserved by signed permutations of (X, Y) modulo q.

For independent t, w and integer X, Y, set

$$\mathcal{F}_{X,Y}(t,w) = \sum_{(d,e)\in\{(1,3),(-1,2),(-3,-2)\}} \Big(y_{d,e}(X,Y) - y_{-d,-e}(X,Y) \Big), \tag{4.4}$$

$$y_{d,e}(X,Y) = \frac{R_X(d)R_Y(e) - R_X(e)R_Y(d)}{2(c_0-c_d)(c_0-c_e)(c_d-c_e)}.$$

All denominators are nonzero at the specified roots of unity. In particular, each factor $1-t^\ell w^m$ that occurs has $\ell \in \{2,4\}$, so its exponent is not divisible by p; the factors w^2-1, w^4-1, w^6-1 are nonzero for $p \geq 5$.

Lemma 4.3. *The vector $(B_{u,v}(a))_{a \bmod q}$ is zero if and only if $\mathcal{F}_{X,Y}(t,t^p) = 0$.*

Proof. Expand the first row of the determinant in the cyclic character average and apply (4.3). The finite Fourier support in the variable a consists only of the units modulo q. To obtain the coefficient of t^{2a}, the power of a generator associated with its j-th rotation coordinate must be $(1+pa_j)^{-1}$ or its negative. The two choices give equal contributions, since the cosine determinant and the Dirichlet quotients are invariant under simultaneous inversion.

The elementary identity

$$(1+pa_k)(1+pa_j)^{-1} \equiv 1 + p(a_k - a_j) \pmod{p^2}$$

normalizes the first rotation coordinate to t^2. For $a_j = 0, 1, 3$, the remaining normalized pairs are respectively $(d,e) = (1,3), (-1,2), (-3,-2)$. The minus family gives their negatives. Since $D(z) = 2\prod_{i<j}(c_i - c_j)$, the coefficient of t^{2a} in $B_{u,v}(a)$ is $4\mathcal{F}_{X,Y}(t,w)/q$. Every other unit-frequency coefficient is its Galois conjugate. Hence all Fourier coefficients vanish exactly when $\mathcal{F}_{X,Y}(t,t^p) = 0$. □

4.3. **Clearing denominators and the cyclotomic criterion.** Put

$$A_{d,e} = (1-w^{2d})(1-w^{2e})(w^{2d}-w^{2e}).$$

A direct factorization gives

$$2(c_0-c_d)(c_0-c_e)(c_d-c_e)S_dS_e = -2t^{-8}w^{-5(d+e)}A_{d,e}F_dF_eF_{d+e}E_dE_e. \tag{4.5}$$

For $(d,e) = (1,3), (-1,2), (-3,-2)$, respectively,

$$\frac{A(w)}{A_{d,e}} = -w^{-2}, \quad w^4, \quad -w^{16}.$$

Moreover $A_{-d,-e} = -w^{-4(d+e)}A_{d,e}$. Substitution into (4.5) gives exactly the six monomials $m_{d,e}$ displayed above, including the subtraction of the minus family. Thus, on setting

$$\mathcal{P}_{X,Y}(t,w) = \sum_{\epsilon,\delta=\pm1} \epsilon\delta\, t^{\epsilon X+\delta Y} J(t,w,w^{\epsilon X}, w^{\delta Y}),$$

one obtains the rational identity

$$\mathcal{F}_{X,Y}(t,w) = \frac{t^8}{2A(w)\Lambda_t(t,w)}\mathcal{P}_{X,Y}(t,w). \tag{4.6}$$

Reducing $\mathcal{P}_{X,Y}(t,t^p)$ modulo $t^{p^2}-1$ gives exactly $\sum_{s=0}^{q-1} C_s(p;X,Y)t^s$.

The minimal polynomial of a primitive p^2-th root is $\Phi_{p^2}(t) = 1 + t^p + \cdots + t^{(p-1)p}$. Indeed, $\Phi_{p^2}(t+1)$ has constant coefficient p and is congruent to $t^{p(p-1)}$ modulo p, so Eisenstein's criterion applies. Thus, for $C(t) \in \mathbb{Q}[t]$ of degree less than p^2 and a primitive p^2-th root ξ,

$$C(\xi) = 0 \quad \Longleftrightarrow \quad C(t) = Q(t)\Phi_{p^2}(t), \quad Q \in \mathbb{Q}[t], \quad \deg Q < p.$$

Comparing coefficients proves the equality condition in Theorem 4.2(1).

For fixed u, v, the spectral decompositions form the periodic quadratic model with $\sigma = 1$, $t_r = r+2$, $c = 4$, $r_0 = u$, and block width H. Both finite prefixes are empty, and the block discrepancies are $B_{u,v}(r+2)$ for every $r \geq u$. Theorem 2.4 gives bounded complete ordered displacement if and only if $B_{u,v} = 0$, with bound H at every index. Proposition 2.7 gives the equivalent bounded commuting correction. The sum of the two perturbation norms proves necessity when corrections are allowed on both sides. This completes the proof of Theorem 4.2.

4.4. **Universal families and two uniform obstructions.**

Proposition 4.4. *For every prime $p \geq 7$, bounded correction is possible on the union of*

(1) the four walls $X = 0$, $Y = 0$, $X = Y$, $X = -Y$ modulo p^2;

(2) the grid $p \mid X$, $p \mid Y$;

(3) the signed-permutation orbits of $\{1,3\}, \{1,5\}, \{3,5\}, \{1,7\}, \{3,7\}$.

For $p \geq 11$ this union has exactly $5p^2 - 4p + 40$ residue pairs. For every prime $p \geq 7$, the types $\tau_{3,0}$ and $\tau_{2,2}$ admit no bounded self-adjoint correction. At $p = 5$ the two lens spaces are isometric and all types admit the zero correction after the induced unitary identification.

Proof. For $X = 0$ or $Y = 0$ the corresponding Dirichlet numerator vanishes. For $X = Y$ or $X = -Y$ the two Dirichlet rows are proportional. These four cases give the wall identities. If $p \mid X, Y$, then $w^X = w^Y = 1$, and each skew monomial in (4.1) vanishes after every sign substitution. The grid therefore vanishes term by term in the uniform numerator.

For the five small pairs, the following are identities in two independent Laurent variables:

$$\mathcal{P}_{3,1} = \mathcal{P}_{5,1} = \mathcal{P}_{5,3} = \mathcal{P}_{7,1} = \mathcal{P}_{7,3} = 0.$$

For each of the six terms in (4.1), expand the remaining binomials in the expression for $Q_{d,e}$. For each monomial $jt^a w^b x^c y^d$ add

$$\epsilon\delta j \quad \text{at the Laurent exponent} \quad (a + \epsilon X + \delta Y,\ b + \epsilon c X + \delta d Y)$$

for all four choices of ϵ, δ. At each of the five displayed pairs every resulting integer coefficient is zero. Thus these are identities in $\mathbb{Z}[t^{\pm 1}, w^{\pm 1}]$ before specialization at roots of unity.

The corresponding products for $(9,1)$ and $(7,5)$ give

$$\mathcal{F}_{9,1}(t,w) = \frac{(t^4-1)(w^2-1)(w^8-1)(w^{10}-1)}{2t^2 w^{10}}, \tag{4.7}$$

$$\mathcal{F}_{7,5}(t,w) = -\frac{(t^8-1)(w^2-1)(w^8-1)(w^{10}-1)}{2t^4 w^{10}}. \tag{4.8}$$

For t of order p^2 and $w = t^p$ of order $p \geq 7$, every factor in these numerators is nonzero. Lemma 4.3 and Theorem 4.2 prove the two obstructions. At $p = 5$ the factor $w^{10} - 1$ vanishes.

The walls have $4p^2 - 3$ points. The grid contributes $p^2 - (4p-3) = (p-1)(p-3)$ additional points. For $p \geq 11$ the five small orbits are disjoint from the grid and walls and have eight points each. Their total is therefore $4p^2 - 3 + (p-1)(p-3) + 40 = 5p^2 - 4p + 40$. At $p = 5$, multiplication of $(1, 6, 16)$ by the unit 21 modulo 25 gives $(21, 1, 11)$, a permutation of the minus triple $(1, 21, 11)$. This gives the isometry and intertwines all natural Casimirs. □

4.5. **Vanishing in the cyclic group algebra.**

Lemma 4.5. *Let $C(z) \in \mathbb{Q}[z]$ have degree less than p^2. If $C \equiv 0 \pmod{z^p - 1}$ and $C(\zeta_{p^2}) = 0$, then $C = 0$.*

Proof. The coprime polynomials $z^p - 1$ and $\Phi_{p^2}(z)$ both divide C. Their product is $z^{p^2} - 1$, whose degree exceeds that of C. □

Each skew monomial in (4.1) vanishes at $x = y = 1$. Consequently $J(t, 1, 1, 1) = 0$, and the reduced polynomial $\sum_{s \bmod p^2} C_s(p; X, Y)t^s$ vanishes modulo $t^p - 1$. Lemma 4.5 strengthens Theorem 4.2 to

$$\sup_j |\lambda_j(\Delta^+_{u,v}) - \lambda_j(\Delta^-_{u,v})| < \infty \quad \Longleftrightarrow \quad C_s(p; X, Y) = 0 \quad (s \bmod p^2). \tag{4.9}$$

The coefficient equalities here are equalities of integers.

4.6. **Affine cancellations over the prime field.** Write $J_0(w, x, y)$ for the coefficient of t^0 in J. Multiplication of the six products in (4.1) gives

$$\begin{gathered} J_0(w, x, y) = w^6 D(w^3 x, w^3 y), \\ D(z, h) = \frac{(z-1)(z-h)(h-1)(zh-1)(zh^2-1)(z^2h-1)}{z^3h^3}. \end{gathered} \tag{4.10}$$

The two Laurent polynomials have the same twelve nonzero coefficients. This is the Weyl denominator of G_2 in the lattice whose simple roots are $(1, -1)$ and $(0, 1)$ and whose inner product has matrix $\left(\begin{smallmatrix} 1 & -1/2 \\ -1/2 & 1 \end{smallmatrix}\right)$. The six positive roots are $(1, -1), (0, 1), (1, 0), (1, 1), (1, 2), (2, 1)$ and their half-sum is $(3, 2)$.

Set $x = X \bmod p$ and $y = Y \bmod p$. For $p > 28$, two different first-residue functions

$$a + \epsilon x + \delta y, \qquad a \in \{0, 2, \dots, 28\}, \quad \epsilon, \delta \in \{1, -1\},$$

can coincide only on the following 116 lines:

$$x = k, \quad y = k, \quad x + y = k, \quad x - y = k, \qquad -14 \le k \le 14. \tag{4.11}$$

Suppose (x, y) belongs to none of these lines. Then the four first-residue groups with $a = 0$ are separate. Their vanishing and (4.10), evaluated at a primitive pth root, imply that each $(\epsilon x, \delta y)$ belongs to the union of

$$x = -3, \quad y = -3, \quad x = y, \quad x + y = -6, \quad x + 2y = -9, \quad 2x + y = -9.$$

Choose one line from each of the four signed copies. Solving the 6^4 resulting systems over $\mathbb{Q}$ gives the points

$$(0, 0), \qquad (\pm 3, \pm 3), \qquad (\pm 9, \pm 9).$$

All nonzero minors of order at most three in these systems have prime divisors in $\{2, 3, 5, 7, 11\}$. Outside this set the same systems have the same ranks and solutions after reduction. Every resulting point lies on $x = y$ or $x = -y$, a contradiction. Thus (x, y) lies on (4.11).

The nonzero minors of order at most three in the augmented matrix of (4.11) have prime divisors in

$$\mathcal{S}_0 = \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53\}.$$

The largest absolute minor of order three is 56. The arrangement therefore has its rational intersection pattern over $\mathbb{F}_p$ for $p \notin \mathcal{S}_0$. Its 2101 intersection points have 281 representatives under signed permutations, chosen with $x \ge y \ge 0$. Its open lines have the 30 representatives

$$x = k \quad \text{and} \quad x + y = k, \qquad 0 \le k \le 14,$$

with intersection points removed.

For an open line write $(x, y) = (a_x z + b_x, a_y z + b_y)$, where the two representative parameterizations have coefficients $(0, k, 1, 0)$ and $(1, 0, -1, k)$. Choose a lift $\widehat{z}$ modulo p^2 and put

$$X = a_x \widehat{z} + b_x + pU, \qquad Y = a_y \widehat{z} + b_y + pV.$$

For a term (a, b, c, d) of J and signs (ϵ, δ), the first-residue function is $(\epsilon a_x + \delta a_y)z + a + \epsilon b_x + \delta b_y$. Within one such group the second-residue form is

$$(\epsilon c a_x + \delta d a_y)z + \epsilon U + \delta V + b + \epsilon c b_x + \delta d b_y. \tag{4.12}$$

The first-residue functions in a group agree identically, so their common carry cancels in differences. Different first-residue functions do not meet on the open stratum. A change of $\widehat{z}$ changes only the coordinates U, V of the same fiber.

At a rational intersection point (x, y), whose denominators divide two, take its rational lift modulo p^2 and write $X = x + pU$, $Y = y + pV$. The first-residue group is indexed by $a + \epsilon x + \delta y$; its second-residue forms are

$$\epsilon U + \delta V + b + \epsilon c x + \delta d y. \tag{4.13}$$

Multiplication by two makes all coefficients integral.

Represent affine subspaces by reduced row echelon equations, in the variable order (z, U, V) on a line and (U, V) at a point. For a group G and an affine subspace S, restrict its forms to S, combine equal forms, and add their integer coefficients. If a group has nonzero total coefficient, there is no solution on S: partitioning its forms by their values preserves that integer total. If every coefficient in every group is zero, retain S. Otherwise let $F_+(G, S)$ and $F_-(G, S)$ be the lexicographically ordered lists of forms with positive and negative coefficients. If $|F_-| \le |F_+|$, put $f = \min F_+$ and $\mathcal{H} = F_-$; otherwise put $f = \min F_-$ and $\mathcal{H} = F_+$. Among the nonzero groups choose the least tuple $(|\mathcal{H}|, f, \mathcal{H})$ in lexicographic order. At a point where all coefficient sums vanish, the coefficient of f must be cancelled by a form in $\mathcal{H}$. The remaining affine subspaces are therefore

$$S \cap \{f = g\}, \qquad g \in \mathcal{H},$$

with the inconsistent systems omitted. Every nonempty intersection has smaller dimension: f and g have distinct restrictions to S. Thus induction on $\dim S$ proves that the retained subspaces give all solutions; every sequence has length at most three. A zero-dimensional subspace is retained precisely when all its coefficient sums vanish. On an open line remove the solutions whose parameter is an intersection point of the original arrangement.

Applying this recursion over $\mathbb{Q}$ to (4.12) and (4.13) gives Table 4. The recursion contains 871 affine subspaces, of which 73 have all coefficient sums zero, and 5754 inconsistent intersections. The second column gives the complete set of solutions over every prime field outside

$$\mathcal{S} = \{\ell : \ell \text{ is prime and } \ell \le 173\}. \tag{4.14}$$

Indeed, in all branches of the rational row reductions of the systems (4.12) and (4.13), every prime divisor of a pivot numerator or denominator, an equation denominator, or a nonzero constant equation belongs to

$$\{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61,$$
$$67, 71, 73, 79, 89, 97, 101, 103, 107, 109, 113, 127, 131, 157, 173\}.$$

Equations with equal restricted forms are combined before each new intersection. Every division therefore remains valid over $\mathbb{F}_p$ for $p > 173$, and every discarded nonzero constant equation remains inconsistent.

TABLE 4. Solutions of the second-residue equations.

First-residue stratum	Second-residue solutions
Open line $x = 0$	$U = 0$
Open line $x + y = 0$	$U + V = 0$
Other open representative lines	$\emptyset$
Point $(0, 0)$	All (U, V)
Points $(x, 0)$ with $x > 0$	$V = 0$
Points (x, x) with $x > 0$	$U = V$
$(3, 1), (5, 1), (5, 3), (7, 1), (7, 3)$	$U = V = 0$
Other representative intersection points	$\emptyset$

The coefficients j_ν and their sums remain integers throughout; only the affine forms are reduced modulo p. At a solution of a reduced branch, the chosen nonzero coefficient still requires an opposite-sign contribution at the same value. One of the child equations therefore holds, even if further forms coincide after reduction. Induction down the tree proves that the reduced retained subspaces contain every solution. Conversely, the zero coefficient sums at every retained subspace are integer identities and survive reduction. This proves equality of solution sets over $\mathbb{F}_p$. After signed permutations and the lift formulas, its solution set is $\mathcal{W}_{p^2} \cup \mathcal{G}_p \cup \mathcal{O}_{p^2}$.

4.7. **The exceptional primes.** For the remaining primes, take representatives $X = x + pU$, $Y = y + pV$ with $0 \le x, y, U, V < p$. Signed permutations reduce the base pairs to $0 \le y \le x \le (p-1)/2$. Their orbits have size 1 at $(0,0)$, size 4 when $y = 0 < x$ or $x = y > 0$, and size 8 otherwise; the induced maps of full fibers are bijections. For a monomial and sign pair, set

$$r \equiv a + \epsilon x + \delta y \pmod{p}, \quad 0 \le r < p, \qquad h = \frac{a + \epsilon x + \delta y - r}{p}.$$

Its actual second-residue exponent is

$$h + b + \epsilon c x + \delta d y + \epsilon U + \delta V \pmod{p}. \tag{4.15}$$

For each of the p^2 base pairs, group these forms by r and add equal integer coefficients. A nonzero group requires a collision between opposite-sign forms. The corresponding equations are affine lines in (U, V). On each line, combine identical affine functions of its parameter. Either every group vanishes on the line, or any solution is one of the intersections of opposite-sign affine functions. Evaluation in all original groups retains exactly the solutions of (4.9).

Exact enumeration with the carries in (4.15) gives the solutions in all 45069 representative fibers for $5 \le p \le 173$. The orbit sizes give Table 5. At each prime $p \ge 11$ in the table every solution belongs to $\mathcal{W}_{p^2} \cup \mathcal{G}_p \cup \mathcal{O}_{p^2}$. At $p = 7$ the additional solutions are exactly the signed-permutation orbits of $(14,1), (14,3), (21,1), (21,3)$. At $p = 5$ every pair is a solution.

TABLE 5. Solutions modulo p^2 for $5 \le p \le 173$.

p	cardinality	p	cardinality
5	625	79	30929
7	289	83	34153
11	601	89	39289
13	833	97	46697
17	1417	101	50641
19	1769	103	52673
23	2593	107	56857
29	4129	109	59009
31	4721	113	63433
37	6737	127	80177
41	8281	131	85321
43	9113	137	93337
47	10897	139	96089
53	13873	149	110449
59	17209	151	113441
61	18401	157	122657
67	22217	163	132233
71	24961	167	138817
73	26393	173	148993

The reverse inclusions were proved in Proposition 4.4; the four extra orbits at $p = 7$ also follow from Lemma 3.4. Since $\mathcal{S}$ has no prime greater than 173, the two finite eliminations prove Theorem 4.1

for every prime. For $p \geq 11$, the count is given in Proposition 4.4. At $p = 7$ the four additional eight-point orbits give $257 + 32 = 289$. At $p = 5$ that proposition gives all $25^2 = 625$ residue pairs.

For $p = 7$, multiplication by 6 sends $(1, 8, 22)$ to $(6, -1, -15)$ modulo 49. Interchanging the first two coordinates and conjugating the first and third give $(1, 6, 15)$. Conjugation of the last two coordinates sends $(1, -6, -20)$ to $(1, 6, 20)$. These transformations preserve the real orientation. They identify the $p = 7$ family, including its natural SO(6) operators, with the pair of Section 1.

5. Polynomial counting for prime-dependent affine congruences

For fixed integral matrices, the central and affine counting formulas are due to Kamiya, Takemura and Terao [8, Lemma 2.1 and Theorem 2.4] and [9, Lemma 2.3 and Theorem 3.2]. Dolzmann and Sturm determine feasibility conditions and guarded p-adic sample solutions for prime-power congruences [6, Sections 5–6].

Let $d, N \geq 1$, and fix rows

$$(a_i(T), b_i(T)) \in \mathbb{Z}[T]^d \times \mathbb{Z}[T], \qquad 1 \leq i \leq N.$$

For a prime p and an integer $k \geq 1$, consider the predicates

$$H_i(p, k; x) : \quad a_i(p)x = b_i(p) \pmod{p^k}, \qquad x \in (\mathbb{Z}/p^k\mathbb{Z})^d.$$

Fix one Boolean function of these N predicates for all p and k.

For each fixed k, the analogous count for an integer parameter $t \geq 1$ is eventually a quasipolynomial by Bogart, Goodrick and Woods [4, Property 2 and Theorem 1.15]. Indeed, on the box $0 \leq x_j < t^k$, each congruence is equivalent to

$$\exists z \in \mathbb{Z} : \qquad a_i(t)x - b_i(t) = t^k z.$$

The chosen Boolean combination therefore defines a finite parametric Presburger family. Theorem 5.1 gives one integral polynomial on the primes, with an explicit exceptional set independent of k.

For a nonzero polynomial $f(T) \in \mathbb{Z}[T]$, write

$$\operatorname{ord}_T f = v, \qquad f(T) = T^v(c(f) + Tg(T)), \qquad c(f) \neq 0.$$

Let $\mathcal{M}(T)$ be the $N \times (d + 1)$ matrix with rows $(a_i(T), b_i(T))$. Define the finite prime set

$$\mathcal{S}(\mathcal{M}) = \{p : \ p \mid c(f) \text{ for some nonzero square minor } f(T) \text{ of } \mathcal{M}(T)\}. \tag{5.1}$$

Only minors of sizes at most $d + 1$ occur. The set (5.1) is finite.

Theorem 5.1. *For every fixed $k \geq 1$, there is a polynomial $P_k(Z) \in \mathbb{Z}[Z]$, of degree at most kd, such that the number of x satisfying the chosen Boolean function is $P_k(p)$ for every prime $p \notin \mathcal{S}(\mathcal{M})$. The same exceptional set works simultaneously for all $k \geq 1$.*

Proof. For every minor f in (5.1) and every prime outside that set,

$$v_p(f(p)) = \operatorname{ord}_T f,$$

since $c(f)$ is a p-adic unit. Zero minors remain zero. Thus specialization also preserves the rank of every row submatrix of $\mathcal{M}$ and of its coefficient matrix.

Take an arbitrary subset of the predicates, and write its system as $A(T)x = b(T)$. For a matrix $G(T)$, let $\nu_j(G)$ be the minimum T-order of its nonzero $j \times j$ minors, set $\nu_0(G) = 0$, and write $r(G)$ for its rank over $\mathbb{Q}(T)$. The T-adic Smith valuations are

$$\alpha_j(G) = \nu_j(G) - \nu_{j-1}(G), \qquad 1 \leq j \leq r(G).$$

Let $\mathbb{Z}_p$ denote the ring of p-adic integers. For a nonzero matrix over $\mathbb{Z}_p$, choose an entry of least finite valuation, move it to the upper left, and multiply its row by a unit so that it is p^α. Every other entry is divisible by it. Subtract multiples of its row and column to leave a zero first row and column

off the diagonal, and repeat until the remaining submatrix is zero. Each later diagonal entry has valuation at least α. Invertible row and column operations preserve the ideals generated by minors, so the sum of the first j diagonal valuations equals the least valuation of a nonzero j-minor. The preceding minor identity therefore gives the displayed $\alpha_j(G)$ as the diagonal valuations of $G(p)$, for both $G = A$ and $G = (A, b)$. Define

$$\kappa_k(G) = \sum_{j=1}^{r(G)} \max\{k - \alpha_j(G), 0\}.$$

The image of multiplication by p^α modulo p^k has $p^{\max(k-\alpha,0)}$ elements. Thus the diagonal form gives

$$|\operatorname{im}(G(p) \bmod p^k)| = p^{\kappa_k(G)}.$$

Moreover, the image of $A(p)$ is contained in that of $(A(p), b(p))$. The system is soluble precisely when these two finite images have the same cardinality: equality of the images is equivalent to $b(p)$ belonging to the image of $A(p)$. Every nonempty fiber of $A(p)$ is a coset of its kernel. Thus its number of solutions is

$$\begin{cases} p^{kd-\kappa_k(A)}, & \kappa_k(A) = \kappa_k((A, b)), \\ 0, & \text{otherwise.} \end{cases} \tag{5.2}$$

All exponents and the solubility condition in (5.2) are independent of p outside $\mathcal{S}(\mathcal{M})$. For the empty subset of predicates both matrices have rank zero, $\kappa_k = 0$, and the count is p^{kd}.

For a Boolean function $F : \{0,1\}^N \to \{0,1\}$, put

$$c_I = \sum_{J \subset I} (-1)^{|I|-|J|} F(\mathbf{1}_J) \qquad (I \subset \{1, \dots, N\}).$$

Then $F(e) = \sum_I c_I \prod_{i \in I} e_i$: for $e = \mathbf{1}_K$, the coefficient of $F(\mathbf{1}_J)$ in the right-hand side is $\sum_{J \subset I \subset K} (-1)^{|I|-|J|}$, equal to 1 if $J = K$ and to 0 otherwise. Substitution of the predicate indicators expresses the desired count as an integral linear combination of the intersection counts (5.2). Each nonzero term is a monomial of degree between 0 and kd. This proves the theorem. □

Corollary 5.2. *Suppose the entries of $\mathcal{M}(T)$ have degree at most an integer $h \geq 0$. The polynomials in Theorem 5.1 can be chosen so that*

$$\sum_{k \geq 1} P_k(Z) U^k \in \mathbb{Z}\big[Z, U, (1-U)^{-1}, (1-ZU)^{-1}, \dots, (1-Z^d U)^{-1}\big].$$

For every $p \notin \mathcal{S}(\mathcal{M})$, substituting $Z = p$ gives the counting series for all moduli p^k.

Proof. Put $M = (d+1)h$. For a matrix G of rank r, every nonzero r-minor has degree at most rh. Hence

$$0 \leq \alpha_j(G) \leq v_r(G) \leq rh \leq M, \qquad \kappa_k(G) = kr - v_r(G) \quad (k > M).$$

For a system $Ax = b$, put $A' = (A, b)$ and $r = \operatorname{rank} A$. If $\operatorname{rank} A' = r + 1$, then, for $k > M$,

$$\kappa_k(A') - \kappa_k(A) = k - v_{r+1}(A') + v_r(A) \geq k - M > 0,$$

so the system has no solutions. If the ranks agree, the equality of the two image exponents is equivalent to $v_r(A') = v_r(A)$. If these orders differ the contribution is zero. If they agree, the contribution of $k > M$ to the counting series is

$$\sum_{k > M} Z^{k(d-r)+v_r(A)} U^k = \frac{Z^{(M+1)(d-r)+v_r(A)} U^{M+1}}{1 - Z^{d-r} U}.$$

The terms with $k \leq M$ are polynomials in Z, U. Taking the integral linear combination with coefficients c_I gives the displayed rational form of $\sum_{k \geq 1} P_k(Z) U^k$. □

For the coefficient equations (4.2), Proposition A.1 gives the bound 14400 for the exceptional primes in (5.1). Corollary A.2 gives the corresponding five-prime interpolation criterion.

Part 3. Flat bundles and lens orbifolds

6. Flat bundles on lens spaces with two weights

Fix an odd integer $q \geq 3$, an integer $m \geq 2$, and a unit $a \in (\mathbb{Z}/q\mathbb{Z})^\times$. Let

$$Q = L(q; 1^m, a) = S^{2m+1}/\langle h \rangle, \qquad h(z_1, \dots, z_m, w) = (\zeta z_1, \dots, \zeta z_m, \zeta^a w), \qquad \zeta = e^{2\pi i/q}.$$

The twisted spherical multiplicities are given in [10, Theorems 2.3 and 3.1]; for line twists, the one-norm criterion is [10, Theorem 4.2]. Equip Q with the curvature-one round quotient metric. The action is free because $h^s(z, w) = (z, w)$ at a point of the sphere implies $s = 0$ or $as = 0$ modulo q, and a is a unit. For a finite-dimensional unitary representation ρ of C_q, let Δ_ρ be its flat connection Laplacian and set $P_\rho = (\Delta_\rho + m^2)^{1/2}$. Its frequencies are integers $k + m$, $k \geq 0$, with multiplicities $d_\rho(k)$.

Define the collision indices

$$\mathcal{C} = \{1 \leq s < q \,:\, (a-1)s \equiv 0 \pmod q \text{ or } (a+1)s \equiv 0 \pmod q\}, \qquad \mathcal{N} = \{1, \dots, q-1\} \setminus \mathcal{C}.$$

The two alternatives defining $\mathcal{C}$ are disjoint: both would imply $2s \equiv 0$, hence $s \equiv 0$, since q is odd. The set $\mathcal{C}$ consists exactly of the nonidentity powers that are Clifford translations of the round sphere. Indeed the cosine of their displacement is $\cos(2\pi s/q) \sum_{j=1}^m |z_j|^2 + \cos(2\pi as/q)|w|^2$, which is independent of the point exactly in the displayed congruence cases.

Proposition 6.1. *Put $A_\pm = \ker(a \pm 1 : \mathbb{Z}/q\mathbb{Z} \to \mathbb{Z}/q\mathbb{Z})$. Then*

$$\mathcal{C} = (A_- \cup A_+) \setminus \{0\}, \qquad A_- \cap A_+ = \{0\}, \qquad |\mathcal{C}| = \gcd(q, a-1) + \gcd(q, a+1) - 2.$$

The set $\mathcal{C} \cup \{0\}$ is a subgroup exactly when at least one of A_- and A_+ is trivial.

Proof. The kernel description is the definition of $\mathcal{C}$. An index in both kernels is killed by two and hence is zero. For $g_\pm = \gcd(q, a \pm 1)$, the corresponding kernel is $\{jq/g_\pm \,:\, 0 \leq j < g_\pm\}$, which gives the cardinality formula. If both kernels contain nonzero elements u, v, then $u + v$ belongs to neither kernel: membership in one would put the other element in their zero intersection. Their union is therefore not a subgroup. If one kernel is trivial, the union is the other kernel and is a subgroup. For $q = 15$, $a = 4$, the two kernels are $\{0, 5, 10\}$ and $\{0, 3, 6, 9, 12\}$; their union omits $3 + 5 = 8$. □

Theorem 6.2 (Flat spectral rigidity). *Let $q \geq 3$ be odd, let $m \geq 2$, and let a be a unit modulo q. For any two finite-dimensional unitary representations ρ, σ of C_q, the following are equivalent:*

(1) For a fixed $0 \leq \eta < 1$ and a sequence $R_\nu \to \infty$, $E_\eta(R_\nu; P_\rho, P_\sigma) = o(R_\nu^m)$.

(2) $\operatorname{Re} \chi_\rho(h^s) = \operatorname{Re} \chi_\sigma(h^s)$ for every s.

(3) $\rho \oplus \overline{\rho} \cong \sigma \oplus \overline{\sigma}$.

(4) Δ_ρ and Δ_σ are exactly isospectral.

Suppose their ranks agree, and put $c_s = \operatorname{Re} \chi_\rho(h^s) - \operatorname{Re} \chi_\sigma(h^s)$. If these characters differ, set

$$r = \begin{cases} m+1, & \text{some } c_s \neq 0 \text{ with } s \in \mathcal{C}, \\ m, & c_s = 0 \text{ on } \mathcal{C} \text{ and some } c_s \neq 0 \text{ on } \mathcal{N}. \end{cases}$$

Then for every fixed $0 \leq \eta < 1$,

$$E_\eta(R; P_\rho, P_\sigma) = \kappa_r R^r + O(R^{r-1}), \qquad \kappa_r > 0.$$

In particular, among equal-rank representations, $E_\eta(R) = o(R^{m+1})$ is equivalent to $c_s = 0$ on $\mathcal{C}$. The coefficient κ_r is given explicitly in (6.6) below. If the ranks differ, the defect has order $\Theta(R^{2m+1})$.

Proof. All integer frequencies $k + m$ arise on the sphere before taking equivariants. Let $\mathcal{P}_k$ be the complex homogeneous polynomials of degree k on $\mathbb{R}^{2m+2}$, put $\mathcal{D} = \sum_{i=1}^{2m+2} \partial_{x_i}^2$ and $\mathcal{H}_k = \ker(\mathcal{D}|_{\mathcal{P}_k})$. For $\phi \in \mathcal{H}_\ell$,

$$\mathcal{D}(|x|^{2j}\phi) = 4j(\ell + m + j)|x|^{2j-2}\phi \qquad (j \geq 1).$$

Starting with $\mathcal{P}_k = \mathcal{H}_k$ for $k = 0, 1$, induction gives $\mathcal{P}_k = \bigoplus_{j=0}^{\lfloor k/2 \rfloor} |x|^{2j}\mathcal{H}_{k-2j}$: the displayed nonzero scalars make the restriction of $\mathcal{D}$ from $|x|^2\mathcal{P}_{k-2}$ onto $\mathcal{P}_{k-2}$ invertible. The decomposition is orthogonally equivariant. For a diagonal transformation with eigenvalues α_i on linear polynomials, the monomial basis gives

$$\sum_{k\geq 0} \operatorname{tr}(h|_{\mathcal{P}_k}) z^k = \prod_i (1-\alpha_i z)^{-1}.$$

Taking the trace of the projection $q^{-1}\sum_{j=0}^{q-1} h^j$ on $\mathcal{H}_k \otimes \rho^*$ yields

$$\sum_{k\geq 0} d_\rho(k) z^k = \frac{1-z^2}{q} \sum_{j=0}^{q-1} \frac{\operatorname{Re}\chi_\rho(h^j)}{(1-z\zeta^j)^m(1-z\zeta^{-j})^m(1-z\zeta^{aj})(1-z\zeta^{-aj})}. \tag{6.1}$$

Pairing j and $-j$ replaces the character by its real part. Equality of real characters therefore implies exact equality of the spectra.

Distinct shifted frequencies are separated by one. For $\eta < 1$ an optimal matching can only pair equal frequency values, so

$$E_\eta(R) = \sum_{k+m\leq R} |d_\rho(k) - d_\sigma(k)|. \tag{6.2}$$

Put $c_0 = \dim\rho - \dim\sigma$. The identity contribution is

$$\frac{c_0}{q}\frac{1+z}{(1-z)^{2m+1}}.$$

Every nonidentity pole has order at most $m+1$. Hence

$$d_\rho(k) - d_\sigma(k) = \frac{2c_0}{q(2m)!}k^{2m} + O(k^{2m-1}).$$

If $c_0 \neq 0$, summation of absolute values gives

$$E_\eta(R) = \frac{2|c_0|}{q(2m+1)!}R^{2m+1} + O(R^{2m}).$$

Thus (1) forces $c_0 = 0$. Assume this equality below.

Put $D(z) = \sum_{k\geq 0}(d_\rho(k) - d_\sigma(k))z^k$. The identity term in its Molien formula vanishes, and every summand is $O(z^{-2m})$ at infinity. At $z = \zeta^s$ the only possible contributing indices are $j = \pm s$ and $j = \pm a^{-1}s$. If $s \in \mathcal{N}$, these four indices are distinct; the first pair has poles of order m, and the second pair has poles of order one. In the local coordinate $1 - z/\zeta^s$, the coefficient of the order-m term is therefore

$$C_{s,m} = \frac{2c_s}{q(1-\zeta^{2s})^{m-1}(1-\zeta^{(a+1)s})(1-\zeta^{(1-a)s})}. \tag{6.3}$$

Every denominator is nonzero by $s \in \mathcal{N}$ and oddness of q.

If $s \in \mathcal{C}$, the two pairs of indices coincide. At these indices the repeated and exceptional rotation eigenvalues coincide, up to sign. Their denominators are $(1-z\zeta^s)^{m+1}(1-z\zeta^{-s})^{m+1}$. Consequently the coefficient of the order-$(m+1)$ term is

$$C_{s,m+1} = \frac{2c_s}{q(1-\zeta^{2s})^m}. \tag{6.4}$$

It is nonzero exactly when $c_s \neq 0$.

If $c_s = 0$ on $\mathcal{C}$, the complete Molien summands with these indices vanish, not merely their leading pole coefficients. Thus, when c is nonzero, the largest pole order of D is precisely the integer r in the theorem. Define $S_{m+1} = \mathcal{C}$ and $S_m = \mathcal{N}$, and, in the applicable case, define

$$\tau_{s,m+1} = \frac{2\zeta^{sm}}{q\,m!(1-\zeta^{2s})^m}, \qquad \tau_{s,m} = \frac{2\zeta^{sm}}{q(m-1)!(1-\zeta^{2s})^{m-1}(1-\zeta^{(a+1)s})(1-\zeta^{(1-a)s})},$$

$$\gamma_b = \sum_{s\in S_r} \tau_{s,r} c_s \zeta^{-sb}, \qquad b \in \mathbb{Z}/q\mathbb{Z}. \tag{6.5}$$

Partial fractions give

$$d_\rho(\mu-m) - d_\sigma(\mu-m) = \gamma_{\mu \bmod q}\mu^{r-1} + O(\mu^{r-2}).$$

Indeed $(1-z/\zeta^s)^{-r}$ has coefficient $\zeta^{-sk}\binom{k+r-1}{r-1}$, and $\mu = k+m$. Along $\mu \equiv b \pmod q$, multiplication by μ^{1-r} and passage to the limit show that $\gamma_b \in \mathbb{R}$. Since $r \geq 2$, summing the error along each residue class contributes $O(R^{r-1})$. Moreover

$$\sum_{\substack{\mu\leq R\\ \mu\equiv b\ (q)}} \mu^{r-1} = \frac{R^r}{rq} + O(R^{r-1}).$$

The inequality $\big||x+y|-|x|\big| \leq |y|$ and (6.2) now yield

$$\kappa_r = \frac{1}{rq}\sum_{b=0}^{q-1} |\gamma_b|. \tag{6.6}$$

For $s \in S_r$, Fourier inversion gives

$$\tau_{s,r}c_s = \frac{1}{q}\sum_{b=0}^{q-1} \gamma_b \zeta^{sb},$$

because $q^{-1}\sum_b \zeta^{(s-t)b}$ is 1 for $s = t$ modulo q and 0 otherwise. Every $\tau_{s,r}$ is nonzero, and some c_s on S_r is nonzero by the definition of r. Hence $\kappa_r > 0$.

This defect law proves that (1) forces $c = 0$, even along a selected unbounded sequence of cutoffs. Finally, writing $\rho = \bigoplus_t n_t\chi_t$ and $\sigma = \bigoplus_t n'_t\chi_t$, Fourier inversion shows that equal real characters are equivalent to $n_0 = n'_0$ and $n_t + n_{-t} = n'_t + n'_{-t}$ for $t \neq 0$. These are exactly the equalities in (3). Exact isospectrality implies (1), so conditions (1)–(4) are equivalent. □

6.1. **Two discrepancy orders.** On $L(15;1,1,4)$, the collision set is $\{3,5,6,9,10,12\}$. For $\rho = \chi_1$ and $\sigma = \chi_2$, the nonzero Fourier coefficients are $\tau_{3,3}c_3 = \tau_{12,3}c_{12} = (1-\sqrt{5})/60$ and $\tau_{6,3}c_6 = \tau_{9,3}c_9 = (1+\sqrt{5})/60$. Thus the sequence in (6.5) is 5-periodic, with values

$$(\gamma_0,\dots,\gamma_4) = (1/15, -1/10, 1/15, 1/15, -1/10).$$

Its absolute sum over a period of length 15 is $6/5$. Therefore

$$E_\eta(R) = \frac{2}{75}R^3 + O(R^2) \qquad (0 \leq \eta < 1).$$

For the equal-rank pair

$$\rho = 4\chi_0 + \chi_2 + \chi_7 + \chi_8 + \chi_{13}, \qquad \sigma = 2\chi_3 + 2\chi_{12} + 2\chi_5 + 2\chi_{10},$$

the real-character difference is

$$c_s = (2 - 2\cos(2\pi\cdot 3s/15))(2 - 2\cos(2\pi\cdot 5s/15)).$$

It vanishes on the collision set. For $s \in \mathcal{N}$,

$$\tau_{s,2}c_s = \frac{2}{15}\zeta^{2s}(1-\zeta^{3s})\sum_{j=0}^{4}\zeta^{2js}.$$

The expression on the right also vanishes for $s \in \mathcal{C}\cup\{0\}$. Fourier inversion therefore gives $\gamma_b = 2$ for $b = 2,4,6,8,10$, $\gamma_b = -2$ for $b = 5,7,9,11,13$, and zero at the other residues modulo 15. Consequently

$$E_\eta(R) = \frac{2}{3}R^2 + O(R) \qquad (0 \leq \eta < 1).$$

7. The three-dimensional flat-bundle kernel

For an odd $q \geq 3$ and a unit a modulo q, set

$$\mathcal{C} = \{s \in \mathbb{Z}/q\mathbb{Z} \setminus \{0\} : (a-1)s = 0 \text{ or } (a+1)s = 0\}, \qquad \mathcal{N} = (\mathbb{Z}/q\mathbb{Z} \setminus \{0\}) \setminus \mathcal{C}.$$

Both sets are invariant under multiplication by a and under sign. Let ρ, σ be finite-dimensional unitary representations of C_q. Write $\chi_b(h^s) = e^{2\pi i b s/q}$, $c_s = \operatorname{Re}\chi_\rho(h^s) - \operatorname{Re}\chi_\sigma(h^s)$, and let $d_\rho(k)$ be the multiplicity of $k(k+2)$ on the curvature-one quotient $L(q;1,a)$. The shifted frequency is $k+1$.

Set

$$\mathscr{R}_q = \big(\mathbb{Q}[X]/(X^q-1)\big)^\iota, \qquad \iota(X) = X^{-1}, \qquad \mathscr{R}_q^0 = \{f \in \mathscr{R}_q : f(1) = 0\}.$$

Through $X^b \mapsto \chi_b$, the algebra $\mathscr{R}_q$ identifies with the rational span of the real parts of characters. The augmentation $f \mapsto f(1)$ is the rank.

Theorem 7.1 (The three-dimensional spectral kernel). *On $L(q;1,a)$, two flat connection Laplacians are isospectral if and only if their ranks agree and their real-character difference satisfies*

$$c_s = 0 \quad (s \in \mathcal{C}), \qquad \frac{c_s}{t_s - t_{as}} = \frac{c_{a^{-1}s}}{t_{a^{-1}s} - t_s} \quad (s \in \mathcal{N}), \qquad t_s = \cos(2\pi s/q).$$

For each divisor $d > 1$ of q, put $\zeta_d = e^{2\pi i/d}$, $K_d = \mathbb{Q}(\zeta_d + \zeta_d^{-1})$, and $\sigma_a(\zeta_d + \zeta_d^{-1}) = \zeta_d^a + \zeta_d^{-a}$. The spectral kernel on $\mathscr{R}_q^0$, under its cyclotomic decomposition, is exactly the product of the following subspaces, one for each $d \mid q$, $d > 1$:

$$\begin{cases} \{0\}, & a \equiv \pm 1 \pmod d, \\ \big(\cos(2\pi/d) - \cos(2\pi a/d)\big) K_d^{\langle \sigma_a \rangle}, & a \not\equiv \pm 1 \pmod d. \end{cases}$$

Consequently its rational dimension is

$$\sum_{\substack{d \mid q, d>1 \\ a \not\equiv \pm 1 \pmod d}} \frac{\varphi(d)}{2 \operatorname{ord}_{(\mathbb{Z}/d\mathbb{Z})^\times/\{\pm 1\}}(a)}.$$

For fixed $0 \leq \eta < 1$, the complete-list defect has the following exhaustive classification:

(1) If the ranks differ, $E_\eta(R) = \Theta(R^3)$.
(2) If the ranks agree and c is nonzero on $\mathcal{C}$, then $E_\eta(R) = \kappa_2 R^2 + O(R)$ with $\kappa_2 > 0$.
(3) If the ranks agree, c vanishes on $\mathcal{C}$, and c is outside the displayed kernel, then $E_\eta(R) = \kappa_1 R + O(1)$ with $\kappa_1 > 0$.
(4) If c belongs to the displayed kernel and the ranks agree, $E_\eta(R) = 0$ for every R.

In the third case, $b_k = d_\rho(k) - d_\sigma(k)$ is exactly q-periodic and $\kappa_1 = q^{-1}\sum_{k=0}^{q-1}|b_k|$. In the second case, putting $\zeta = e^{2\pi i/q}$ gives the exact leading coefficient

$$\gamma_b = \sum_{s \in \mathcal{C}} \frac{2\zeta^s c_s}{q(1-\zeta^{2s})} \zeta^{-sb}, \qquad \kappa_2 = \frac{1}{2q}\sum_{b=0}^{q-1}|\gamma_b|.$$

The γ_b are real. Thus $E_\eta = o(R)$ is equivalent to exact isospectrality also for every odd composite modulus.

Proof. Put $\zeta = e^{2\pi i/q}$ and $D_s(z) = 1 - 2t_s z + z^2$. The harmonic decomposition in the proof of Theorem 6.2 gives, with $b_k = d_\rho(k) - d_\sigma(k)$,

$$B(z) := \sum_{k \geq 0} b_k z^k = \frac{1-z^2}{q} \sum_{s=0}^{q-1} \frac{c_s}{D_s(z) D_{as}(z)}. \tag{7.1}$$

The identity term is $c_0(1+z)/(q(1-z)^3)$, where $c_0 = \dim\rho - \dim\sigma$; the nonidentity terms are regular at $z = 1$. Isospectrality therefore forces $c_0 = 0$. Assume this rank equality while determining the

remaining kernel. At a nonidentity root $z = \zeta^s$ with $s \in C$, one has $a^{-1}s = \pm s$; hence only the paired terms indexed by s and $-s$ have a pole. It has order two, and its coefficient in the normalization $(1 - z\zeta^{-s})^{-2}$ is

$$C_{s,2} = \frac{2c_s}{q(1 - \zeta^{2s})}.$$

The denominator is nonzero because q is odd. Vanishing of the rational function therefore forces $c_s = 0$ for every collision index. Conversely, when these values vanish, all collision summands vanish identically. On $\mathcal{N}$, put $u_s = c_s/(t_s - t_{as})$. The denominators are nonzero by the definition of $\mathcal{N}$. The identity

$$\frac{c_s}{D_s D_{as}} = \frac{u_s}{2z}\left(\frac{1}{D_s} - \frac{1}{D_{as}}\right)$$

and the permutation $s \mapsto as$ of $\mathcal{N}$ give, when the collision terms vanish,

$$B(z) = \frac{1 - z^2}{2qz} \sum_{s \in \mathcal{N}} \frac{u_s - u_{a^{-1}s}}{D_s(z)}. \tag{7.2}$$

The factors D_s and $D_{s'}$ have a common root exactly when $s' = \pm s$. Since $u_s = u_{-s}$ and q is odd, neither z nor $1 - z^2$ vanishes at these nonidentity roots. Thus the right-hand side vanishes exactly when $u_s = u_{a^{-1}s}$ on $\mathcal{N}$.

The Chinese remainder theorem gives

$$\mathscr{R}_q \cong \mathbb{Q} \times \prod_{\substack{d \mid q \\ d > 1}} K_d.$$

The first factor is evaluation at $X = 1$, namely rank. Removing that factor gives the augmentation-zero algebra. The dth component of a real character difference is $c_{q/d}$; its other primitive-order-d values are the Galois conjugates. If $a \equiv \pm 1 \pmod{d}$ this component is zero by the double-pole argument. Otherwise divide it by $\cos(2\pi/d) - \cos(2\pi a/d)$ and apply the orbit-invariance criterion. This is precisely the fixed field of σ_a. Since $[K_d : \mathbb{Q}] = \varphi(d)/2$, the dimension formula follows. The real parts of virtual characters form the lattice $\sum_{b \in \mathbb{Z}/q\mathbb{Z}} \mathbb{Z} \operatorname{Re} \chi_b$. Its intersection with the displayed rational subspace gives the real-character differences with vanishing spectral multiplicities.

After removing the rank contribution, the generating function is $O(z^{-2})$ at infinity and has at most double poles at qth roots of unity. Its exact coefficients have the form

$$b_{\mu-1} = \mu\gamma_{\mu \bmod q} + \beta_{\mu \bmod q}, \qquad \mu \geq 1,$$

where β is a real q-periodic sequence. The double-pole coefficient above gives the stated Fourier formula for γ. If a collision value is nonzero, Fourier injectivity makes γ nonzero, and summing absolute values on the residue classes gives $\kappa_2 R^2 + O(R)$ with $\kappa_2 > 0$. If all collision values vanish, $\gamma = 0$, and the discrepancy is exactly q-periodic. Its absolute sum through frequency R is $Rq^{-1}\sum_{b=0}^{q-1} |\beta_b| + O(1)$, which proves the third case; if every β_b is zero, the defect vanishes for every R. For unequal ranks, the identity coefficients are $c_0(k+1)^2/q$, while the other coefficients are $O(k+1)$. Summing their absolute values gives $E_\eta(R) = |c_0|R^3/(3q) + O(R^2)$. Here, as in (6.2), $\eta < 1$ permits matching only equal integer frequencies. □

7.1. **Prime orders and the rational kernel.** Let $p \geq 5$ be prime, let $a \in \mathbb{F}_p^\times \setminus \{1, -1\}$, and put

$$Q = L(p; 1, a), \qquad h(z, w) = (\zeta z, \zeta^a w), \qquad \zeta = e^{2\pi i/p}.$$

The metric is the round quotient metric of curvature one. For a unitary representation ρ of $C_p = \langle h \rangle$, let $d_\rho(k)$ denote the multiplicity of $k(k+2)$ for its flat connection Laplacian. The shifted square root $P_\rho = (\Delta_\rho + 1)^{1/2}$ has frequency $k + 1$. Write

$$\chi_b(h^s) = \zeta^{bs}, \qquad t_s = \cos(2\pi s/p), \qquad c_s = \operatorname{Re} \chi_\rho(h^s) - \operatorname{Re} \chi_\sigma(h^s).$$

Both $a-1$ and $a+1$ are nonzero in $\mathbb{F}_p$. Thus $(a\pm 1)s = 0$ implies $s = 0$, so the collision set is empty. Theorem 7.1 gives isospectrality exactly when the ranks agree and

$$u_s = \frac{c_s}{t_s - t_{as}}, \qquad s \in \mathbb{F}_p^\times,$$

is constant on each a-orbit in $\mathbb{F}_p^\times/\{\pm 1\}$. Every denominator is nonzero. Fix $0 \le \eta < 1$. For equal ranks, $b_k = d_\rho(k) - d_\sigma(k)$ is exactly p-periodic and

$$E_\eta(R) = \kappa R + O(1), \qquad \kappa = \frac{1}{p}\sum_{k=0}^{p-1} |b_k|.$$

Here $\kappa > 0$ if b is nonzero, while unequal ranks give $E_\eta(R) = \Theta(R^3)$. Thus $E_\eta(R_\nu) = o(R_\nu)$ along an unbounded sequence is equivalent to exact isospectrality.

7.2. **The fixed-field form of the complete rational kernel.** Put

$$K = \mathbb{Q}(\zeta + \zeta^{-1}), \quad M = (p-1)/2, \quad \tau = \operatorname{ord}_{\mathbb{F}_p^\times/\{\pm 1\}}(a), \quad r = M/\tau.$$

Let $\mathcal{R}_p$ be the rational vector space spanned by the real parts of all characters of C_p, and let $\mathcal{R}_p^0 = \{c \in \mathcal{R}_p \,:\, c(1) = 0\}$. The notation $c(1)$ denotes the value at the group identity, namely the virtual rank; values at h^s are denoted c_s.

Corollary 7.2. *Evaluation $c \mapsto c_1 = c(h)$ identifies $\mathcal{R}_p^0$ with K. Under this identification the kernel of the twisted spectral multiplicity map is*

$$(t_1 - t_a)K^{\langle \sigma_a \rangle}, \qquad \sigma_a(\zeta + \zeta^{-1}) = \zeta^a + \zeta^{-a}. \tag{7.3}$$

In particular its rational dimension is exactly r.

Let $H = \langle -1, a\rangle \subset \mathbb{F}_p^\times$. For each multiplicative coset O of H, the virtual character

$$V_O = (\chi_1 + \chi_{-1} - \chi_a - \chi_{-a}) \sum_{b\in O} \chi_b$$

has rank zero and zero twisted spectral multiplicities. The r characters V_O form a basis of the rational real-character kernel. The real parts of virtual characters in this kernel form its intersection with $\sum_{b\in\mathbb{F}_p} \mathbb{Z}\operatorname{Re}\chi_b$.

Proof. For $q = p$, the Chinese remainder decomposition in Theorem 7.1 has the two factors $\mathbb{Q}$ and K, given by $c \mapsto c(1)$ and $c \mapsto c_1$. Thus evaluation identifies the zero-rank subspace with K. The component $d = p$ in that theorem gives (7.3); its dimension is $[K : \mathbb{Q}]/\tau = M/\tau = r$.

The Gaussian periods $\eta_O = \sum_{b\in O} \zeta^b$ lie in $K^{\langle\sigma_a\rangle}$. They are rationally independent: a relation among them gives a polynomial of degree at most $p-1$ vanishing at ζ with constant coefficient zero. Such a polynomial must be the zero multiple of the cyclotomic polynomial. There are r periods, so they form a rational basis of the fixed field. Evaluation of V_O at h is $2(t_1 - t_a)\eta_O$. Thus the V_O form the indicated basis and have zero spectral multiplicities. □

7.3. **Isospectral flat bundles with distinct realifications.** The case $a = 2$ for odd q occurs in [10, Example 4.10].

Corollary 7.3. *For every integer $q \ge 3$ and every unit $a \in (\mathbb{Z}/q\mathbb{Z})^\times$, the flat line Laplacians on the round lens space $L(q;1,a)$ twisted by χ_1 and χ_a are isospectral. If $a \not\equiv \pm 1 \pmod q$, their realified holonomy representations of C_q are inequivalent.*

Proof. Harmonic decomposition and averaging over C_q give (7.1) for every $q \ge 3$. Put $t_s = \cos(2\pi s/q)$ and $D_s(z) = 1 - 2t_s z + z^2$. For this pair, $c_s = t_s - t_{as}$. The identity

$$\frac{t_s - t_{as}}{D_s D_{as}} = \frac{1}{2z}\left(\frac{1}{D_s} - \frac{1}{D_{as}}\right)$$

holds without dividing by $t_s - t_{as}$, even when it is zero. Sum over all indices modulo q. Since multiplication by a permutes these indices, the sum vanishes. The Molien difference is zero. The realified holonomy representations have characters $2t_s$ and $2t_{as}$; these coincide for all s exactly when $a \equiv \pm 1 \pmod q$. □

For instance, on $L(13;1,3)$ the class of 3 has order 3 modulo sign, and the kernel dimension is $6/3 = 2$. One relation is $\operatorname{Re}\chi_1 - \operatorname{Re}\chi_3$. A second, independent relation is

$$\operatorname{Re}\chi_2 + \operatorname{Re}\chi_3 + \operatorname{Re}\chi_5 - \operatorname{Re}\chi_1 - 2\operatorname{Re}\chi_6. \tag{7.4}$$

Indeed, with $t_b = \cos(2\pi b/13)$,

$$2(t_1 - t_3)(t_1 + t_3 + t_4) = t_2 + t_3 + t_5 - t_1 - 2t_6,$$

and $t_1 + t_3 + t_4$ is invariant under σ_3. Consequently the rank-three bundles

$$\rho = \chi_2 \oplus \chi_3 \oplus \chi_5, \qquad \sigma = \chi_1 \oplus \chi_6 \oplus \chi_6$$

are exactly isospectral, with different realified holonomy. The second relation is independent of the first because its $\operatorname{Re}\chi_2$ coefficient is nonzero.

7.4. **Three-dimensional discrepancy coefficients.** On $L(9;1,2)$, the collision indices are $3, 6$ and $c_3 = c_6 = 3/2$ for the twists χ_0, χ_1. Thus $\gamma_b = 2\sin(2\pi b/3)/(3\sqrt{3})$ is the 3-periodic sequence $(0, 1/3, -1/3)$, and

$$E_\eta(R) = \frac{1}{9}R^2 + O(R) \qquad (0 \le \eta < 1).$$

For the twists χ_1, χ_4 on the same space, the discrepancy has generating function

$$\sum_{k\ge 0} b_k z^k = \frac{z(1-z^2)}{1+z^3+z^6} = \frac{z - z^3 - z^4 + z^6}{1 - z^9}.$$

Thus the discrepancy at frequency μ is the 9-periodic sequence

$$(b_{\mu-1})_{\mu \bmod 9} = (0, 0, 1, 0, -1, -1, 0, 1, 0).$$

Thus $E_\eta(R) = 4R/9 + O(1)$ for $0 \le \eta < 1$. On $L(15;1,4)$, the twists χ_1, χ_2 satisfy $c_3 = c_{12} = \sqrt{5}/2$, $c_6 = c_9 = -\sqrt{5}/2$, and $c_5 = c_{10} = 0$. Their leading sequence is

$$\gamma_b = \frac{\sqrt{5}}{15}\left(\frac{\sin(2\pi b/5)}{\sin(2\pi/5)} - \frac{\sin(4\pi b/5)}{\sin(4\pi/5)}\right),$$

whose values are $(0, 0, 1/3, -1/3, 0)$ modulo 5. Hence

$$E_\eta(R) = \frac{1}{15}R^2 + O(R) \qquad (0 \le \eta < 1).$$

8. Prime-square lens orbifolds and corresponding covers

Lauret proved that scalar-isospectral lens manifolds have scalar-isospectral corresponding covers [12, Theorem 4.3]. The extension to lens orbifolds was proposed in [12, Remark 4.4] and restated as [16, Conjecture 7.3]. For $d \mid q$, the corresponding covers are the quotients $L(q/d; w)$ and $L(q/d; w')$ by the respective subgroups of order q/d. At orders 9 and 25, scalar-isospectral effective lens orbifolds have isometric corresponding proper covers. For an odd prime p, the corresponding order-p covers of scalar-isospectral effective order-p^2 lens orbifolds are scalar isospectral if $s = 0$ or $2s + 4 > r(p-3)$, with r, s as in (8.1). In dimension three, scalar isospectrality already implies isometry for every group order [3, Theorem 3.1]. The scalar spectrum also determines the complete gcd multiset at every prime square. Lauret's reduction in [12, Remark 4.6] then shows that isospectrality in form degrees $0, \dots, n-2$ implies isospectrality in all degrees. All metrics in this section are round quotient metrics.

8.1. **The Molien series and the isotropy filtration.** Let p be prime, let $q = p^2$, and let $n \geq 1$. For $w = (w_1, \dots, w_n) \in (\mathbb{Z}/q\mathbb{Z})^n$, let $L(q; w)$ denote the quotient of the unit sphere in $\mathbb{C}^n$ by the cyclic action generated by

$$g_w(z_1, \dots, z_n) = (\zeta_q^{w_1} z_1, \dots, \zeta_q^{w_n} z_n), \qquad \zeta_q = e^{2\pi i/q}.$$

Assume that the action is effective, equivalently $\gcd(q, w_1, \dots, w_n) = 1$. Weights equal to zero are permitted. Write

$$\begin{aligned} b &= \#\{j \,:\, w_j = 0 \pmod{p^2}\}, \\ s &= \#\{j \,:\, p \mid w_j,\ w_j \neq 0 \pmod{p^2}\}, \\ r &= \#\{j \,:\, p \nmid w_j\}. \end{aligned} \tag{8.1}$$

Thus $n = b + s + r$ and $r \geq 1$. Each of b, s, r counts complex coordinate lines; the corresponding real dimensions are $2b, 2s, 2r$. Let $U = (u_1, \dots, u_r)$ be the reductions modulo p of the unit weights.

The Hilbert series of invariant complex polynomials on $\mathbb{R}^{2n}$ is

$$H_w(t) = \frac{1}{p^2} \sum_{k=0}^{p^2-1} \prod_{j=1}^{n} \frac{1}{(1 - t\zeta_{p^2}^{kw_j})(1 - t\zeta_{p^2}^{-kw_j})}. \tag{8.2}$$

The scalar spherical harmonic generating function is $(1 - t^2)H_w(t)$. Indeed, the harmonic decomposition in Theorem 6.2 holds on $\mathbb{R}^{2n}$: the corresponding coefficients $4j(\ell + n + j - 1)$ are positive for $j \geq 1$, including $n = 1$. It is equivariant under the cyclic action. A harmonic polynomial of degree k restricts to an eigenfunction with eigenvalue $k(k + 2n - 2)$. These spectra determine n and q. Indeed, the identity summand of the harmonic generating function is

$$\frac{1 + t}{q(1 - t)^{2n-1}}.$$

Its cumulative coefficient through degree K is $2K^{2n-1}/(q(2n-1)!) + O(K^{2n-2})$. Every nonidentity summand has a pole of order at most $\max(0, 2n-3)$ at 1; for $n = 1$ it vanishes there. At a root different from 1, its pole order is at most $2n - 1$: a root other than -1 occurs at most once in each paired coordinate factor, while the numerator $1 - t^2$ removes one factor at -1. Division by $1 - t$ therefore makes its cumulative coefficient $O(K^{2n-2})$. Since $k(k + 2n - 2) = k^2 + O(k)$, the scalar counting function is

$$N(\Lambda) = \frac{2}{q(2n-1)!}\Lambda^{n-1/2} + O(\Lambda^{n-1}).$$

Its exponent and leading coefficient determine n and q. The recovered value of n assigns each scalar multiplicity to its degree k, since $k(k+2n-2)$ is strictly increasing for $k \geq 0$. Thus scalar isospectrality is equivalent to equality of the Hilbert series; see also [12, Section 2.3] and [16, Section 3].

Define

$$R_U(t) = \sum_{a=1}^{p-1} \prod_{j=1}^{r} \frac{1}{(1 - t\zeta_p^{au_j})(1 - t\zeta_p^{-au_j})}. \tag{8.3}$$

Separate in (8.2) the identity, the elements of order p, and the elements of order p^2. This gives

$$p^2(1 - t)^{2b} H_w(t) = (1 - t)^{-2(r+s)} + (1 - t)^{-2s} R_U(t) + T_w(t), \tag{8.4}$$

where T_w is the sum over k prime to p after deleting the b zero-weight coordinate lines. The function T_w is analytic at $t = 1$: for such k, every nonzero weight w_j satisfies $kw_j \neq 0 \pmod{p^2}$. Both R_U and T_w can have poles at other roots of unity.

Theorem 8.1. *The scalar spectrum of an effective order-p^2 lens orbifold determines b, s, r. In particular, scalar-isospectral lens orbifolds $L(p^2; w)$ and $L(p^2; w')$ have equal multisets*

$$\{\!\{\gcd(p^2, w_j) \,:\, 1 \leq j \leq n\}\!\} = \{\!\{1^r, p^s, (p^2)^b\}\!\}. \tag{8.5}$$

Proof. The coefficient of t in H_w is the dimension of the invariant linear polynomials, namely $2b$. Thus b is determined. Remove the known identity contribution and set

$$K_w(t) = p^2(1-t)^{2b}H_w(t) - (1-t)^{-2(n-b)}. \tag{8.6}$$

By (8.4),

$$K_w(t) = (1-t)^{-2s}R_U(t) + T_w(t).$$

The function R_U is analytic at 1, and

$$R_U(1) = \sum_{a=1}^{p-1}\prod_{j=1}^{r}\frac{1}{4\sin^2(\pi a u_j/p)} > 0. \tag{8.7}$$

If $s > 0$, K_w therefore has a pole of exact order $2s$ at 1. If $s = 0$, it is analytic there. This determines s, and then $r = n - b - s$. □

8.2. **The reciprocal residual polynomial.** The order-p subgroup is generated by g_w^p. Its weight list consists of U and $b + s$ zero weights, so the Hilbert series of its spherical quotient is

$$H_w^{(p)}(t) = \frac{1}{p}\left[(1-t)^{-2n} + (1-t)^{-2(b+s)}R_U(t)\right]. \tag{8.8}$$

Suppose $H_w = H_{w'}$. Theorem 8.1 gives common values of r, s, b. Write $\Delta R = R_U - R_{U'}$ and $\Delta T = T_w - T_{w'}$. Equation (8.4) implies

$$\Delta R(t) = -(1-t)^{2s}\Delta T(t). \tag{8.9}$$

In particular, if $s > 0$, the function ΔR has a zero of order at least $2s$ at $t = 1$.

Theorem 8.2. *Let p be odd, and let $L(p^2; w)$ and $L(p^2; w')$ be scalar-isospectral effective lens orbifolds with $s > 0$. Put $\Phi_p(t) = 1 + t + \cdots + t^{p-1}$ and*

$$E = r(p-3) - 2s - 4.$$

There is a polynomial $P \in \mathbb{Q}[t]$ such that

$$\Delta R(t) = \frac{t^2(1-t)^{2s}P(t)}{\Phi_p(t)^r}, \qquad P(t) = t^E P(1/t). \tag{8.10}$$

If $E < 0$, then $P = 0$. If $P \neq 0$, then $E \geq 0$ and $\deg P \leq E$. In particular, the corresponding order-p covers are scalar isospectral whenever

$$2s + 4 > r(p-3). \tag{8.11}$$

Their Hilbert-series difference is always

$$H_w^{(p)}(t) - H_{w'}^{(p)}(t) = \frac{t^2P(t)}{p(1-t)^{2b}\Phi_p(t)^r}. \tag{8.12}$$

Proof. Every root of each denominator in (8.3) is a primitive p-th root. The two roots contributed by one coordinate line are distinct, since p is odd and $u_j \neq 0$. The multiplicity at a fixed primitive root is at most r. Hence

$$N(t) = \Phi_p(t)^r\Delta R(t)$$

is a polynomial. The Galois group of $\mathbb{Q}(\zeta_p)/\mathbb{Q}$ permutes the summands of R_U, so $N \in \mathbb{Q}[t]$. Repeated weights are counted with multiplicity in the bound r.

Each summand in (8.3) is $O(t^{-2r})$ at infinity. Therefore

$$\deg N \leq D := r(p-1) - 2r = r(p-3). \tag{8.13}$$

Each paired denominator is reciprocal. Consequently

$$\Delta R(1/t) = t^{2r}\Delta R(t), \qquad N(t) = t^D N(1/t).$$

Moreover,

$$R_U(0) = p - 1, \qquad R'_U(0) = \sum_{j=1}^{r} \sum_{a=1}^{p-1} (\zeta_p^{au_j} + \zeta_p^{-au_j}) = -2r. \tag{8.14}$$

The same identities hold for U'. Thus t^2 divides N. Its reciprocity forces the two highest formal coefficients to vanish as well; a nonzero N has degree at most $D - 2$.

Since $\Phi_p(1) = p$, equation (8.9) implies that $(1-t)^{2s}$ divides N. The two factors t^2 and $(1-t)^{2s}$ are coprime, and hence $N = t^2(1-t)^{2s}P$ for $P \in \mathbb{Q}[t]$. Substitution into the reciprocity identity for N gives $P(t) = t^{D-2s-4}P(1/t)$. If $P \neq 0$, comparison of the smallest exponents on the two sides gives

$$\operatorname{ord}_0 P + \deg P = D - 2s - 4 = E.$$

Both terms on the left are nonnegative. Thus $E \geq 0$ and $\deg P \leq E$. This proves (8.10) and (8.11). Finally, (8.8) gives (8.12). □

Remark 8.3. Assume p is odd. For $c \in \mathbb{F}_p^*/\{\pm 1\}$, let $m_c(U)$ denote the multiplicity of the corresponding real character in U. Set

$$M = \max_c \{m_c(U), m_c(U')\}.$$

Multiplication by $a \in \mathbb{F}_p^\times$ permutes the real characters, so each summand denominator in (8.3) divides Φ_p^M. Consequently $N_M = \Phi_p^M \Delta R \in \mathbb{Q}[t]$ satisfies

$$\deg N_M \leq M(p-1) - 2r, \qquad N_M(t) = t^{M(p-1)-2r} N_M(1/t).$$

Equations (8.9) and (8.14) give $t^2(1-t)^{2s} \mid N_M$. Hence

$$\Delta R(t) = \frac{t^2(1-t)^{2s} P_M(t)}{\Phi_p(t)^M}, \qquad P_M(t) = t^{E_M} P_M(1/t),$$

where $E_M = M(p-1) - 2r - 2s - 4$. The criterion $E_M < 0$ can be stronger than (8.11); the latter depends only on the spectrally determined integers r, s. At the first boundary $E = 0$ in (8.10), P must be constant.

Proposition 8.4. *If $s = 0$, scalar-isospectral effective order-p^2 lens orbifolds have scalar-isospectral corresponding order-p covers.*

Proof. Remove the common b-dimensional trivial complex blocks. Every remaining weight is a unit modulo p^2. The poles of R_U and $R_{U'}$ are primitive pth roots of unity; the poles of T_w and $T_{w'}$ are primitive p^2-th roots. Equation (8.9), with $s = 0$, gives $\Delta R = -\Delta T$. The two possible pole sets are disjoint, so this common rational function has no finite pole. Each summand tends to zero at infinity; therefore $\Delta R = 0$. Formula (8.8) gives equality of the cover Hilbert series. □

8.3. **Orders nine and twenty-five.**

Theorem 8.5 (Isometric corresponding covers). *Let $q = 9$ or $q = 25$. If two effective lens orbifolds $L(q; w)$ and $L(q; w')$ are scalar isospectral, then their corresponding proper orbifold covers are isometric. In particular, the corresponding-cover conjecture holds for these group orders in every odd dimension.*

Proof. The trivial subgroup gives the common round sphere. Let H be the subgroup of order p. When $p = 3$, there is only one nontrivial real character of C_3. Thus the order-3 representation is determined by its r nontrivial and $b + s$ trivial complex coordinate lines. These numbers coincide by Theorem 8.1. Permute the matching coordinate lines and conjugate any complex coordinate whose nonzero weight has the opposite sign. This real orthogonal map conjugates the generators, and hence induces an isometry of the quotients.

Let $p = 5$. Up to individual signs, the nontrivial real characters are represented by 1 and 2. Write their multiplicities in U as x and $r - x$. Set

$$A = 4\sin^2(\pi/5) = \frac{5 - \sqrt{5}}{2}, \qquad B = 4\sin^2(2\pi/5) = \frac{5 + \sqrt{5}}{2}.$$

The two pairs of nonidentity elements either preserve or exchange the two character classes. The order-five expression of Álzaga and Lauret [1, Section 4, equation (4.3)] therefore gives

$$\begin{aligned} R_U(1) &= 2\big(A^{-x}B^{-(r-x)} + A^{-(r-x)}B^{-x}\big) \\ &= 4(AB)^{-r/2}\cosh\big((x-r/2)\log(B/A)\big). \end{aligned} \tag{8.15}$$

For fixed r, this determines $|x - r/2|$ and therefore the unordered pair $\{x, r-x\}$, because $B > A > 0$ and cosh is strictly increasing on the nonnegative real axis.

If $s > 0$, equation (8.9) gives $R_U(1) = R_{U'}(1)$. If $s = 0$, Proposition 8.4 gives equality of the entire R-functions. Thus in either case the unordered pairs of character multiplicities agree. Multiplication of a generator of C_5 by 2 exchanges these two real characters. The trivial-block dimensions also agree by Theorem 8.1. After replacing one generator by its square if the multiplicities are exchanged, a permutation and individual complex conjugations match all coordinate weights. The resulting real orthogonal map conjugates the order-5 subgroups and induces an isometry of the corresponding covers. □

8.4. **Recovery of the missing form degree.**

Corollary 8.6. *Let $n \geq 2$ and let p be any prime. Suppose two effective $(2n-1)$-dimensional lens orbifolds of order p^2 are isospectral on differential forms in every degree $0, 1, \dots, n-2$. Then they are isospectral on differential forms in every degree.*

Proof. For a weight list w, let

$$\Lambda_w = \left\{ a \in \mathbb{Z}^n \,:\, \sum_{j=1}^{n} a_j w_j = 0 \pmod{p^2} \right\}.$$

For $0 \leq \ell \leq n$, define the coordinate-refined one-norm generating series

$$Z_w^{(\ell)}(z) = \sum_{\substack{a \in \Lambda_w \\ \#\{j\,:\,a_j=0\}=\ell}} z^{\|a\|_1}.$$

The series $Z_w^{(n)}$ equals 1. The vectors counted by $Z_w^{(n-1)}$ are the nonzero axis vectors ke_j. Such a vector belongs to Λ_w precisely when $p^2/\gcd(p^2, w_j)$ divides k. Therefore

$$Z_w^{(n-1)}(z) = 2\sum_{j=1}^{n} \frac{z^{p^2/\gcd(p^2,w_j)}}{1 - z^{p^2/\gcd(p^2,w_j)}}. \tag{8.16}$$

Theorem 8.1, applied to the scalar spectra included in the hypothesis, proves equality of these series for w and w'.

Put $\delta_\ell = Z_w^{(\ell)} - Z_{w'}^{(\ell)}$. Thus $\delta_n = \delta_{n-1} = 0$. By the form-spectrum criterion for lens orbifolds [11, Corollary 2.3], the assumed form-spectrum equalities imply

$$\sum_{\ell=0}^{n} \ell^h \delta_\ell(z) = 0, \qquad 0 \leq h \leq n-2, \tag{8.17}$$

where the $h = 0$ coefficient is 1 for every ℓ. After deleting the two zero terms, this is the Vandermonde system at $0, 1, \dots, n-2$. Its determinant is $\prod_{0 \leq i < j \leq n-2}(j-i) \neq 0$, so every δ_ℓ vanishes. For $n = 2$ this is the one-by-one matrix (1), with empty determinant product equal to 1. The criterion gives equality of all form spectra in degrees $0, \dots, n-1$. The cyclic actions preserve orientation, so the Hodge star descends to each quotient and satisfies $* \, \Delta_j = \Delta_{2n-1-j} \, *$. It identifies the eigenspaces in complementary degrees, giving equality of the remaining spectra. □

Appendix A. Coefficient bounds for prime-square congruences

For each prime p, put

$$\mathcal{T}_p = \{(X, Y) \in (\mathbb{Z}/p^2\mathbb{Z})^2 \,:\, C_s(p; X, Y) = 0 \text{ for every } s \in \mathbb{Z}/p^2\mathbb{Z}\},$$

where C_s is given by (4.2). The factorization of the six products $m_{d,e}Q_{d,e}$ in (4.1) gives the following ranges of w-exponents:

(d, e)	Range of the w-exponents in $m_{d,e}Q_{d,e}$
$\pm(1, 3)$	$[-18, 30]$
$\pm(-1, 2)$	$[-23, 35]$
$\pm(-3, -2)$	$[-15, 27]$

Consequently the exponents of the nonzero monomials of J satisfy

$$0 \le a_\nu \le 28, \qquad -23 \le b_\nu \le 35, \qquad |c_\nu|, |d_\nu| \le 3.$$

Proposition A.1. *There is a polynomial $P(Z) \in \mathbb{Z}[Z]$ of degree at most four such that $|\mathcal{T}_p| = P(p)$ for every prime $p > 14400$.*

Proof. For two signed terms, equality of their exponents modulo p^2 is an affine congruence whose coefficient row, with p replaced by T, is

$$\begin{aligned}&(\epsilon - \epsilon' + T(\epsilon c_\nu - \epsilon' c_\mu),\\ &\delta - \delta' + T(\delta d_\nu - \delta' d_\mu),\\ &a_\mu - a_\nu + T(b_\mu - b_\nu)).\end{aligned}$$

The absolute values of its coefficients, in increasing powers of T, are bounded by the corresponding coefficients of

$$2 + 6T, \qquad 2 + 6T, \qquad 28 + 58T.$$

Every coefficient of a three-row determinant is therefore bounded by the corresponding coefficient of

$$6(2 + 6T)^2(28 + 58T) = 672 + 5424T + 14400T^2 + 12528T^3.$$

For a two-row minor involving the last column, the corresponding bound is

$$2(2 + 6T)(28 + 58T) = 112 + 568T + 696T^2.$$

A two-row minor in the first two columns is bounded by $2(2 + 6T)^2 = 8 + 48T + 72T^2$, and a one-row minor has every coefficient bounded by 58. Thus the first nonzero coefficient of every nonzero square minor has absolute value at most 14400. No prime greater than 14400 belongs to its exceptional set.

The truth values of all pairwise exponent equalities determine a partition of the signed terms into equal-exponent classes. Membership in $\mathcal{T}_p$ means that the integer coefficient sum in every class is zero. It is consequently one fixed Boolean function of the pairwise equality predicates. Theorem 5.1, with $d = k = 2$, gives the polynomial P. □

The same coefficient bounds hold after adjoining the four wall equations $X = 0$, $Y = 0$, $X = Y$, $X = -Y$, the grid equations $TX = 0$, $TY = 0$ modulo p^2, and the equations for the signed permutations of $(3, 1), (5, 1), (5, 3), (7, 1), (7, 3)$. A grid equation specialized at $T = p$ is equivalent to divisibility of the corresponding coordinate by p.

Corollary A.2. *For every prime $p > 14400$, let $S_p \subset \mathcal{T}_p$. Suppose that $|S_p|$ is a polynomial in p of degree at most four. If $|S_p| = |\mathcal{T}_p|$ at five distinct primes greater than 14400, then $S_p = \mathcal{T}_p$ for every prime greater than 14400.*

Proof. The difference of the two counting polynomials has degree at most four and has five distinct roots. Hence the polynomials are equal. The inclusion and equality of finite cardinalities give $S_p = \mathcal{T}_p$. □

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