LOCAL SPECTRAL SECTIONS AND TOPOLOGY OF EIGENSPACES

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Abstract. Let $\lambda > 0$ be an isolated Laplace eigenvalue on a closed connected smooth manifold of dimension at least two. We construct a local right inverse of its symmetric spectral-cluster matrix in the reference conformal class. The inverse is $1/2$-Hölder continuous and smooth off the initial matrix. The exponent is optimal at a conformally unstable reference. Metrics at which the cluster derivative is surjective approximate the reference while preserving both λ and its multiplicity. This proves weak conformal stability. A separate signature bound gives unrestricted strong stability through multiplicity seven, with failure possible at eight. In either the full metric space or a conformal class, the fixed-cluster multiplicity locus is a C^1 embedded submanifold precisely at strongly stable points.

The same second-variation argument gives local matrix sections for $\Delta_g + V$ under scalar potentials supported in any prescribed nonempty interior open set, with the metric and the Dirichlet or Neumann realization fixed. Here the reference eigenvalue may be any real number. The analytic step is a sign-changing principal symbol for every nonzero contracted cokernel Hessian; each resulting quadratic form has infinite positive and negative index. Finite-dimensional regular-zero arguments then give exact realizations.

The local sections realize Grassmannians as retracts of internal spectral-gap spaces. Localized, volume-preserving metric variations on round spheres yield prescribed spectral subbundles; a pairwise-isometric family of metrics near a scaled round metric on the four-sphere has fixed exterior geometry, positive Ricci curvature, and a nontrivial rank-three bundle for the first three positive eigenvalues, counted with multiplicity. For the fixed round two-sphere, localized potential sections are smooth at zero and realize every smooth real vector bundle of positive finite rank over a compact smooth manifold in a suitable finite spectral cluster.

Contents

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1. Introduction

An isolated eigenvalue of multiplicity m determines a finite-dimensional spectral projection. After choosing a local orthonormal frame, the restriction of the operator to its range is a map to $\mathrm{Sym}_m(\mathbb{R})$. Its derivative, its local image, and the bundles defined by its separated spectral projections are different objects. This article studies all three for scalar Laplacians and for Schrödinger operators.

At an arbitrary reference metric, the Laplace cluster map admits a local $1/2$-Hölder section within the conformal class. For a fixed Schrödinger operator, the corresponding section can be chosen among potentials supported in any prescribed nonempty interior open set. The construction preserves the complete cluster matrix, rather than only its unordered eigenvalues. We also prove a sharp signature restriction on first-order metric obstructions and use the local sections to realize vector bundles as spectral subbundles of the scalar operators.

1.1. **Perturbation, stability, and prescription.** Arnol'd formulated the comparison between multiple eigenvalues and the strata of real symmetric matrices [4]. Uhlenbeck established generic simplicity and eigenfunction transversality for variable metrics [25]. Bleecker–Wilson [9] and Bando–Urakawa [6] developed perturbation-theoretic criteria for splitting multiple eigenvalues. Colin de Verdière introduced strong and weak spectral stability and exhibited failures of the strong condition [11]. The quadratic obstruction criterion originates in Colin de Verdière [11]; Besson reduced it to the pointwise identities used below [8, Sections 4A and 7A]. The relation between these formulations and the preceding perturbation criteria is discussed in [14, Sections 1.1 and 2.2]. We recall that reduction, with our sign convention, in Proposition 2.1.

Teytel establishes codimension and noncrossing results for differentiable self-adjoint families under a transversality hypothesis [24]. Micheletti–Pistoia gave a sufficient condition for the local multiplicity locus to be a manifold of codimension $m(m+1)/2-1$ and verified it for double eigenvalues on surfaces [21]. Greilhuber–Kepplinger prove that the exceptional metrics at which strong stability fails have infinite transversality codimension [14]. Their unrestricted result covers multiplicities at most six, and their examples include multiplicity eight. Theorem A excludes the remaining multiplicity-seven case of [14, Question 4.7]. Theorem C addresses the possibility in [14, Question 4.8] that an unstable point nevertheless lies on a smooth multiplicity locus. For the first positive eigenvalue of the square torus, [14, Proposition 4.3] proves that the conformal multiplicity locus is not a submanifold. Their Lemma 4.5 proves weak conformal stability in dimension two when the conformal cokernel is one-dimensional and an isometric involution reverses its two-point kernel. Theorems B and C do not require either of these additional hypotheses. The generic avoidance theorem of [14] is distinct from the local-section assertion proved here. All numbered comparisons refer to the fourth arXiv version.

Stability also enters inverse spectral constructions. Colin de Verdière prescribes finite parts of the Laplace spectrum, with multiplicities, on a fixed closed manifold of dimension at least three [10, Theorem 1.2]. Lohkamp refines this prescription with additional geometric constraints [20]. For a compact manifold of dimension at least three with nonempty boundary, He–Wang prescribe total volume and an initial Dirichlet list $0 < a_1 < a_2 \leq \cdots \leq a_N$ [17, Theorem 1.1]. The strict first inequality is required by simplicity of the first Dirichlet eigenvalue. These existence theorems do not require the resulting metric to approximate an arbitrary specified metric. Theorem B is local at every reference: it prescribes the entire cluster matrix and has an exact level-preserving approximation statement. For the boundary problem, the generic simplicity and double-eigenvalue results of Gomes–Marrocos [13] are likewise distinct from the interior-supported all-multiplicity realization in Section 9.

1.2. **The spectral map.** Let (M^n, g) be closed, connected, and smooth, $n \geq 2$. Put

$$\Delta_g = -\operatorname{div}_g \nabla_g, \qquad 0 = \lambda_0(g) < \lambda_1(g) \leq \lambda_2(g) \leq \cdots. \tag{1.1}$$

For an integer $k \geq 4$, the C^k metrics form an open subset of the Banach space of C^k symmetric two-tensors. Local smoothness assertions are taken in these Banach spaces at a smooth reference metric. All constructed finite-dimensional families consist of smooth metrics. Smooth metrics carry the C^∞ topology. On a fixed compact support, smooth functions and tensors carry the induced C^∞ topology. A statement involving all C^k norms permits constants depending on k; it does not assert a bound uniform in k.

Fix $\lambda > 0$ of multiplicity m and a bounded interval $I \Subset (0, \infty)$ containing λ and no other eigenvalue of g. On a sufficiently small neighborhood $\mathcal{U}$ of g, a smooth orthonormal frame of the entire spectral cluster defines

$$\mathsf{L} : \mathcal{U} \longrightarrow \operatorname{Sym}_m(\mathbb{R}), \qquad \mathsf{L}(g) = \lambda I_m, \qquad \operatorname{spec} \mathsf{L}(g') = \operatorname{spec}(\Delta_{g'}) \cap I, \tag{1.2}$$

with multiplicities. The construction is given in Section 2. We write

$$\Gamma(g') = \mathsf{L}(g') - \frac{\operatorname{tr} \mathsf{L}(g')}{m} I_m. \tag{1.3}$$

Definition 1.1 (Colin de Verdière; cf. [14, Definition 4.4]). For a metric parameter space $\mathcal{M}$, the pair (g, λ) satisfies the *Strong Arnold Hypothesis* if $D\mathsf{L}(g) : T_g\mathcal{M} \to \mathrm{Sym}_m(\mathbb{R})$ is onto. It satisfies the *Weak Arnold Hypothesis* if, in every sufficiently small neighborhood $\mathcal{U}$ of g in $\mathcal{M}$, there is $\delta > 0$ such that each continuous map $\widetilde{\mathsf{L}} : \mathcal{U} \to \mathrm{Sym}_m(\mathbb{R})$ with $\sup_{\mathcal{U}} \left\|\widetilde{\mathsf{L}} - \mathsf{L}\right\| < \delta$ attains λI_m.

The terms *stable* and *unstable* mean, respectively, that the Strong Arnold Hypothesis holds or fails in the specified parameter space. The full metric space and a fixed conformal class are distinguished throughout.

1.3. **The sharp signature bound.**

Theorem A (Signature bound). *Let $u_1, \dots, u_m$ be linearly independent real λ-eigenfunctions. If a nonzero $C \in \mathrm{Sym}_m(\mathbb{R})$ satisfies*

$$\sum C_{ij} u_i u_j = 0, \qquad \sum C_{ij}\, du_i \otimes du_j = 0, \tag{1.4}$$

then C has at least four positive and four negative eigenvalues. Consequently every positive eigenvalue of multiplicity at most seven satisfies the Strong Arnold Hypothesis for unrestricted metric variations. Multiplicity eight admits failure.

The additional condition supplied by the eigenfunction equation is a positive weighted trace condition on the second fundamental form of a spherical immersion. The Gauss equation then excludes signatures with one index at most three. The proof occupies Section 4.

1.4. **Local sections and multiplicity loci.**

Theorem B (Local sections and exact approximation). *Let (M^n, g) be closed, connected and smooth, $n \geq 2$, and let $\lambda > 0$ have multiplicity m. For the cluster matrix* (1.2)*, there are $\delta > 0$ and a map*

$$\mathscr{S} : B_\delta(0) \subset \mathrm{Sym}_m(\mathbb{R}) \longrightarrow \mathcal{G}(M, g)$$

satisfying

$$\mathscr{S}(0) = g, \qquad \mathsf{L}(\mathscr{S}(Y)) = \lambda I_m + Y, \qquad \left\|\mathscr{S}(Y) - \mathscr{S}(Y')\right\|_{C^k} \leq C_k \left\|Y - Y'\right\|^{1/2}. \tag{1.5}$$

Here $\mathcal{G}(M, g)$ is the smooth conformal class of g. The estimate holds for every fixed integer $k \geq 4$, and $\mathscr{S}$ is smooth off zero. At a strongly conformally stable reference it may be chosen smooth at zero. At an unstable conformal reference no exponent greater than $1/2$ is possible in a uniform estimate at zero.

There are also smooth conformal metrics $g_j \to g$ in C^∞ with $\mathsf{L}(g_j) = \lambda I_m$, at which the conformal cluster derivative is surjective. In particular, every positive eigenvalue satisfies the Weak Arnold Hypothesis in its conformal class.

Theorem C (Multiplicity loci). *Let $\mathcal{S} = \{g' \in \mathcal{U} : \Gamma(g') = 0\}$, in the full metric space or in a fixed conformal class. At a strongly stable pair it is a smooth submanifold of codimension $m(m+1)/2 - 1$. At an unstable pair it is not a C^1 embedded submanifold. The latter conclusion holds in the C^k metric spaces for each integer $k \geq 4$, at a smooth reference metric.*

The operator with kernel

$$R(x, y) \sum_{i,j} C_{ij} u_i(x) u_j(y)$$

and the use of unique continuation to exclude infinite-order diagonal vanishing occur in [14, Proposition 3.1 and Lemma 4.5]. We retain this operator and identify the first nonzero homogeneous diagonal term, prove that it is harmonic for the frozen metric, and compute the resulting principal symbol. Its two signs imply infinite positive and negative index for every nonzero scalar component of the cokernel Hessian. The convention in [14] is the nonpositive Laplacian; throughout this article the Laplacian is nonnegative.

The passage from an indefinite Hessian to local solvability belongs to the theory of quadratic maps developed by Agrachev [1]. Agrachev–Sarychev give a finite-dimensional reduction, a regular-zero criterion, and a rescaling argument for nonlinear level sets [3, Lemmas 9.2–9.3 and Proposition 9.4]. Section 7 proves the version needed here, including surjectivity after finite-codimensional restrictions and two regular zeros with prescribed vanishing mixed terms. The analytic input specific to the Laplacian is the assertion that every nonzero scalar cokernel Hessian has both indices infinite. Section 8 then gives a single pairwise $1/2$-Hölder section, exact approximation on the same spectral level, and the tangent-space obstruction for the multiplicity locus.

There are interior-supported Dirichlet and Neumann versions. If Z is closed and $Z \cap M^\circ$ has empty interior, the construction uses a fixed compact set $K \Subset M^\circ \setminus Z$. It realizes every sufficiently small cluster-matrix displacement Y with

$$\|g_Y - g_{Y'}\|_{C^k} \le C_k \left\|Y - Y'\right\|^{1/2}, \qquad g_0 = g. \tag{1.6}$$

The exponent is optimal at a conformally unstable reference. The support set and constants depend on the reference eigenvalue. The hypothesis that $Z \cap M^\circ$ has empty interior is used in the global annihilator equation. No pointwise volume constraint is asserted for these conformal variations.

1.5. **Spectral subbundles and internal gaps.** Nowaczyk uses nontrivial eigenspinor bundles over loops of metrics to force Dirac eigenvalue multiplicities and transports those families through surgery [23]. His argument already uses the obstruction to extending a nontrivial spectral bundle across a contractible parameter space. Below, local sections realize prescribed real bundles for scalar operators, with explicit support and geometric constraints. These conclusions concern realization of the bundle, not a new obstruction principle.

Grassmannians already occur in the topology of spaces of symmetric operators. Agrachev relates multiplicity filtrations to Thom spaces over Grassmannians [2, Theorem 1]. Berkolaiko–Zelenko express the local Morse contribution of an eigenvalue crossing in terms of Grassmannian homology [7]. Our family statements realize such eigenspace topology by scalar Laplacians. The local section transfers the tautological bundle to a space of metrics with an internal spectral gap. A separate submersion at a round sphere permits exact preservation of the volume form and localization in a prescribed open set. These additional metric constraints are not conclusions about arbitrary symmetric-matrix families.

Theorem D (Spectral bundles). *Let $m \ge 2$ and $1 \le r < m$. A local section of Γ, continuous at zero and smooth away from zero, realizes the tautological rank-r bundle over $\mathrm{Gr}_r(\mathbb{R}^m)$ as the lower spectral bundle of a family with a gap after the r-th eigenvalue in the cluster. The Grassmannian is a retract of the corresponding local gap space. Such sections exist at every reference in Theorem B, including unstable references.*

On a round S^n, every smooth rank-r subbundle of a trivial $(n+1)$-plane bundle over a compact smooth manifold, $1 \le r \le n$, is realized by the spectral subspaces corresponding to the first r positive eigenvalues of nearby metrics, counted with multiplicity. Their volume form is fixed, and their perturbations are supported in any specified nonempty open subset of the sphere.

Let γ be the unit-round metric on S^4 and put $g_* = \gamma/2$. There are a compact set K contained in a polar cap and a smooth S^2-family of metrics g_b such that

$$dv_{g_b} = dv_{g_*}, \qquad g_b = g_* \text{ on } S^4 \setminus K, \qquad \mathrm{Ric}_{g_b} > 3g_b.$$

The metrics have the same complete scalar spectrum. The rank-three spectral bundle corresponding to their first three positive eigenvalues, counted with multiplicity, is $\underline{\mathbb{R}} \oplus L_{\mathbb{R}}$, where $L \to \mathbb{CP}^1$ is the tautological complex line. Its second Stiefel–Whitney class is nonzero.

The metrics in the explicit four-sphere family are pairwise isometric. The family is nontrivial in a marked metric space with a spectral gap, not in the quotient by isometries. Its rank-three spectral bundle cannot split globally into real eigenline bundles over S^2. We do not identify its w_2 with an unweighted parity of internal crossings.

Appendix A computes the obstruction for every fixed even differential order on a flat two-torus. In particular, allowing all self-adjoint second-order scalar differential variations does not remove every obstruction. This treats the suggested differential enlargement in [14, Question 4.9], not every possible enlarged parameter space. The Grassmannian retractions do not determine the connected components of global multiplicity loci, as asked in [14, Question 4.11].

1.6. **Potentials on a fixed manifold.** Let $P = \Delta_g + V$ on a compact connected smooth manifold of dimension at least two, with a fixed closed, Dirichlet, or Neumann realization, and let $O \subset M^\circ$ be nonempty and open. Theorem 12.1 treats any real isolated eigenvalue of P. It constructs a local section of its full cluster matrix with values in one finite-dimensional subspace of $C_c^\infty(O)$, with the pairwise exponent $1/2$. The compact support is selected inside the prescribed O. Neither the background metric nor the boundary realization changes. This support statement is stronger than the relative metric assertion of Section 9, where the allowed region is the complement of a closed set with empty interior.

The round-sphere refinement uses an established fact. Colin de Verdière proves strong conformal stability for every nonzero round S^2 eigenvalue [11, p. 186]; [14, Proposition 4.6] reproduces the argument through multiplication of spherical harmonics. Lemma 13.1 gives the required product isomorphism explicitly. Analyticity and a compactly supported Gram construction then give a potential section smooth at zero, with a linear estimate. Theorem 13.3 uses that section to realize every smooth real vector bundle of positive finite rank over a compact smooth manifold in a suitable finite cluster on this fixed surface, with support in a prescribed disk. The product isomorphism and the underlying round-sphere stability are not claimed as new.

1.7. **Proofs and conventions.** Section 2 constructs the isolated-cluster projection and its orthonormal frame, proves the frame derivative formulas, and determines the metric-variation annihilator. Section 3 proves the volume-preserving exponential parametrization, the finite-dimensional local inverse used in the realization arguments, and a quantitative estimate for the change in Ricci curvature.

The signature theorem is proved in Section 4. Sections 5–8 compute the conformal Hessian, prove its infinite indices, and solve the resulting finite-dimensional quadratic equations. Sections 9–11 give the relative metric statements and the geometrically constrained spectral-bundle constructions. Section 12 treats potentials supported in a prescribed open set. Section 13 proves the independent round-sphere refinement and the fixed-surface bundle theorem. Appendix A computes the flat-torus obstructions and second variations.

2. Spectral projections and metric derivatives

2.1. **A fixed Hilbert space.** Fix a reference density dv_g. For g' near g, put $\rho_{g'} = dv_{g'}/dv_g$ and

$$J_{g'}u = \rho_{g'}^{1/2}u, \qquad T_{g'} = J_{g'}\Delta_{g'}J_{g'}^{-1} \quad \text{on } L^2(M, dv_g). \tag{2.1}$$

The map $J_{g'}$ is unitary from $L^2(M, dv_{g'})$ to $L^2(M, dv_g)$, since $\int |J_{g'}u|^2\, dv_g = \int |u|^2\, dv_{g'}$.

For closed M, the operator domain is $H^2(M)$. For a smooth boundary, write

$$\mathcal{D}_D = H^2(M) \cap H_0^1(M), \qquad \mathcal{D}_N = \{u \in H^2(M) \,:\, \partial_\nu u = 0\}.$$

Every construction below on a manifold with boundary leaves the metric unchanged on a collar of ∂M. On that collar $J_{g'} = I$, and the conormal derivative is independent of g'. Hence the domains of $T_{g'}$ are the fixed spaces $\mathcal{D}_D$ or $\mathcal{D}_N$.

The form $q_g(u) = \int_M |du|_g^2\, dv_g$, with domain $H_0^1(M)$ for Dirichlet data and $H^1(M)$ otherwise, is closed and nonnegative. Indeed, $q_g(u) + \|u\|_2^2$ is an equivalent squared norm on the indicated form domain. Its inclusion into L^2 is compact. The associated self-adjoint operator therefore has compact resolvent. For closed M, the domain and regularity assertions follow from [12, Theorems 15.1 and 16.1]. For the boundary domains, see [16, (2.4) and Theorem 2.2, p. 833]; the elliptic regularity implication on compact manifolds is [15, Theorem 6.8, 3°, and Corollary 6.10, pp. 40–41].

The boundary ellipticity hypotheses in these references can be checked directly. In inward boundary normal coordinates, the principal part of Δ_g at a boundary point is

$$-\partial_t^2 - h^{\alpha\beta}(y,0)\partial_{y_\alpha}\partial_{y_\beta}.$$

For a nonzero tangential covector ξ', set $\kappa = (h^{\alpha\beta}\xi'_\alpha\xi'_\beta)^{1/2} > 0$. The solutions of $-w'' + \kappa^2 w = 0$ decaying as $t \to +\infty$ are $w(t) = ce^{-\kappa t}$. Their Dirichlet data are c, and their outward normal derivatives are κc. Either homogeneous boundary condition therefore forces $c = 0$. Both boundary operators are normal. Applying the cited regularity theorem to $1 + \Delta_g$ meets its invertible Dirichlet-realization hypothesis: the form $q_g + \|\cdot\|_2^2$ is coercive. Adding the constant 1 does not change the principal or boundary symbols.

For the closed, Dirichlet, and Neumann realizations, and every integer $s \geq 0$, the resulting estimate is

$$\|u\|_{H^{s+2}} \leq C_s\big(\|\Delta_g u\|_{H^s} + \|u\|_{L^2}\big) \qquad (u \in H^{s+2}(M) \cap \mathcal{D}). \tag{2.2}$$

Here $\mathcal{D}$ denotes the fixed H^2 domain. To obtain the norm estimate from regularity, let $R_+ = (1+\Delta_g)^{-1}$. Regularity gives $R_+(H^s) \subset H^{s+2} \cap \mathcal{D}$. This map has closed graph: convergence in H^{s+2} implies convergence in L^2, where R_+ is bounded. Thus $R_+ : H^s \to H^{s+2}$ is bounded. For $s > 0$, the Sobolev interpolation inequality

$$\|u\|_{H^s} \leq \epsilon\|u\|_{H^{s+2}} + C_{s,\epsilon}\|u\|_{L^2}$$

absorbs the additional $\|u\|_{H^s}$ term in the bound for $R_+(1+\Delta_g)u$; the case $s = 0$ is immediate. Repeated application to $\Delta_g u = \lambda u$ makes every eigenfunction smooth up to the boundary.

Write $m' = \sqrt{\det g'}$ in a coordinate chart. The differential expression on the fixed Hilbert space is

$$T_{g'}f = -\rho_{g'}^{1/2}(m')^{-1}\partial_i\Big(m' g'^{ij}\partial_j(\rho_{g'}^{-1/2}f)\Big). \tag{2.3}$$

For each $k \geq 4$, the maps $g' \mapsto g'^{-1}$, m', and $\rho_{g'}^{\pm 1/2}$ are smooth between the corresponding C^k spaces on a uniformly positive neighborhood. After expanding the two derivatives in (2.3), each coefficient is a product of these functions and at most two metric derivatives. Thus $g' \mapsto T_{g'}$ is smooth into $\mathcal{L}(\mathcal{D}, L^2)$, where $\mathcal{D}$ is the fixed H^2 domain stated above. Multiplication by a bounded coefficient has L^2 norm at most its supremum norm, and each coordinate derivative of order at most two maps H^2 continuously to L^2. A finite partition of unity sums these estimates. In particular, on a smaller neighborhood,

$$\left\|T_{g'} - T_g\right\|_{\mathcal{D}\to L^2} \leq C\left\|g' - g\right\|_{C^2}. \tag{2.4}$$

The same expansion gives all parameter derivatives on finite-dimensional smooth families. For a fixed high Sobolev exponent, replace $H^2 \to L^2$ by $H^{s+2} \to H^s$ and require the corresponding finite number of bounded coefficient derivatives.

Choose a positively oriented rectangular contour γ, invariant as a set under complex conjugation, whose real intersections are the endpoints of I and whose interior contains no reference eigenvalue other than λ. Let

$$d_\gamma = \min_{z\in\gamma}\operatorname{dist}(z, \operatorname{spec} T_g) > 0, \qquad R_0(z) = (z - T_g)^{-1}.$$

For every $\nu \in \operatorname{spec} T_g$ and $z \in \gamma$,

$$\frac{1+\nu}{|z-\nu|} \leq 1 + \frac{1+|z|}{d_\gamma}.$$

Expansion in the reference eigenbasis and (2.2) therefore give

$$\sup_{z\in\gamma}\|R_0(z)\|_{L^2\to\mathcal{D}} < \infty. \tag{2.5}$$

The perturbation bound (2.4) now applies uniformly on γ. If $\sup_\gamma \left\|(T_{g'} - T_g)R_0(z)\right\|_{L^2\to L^2} < 1/2$, then

$$(z - T_{g'})^{-1} = R_0(z) \sum_{j=0}^{\infty} \big((T_{g'} - T_g)R_0(z)\big)^j. \tag{2.6}$$

The identity

$$z - T_{g'} = \big[I - (T_{g'} - T_g)R_0(z)\big](z - T_g)$$

proves that (2.6) is a two-sided inverse with domain $\mathcal{D}$. The geometric-series bound and (2.5) imply uniform convergence in $\mathcal{L}(L^2, \mathcal{D})$. To verify parameter differentiability, subtract the inverses at two parameters:

$$R_{g''}(z) - R_{g'}(z) = R_{g''}(z)(T_{g''} - T_{g'})R_{g'}(z).$$

Division by the parameter increment, followed by the operator-norm limit, gives $DR[h] = R\,DT[h]\,R$. In particular,

$$\begin{aligned} D^2R[h_1, h_2] = {} & R\,D^2T[h_1, h_2]R + R\,DT[h_1]R\,DT[h_2]R \\ & + R\,DT[h_2]R\,DT[h_1]R. \end{aligned}$$

Induction by the product rule gives each higher derivative as a finite sum of products of these bounded operators and derivatives of T. The estimates are uniform on the compact contour. This proves smoothness and justifies differentiation under its integral. In particular, no eigenvalue crosses either endpoint of I in this neighborhood.

Define

$$P_{g'} = \frac{1}{2\pi i} \int_\gamma (z - T_{g'})^{-1}\, dz. \tag{2.7}$$

The Riesz projection theorem [19, Chapter III, Section 6.4], applied to the self-adjoint realization above, identifies $P_{g'}$ with the orthogonal projection onto the indicated cluster. It commutes with complex conjugation and therefore preserves real functions. Its rank is m. Indeed, if P' and P are orthogonal projections with $\|P' - P\| < 1$ and $v \in \operatorname{im} P$ satisfies $P'v = 0$, then

$$\|v\| = \left\|(P - P')v\right\| \le \left\|P - P'\right\| \|v\|,$$

so $v = 0$. Interchanging the projections proves injectivity in the other direction. Since $\operatorname{im} P$ has dimension m, both inequalities on the dimensions follow.

Let $U : \mathbb{R}^m \to L^2(M, dv_g)$ have columns an orthonormal real eigenbasis $u_1, \dots, u_m$. Put

$$U_{g'} = P_{g'}U\,(U^*P_{g'}U)^{-1/2}, \qquad \mathsf{L}(g') = U_{g'}^*\,T_{g'}\,U_{g'}. \tag{2.8}$$

The Gram matrix equals I_m at g and remains positive. Its positive inverse square root is smooth by the convergent power series for $(I + X)^{-1/2}$ when $\|X\| < 1$. Direct multiplication gives $U_{g'}^* U_{g'} = I_m$. Its image has dimension m and equals $\operatorname{im} P_{g'}$. Consequently

$$T_{g'}U_{g'} = U_{g'}\mathsf{L}(g'), \qquad \mathsf{L}(g')^T = \mathsf{L}(g'). \tag{2.9}$$

The columns in (2.8) are not required to be separate eigenfunctions. The corresponding frame of functions for the original metric is

$$V_{g'} = J_{g'}^{-1}U_{g'}, \qquad \Delta_{g'}V_{g'} = V_{g'}\mathsf{L}(g'), \qquad V_{g'}^* V_{g'} = I_m, \tag{2.10}$$

where the last adjoint is taken in $L^2(M, dv_{g'})$. Thus $V_{g'}$ sends each eigenspace of $\mathsf{L}(g')$ to the corresponding eigenspace of $\Delta_{g'}$. An arbitrary invariant subspace is sent to an invariant subspace; it need not consist of eigenfunctions with one common eigenvalue.

For later bundle constructions we record the dependence in the spatial variables as well. Let t range over a finite-dimensional ball and let g_t have coefficients smooth in (t, x), with the same boundary realization as above. For each nonnegative integer s, put

$$\mathcal{D}_s = \{u \in H^{s+2}(M) \,:\, u \text{ satisfies the fixed boundary condition}\};$$

in the closed case there is no boundary condition. On a smaller parameter ball, the elliptic estimate and the resolvent bound give

$$\sup_{\substack{t\in B_\epsilon(0)\\ z\in\gamma}} \|(z-T_t)^{-1}\|_{H^s\to\mathcal{D}_s} < \infty. \tag{2.11}$$

Choose a compact parameter ball $\mathcal{P}$ contained in the neighborhood where the contour avoids the spectrum. Fix s and $(t_0, z_0) \in \mathcal{P}\times\gamma$. Regularity and the closed graph theorem give

$$(z_0 - T_{t_0})^{-1} \,:\, H^s \longrightarrow \mathcal{D}_s$$

as a bounded map. The coefficients of T_t depend continuously on t in the norms required for $\mathcal{L}(\mathcal{D}_s, H^s)$. Therefore, near (t_0, z_0),

$$\big\|\big((T_t - T_{t_0}) + (z_0 - z)I\big)(z_0 - T_{t_0})^{-1}\big\|_{H^s\to H^s} < \tfrac{1}{2}.$$

The Neumann series gives a uniform $H^s \to \mathcal{D}_s$ bound on this neighborhood. A finite cover of $\mathcal{P}\times\gamma$ proves (2.11). The compact ball $\mathcal{P}$ was chosen using only the L^2 contour condition, so it is the same for every s; the bounds and finite covers may depend on s.

For fixed smooth f, let $u_t(z) = (z-T_t)^{-1}f$. Differentiating its equation gives, for a nonzero parameter multi-index α,

$$(z-T_t)\partial_t^\alpha u_t = \sum_{0<\eta\le\alpha} \binom{\alpha}{\eta} (\partial_t^\eta T_t)\partial_t^{\alpha-\eta} u_t. \tag{2.12}$$

The resolvent identity first proves this equality in $\mathcal{D}_0$ by difference quotients. Estimate (2.11) and induction on $|\alpha|$ prove it in every $\mathcal{D}_s$. Thus $t \mapsto u_t(z)$ is smooth with values in each Sobolev space, uniformly on the contour. Sobolev embedding, with s chosen arbitrarily large, makes $u_t(z,x)$ jointly smooth in (t,x). Integration over γ, followed by the finite-dimensional Gram normalization, proves the same assertion for the physical cluster frames used in Sections 10–13.

Let g_t be a smooth one-parameter family through g. Differentiating $(z-T_t)(z-T_t)^{-1} = I$ gives

$$\frac{d}{dt}(z-T_t)^{-1} = (z-T_t)^{-1}\dot{T}_t(z-T_t)^{-1}. \tag{2.13}$$

The series (2.6) converges uniformly on the contour, and the differentiated series does so on any smaller parameter neighborhood. Hence

$$\dot{P} = \frac{1}{2\pi i}\int_\gamma (z-T_g)^{-1}\dot{T}(z-T_g)^{-1}\,dz. \tag{2.14}$$

For example,

$$\|\dot{P}\|_{L^2\to\mathcal{D}} \le \frac{\text{length}(\gamma)}{2\pi} \sup_{z\in\gamma} \|(z-T_g)^{-1}\|^2_{L^2\to\mathcal{D}} \,\|\dot{T}\|_{\mathcal{D}\to L^2}\,.$$

The embedding $\mathcal{D} \hookrightarrow L^2$ is incorporated into the fixed norm. This estimate uses only the contour separating the chosen cluster from the excluded spectrum. No difference between eigenvalues within the cluster occurs.

Differentiating $P_t^2 = P_t$ and multiplying on both sides by P gives

$$P\dot{P}P = 0.$$

Since $PU = U$, it follows that $\frac{d}{dt}(U^*P_tU)|_0 = 0$. The derivative of its inverse square root at the identity is therefore zero, and

$$\dot{U} = \dot{P}U, \qquad P\dot{U} = 0, \qquad U^*\dot{U} = 0. \tag{2.15}$$

These assertions hold for the chosen frame on the fixed Hilbert space. They do not assert that the derivatives of arbitrary moving eigenbases are orthogonal to the whole cluster.

The same construction applies when the reference cluster has eigenvalues $\theta_1, \dots, \theta_m$, not necessarily equal. Choose U so that $T_g U = U\Theta$, $\Theta = \mathrm{diag}(\theta_1, \dots, \theta_m)$. Equations (2.9) and (2.15) give

$$\dot{\mathsf{L}} = U^* \dot{T} U, \qquad (T_g - \theta_a)\dot{u}_a = -(I-P)\dot{T} u_a. \tag{2.16}$$

Indeed, differentiation of $\mathsf{L} = U_t^* T_t U_t$ gives two frame terms. They are $(U^*\dot{U})^*\Theta$ and $\Theta U^* \dot{U}$, both zero. Differentiation of $T_t U_t = U_t \mathsf{L}_t$ and projection to $\ker P$ gives the second equality. Define

$$R_a = (T_g - \theta_a)^{-1}|_{\ker P}, \qquad R_a|_{\operatorname{im} P} = 0.$$

The excluded spectrum is disjoint from every θ_a, so R_a exists and

$$\dot{u}_a = -R_a \dot{T} u_a. \tag{2.17}$$

Both formulas remain valid when some of the θ_a coincide.

At a smooth reference metric, take a complete orthonormal eigenbasis e_j with eigenvalues $\nu_j \geq 0$, and use the Hilbert scale

$$\|f\|_{\mathcal{H}^s}^2 = \sum_j (1+\nu_j)^s |\langle f, e_j \rangle|^2.$$

Let $d_a = \inf\{|\nu_j - \theta_a| : e_j \in \ker P\} > 0$. For every excluded eigenvalue,

$$\frac{1+\nu_j}{|\nu_j - \theta_a|} \leq 1 + \frac{1+\theta_a}{d_a}.$$

Squaring this inequality, multiplying by $(1+\nu_j)^{-1}|f_j|^2$, and summing proves

$$\|R_a f\|_{\mathcal{H}^1} \leq \left(1 + \frac{1+\theta_a}{d_a}\right) \|f\|_{\mathcal{H}^{-1}}. \tag{2.18}$$

The $\mathcal{H}^1$ norm is the norm of the Dirichlet energy plus L^2 mass, on H^1 in the closed and Neumann cases and on H_0^1 in the Dirichlet case. This follows first on finite eigenfunction sums and then by completion. The extension in (2.18) is self-adjoint in the associated dual pairing, since its real spectral multipliers are symmetric.

Another smooth real orthonormal frame is $U_{g'} O(g')$, with $O(g') \in O(m)$. After constant conjugation, assume $O(g) = I_m$. Along a curve through g, set $A = \dot{O}(0)$ and $L(t) = \mathsf{L}(g_t)$. Since $O^T O = I_m$ and $L(0) = \lambda I_m$,

$$\frac{d}{dt}(O^T L O)\bigg|_0 = \dot{L}(0), \qquad \frac{d^2}{dt^2}(O^T L O)\bigg|_0 = \ddot{L}(0) + 2A^T \dot{L}(0) + 2\dot{L}(0)A.$$

Thus the second derivative is unchanged when $\dot{g}_0 \in \ker D\mathsf{L}(g)$. For a parameter chart θ with $\theta(0) = g$, the chain rule gives

$$D^2(\mathsf{L} \circ \theta)(0)[a,a] = D^2\mathsf{L}(g)[D\theta(0)a, D\theta(0)a] + D\mathsf{L}(g) D^2\theta(0)[a,a].$$

Pairing with a cokernel matrix removes the last term. This proves the asserted frame and chart independence.

2.2. **The metric derivative.** For $g_t = g + th + O(t^2)$,

$$\frac{d}{dt} g_t^{-1}\bigg|_0 = -h^{\sharp\sharp}, \qquad \frac{d}{dt} dv_{g_t}\bigg|_0 = \tfrac{1}{2} \operatorname{tr}_g(h)\, dv_g. \tag{2.19}$$

The first identity follows by differentiating $g_t g_t^{-1} = I$. The second follows from $\frac{d}{dt} \det g_t = (\det g_t) \operatorname{tr}(g_t^{-1} \dot{g}_t)$.

Let $v_i(t) = J_{g_t}^{-1} u_i(t)$ be the columns of the physical cluster frame. They are orthonormal in $L^2(g_t)$. At $t = 0$, differentiation gives

$$\int_M (\dot{v}_i u_j + u_i \dot{v}_j)\, dv_g = -\tfrac{1}{2} \int_M \operatorname{tr}_g h\, u_i u_j\, dv_g. \tag{2.20}$$

The matrix entry is $\int g_t^{ab}\partial_a v_i(t)\partial_b v_j(t)\,dv_{g_t}$. Its derivative has the inverse-metric term, the density term, and the two terms

$$\int_M \big(\big\langle d\dot v_i, du_j\big\rangle + \big\langle du_i, d\dot v_j\big\rangle\big)\,dv_g = \lambda\int_M (\dot v_i u_j + u_i\dot v_j)\,dv_g.$$

Green's formula proves this equality. For Dirichlet data, $\dot v_i|_{\partial M} = 0$; for Neumann data, $\partial_\nu u_j = 0$. These identities remove the boundary terms. There is no boundary term when M is closed. Substituting (2.20) therefore gives

$$\begin{aligned} D\mathsf{L}(g)[h]_{ij} = \int_M \Big[&-h(\nabla u_i,\nabla u_j) \\ &+\tfrac12 \operatorname{tr}_g h\,\big(\big\langle du_i, du_j\big\rangle_g - \lambda u_i u_j\big)\Big]\,dv_g. \end{aligned} \tag{2.21}$$

The formulas hold for all closed-manifold variations and for the interior-supported boundary variations considered here.

For a family with $dv_{g_t} = dv_g$ and initial derivative h, the coordinate divergence expression gives

$$\dot\Delta f = \operatorname{div}_g(h^{\sharp\sharp}df), \qquad \big\langle \dot\Delta f, z\big\rangle = -\int_M h(\nabla f,\nabla z)\,dv_g. \tag{2.22}$$

The second identity follows by integration by parts. It is valid for $z\in H^1$ when h has interior support, by density of smooth functions in H^1 and continuity of both sides. Here $h^{\sharp\sharp}df$ has components $g^{ia}g^{jb}h_{ab}\partial_j f$. For such a family the density conjugation is the identity. Thus (2.16) and (2.17) become

$$\dot{\mathsf{L}}_{ab} = -\int_M h(\nabla u_a,\nabla u_b)\,dv_g, \qquad \dot u_a = -R_a\dot\Delta u_a. \tag{2.23}$$

If $\eta\in\mathcal{H}^{-1}$ annihilates the cluster and $w_a = R_a\eta$, equations (2.18) and (2.22) also give the weak pairing identity

$$\langle \eta, \dot u_a\rangle = -\big\langle R_a\eta, \dot\Delta u_a\big\rangle = \int_M h(\nabla u_a,\nabla w_a)\,dv_g. \tag{2.24}$$

The integral is finite because u_a and h are smooth and $w_a\in\mathcal{H}^1$. Finite spectral sums prove the dual self-adjointness equality first; (2.18) extends it to the stated data.

For $C\in\operatorname{Sym}_m(\mathbb{R})$ set

$$F_C = \sum C_{ij}u_iu_j, \qquad T_C = \sum C_{ij}du_i\otimes du_j, \qquad \tau_C = \operatorname{tr}_g T_C. \tag{2.25}$$

The tensor T_C is symmetric. For unrestricted variations, the annihilator equation obtained from (2.21) is

$$T_C = \tfrac12(\tau_C - \lambda F_C)g. \tag{2.26}$$

Localization is valid because all smooth symmetric tensors supported in any coordinate ball are allowed as test variations.

Taking the trace in (2.26), and then applying the product rule, gives

$$(n-2)\tau_C = n\lambda F_C, \qquad \Delta_g F_C = 2\lambda F_C - 2\tau_C. \tag{2.27}$$

For $n>2$, these imply $\Delta_g F_C = -4\lambda F_C/(n-2)$. Multiplication by F_C and integration gives

$$\int_M |dF_C|^2\,dv_g = -\frac{4\lambda}{n-2}\int_M F_C^2\,dv_g. \tag{2.28}$$

Both sides force $F_C = 0$. Then $\tau_C = 0$ and $T_C = 0$. For $n=2$, the first equation in (2.27) gives $F_C = 0$; its second gives $\tau_C = 0$, and (2.26) gives $T_C = 0$.

Proposition 2.1 (Colin de Verdière–Besson characterization). *On a closed connected manifold, the cokernel of $D\mathsf{L}(g)$ for unrestricted metric variations consists precisely of the matrices C for which $F_C = T_C = 0$. Every such matrix is trace-free.*

Proof. The preceding calculation proves necessity. Substitution into (2.21) proves sufficiency. Finally, $\int_M F_C \, dv_g = \sum C_{ij} \left\langle u_i, u_j \right\rangle_{L^2} = \operatorname{tr} C$.

When $n > 2$, the sign in (2.28) is essential: its left-hand side is nonnegative, while its right-hand side is nonpositive and vanishes only when $F_C = 0$, because $\lambda > 0$. No conclusion for the zero Neumann eigenvalue is used. □

The attribution in the proposition concerns this characterization, not the signature theorem below; see [8, Sections 4A and 7A] and [14, Proposition 2.3].

For $g_t = (1+t)g$, $\Delta_{g_t} = (1+t)^{-1}\Delta_g$. Therefore

$$D\mathsf{L}(g)[g] = -\lambda I_m. \tag{2.29}$$

For the full metric space and the full conformal class, $D\mathsf{L}(g)$ is onto if and only if $D\Gamma(g)$ is onto Sym^0_m. Indeed, if $D\Gamma(g)h = B \in \mathrm{Sym}^0_m$, write $D\mathsf{L}(g)h = B + cI_m$. Equation (2.29) gives

$$D\mathsf{L}(g)\big[h + (c/\lambda)g\big] = B.$$

Adding a further multiple of g gives every scalar component. Conversely, surjectivity of $D\mathsf{L}(g)$ implies surjectivity after projection onto Sym^0_m.

The ranges computed using smooth variations agree with the ranges on C^k variations. In fact, smooth functions and symmetric tensors are dense in the corresponding C^k spaces. The derivative is continuous, and its image on the smooth subspace is a finite-dimensional, hence closed, subspace of Sym_m. Taking limits therefore gives equality of the two ranges. This observation applies both to the unrestricted space and to the full conformal class.

Aronszajn's theorem [5, pp. 235–239] implies that a smooth solution of $(\Delta_g - \lambda)u = 0$ which vanishes to infinite order at an interior point vanishes on its connected interior component. In an interior chart the principal matrix is smooth and uniformly positive definite on compact subsets, and the lower-order coefficients are bounded there; thus the equation satisfies the hypotheses of that theorem. Every application below concerns one scalar eigenfunction or a constant linear combination with the same eigenvalue. Vanishing on a nonempty open set implies infinite-order vanishing at each of its points.

The interior of a connected smooth manifold with boundary is connected. Indeed, each boundary point has a half-ball neighborhood whose interior part is connected, and hence belongs to one component of M°. Assign the boundary point to that component; nested half-ball neighborhoods show that the assignment is independent of the chart. Each interior component, together with its assigned boundary points, is open in M. These sets form a disjoint open partition of M. Connectedness leaves a single set. Unique continuation on M° and continuity on M therefore imply global vanishing.

3. Metric variations and local inversion

All norms of tensors on a fixed compact manifold are computed in a fixed finite smooth atlas; changing that atlas changes the constants, not the assertions.

3.1. **The trace-free exponential.**

Proposition 3.1. *Let $h_1, \dots, h_d$ be smooth symmetric two-tensors on a compact Riemannian manifold (M, g). Suppose $\operatorname{tr}_g h_j = 0$ and that their supports have compact union K. For $t \in \mathbb{R}^d$, put*

$$H(t) = \sum_{j=1}^{d} t_j g^{-1} h_j, \qquad g_t(X, Y) = g(e^{H(t)}X, Y). \tag{3.1}$$

Then g_t is a smooth metric, $dv_{g_t} = dv_g$, and $g_t = g$ on $M \setminus K$. Moreover,

$$\partial_{t_j} g_t\big|_{t=0} = h_j, \qquad \|g_t - g\|_{C^k} \le C_k|t| \quad (|t| \le 1), \tag{3.2}$$

for every nonnegative integer k.

Proof. At each point, $g^{-1}h_j$ is self-adjoint: $g(g^{-1}h_jX,Y) = h_j(X,Y) = h_j(Y,X)$. Hence $H(t)$ is self-adjoint. An orthonormal eigenbasis gives

$$H(t) = \mathrm{diag}(\theta_1,\dots,\theta_n), \qquad e^{H(t)} = \mathrm{diag}(e^{\theta_1},\dots,e^{\theta_n}).$$

Every exponential eigenvalue is positive. This proves symmetry and positive definiteness of g_t. In arbitrary coordinates, $\det g_t = (\det g)\det e^{H(t)}$. Since

$$\det e^{H(t)} = \prod_i e^{\theta_i} = e^{\operatorname{tr} H(t)} = 1,$$

the positive square roots of these determinants agree. If $x \notin K$, then $H(t,x) = 0$, and $g_t(x) = g(x)$.

The matrix exponential is defined by its locally uniformly convergent power series. Its derivative at zero is the identity, so differentiating (3.1) gives the first formula in (3.2). For every fixed k, the Leibniz rule gives

$$\|AB\|_{C^k} \le c_k\,\|A\|_{C^k}\,\|B\|_{C^k}$$

for smooth endomorphisms. Multiplying this norm by c_k makes it submultiplicative. In that equivalent norm, $\|H(t)\|_{C^k} \le M_k|t|$ and

$$\left\|e^{H(t)} - I\right\|_{C^k} \le \sum_{j\ge1} \frac{\|H(t)\|_{C^k}^j}{j!} \le M_k|t|e^{M_k|t|}.$$

Multiplication by the fixed metric and return to the original norm prove the second estimate in (3.2). The same uniformly convergent series, differentiated term by term on bounded parameter sets, proves smooth dependence on t in every C^k space. □

3.2. A local inverse with parameters.

Lemma 3.2. *Let $f : \overline{B}_r(0) \subset \mathbb{R}^d \to \mathbb{R}^d$ extend to a C^1 map on a neighborhood of the closed ball. Suppose $L = Df(0)$ is invertible and*

$$\sup_{|x|\le r} \left\|I - L^{-1}Df(x)\right\| \le \tfrac12. \tag{3.3}$$

If $\left\|L^{-1}(y - f(0))\right\| \le r/2$, there is a unique $x \in \overline{B}_r(0)$ with $f(x) = y$, and

$$|x| \le 2\left\|L^{-1}(y - f(0))\right\|. \tag{3.4}$$

The inverse is as smooth as f on the open target neighborhood with strict inequality. The same assertion holds smoothly with additional finite-dimensional parameters, when the hypotheses hold uniformly.

Proof. Set $T_y(x) = x - L^{-1}(f(x) - y)$. The integral mean-value formula and (3.3) give

$$|T_y(x) - T_y(x')| \le \tfrac12|x - x'|, \qquad |T_y(0)| = \left\|L^{-1}(y - f(0))\right\| \le r/2.$$

Thus T_y maps $\overline{B}_r$ into itself and contracts distances by at most $1/2$. The iterates $x_{j+1} = T_y(x_j)$, $x_0 = 0$, satisfy

$$|x_{j+1} - x_j| \le 2^{-j}|T_y(0)|.$$

They are Cauchy, and their limit x is a fixed point by continuity. Summing the geometric series proves (3.4). If x' is another fixed point in the ball, the contraction inequality gives $|x - x'| \le |x - x'|/2$, so $x = x'$.

For smoothness, write $Df(x) = L(I + E(x))$, where $\|E(x)\| \le 1/2$. The inverse exists by the matrix series $\sum_{j\ge0}(-E)^j$. For two targets y, y' and their fixed points x, x', subtraction gives

$$|x - x'| \le \tfrac12|x - x'| + \left\|L^{-1}\right\|\,|y - y'|, \qquad |x - x'| \le 2\left\|L^{-1}\right\|\,|y - y'|.$$

Apply the differentiability remainder of f to

$$f(x(y+h)) - f(x(y)) = h.$$

The Lipschitz bound makes that remainder $o(|h|)$, so

$$D_y x = (Df(x))^{-1}.$$

For a parameter p in $f(x,p) = y$, subtraction at nearby (y,p) first gives continuity and a local Lipschitz bound, and the same calculation yields

$$D_y x = (D_x f)^{-1}, \qquad D_p x = -(D_x f)^{-1} D_p f. \tag{3.5}$$

Differentiating these identities inductively proves every higher derivative for which f is differentiable. This establishes the parameter assertion. □

Proposition 3.3. *Let $\mathscr{F}$ be a smooth map from a neighborhood of a smooth metric g to a finite-dimensional real vector space W. Suppose*

$$D\mathscr{F}(g) : C_c^\infty(O; S_0^2 T^*M) \longrightarrow W$$

is onto, where $O \subset M^\circ$ is open. There are a compact set $K \Subset O$, a neighborhood V of $\mathscr{F}(g)$, and a smooth family $y \mapsto g_y$, $y \in V$, such that

$$\mathscr{F}(g_y) = y, \qquad g_{\mathscr{F}(g)} = g, \qquad dv_{g_y} = dv_g, \qquad g_y = g \text{ on } M \setminus K. \tag{3.6}$$

For every fixed k, $\|g_y - g\|_{C^k} \le C_k |y - \mathscr{F}(g)|$ after decreasing V.

Proof. Choose a basis $e_1, \dots, e_d$ of W and tensors h_j with $D\mathscr{F}(g)h_j = e_j$. Their finite support union is compact in O; enlarge it to $K \Subset O$. Use (3.1) and let $f(t) = \mathscr{F}(g_t)$ in this basis. The chain rule and (3.2) give $Df(0) = I_d$. Continuity of Df gives a ball satisfying (3.3). Lemma 3.2 gives a smooth solution $t(y)$ and $|t(y)| \le 2|y - \mathscr{F}(g)|$. Define $g_y = g_{t(y)}$. Proposition 3.1 proves every assertion in (3.6) and its norm bound. □

Let $F : E \supset U \to W$ be smooth, where E is Banach, W is finite-dimensional, $0 \in U$, and $A = DF(0)$ is onto. Choose a linear right inverse $R : W \to E$. It is bounded because W is finite-dimensional. The operator RA is a bounded projection, so

$$E = \ker A \oplus R(W).$$

For $x \in \ker A$ and $z \in R(W)$, the derivative in z of $F(x+z)$ at $(0,0)$ is the isomorphism $A|_{R(W)}$. The contraction proof of Lemma 3.2, with the Banach variable x as a parameter, gives a unique smooth solution $z = z(x,y)$ of $F(x+z) = y$ near $(0, F(0))$. The derivative formulas (3.5) hold with Fréchet derivatives in x. Thus

$$F^{-1}(y) = \{x + z(x,y) : x \in \ker A \text{ near } 0\}$$

locally, and this level set has codimension $\dim W$.

Lemma 3.4. *Let F be a smooth map between finite-dimensional manifolds, with rank constantly r near a point. There are local coordinates (s,z) in the source in which $F(s,z)$ depends only on $s \in \mathbb{R}^r$. The local fibres may be taken connected. Every vector in* $\ker dF$ *is the velocity of a smooth curve in its local level set.*

Proof. Work in coordinate charts. For $r = 0$, each component has zero differential on a connected coordinate ball and is constant there by integration along line segments. For $r > 0$, reorder source and target coordinates so that the first r target components have an invertible derivative in the first r source variables. The map

$$x \longmapsto (F_1(x), \dots, F_r(x), x_{r+1}, \dots, x_N)$$

has invertible derivative. Lemma 3.2, after translation and multiplication by that derivative's inverse, supplies local coordinates (s,z). In these coordinates the first r components of F equal s. A nonzero derivative in any z direction of one of the remaining components would add an independent column to the derivative and raise its rank above r. Therefore all those derivatives vanish. Restricting to a product of balls makes each fibre connected and proves that $F(s,z)$ is independent of z. Curves $z(t) = z_0 + tv$, with s fixed, realize all kernel velocities; transforming back proves the last assertion. □

3.3. **Quantitative preservation of a strict Ricci inequality.** Let ∇ be the Levi–Civita connection of g, and write $\widetilde{g} = g + h$. The difference of the two connections is the tensor

$$A^k{}_{ij} = \tfrac{1}{2}\widetilde{g}^{k\ell}(\nabla_i h_{j\ell} + \nabla_j h_{i\ell} - \nabla_\ell h_{ij}). \tag{3.7}$$

Indeed, the torsion-free identity makes $A^k{}_{ij} = A^k{}_{ji}$. Subtract the two metric-compatibility equations and cycle their three covariant indices; solving for $\widetilde{g}_{k\ell}A^\ell{}_{ij}$ gives (3.7). Expanding the curvature of $\nabla + A$ and contracting its first and third indices gives

$$\mathrm{Ric}_{\widetilde{g},ij} - \mathrm{Ric}_{g,ij} = \nabla_k A^k{}_{ij} - \nabla_j A^k{}_{ik} + A^k{}_{k\ell}A^\ell{}_{ij} - A^k{}_{j\ell}A^\ell{}_{ik}. \tag{3.8}$$

Our curvature convention is $R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$, with sectional curvature $g(R(X,Y)Y,X)$.

Lemma 3.5. *For each compact set K_1 and fixed reference metric g, there are constants $C > 0$ and $\epsilon > 0$ such that*

$$\left\|\mathrm{Ric}_{g+h} - \mathrm{Ric}_g\right\|_{C^0(K_1,g)} \le C\,\|h\|_{C^2(K_1,g)} \quad \text{if } \|h\|_{C^2(K_1,g)} < \epsilon. \tag{3.9}$$

Consequently, if $\mathrm{Ric}_g - cg \ge \delta g$ *on* K_1, *with* $\delta > 0$, *then*

$$\mathrm{Ric}_{g+h} - c(g+h) \ge \tfrac{\delta}{2} g > 0 \tag{3.10}$$

there whenever $(C + |c|)\,\|h\|_{C^2} < \delta/2$ *and* $\|h\|_{C^2} < \epsilon$.

Proof. Assume $\|h\|_{C^0(g)} \le 1/2$. Then $\frac{1}{2}g \le \widetilde{g} \le \frac{3}{2}g$, and the inverse metric is uniformly bounded in the g norm. Differentiating $\widetilde{g}^{-1}\widetilde{g} = I$ gives

$$\nabla\widetilde{g}^{-1} = -\widetilde{g}^{-1}(\nabla h)\widetilde{g}^{-1}.$$

Equation (3.7) therefore implies

$$|A|_g \le C_n|\nabla h|_g, \qquad |\nabla A|_g \le C_n(|\nabla^2 h|_g + |\nabla h|_g^2).$$

Substitution into (3.8) gives

$$|\,\mathrm{Ric}_{g+h} - \mathrm{Ric}_g|_g \le C_n(|\nabla^2 h|_g + |\nabla h|_g^2).$$

For $\|h\|_{C^2} \le 1$, the quadratic term is bounded by $\|h\|_{C^2}$. This proves (3.9). For a g-unit vector X, the difference between $(\mathrm{Ric}_{g+h} - c(g+h))(X,X)$ and $(\mathrm{Ric}_g - cg)(X,X)$ has absolute value at most $(C + |c|)\,\|h\|_{C^2}$. The assumed bound proves (3.10). Positivity as a quadratic form does not depend on which of the two equivalent metric norms is used. $\square$

If $\mathrm{Ric}_g > cg$ on a compact neighborhood K_1 of the support of a perturbation, the continuous function

$$(x,X) \longmapsto (\mathrm{Ric}_g - cg)_x(X,X), \qquad x \in K_1, \quad |X|_g = 1,$$

has a positive minimum δ. The unit-sphere bundle over K_1 is compact, so the minimum is attained. Lemma 3.5 and the C^2 estimate of Proposition 3.3 give one parameter neighborhood on which the strict inequality is retained throughout K_1. Outside the support the two metrics and their Ricci tensors are equal.

4. A sharp signature bound

4.1. **Spherical immersions.**

Lemma 4.1. *Let $F : O \subset S^2 \to S^q$ be a smooth isometric immersion between unit spheres. Let* II *be its second fundamental form inside S^q. Suppose that, at each point of O, a positive-definite contravariant quadratic form Q on TO satisfies* $\mathrm{tr}_Q\,\mathrm{II} = 0$. *Then* $\mathrm{II} = 0$. *Near each point, $F(y) = Ay$ for a constant linear isometric inclusion $A : \mathbb{R}^3 \to \mathbb{R}^{q+1}$.*

Proof. Choose an orthonormal tangent basis and an orthonormal normal basis at a point. Write A_α for the symmetric 2×2 matrix of the normal component of II. The hypothesis is $\operatorname{tr}(QA_\alpha) = 0$. The symmetric matrix $Q^{1/2}A_\alpha Q^{1/2}$ has eigenvalues $b_\alpha, -b_\alpha$. Hence

$$\det A_\alpha = -\frac{b_\alpha^2}{\det Q} \leq 0, \tag{4.1}$$

with equality only if $A_\alpha = 0$.

Write D for the Euclidean connection of $\mathbb{R}^{q+1}$. For tangent fields X, Y on O,

$$D_X(dF(Y)) = dF(\nabla_X Y) + \mathrm{II}(X, Y) - \langle X, Y\rangle F.$$

The radial term follows by differentiating $\langle dF(Y), F\rangle = 0$. Put $\mathbb{B}(X, Y) = \mathrm{II}(X, Y) - \langle X, Y\rangle F$. Commuting the Euclidean covariant derivatives and taking tangent components gives, on an orthonormal pair e_1, e_2,

$$K_O(e_1, e_2) = \langle \mathbb{B}(e_1, e_1), \mathbb{B}(e_2, e_2)\rangle - |\mathbb{B}(e_1, e_2)|^2 = 1 + \sum_\alpha \det A_\alpha.$$

Here II is perpendicular to F and $|F| = 1$. Since O has the unit-sphere metric, $K_O = 1$, and the determinant sum is zero. Equation (4.1) makes every summand nonpositive; thus each $A_\alpha = 0$. If the normal space is zero-dimensional, $\mathrm{II} = 0$ by its definition.

Fix $y_0 \in O$ and define A on $\mathbb{R}y_0 \oplus T_{y_0}S^2$ by $Ay_0 = F(y_0)$ and $A|_{T_{y_0}S^2} = dF_{y_0}$. These orthogonal prescriptions make A an isometric inclusion. A sufficiently short unit-speed great circle γ from y_0 satisfies $\gamma'' = -\gamma$. Since $\mathrm{II} = 0$, $F \circ \gamma$ satisfies the same equation in $\mathbb{R}^{q+1}$, with the initial data of $A\gamma$. Uniqueness of this linear differential equation gives $F \circ \gamma = A\gamma$. Such segments cover a neighborhood of y_0. □

4.2. **Quadratic relations.**

Proof of Theorem A. Diagonalize C by a constant orthogonal matrix. Multiply each resulting eigenfunction by the square root of the absolute value of its nonzero coefficient, and omit zero coefficients. The resulting functions remain linearly independent. They form two vector-valued eigenfunctions $U : M \to \mathbb{R}^p$ and $V : M \to \mathbb{R}^q$ satisfying

$$|U|^2 = |V|^2, \qquad dU^T dU = dV^T dV, \qquad \Delta_g U = \lambda U, \quad \Delta_g V = \lambda V. \tag{4.2}$$

Both p, q are positive, since a sum of nonzero squares cannot vanish identically.

A constant linear combination of the components in (4.2) is a λ-eigenfunction. If it vanishes on a nonempty open set, Aronszajn's theorem [5], with the scalar elliptic hypotheses verified in Section 2, makes it zero on M. Linear independence therefore excludes every such local relation with nonzero coefficients.

Suppose that $p \leq 3$. On a nonempty open set where $r = |U| = |V| > 0$, put $\phi = U/r$ and $\psi = V/r$. Their values lie in S^{p-1} and S^{q-1}. Since $\phi \cdot d\phi = \psi \cdot d\psi = 0$,

$$dU^T dU = dr \otimes dr + r^2 d\phi^T d\phi, \qquad \phi^* g_{S^{p-1}} = \psi^* g_{S^{q-1}}. \tag{4.3}$$

Consequently $\ker d\phi = \ker d\psi$ and their ranks agree.

Choose a point where this rank is maximal on the chosen open set. If the maximal rank is positive, one minor of that order is nonzero on a neighborhood; maximality excludes larger rank there. If the maximal rank is zero, both differentials vanish on the chosen open set. The common rank is constant and belongs to $\{0, 1, 2\}$.

At rank zero, ϕ and ψ are constant on a connected smaller neighborhood. All components of U, V are constant multiples of r. Since $p + q \geq 2$, this contradicts the absence of a local constant relation.

At rank one, the coordinates of Lemma 3.4 for ϕ have the form (t, z), and the fibres with fixed t may be taken connected. Equality of kernels makes ψ constant on these fibres. Thus $\phi = \gamma(t)$ and $\psi = \delta(t)$. Reparametrizing by the arc length of γ gives $|\gamma'| = 1$. Equation (4.3) then gives $|\delta'| = 1$, and $dt \neq 0$.

The equation for $U = r\gamma(t)$ is

$$(\Delta r - \lambda r)\gamma + (r\Delta t - 2\langle dr, dt\rangle)\gamma' - r|dt|^2\gamma'' = 0. \tag{4.4}$$

The identities $|\gamma| = |\gamma'| = 1$ imply $\gamma \cdot \gamma' = 0$, $\gamma \cdot \gamma'' = -1$, and $\gamma' \cdot \gamma'' = 0$.

Taking inner products of (4.4) with γ and γ' gives

$$\Delta r - \lambda r + r|dt|^2 = 0, \qquad r\Delta t - 2\langle dr, dt\rangle = 0. \tag{4.5}$$

Since $r|dt|^2 > 0$, substitution yields $\gamma'' + \gamma = 0$. The equation for $V = r\delta(t)$ has the same scalar coefficients, so it gives $\delta'' + \delta = 0$.

There are constant vectors a, b, c, d with $\gamma = a\cos t + b\sin t$ and $\delta = c\cos t + d\sin t$. All components of U, V lie in the span of $r\cos t$ and $r\sin t$. Rank one requires $p, q \geq 2$, so $p + q \geq 4$. A nonzero constant relation follows, a contradiction.

At rank two, $p = 3$. The map ϕ is a submersion onto an open subset $O \subset S^2$. Apply Lemma 3.4 to obtain product coordinates with connected fibres. Equality of kernels implies

$$\psi = F \circ \phi, \qquad F : O \longrightarrow S^{q-1}. \tag{4.6}$$

For tangent vectors of S^2, choose arbitrary lifts through $d\phi$. Equation (4.3) shows that F preserves their inner products. Hence F is an isometric immersion.

The definition of F is independent of the chosen lift: in a product neighborhood the fibre is connected, and the derivative of ψ vanishes on each tangent vector to it. Integration along paths inside the fibre makes ψ constant there. If $\xi, \eta \in T_{\phi(x)}S^2$ and $d\phi(X) = \xi$, $d\phi(Y) = \eta$, then

$$\langle dF(\xi), dF(\eta)\rangle = \langle d\psi(X), d\psi(Y)\rangle = \langle d\phi(X), d\phi(Y)\rangle = \langle \xi, \eta\rangle.$$

In particular dF is injective.

Fix a point x. Choose a local g-orthonormal frame satisfying $\nabla_{e_i} e_j(x) = 0$. All identities in the following calculation are evaluated at x. Write $\xi_i = d\phi(e_i)$ and let ∇^ϕ be the connection on ϕ^*TS^2. The Euclidean derivative of F gives

$$\begin{aligned} D_{e_i}D_{e_i}(rF(\phi)) = {} & e_ie_i(r)F(\phi) + 2e_i(r)dF(\xi_i) \\ & + r\, dF(\nabla^\phi_{e_i}\xi_i) + r\,\mathrm{II}(\xi_i, \xi_i) - r|\xi_i|^2F(\phi). \end{aligned}$$

Let pr_{N^S} denote projection onto the normal space of $F(O)$ inside S^{q-1}. The first and last terms belong to $\mathbb{R}F(\phi(x))$; the second and third belong to $dF(T_{\phi(x)}O)$. Therefore

$$\mathrm{pr}_{N^S}\,\Delta_g(rF \circ \phi) = -r\sum_i \mathrm{II}(\xi_i, \xi_i).$$

The normal projection of $\lambda rF(\phi)$ is zero. Since $r > 0$, the common eigenfunction equation implies

$$\sum_{i=1}^n \mathrm{II}\big(d\phi(e_i), d\phi(e_i)\big) = 0. \tag{4.7}$$

Choose a smooth local section $s : O \to M$ of the submersion ϕ, after decreasing O. Define

$$Q_y = \sum_{i=1}^n d\phi_{s(y)}(e_i) \otimes d\phi_{s(y)}(e_i), \qquad y \in O,$$

where e_i is any g-orthonormal basis at $s(y)$. This tensor is independent of that basis and is smooth in y. If $0 \neq \xi \in T_y^*S^2$, surjectivity of $d\phi_{s(y)}$ gives

$$Q_y(\xi, \xi) = \sum_i \big(\xi(d\phi_{s(y)}e_i)\big)^2 > 0.$$

Equation (4.7), evaluated at $s(y)$, gives $\mathrm{tr}_{Q_y}\mathrm{II}_y = 0$. Lemma 4.1 applies.

The lemma gives $F(y) = Ay$ on an open set. Therefore $V = rF(\phi) = AU$ there. Each component of $V - AU$ is a constant linear combination of the independent components of U, V, with a nonzero coefficient on a component of V. Unique continuation contradicts their independence. All ranks have been excluded when $p \leq 3$. Interchanging U, V gives the exclusion when $q \leq 3$. Hence $p, q \geq 4$. The cokernel characterization in Proposition 2.1 proves the claimed Strong Arnold consequence. □

Example 4.2. On $\mathbb{R}^2/(2\pi\mathbb{Z})^2$, the shell $|k|^2 = 5$ is $\{\pm(1,2), \pm(2,-1), \pm(1,-2), \pm(2,1)\}$. These are all solutions, since each coordinate square is at most five and the only sum of two integer squares equal to five is $1 + 4$. The real eigenspace has dimension eight. Put

$$\begin{aligned} U &= (\cos(x+2y), \sin(x+2y), \cos(2x-y), \sin(2x-y)), \\ V &= (\cos(x-2y), \sin(x-2y), \cos(2x+y), \sin(2x+y)). \end{aligned}$$

Distinct Fourier frequencies give independence, while

$$|U|^2 = |V|^2 = 2, \qquad dU^T dU = dV^T dV = 5(dx^2 + dy^2). \tag{4.8}$$

Thus $\mathrm{diag}(I_4, -I_4)$ is a nonzero cokernel element. This is the known sharp torus example; see [14, Section 4.2].

5. Conformal variations and the cokernel Hessian

The arguments of this section apply on a closed manifold and to Dirichlet or Neumann realizations when every parameter is supported in the interior.

5.1. **The generalized eigenvalue equation.** For $n \geq 3$, put $\beta = 4/(n-2)$ and

$$H = 1 + v > 0, \qquad g_v = H^\beta g. \tag{5.1}$$

For $n = 2$, put $g_v = (1+v)g$ with $1 + v > 0$. All parameters are real and smooth. On a manifold with boundary, v is compactly supported in the interior.

For $n \geq 3$, $dv_{g_v} = H^{2n/(n-2)} dv_g$ and $g_v^{-1} = H^{-4/(n-2)} g^{-1}$. Hence

$$\Delta_{g_v} u = H^{-(\beta+2)} \big(H^2 \Delta_g u - 2H \langle dH, du \rangle \big). \tag{5.2}$$

The product rule for the positive Laplacian gives $\Delta_g(Hu) = H\Delta_g u + u\Delta_g H - 2\langle dH, du\rangle$. Substituting $\Phi = Hu$ into (5.2) yields

$$A_v \Phi = \theta B_v \Phi, \qquad A_v = \Delta_g - H^{-1}\Delta_g H, \qquad B_v = H^\beta. \tag{5.3}$$

Here $H^{-1}\Delta_g H$ acts by multiplication, not composition.

The mass identity is

$$\int_M u^2 \, dv_{g_v} = \int_M H^\beta \Phi^2 \, dv_g. \tag{5.4}$$

Thus $B_v^{-1/2} A_v B_v^{-1/2}$ is the fixed-L^2 self-adjoint realization of the pencil. It agrees with (2.1), since $B_v^{1/2}\Phi = H^{n/(n-2)} u$. Near a boundary $H = 1$, so the Dirichlet and Neumann domains are unchanged. In dimension two, use $A_v = \Delta_g$ and $B_v = 1 + v$.

Along $v = ta$, define

$$\mathcal{A} = \begin{cases} \Delta_g + \beta\lambda, & n \geq 3, \\ \lambda I, & n = 2. \end{cases} \tag{5.5}$$

At $t = 0$, $A_0 = \Delta_g$ and $B_0 = I$. Differentiation of (5.3) gives

$$A_0' - \lambda B_0' = -M_{\mathcal{A}a}. \tag{5.6}$$

For $n \geq 3$ its direct second derivative is

$$A_0'' - \lambda B_0'' = M_{2a\Delta_g a - \lambda\beta(\beta-1)a^2}. \tag{5.7}$$

In dimension two this direct second derivative is zero.

Let $\Phi(v)$ be the frame corresponding to (2.8), normalized by $\Phi(v)^* B_v \Phi(v) = I_m$. Differentiating $A_v\Phi(v) = B_v\Phi(v)\mathsf{L}(v)$ at zero and taking inner products with E_λ gives

$$D\mathsf{L}(0)[a]_{ij} = -\int_M (\mathcal{A}a)u_i u_j \, dv_g. \tag{5.8}$$

The terms $\Delta_g\Phi'$ and $\lambda\Phi'$ cancel on projection. This derivation includes the mass derivative B_0'.

More explicitly, left multiplication of the differentiated pencil equation by U^* gives

$$U^*(A_0' - \lambda B_0')U + U^*(\Delta_g - \lambda)\Phi' = \mathsf{L}'.$$

The second summand is zero because the columns of U solve $\Delta_g u_i = \lambda u_i$ and Φ' satisfies the fixed boundary realization. Thus no condition on the component $\Pi\Phi'$ is needed in the first variation.

5.2. **The relative cokernel.** Let $Z \subset M$ be closed, with $Z \cap M^\circ$ having empty interior. Set $O = M^\circ \setminus Z$ and $\mathcal{V} = C_c^\infty(O)$. For closed M this notation also permits $Z = \emptyset$ and $\mathcal{V} = C^\infty(M)$. The allowed region O is dense in M°.

If C annihilates (5.8) on $\mathcal{V}$, then $\int a\,\mathcal{A}F_C \, dv_g = 0$ for all $a \in \mathcal{V}$. Therefore $\mathcal{A}F_C = 0$ on O. This function is smooth; density extends the equality to M°.

For Dirichlet eigenfunctions, $F_C|_{\partial M} = 0$. For Neumann eigenfunctions, $\partial_\nu F_C = \sum C_{ij}(u_i\partial_\nu u_j + u_j\partial_\nu u_i) = 0$. For $n \geq 3$, integration by parts therefore yields

$$0 = \int_M F_C \mathcal{A} F_C \, dv_g = \int_M |dF_C|^2 \, dv_g + \beta\lambda \int_M F_C^2 \, dv_g. \tag{5.9}$$

The boundary term vanishes in each realization. Thus $F_C = 0$. For $n = 2$, the equation $\mathcal{A}F_C = \lambda F_C = 0$ gives the same conclusion.

We have proved

$$\mathcal{C} = \operatorname{coker} D\mathsf{L}(0) = \{C \in \mathrm{Sym}_m(\mathbb{R}) \,:\, F_C = 0\}. \tag{5.10}$$

The converse follows by substituting $F_C = 0$ into the integral in (5.8). Orthonormality gives $\operatorname{tr} C = \int_M F_C \, dv_g = 0$ for every $C \in \mathcal{C}$.

Put

$$\mathcal{K} = \{a \in \mathcal{V} \,:\, PM_{\mathcal{A}a}P = 0\}, \qquad P = P_g, \tag{5.11}$$

and let

$$R = (\Delta_g - \lambda)^{-1}|_{E_\lambda^\perp}, \qquad R|_{E_\lambda} = 0. \tag{5.12}$$

The inverse is bounded on L^2 because the excluded spectrum has positive distance from λ; it maps L^2 to the elliptic domain. It is real and self-adjoint.

5.3. **The second variation.** For $C \in \mathcal{C}$, define

$$B_C = \sum_{i,j} C_{ij} M_{u_i} R M_{u_j}. \tag{5.13}$$

This contracted operator occurs in [14, Proposition 3.1 and Lemma 4.5]. It is real and self-adjoint: taking its adjoint exchanges i, j, while $C_{ij} = C_{ji}$. Its Schwartz kernel relative to $dv_g(y)$ is $b_C(x,y)R(x,y)$, where

$$b_C(x,y) = \sum_{i,j} C_{ij} u_i(x) u_j(y). \tag{5.14}$$

For $a \in \mathcal{K}$, $\mathsf{L}'(0) = 0$. Differentiating $A_t\Phi_t = B_t\Phi_t\mathsf{L}_t$ once at $t = 0$ gives

$$(\Delta_g - \lambda)\Phi' = -(A_0' - \lambda B_0')U + U\mathsf{L}'.$$

Since $\mathsf{L}' = 0$ and $A_0' - \lambda B_0' = -M_{\mathcal{A}a}$, projection to $E_\lambda^\perp$ and the definition of R give

$$(I - P)\Phi'(0) = RM_{\mathcal{A}a}U. \tag{5.15}$$

A component of Φ' in E_λ is permitted; it need not vanish in the pencil normalization.

The second differentiated pencil equation, before any projection, is

$$A_0''U + 2A_0'\Phi' + \Delta_g\Phi'' = \lambda B_0''U + 2\lambda B_0'\Phi' + \lambda\Phi'' + 2(B_0'U + \Phi')\mathsf{L}' + U\mathsf{L}''.$$

Here $\mathsf{L}' = 0$. Moreover, $U^*(\Delta_g - \lambda)\Phi'' = 0$ by self-adjointness and $\Delta_g U = \lambda U$. Multiplication by U^* therefore gives

$$\mathsf{L}''(0) = U^*(A_0'' - \lambda B_0'')U + 2U^*(A_0' - \lambda B_0')\Phi'(0). \tag{5.16}$$

There is no L' term. Since $P(A_0' - \lambda B_0')P = 0$, the E_λ component of Φ' contributes zero. Equation (5.15) determines the other component.

By (5.7), the direct term in (5.16) is multiplication by a smooth scalar function, or zero if $n = 2$. Its contraction with C is the integral of that scalar function times F_C, hence is zero. Substituting (5.6) and (5.15) gives

$$\left\langle C, D^2\mathsf{L}(0)[a,a]\right\rangle = -2\left\langle \mathcal{A}a, B_C\mathcal{A}a\right\rangle_{L^2} \qquad (a \in \mathcal{K}). \tag{5.17}$$

The matrix pairing is $\langle C, D\rangle = \mathrm{tr}(CD)$. Polarization determines the contracted mixed Hessian on $\mathcal{K}\times\mathcal{K}$.

On a closed manifold, every annihilator for the full metric derivative lies in (5.10). Every $a \in \mathcal{K}$ determines a conformal tangent vector in the kernel of the full derivative. The contracted Hessian (5.17) is therefore available for full-metric annihilators as well. Its value is independent of the acceleration of the conformal chart by the cokernel observation following (2.9).

6. Infinite positive and negative index of the cokernel Hessians

Theorem 6.1. *For every nonzero $C \in \mathcal{C}$, the quadratic form $a \mapsto \left\langle C, D^2\mathsf{L}(0)[a,a]\right\rangle$ has infinite positive index and infinite negative index on $\mathcal{K}$. This holds with the support restrictions and boundary realizations of Section 5.*

Here the positive index is the supremum of the dimensions of subspaces on which the quadratic form is positive definite; the negative index is defined with the opposite sign.

6.1. The first nonzero diagonal term.

Let $C \neq 0$. Orthonormality gives

$$\int_{M\times M} b_C(x,y)^2\, dv_g(x)dv_g(y) = \sum_{i,j} C_{ij}^2 > 0. \tag{6.1}$$

Also $b_C(x,y) = b_C(y,x)$, $b_C(x,x) = 0$, and $(\Delta_y - \lambda)b_C(x,y) = 0$.

There is a finite least order ℓ for which a transverse derivative along the diagonal is not identically zero on the allowed region O. Otherwise, for each $x \in O$ the function $y \mapsto b_C(x,y)$ would vanish to infinite order at $y = x$. Aronszajn's theorem [5] would make it zero for all y in the connected interior; its equation has the smooth elliptic coefficients and constant potential verified in Section 2. Continuity gives the same conclusion up to the boundary. Taking its L^2 pairing with $u_j(y)$ gives $\sum_i C_{ij}u_i(x) = 0$ on O. Unique continuation in x then gives this relation globally for every j, and orthonormality gives $C = 0$.

Lemma 6.2. *At each fixed reference (g,λ) and allowed dense open set O, there is an integer L such that every nonzero $C \in \mathcal{C}$ has a nonzero transverse diagonal derivative of order at most L at some point of O.*

Proof. If $\mathcal{C} = 0$, take $L = 0$; the assertion is vacuous. Suppose $\dim\mathcal{C} > 0$. For every $x \in O$ and multi-index α in a chart, evaluation of $\partial_y^\alpha b_C(x,y)|_{y=x}$ is a linear functional of $C \in \mathcal{C}$. The common kernel of all these functionals is zero by the preceding unique-continuation argument. Their span is therefore the whole dual of the finite-dimensional space $\mathcal{C}$: a proper span would have a nonzero annihilator. Choose finitely many of these functionals forming a basis of that dual, and let L be the largest derivative order among them. If $C \neq 0$, at least one selected functional is nonzero on C, proving the assertion. If the selected functionals are $\ell_1, \dots, \ell_d$, their joint map is an isomorphism from $\mathcal{C}$ to $\mathbb{R}^d$. Hence, for constants $a, b > 0$ depending on the reference,

$$a\,\|C\| \le \left(\sum_{j=1}^{d} |\ell_j(C)|^2\right)^{1/2} \le b\,\|C\| \qquad (C \in \mathcal{C}). \tag{6.2}$$

The lower constant is the positive minimum of that expression on the unit sphere of $\mathcal{C}$; the upper constant is its maximum. This is a finite bound for one fixed reference; no bound uniform over varying metrics is asserted. ☐

Choose an allowed coordinate chart containing a point where the order-ℓ coefficient is nonzero. In coordinates $y = x + z$,

$$b_C(x, x+z) = P_x(z) + O(|z|^{\ell+1}), \qquad P_x(z) = \sum_{|\alpha|=\ell} \frac{z^\alpha}{\alpha!} \sum_{i,j} C_{ij} u_i(x) \partial^\alpha u_j(x). \tag{6.3}$$

All lower coefficient functions vanish throughout the chart.

Write $\widetilde{b}(x,z) = b_C(x, x+z)$. On a smaller product neighborhood, the remainder has the integral representation

$$E(x,z) = (\ell+1) \sum_{|\gamma|=\ell+1} \frac{z^\gamma}{\gamma!} \int_0^1 (1-t)^\ell \partial_z^\gamma \widetilde{b}(x, tz)\, dt. \tag{6.4}$$

Indeed, the one-dimensional Taylor formula applied to $t \mapsto \widetilde{b}(x,tz)$ expands its $(\ell+1)$-st derivative as $\sum_{|\gamma|=\ell+1} (\ell+1)! z^\gamma \partial_z^\gamma \widetilde{b}/\gamma!$. Differentiating (6.4) under the fixed integral gives the remainder estimates below; the required derivatives are bounded on a compact product set. After restricting its closure to a compact subset of O, the Taylor remainder $E(x,z) = b_C(x,x+z) - P_x(z)$ satisfies

$$|\partial_x^\alpha \partial_z^\beta E(x,z)| \le C_{\alpha\beta} |z|^{\ell+1-|\beta|}, \qquad |\beta| \le 2, \tag{6.5}$$

for every fixed finite list of α. Apply the one-variable integral Taylor formula to $t \mapsto b_C(x, x+tz)$, or to its indicated derivatives; boundedness of the derivatives of order $\ell + 1 + |\alpha|$ on a compact set gives this estimate. Under a change $z = L\widetilde{z} + O(|\widetilde{z}|^2)$, all lower coefficients are zero, so the leading polynomial changes to $P_x(L\widetilde{z})$. Thus the order and the leading homogeneous tensor are independent of the chosen chart.

Symmetry and the Taylor formula give

$$b_C(x, x+z) = b_C(x+z, x) = P_{x+z}(-z) + O(|z|^{\ell+1}).$$

The coefficient tensors depend smoothly on their base point, so $P_{x+z}(-z) = P_x(-z) + O(|z|^{\ell+1})$. Comparing homogeneous terms yields $P_x(z) = P_x(-z)$. At a point where $P_x \ne 0$, homogeneity gives $(-1)^\ell = 1$. Thus ℓ is even. Since $b_C(x,x) = 0$, $\ell \ge 2$.

Apply $(\Delta_y - \lambda)$ to (6.3). Terms of degree $\ell - 2$ arise only from the principal second-order part applied to P_x. Thus

$$\sum_{a,b} g^{ab}(x) \partial_{z_a} \partial_{z_b} P_x(z) = 0. \tag{6.6}$$

In coordinates, write $\Delta_y = -g^{ab}(y)\partial_{y_a}\partial_{y_b} + b^j(y)\partial_{y_j}$. The principal coefficients satisfy $g^{ab}(x+z) = g^{ab}(x) + O(|z|)$. Since $\partial^2 P_x$ has degree $\ell - 2$, their nonconstant parts contribute $O(|z|^{\ell-1})$. The first-order term is $O(|z|^{\ell-1})$, and $\lambda P_x = O(|z|^\ell)$. Equation (6.5) bounds the twice-differentiated remainder by $O(|z|^{\ell-1})$. The only homogeneous term of degree $\ell - 2$ is therefore $-g^{ab}(x)\partial_{z_a}\partial_{z_b} P_x$, which must be zero.

At a point where $P_x \ne 0$, choose coordinates orthonormal for g_x and put $M(r) = \int_{S^{n-1}} P_x(r\omega)\, dS(\omega)$. The divergence theorem and (6.6) give

$$r^{n-1} M'(r) = \int_{\partial B_r} \partial_\nu P_x \, dS = \int_{B_r} \sum_j \partial_j^2 P_x \, dz = 0.$$

Thus $M(r) = M(0) = |S^{n-1}|P_x(0) = 0$. If P_x were nonnegative on the unit sphere and nonzero there, continuity would make its integral strictly positive. Applying the same argument to $-P_x$ excludes nonpositivity. A homogeneous polynomial that vanished on the unit sphere would vanish on every nonzero ray and hence be zero. Thus both signs occur.

6.2. **The first nonzero symbol.** On a closed manifold,

$$R = (\Delta_g - \lambda + P)^{-1} - P. \tag{6.7}$$

This follows by applying both sides to each eigenfunction: the multiplier of the inverse is one on E_λ and $(\nu - \lambda)^{-1}$ on every excluded eigenspace of eigenvalue ν. The finite-rank operator P has smooth kernel $\sum_i u_i(x)u_i(y)$.

The elliptic parametrix construction [18, Section 18.1], [12, Theorem 14.17] applies to $A = \Delta_g - \lambda$: its principal symbol is $g^{ij}(x)\xi_i\xi_j > 0$ for $\xi \neq 0$. It gives properly supported right and left classical parametrices Q_+ and Q_- of order -2, both with principal symbol $|\xi|_g^{-2}$, such that $AQ_+ - I$ and $Q_-A - I$ have smooth kernels. Their difference is smoothing, since

$$Q_- - Q_+ = Q_-(I - AQ_+) + (Q_-A - I)Q_+.$$

Composition of a properly supported pseudodifferential operator with a smooth-kernel remainder is smoothing on the smaller coordinate chart; on a closed manifold the same assertion is global. Thus $Q = Q_+$ satisfies both $AQ - I \in \Psi^{-\infty}$ and $QA - I \in \Psi^{-\infty}$. Here $\Psi^{-\infty}$ denotes operators with smooth kernels. The construction is local in interior charts and global when M is closed.

We verify that the same interior symbol describes the Dirichlet and Neumann reduced inverses. Choose nested coordinate domains $U_0 \Subset U_1 \Subset M^\circ$ and cutoffs supported in U_1 and equal to one on a neighborhood of $\overline{U}_0$. Conjugate by $m^{1/2}$, where $dv_g = m(x)\,dx$, so that kernels are taken relative to Lebesgue measure. Construct the parametrices on U_1; all subsequent kernel equations are restricted to $U_0 \times U_0$. Derivatives of the cutoffs are supported outside this smaller product and do not enter those equations. Continue to denote the conjugated operators by A, R, P. Let K_R be the distribution kernel of R. The spectral identities $AR = RA = I - P$, tested against compactly supported smooth functions, imply

$$A_x K_R = \delta_{\mathrm{diag}} - K_P, \qquad A_y^{\mathrm{t}} K_R = \delta_{\mathrm{diag}} - K_P. \tag{6.8}$$

Here $^{\mathrm{t}}$ denotes the formal transpose with respect to dy. The two-sided local parametrix has analogous identities with smooth remainders. Consequently, for $K = K_R - K_Q$,

$$(A_x + A_y^{\mathrm{t}})K \in C^\infty \tag{6.9}$$

near the interior diagonal. The principal symbol of the product operator is

$$g^{ij}(x)\xi_i\xi_j + g^{ij}(y)\eta_i\eta_j,$$

which is positive whenever $(\xi, \eta) \neq 0$. Interior elliptic regularity [12, Theorem 15.1] therefore gives $K \in C^\infty$ there. Away from the diagonal, (6.8) gives the same conclusion for K_R itself. Multiplying on both sides by interior cutoffs now shows that the localized reduced inverse is a classical operator of order -2, modulo an operator with smooth kernel. This proof uses only interior equations; it applies to either boundary condition because compactly supported test functions satisfy both boundary conditions.

The conjugation by $m^{1/2}$ does not change the principal symbol. Its two scalar factors $m^{1/2}(x)$ and $m^{-1/2}(x)$ multiply to one in the principal term. They also commute with every M_{u_i}. Thus the calculation for B_C may be made in these Lebesgue coordinates and then returned to $L^2(dv_g)$ without changing its first nonzero principal symbol. All quadratic pairings below are taken with the Riemannian density.

Use left Kohn–Nirenberg symbols, with $D_x = i^{-1}\partial_x$. On a compact coordinate set, a symbol $p \in S^d$ satisfies

$$|\partial_x^\alpha \partial_\xi^\beta p(x, \xi)| \leq C_{\alpha\beta}(1 + |\xi|)^{d-|\beta|}.$$

For properly supported operators with symbols $p \in S^d$ and $q \in S^e$, the composition theorem [18, Section 18.1] gives

$$\sigma(P_1 P_2) - \sum_{|\alpha|<N} \frac{i^{-|\alpha|}}{\alpha!}(\partial_\xi^\alpha p)(\partial_x^\alpha q) \in S^{d+e-N}. \tag{6.10}$$

The inserted compactly supported cutoffs make every operator in the following local calculation properly supported. If $r(x,\xi)$ is a local symbol of R, apply (6.10) with $q = u_j$ and then multiply on the left by u_i. Summation gives

$$\sigma(B_C) \sim \sum_\alpha \frac{i^{-|\alpha|}}{\alpha!}\left(\sum_{i,j} C_{ij} u_i(x)\partial^\alpha u_j(x)\right)\partial_\xi^\alpha r(x,\xi). \tag{6.11}$$

This follows by Taylor expanding the right multiplier in the oscillatory kernel of R; each factor $(y-x)^\alpha$ is transferred by integration by parts to $i^{-|\alpha|}\partial_\xi^\alpha r$. Truncation to $|\alpha| < N$ leaves a remainder in S^{-2-N} by (6.10).

Every coefficient in (6.11) with $|\alpha| < \ell$ vanishes identically. The first possible nonzero degree is $-2-\ell$, obtained only from $|\alpha| = \ell$ and the principal symbol of R. Thus, in coordinates orthonormal at the point,

$$\sigma_{-\ell-2}(B_C) = i^{-\ell} P_x(\partial_\xi)|\xi|^{-2}. \tag{6.12}$$

No lower resolvent symbol can contribute at this degree: its order is at most -3, and the smaller $|\alpha|$ coefficients are zero.

Let $P(z) = A_{i_1\dots i_\ell} z_{i_1}\cdots z_{i_\ell}$ be homogeneous and harmonic, with A symmetric. Since

$$\sum_j \partial_j^2 P = \ell(\ell-1)\sum_j A_{jji_3\dots i_\ell} z_{i_3}\cdots z_{i_\ell},$$

every trace of A is zero. Induction on p gives

$$\partial_{i_1}\cdots\partial_{i_p}|\xi|^{-2} = c_p\, \xi_{i_1}\cdots\xi_{i_p}|\xi|^{-2p-2} + E_{i_1\dots i_p}(\xi),$$

where every term of E contains a factor $\delta_{i_a i_b}$ joining two of the free indices. Differentiating the displayed principal term either differentiates a numerator factor, producing such a delta, or differentiates $|\xi|^{-2p-2}$. The latter gives

$$c_0 = 1, \qquad c_{p+1} = -(2p+2)c_p, \qquad c_p = (-2)^p p!.$$

Differentiation of a term already containing a delta retains it. Contraction with A therefore annihilates E when $p = \ell$, and yields

$$P(\partial_\xi)|\xi|^{-2} = (-2)^\ell \ell!\,\frac{P(\xi)}{|\xi|^{2\ell+2}}. \tag{6.13}$$

Since ℓ is even,

$$\sigma_{-\ell-2}(B_C)(x,\xi) = (-1)^{\ell/2} 2^\ell \ell!\,\frac{P_x(\xi)}{|\xi|^{2\ell+2}}. \tag{6.14}$$

The expression is written in coordinates orthonormal at the point in question. Intrinsically, the numerator is $P_x(\xi^{\sharp g})$, where $P_x \in \mathrm{Sym}^\ell T_x^*M$ is the leading diagonal tensor and $\xi^{\sharp g}$ is the vector corresponding to ξ. The denominator is $|\xi|_g^{2\ell+2}$. Thus its sign is independent of the chart and varies continuously on the nonzero cotangent bundle. The symbol is real and even in ξ. The local operator defining (5.17) is

$$\mathcal{T}_C = -2\mathcal{A}B_C\mathcal{A}. \tag{6.15}$$

For $n \geq 3$ it has order $2-\ell \leq 0$ and principal symbol $-2|\xi|^4$ times (6.14). For $n = 2$ it has order $-2-\ell$ and principal symbol $-2\lambda^2$ times that expression. The sign takes both values by the preceding harmonic-polynomial argument.

6.3. **Oscillatory test functions.**

Lemma 6.3. *Let T be a real, self-adjoint classical pseudodifferential operator of order $d \le 0$ on a coordinate neighborhood, with even principal symbol $t_d(x,\xi)$. If $a \in C_c^\infty$ is real and $\xi_0 \neq 0$ is constant in these coordinates, then*

$$\langle T(a\cos(Nx\cdot\xi_0)), a\cos(Nx\cdot\xi_0)\rangle = \frac{N^d}{2}\int a^2 t_d(x,\xi_0)\,dv_g + O(N^{d-1}). \tag{6.16}$$

For two amplitudes with disjoint compact supports, the cross pairing is $O(N^{-L})$ for every L.

Proof. Conjugate first by the positive density factor, as in the density-conjugation calculation above, and absorb its smooth square root into the amplitude. It suffices to work with Lebesgue measure. Insert fixed cutoffs equal to one near the amplitude supports. A smoothing remainder contributes $O(N^{-L})$ in every fixed C^j norm, for every L, by integration by parts in its smooth kernel.

For the left symbol t of the remaining properly supported operator,

$$e^{-iNx\cdot\xi_0}T(e^{iNx\cdot\xi_0}a) = (2\pi)^{-n}\int e^{ix\cdot\eta}t(x,N\xi_0+\eta)\widehat{a}(\eta)\,d\eta. \tag{6.17}$$

On $|\eta| \le N|\xi_0|/2$, homogeneity of t_d and the order-$d-1$ remainder imply, for each fixed multi-index α,

$$\left|\partial_x^\alpha[t(x,N\xi_0+\eta) - N^d t_d(x,\xi_0)]\right| \le C_\alpha N^{d-1}(1+|\eta|).$$

The segment between $N\xi_0$ and $N\xi_0+\eta$ has length at most $N|\xi_0|/2$ and remains in $|\xi| \ge N|\xi_0|/2$; the bound follows from $|\partial_\xi\partial_x^\alpha t| \le C(1+|\xi|)^{d-1}$. When derivatives also act on $e^{ix\cdot\eta}$, they add a fixed power of $|\eta|$.

On the complementary region, every fixed x derivative of t is bounded because $d \le 0$. For all integers $j, L \ge 0$, rapid Fourier decay gives

$$\int_{|\eta|>N|\xi_0|/2}(1+|\eta|)^j|\widehat{a}(\eta)|\,d\eta = O(N^{-L}).$$

The same estimate applies to the subtracted principal term. Integrating the two regions proves, on the fixed output compact set,

$$\left\|e^{-iNx\cdot\xi_0}T(e^{iNx\cdot\xi_0}a) - N^d t_d(x,\xi_0)a\right\|_{C^j} \le C_j N^{d-1}. \tag{6.18}$$

The negative exponential has the analogous formula with ξ_0 replaced by $-\xi_0$.

Put $u_\pm = ae^{\pm iNx\cdot\xi_0}$ and use the Hermitian L^2 pairing for this calculation. Equation (6.18), with $j=0$, gives

$$(Tu_\pm, u_\pm)_{L^2} = N^d\int a^2 t_d(x,\pm\xi_0)\,dx + O(N^{d-1}).$$

The other two pairings have principal terms

$$N^d\int e^{\pm 2iNx\cdot\xi_0}a^2 t_d(x,\pm\xi_0)\,dx.$$

More precisely, put $b_{\pm,N} = e^{\mp iNx\cdot\xi_0}T(u_\pm)$. Equation (6.18) gives $\|b_{\pm,N}\|_{C^j} \le C_j N^d$ for every fixed j. Consequently the full mixed pairing, including its remainder, satisfies

$$\left|\int e^{\pm 2iNx\cdot\xi_0}a(x)b_{\pm,N}(x)\,dx\right| \le C_j N^{d-j}.$$

This follows by j integrations with $(\xi_0\cdot\partial_x)/(\pm 2iN|\xi_0|^2)$; a has compact support. Choosing j arbitrarily large proves rapid decay. Since $a\cos(Nx\cdot\xi_0) = (u_+ + u_-)/2$ and $t_d(x,-\xi_0) = t_d(x,\xi_0)$, the two diagonal pairings prove (6.16). Undoing density conjugation replaces $a^2\,dx$ by $a^2\,dv_g$.

Let $K_T(x, y)$ be the distribution kernel, and let the two compact amplitude supports be disjoint. Pseudolocality of the properly supported operator, included in the parametrix and composition results cited above, makes K_T smooth on a neighborhood of their product. Each cross pairing is a sum of four integrals

$$\iint e^{iN(\epsilon y-\epsilon' x)\cdot\xi_0} a_2(y)a_1(x)K_T(x, y)\,dx\,dy, \qquad \epsilon, \epsilon' \in \{-1, 1\}.$$

Integrate L times in x with $(\xi_0\cdot\partial_x)/(-i\epsilon' N|\xi_0|^2)$. There are no boundary terms because the amplitudes have compact support. The integral is bounded by $C_L N^{-L}$, where C_L is a finite sum of L^1 norms of derivatives of the smooth amplitude-kernel product. This proves the assertion for every L. The same calculation applies to smoothing remainders. $\square$

Choose a point and a covector in the allowed chart where the principal symbol of $\mathcal{T}_C$ is strictly negative. By continuity it remains negative for that constant coordinate covector on a smaller open set. For any positive integer N_0, choose N_0 nonzero real amplitudes with pairwise disjoint compact supports in that set. The lemma makes the diagonal quadratic entries strictly negative of order N^d; all off-diagonal entries are smaller than every inverse power. Write $\langle \mathcal{T}_C a_{j,N}, a_{j,N}\rangle \le -c_j N^d$ for all sufficiently large N, with $c_j > 0$, and let $c_* = \min_{1\le j\le N_0} c_j$. Choose $L > 1 - d$. For large common N, every off-diagonal entry has absolute value at most $c_* N^d/(2N_0)$. For real scalars b_j, the inequality $2|b_i b_j| \le b_i^2 + b_j^2$ then gives

$$\left\langle \mathcal{T}_C \sum_j b_j a_{j,N}, \sum_j b_j a_{j,N} \right\rangle \le -\tfrac{1}{2} c_* N^d \sum_j b_j^2.$$

Thus their span is negative definite, and the functions are independent.

Choose a basis $\ell_1, \dots, \ell_q$ for the independent coordinate functionals of $D\mathsf{L}(0)$ on $\mathcal{V}$. Equation (5.8) and integration by parts write them as $\ell_i(a) = \int_M a\, w_i\, dv_g$ with fixed smooth w_i. Since these functionals are independent, the map $a \mapsto (\ell_i(a))$ is onto $\mathbb{R}^q$: otherwise a nonzero linear combination of the ℓ_i would vanish. Choose $b_j \in \mathcal{V}$ with $\ell_i(b_j) = \delta_{ij}$.

For an oscillatory test a_N, repeated integration by parts gives $\ell_i(a_N) = O(N^{-L})$ for every L. Set

$$\widehat{a}_N = a_N - \sum_{i=1}^{q} \ell_i(a_N) b_i. \tag{6.19}$$

Then $\widehat{a}_N \in \mathcal{K}$ and $\widehat{a}_N - a_N = O(N^{-L})$ in every fixed Sobolev or C^k norm. The support lies in the compact union of the chosen amplitudes and b_i. Choose $\eta \in C_c^\infty(O)$ equal to one on a neighborhood of this finite union. When $O = M$ is closed this notation permits $\eta \equiv 1$. All pairings in question equal pairings with $\eta\mathcal{T}_C\eta$. On a finite cover of $\operatorname{supp}\eta$, the localized kernels have the symbols obtained in (6.14)–(6.15); kernels between disjoint chart supports are smooth by (6.8). A finite partition of unity therefore expresses $\eta\mathcal{T}_C\eta$ as operators of order at most zero in coordinate charts plus smooth-kernel operators. The Sobolev boundedness theorem [12, Proposition 15.3], equivalently the order-zero boundedness theorem in [18, Section 18.1], therefore gives a bounded map $L^2 \to L^2$: both its spatial cutoffs have compact interior support and its symbol satisfies the estimates stated above. If $e_N = \widehat{a}_N - a_N$, the change in a quadratic pairing is bounded by

$$\|\eta\mathcal{T}_C\eta\|_{L^2\to L^2}\left(2\,\|a_N\|_{L^2}\,\|e_N\|_{L^2} + \|e_N\|_{L^2}^2\right).$$

The amplitudes have bounded L^2 norm, whereas $\|e_N\|_{L^2} = O(N^{-L})$ for every L. Taking $L > -d + 1$ makes this change $o(N^d)$. The same estimate applies to each mixed entry in the fixed finite family; hence the displayed definite inequality persists with $c_*/4$ in place of $c_*/2$.

Put $A_N = (N^{-d}\langle \eta\mathcal{T}_C\eta\, a_{i,N}, a_{j,N}\rangle)_{ij}$ and define $\widehat{A}_N$ with $\widehat{a}_{i,N}$ in place of $a_{i,N}$. If every entry of $\widehat{A}_N - A_N$ has absolute value at most ϵ_N, then

$$\|\widehat{A}_N - A_N\|_{\mathrm{op}} \le N_0\epsilon_N.$$

This follows by Cauchy–Schwarz from $|\sum_{ij} b_i(\widehat{A}_N - A_N)_{ij} b_j| \le \epsilon_N(\sum_i |b_i|)^2 \le N_0\epsilon_N \sum_i b_i^2$. Here $\epsilon_N \to 0$. Take N large enough that $N_0\epsilon_N < c_*/4$. The lower definite bound follows on the whole span, not only on its chosen generators.

The positive symbol region yields N_0 corrected test functions with positive-definite quadratic matrix. In either construction, definiteness implies linear independence. Since N_0 was arbitrary, (5.17) has infinite positive and negative index on $\mathcal{K}$. This proves Theorem 6.1.

If $H \subset \mathcal{K}$ has codimension q, intersect an $(N+q)$-dimensional definite subspace with H. Its intersection has dimension at least N and retains definiteness. Thus every scalar cokernel form still has infinite positive and negative index on H.

7. Regular zeros of quadratic maps

Regular-zero criteria for quadratic maps are established in [1] and [3, Proposition 9.4]. We prove a formulation with surjectivity on every finite-codimensional subspace and a pair of regular zeros sharing the same derivative complement. No topology on the source is required until a finite-dimensional subspace is chosen.

Let E be a real vector space and $Q : E \to \mathbb{R}^d$ a quadratic map: each coordinate of Q is a quadratic form. Thus its polarization is bilinear, not merely homogeneous of degree two. Write

$$B(x,y) = \tfrac{1}{2}\big(Q(x+y) - Q(x) - Q(y)\big), \qquad DQ(x)y = 2B(x,y). \tag{7.1}$$

Assume $d \ge 1$ and that every scalar form $c \cdot Q$, $0 \ne c \in \mathbb{R}^d$, has infinite negative index. Replacing c by $-c$ also gives infinite positive index.

Theorem 7.1. *Under this hypothesis, Q maps every finite-codimensional subspace of E onto $\mathbb{R}^d$. There exist $x, y \in E$ and $w_1, \dots, w_d \in E$ such that*

$$Q(x+y) = Q(x-y) = 0, \qquad Q(y) = -Q(x) \ne 0, \tag{7.2}$$

and both derivatives $DQ(x+y)$ and $DQ(x-y)$ map the span of the w_i isomorphically onto $\mathbb{R}^d$. In particular, Q has a nonzero regular zero in a finite-dimensional subspace.

7.1. **Uniform index on a finite-dimensional restriction.** If $H \subset E$ has finite codimension q, an $(N+q)$-dimensional negative-definite subspace for a scalar form meets H in dimension at least N. Thus the hypothesis remains true on H. It suffices first to prove surjectivity on E.

The finite-dimensional restriction follows by the compactness argument of [3, Lemma 9.2], with an explicit index margin. For each unit vector $c \in \mathbb{R}^d$, choose a $(d+2)$-dimensional subspace V_c on which $c \cdot Q$ is negative definite. On the unit sphere of V_c, this form has a strictly negative maximum. The same inequality persists for dual vectors sufficiently close to c. Compactness of the dual unit sphere gives finitely many such subspaces; let V be their sum and choose an inner product on V.

Let $A(c)$ be the symmetric matrix of $c \cdot Q$ on V. The min–max principle gives

$$|\lambda_j(A(c)) - \lambda_j(A(c'))| \le \|A(c) - A(c')\|_{\mathrm{op}}.$$

The right side tends to zero as $c' \to c$, because $A(c)$ is linear in c. Every unit c has at least $d+2$ negative eigenvalues on V. Therefore

$$\eta = -\max_{|c|=1} \lambda_{d+2}(A(c)) > 0.$$

For $c \ne 0$, the span of eigenvectors corresponding to the first $d+2$ eigenvalues of $A(c)$ satisfies

$$c \cdot Q(v) \le -\eta |c|\, |v|^2. \tag{7.3}$$

7.2. **Surjectivity.** Fix $a \in \mathbb{R}^d \setminus \{0\}$ and choose $R_0^2 > |a|/\eta$. The function $F(z) = \frac{1}{2}|Q(z) - a|^2$ attains a minimum on $\{|z| \le R_0\} \subset V$. Put $p = Q(z) - a$ at a minimizing point. Suppose $p \neq 0$.

At an interior minimum, set $\nu = 0$; then $DF(z) = 0$. At a boundary minimum, restriction to the sphere gives $\nabla F(z) = \alpha z$. The inward derivative satisfies

$$DF(z)[-z] = -\alpha R_0^2 \ge 0,$$

so $\alpha \le 0$. Put $\nu = -\alpha/2 \ge 0$. Thus $\nabla F(z) + 2\nu z = 0$. Taking its inner product with z and using $DQ(z)z = 2Q(z)$ gives

$$\nu R_0^2 = -p \cdot Q(z) = -p \cdot (a + p) \le |p|\,|a| \tag{7.4}$$

at a boundary minimum.

Let $W_p \subset V$ be a subspace of dimension at least $d + 2$ on which (7.3) holds with $c = p$. The linear map

$$W_p \longrightarrow \mathbb{R}^d \oplus \mathbb{R}, \qquad v \longmapsto (DQ(z)v, \langle v, z \rangle)$$

has a nonzero kernel, since its target has dimension $d + 1$. Fix a nonzero vector in that kernel. Then $DQ(z)v = 0$, $v \perp z$, and $p \cdot Q(v) \le -\eta|p||v|^2$. At a boundary minimum it is tangent to the sphere. The Hessian of the Lagrangian on such a tangent vector is nonnegative. For a boundary minimizer z, use the curve

$$z(t) = R_0 \frac{z + tv}{|z + tv|}.$$

Since $|z| = R_0$ and $v \perp z$, it has $z'(0) = v$ and $z''(0) = -|v|^2 z/R_0^2$. Thus

$$0 \le (F \circ z)''(0) = D^2F(z)[v, v] + 2\nu|v|^2,$$

by $DF(z)[w] = -2\nu\langle z, w \rangle$. At an interior minimum the unconstrained Hessian is nonnegative.

Since $D^2Q[v, v] = 2Q(v)$,

$$\begin{aligned} D^2F(z)[v, v] + 2\nu|v|^2 &= |DQ(z)v|^2 + 2p \cdot Q(v) + 2\nu|v|^2 \\ &\le 2|p| \left(-\eta + \frac{|a|}{R_0^2} \right) |v|^2 < 0. \end{aligned} \tag{7.5}$$

For an interior minimum use $\nu = 0$, which gives the same contradiction. Thus $p = 0$ and $Q(z) = a$. The target zero is attained at zero. This proves $Q(V) = \mathbb{R}^d$. Applying the argument to each finite-codimensional H proves its asserted surjectivity as well.

7.3. **A regular point.** Suppose the maximum rank of $DQ|_V$ were $r < d$, and choose a point attaining it. If $r > 0$, a nonzero maximal minor stays nonzero in a neighborhood, and maximality prevents higher rank. If $r = 0$, the derivative is zero throughout V. In both cases the rank is locally constant. Choose $0 \neq c$ annihilating the image of DQ at that point. For every $v \in \ker DQ$, Lemma 3.4 supplies a smooth curve in its local level set with initial velocity v. If the curve is $\zeta(t)$, with $\zeta(0) = z$ and $\zeta'(0) = v$, then

$$0 = \frac{d^2}{dt^2} Q(\zeta(t))\Big|_{t=0} = 2Q(v) + DQ(z)\zeta''(0).$$

Pairing with c, which annihilates $\operatorname{im} DQ(z)$, gives $c \cdot Q(v) = 0$.

Every negative-definite subspace meets $\ker DQ$ only at zero. Its dimension is therefore at most r. This contradicts the negative index at least $d + 2$ in (7.3). There is a regular point x. It can be chosen with $Q(x) \neq 0$: a restriction to a d-dimensional derivative-invertible subspace and Lemma 3.2 put an open neighborhood of $Q(x)$ in the image. A nonzero $d \times d$ minor stays nonzero nearby. Choose a nonzero value in that open image and its preimage on the slice.

7.4. **Two regular zeros.** Choose $w_1, \dots, w_d \in V$ such that $DQ(x)w_i$ are a basis of $\mathbb{R}^d$. Define

$$H = \{y \in E : B(x, y) = 0,\ B(w_i, y) = 0\ (1 \le i \le d)\}. \tag{7.6}$$

The defining map $y \mapsto (B(x, y), B(w_1, y), \dots, B(w_d, y))$ is linear into $(\mathbb{R}^d)^{d+1}$, so $\operatorname{codim} H \le d(d+1)$. Each nonzero scalar form still has infinite negative index on H. The surjectivity assertion already proved, applied with source H and target $-Q(x)$, supplies $y \in H$ with $Q(y) = -Q(x)$. This application may enlarge the finite-dimensional source subspace; only the finitely many vectors subsequently selected are used.

Equation (7.1) gives $Q(x \pm y) = Q(x) + Q(y) \pm 2B(x, y) = 0$, and $DQ(x \pm y)w_i = DQ(x)w_i \pm 2B(y, w_i) = DQ(x)w_i$. Also $Q(y) = -Q(x) \ne 0$. Neither $x + y$ nor $x - y$ is zero: either equality would give $B(x, y) = \pm Q(x) \ne 0$, contradicting (7.6). Both zeros and the common derivative complement belong to $\operatorname{span}\{x, y, w_1, \dots, w_d\}$.

The vectors $w_1, \dots, w_d$ are independent: applying $DQ(x)$ to a relation between them gives the same relation between a basis of $\mathbb{R}^d$. Therefore each restriction in (7.2) is an isomorphism, not only a surjection. The hypotheses were used after each finite-codimensional restriction before choosing the new vector y; no single finite-dimensional subspace with unbounded index is asserted. This proves Theorem 7.1.

The constraints (7.6) are part of the proof. Scalar indefiniteness alone is not being substituted for simultaneous solvability, and no simultaneous diagonalization of the components of Q is assumed.

8. Local sections and singular multiplicity loci

8.1. **Finite-dimensional reduction.** Use the conformal parameter v of Section 5. Let

$$\mathcal{R} = \operatorname{im} D\mathsf{L}(0), \qquad \mathcal{C} = \mathcal{R}^{\perp}, \qquad \mathrm{Sym}_m = \mathcal{R} \oplus \mathcal{C}. \tag{8.1}$$

Orthogonality refers to $\operatorname{tr}(CD)$. If $\mathcal{C} = 0$, choose preimages of a basis of Sym_m and restrict L to their span. Its derivative is an isomorphism, so Lemma 3.2 gives a smooth section through g, exact realization, and an estimate linear in the prescribed matrix displacement. Henceforth suppose $d = \dim \mathcal{C} > 0$.

On $\mathcal{K}$ define

$$Q(a) = \tfrac{1}{2} \operatorname{proj}_{\mathcal{C}} D^2\mathsf{L}(0)[a, a]. \tag{8.2}$$

For every nonzero $C \in \mathcal{C}$, the scalar form $\langle C, Q(a)\rangle$ has infinite negative index by Theorem 6.1. Choose the two regular zeros $z_+ = x + y$, $z_- = x - y$ given by Theorem 7.1. Let $V \subset \mathcal{K}$ be a finite-dimensional space containing them and the vectors with independent derivative images.

Choose $a_1, \dots, a_q \in \mathcal{V}$ whose derivatives form a basis of $\mathcal{R}$. In a basis $v_1, \dots, v_N$ of V, use the metric family with conformal parameter

$$v(z, t) = \sum_{\alpha=1}^{q} z_\alpha a_\alpha + \sum_{j=1}^{N} t_j v_j. \tag{8.3}$$

All functions are fixed before any small parameter is sent to zero.

The derivative in z of $\operatorname{proj}_{\mathcal{R}}(\mathsf{L}(v(z, t)) - \lambda I_m)$ is invertible at zero. Lemma 3.2, with t as parameter, gives a smooth $z = z(t)$ satisfying

$$\operatorname{proj}_{\mathcal{R}}(\mathsf{L}(v(z(t), t)) - \lambda I_m) = 0, \qquad z(0) = 0, \quad Dz(0) = 0. \tag{8.4}$$

The last equality follows because every $v_j \in \ker D\mathsf{L}(0)$.

Put

$$F(t) = \operatorname{proj}_{\mathcal{C}}\big(\mathsf{L}(v(z(t), t)) - \lambda I_m\big). \tag{8.5}$$

Then $F(0) = DF(0) = 0$, and

$$F(t) = Q(t) + O(|t|^3). \tag{8.6}$$

Here Q is restricted to V. In the second derivative, the term containing $D^2 z$ lies in $\operatorname{im} D\mathsf{L}(0) = \mathcal{R}$ and its cokernel projection vanishes. All functions in the finite family are smooth, so the map has the required third and higher parameter derivatives.

The rescaling below is the smooth parameter version of the regular-zero argument in [3, Lemma 9.3]. Define

$$\mathcal{F}(\varepsilon, t) = \begin{cases} \varepsilon^{-2} F(\varepsilon t), & \varepsilon \neq 0, \\ Q(t), & \varepsilon = 0. \end{cases} \tag{8.7}$$

The extension is smooth, since

$$\varepsilon^{-2} F(\varepsilon t) = \int_0^1 (1-s) D^2 F(s\varepsilon t)[t,t]\, ds. \tag{8.8}$$

At $\varepsilon = 0$ the integral is $\frac{1}{2} D^2 F(0)[t,t] = Q(t)$.

At either $z_\pm$, the derivative $DQ(z_\pm)$ is onto. Choose a subspace of V on which it is an isomorphism onto $\mathcal{C}$, hold the remaining coordinates fixed at those of $z_\pm$, and apply Lemma 3.2 to (8.7). It gives a smooth $t_\pm(\varepsilon)$ with

$$t_\pm(0) = z_\pm, \qquad F(\varepsilon t_\pm(\varepsilon)) = 0. \tag{8.9}$$

The resulting metrics satisfy $\mathsf{L} = \lambda I_m$ exactly. A sufficiently small neighborhood retains the original isolating contour. Thus λ has multiplicity exactly m, with no additional eigenvalue entering the cluster.

Indeed, the Riesz projection has rank m throughout the selected neighborhood, and its complementary projection commutes with the self-adjoint operator. The equality $\mathsf{L} = \lambda I_m$ gives exactly m independent eigenfunctions in its range. If another λ-eigenfunction were orthogonal to that range, its spectral projection for the same contour would both equal itself and vanish. This is impossible.

At the constructed nonzero parameters,

$$DF(\varepsilon t_\pm(\varepsilon)) = \varepsilon DQ(z_\pm) + O(\varepsilon^2). \tag{8.10}$$

Let $W \subset V$ be the chosen subspace on which $DQ(z_\pm)$ is invertible, and let $J = (DQ(z_\pm)|_W)^{-1}$. Equation (8.10) gives

$$\left\| J\big(\varepsilon^{-1} DF(\varepsilon t_\pm(\varepsilon))|_W - DQ(z_\pm)|_W\big) \right\| \le C|\varepsilon|.$$

For $C|\varepsilon| < 1/2$, a Neumann series proves invertibility of the restricted derivative.

Write the derivative of the unreduced matrix map in block form

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} : \mathbb{R}^q \oplus V \longrightarrow \mathcal{R} \oplus \mathcal{C}.$$

Here A is invertible and $DF = D - CA^{-1}B$. Given $(a,b) \in \mathcal{R} \oplus \mathcal{C}$, choose $v \in V$ with $(D - CA^{-1}B)v = b - CA^{-1}a$, and put $z = A^{-1}(a - Bv)$. Direct substitution gives

$$Az + Bv = a, \qquad Cz + Dv = b.$$

Thus the full derivative is onto. All parameter directions are conformal to g, and they are fixed smooth functions. Since their coefficients tend to zero, the corresponding metric differences tend to zero in every C^k norm. This proves the approximation assertion of Theorem B.

8.2. **Weak stability.** Let $\mathcal{U}$ be any sufficiently small neighborhood of the original metric. It contains one of the metrics g_* obtained from (8.9). Restrict the cluster map to finitely many conformal directions at g_* on which its derivative is invertible. Lemma 3.2 supplies a smooth section

$$\sigma : \overline{B}_\delta(\lambda I_m) \longrightarrow \mathcal{U}, \qquad \mathsf{L} \circ \sigma = \mathrm{Id}, \tag{8.11}$$

after decreasing $\delta > 0$. The closed ball lies in the domain of the section.

Shrink the target ball so that its closure lies in the open inverse domain. Its image is compact, is contained in $\mathcal{U}'$, and consists of metrics using the same finite collection of conformal directions. This is the only compactness used in the degree argument; no compactness of a neighborhood in the metric space is required.

Suppose $\sup_{\mathcal{U}'} \|\widetilde{\mathsf{L}} - \mathsf{L}\| < \delta/2$. For $0 \le t \le 1$ and $A \in \overline{B}_\delta(\lambda I_m)$, define

$$H_t(A) = A - \lambda I_m + t\big(\widetilde{\mathsf{L}}(\sigma(A)) - A\big).$$

On the boundary, $\|H_t(A)\| \ge \delta - \delta/2 > 0$. Homotopy invariance of Brouwer degree gives

$$\deg(H_1, B_\delta(\lambda I_m), 0) = \deg(A \mapsto A - \lambda I_m, B_\delta(\lambda I_m), 0) = 1.$$

Hence $H_1(A) = 0$ for some A, or $\widetilde{\mathsf{L}}(\sigma(A)) = \lambda I_m$. This proves weak conformal stability. Since σ also takes values in the full metric space, it proves weak stability there as well.

8.3. **The square-root estimate.** Fix $k \ge 4$ and a small target $Y \in \mathrm{Sym}_m$. Write $Y = Y_{\mathcal{R}} + Y_{\mathcal{C}}$. Solve the range equation with right side $Y_{\mathcal{R}}$, obtaining

$$z = z(t, Y_{\mathcal{R}}), \qquad z = O(|t|^2 + |Y_{\mathcal{R}}|). \tag{8.12}$$

Let $\mathcal{F}(z,t) = \mathsf{L}(v(z,t)) - \lambda I_m$ and write $y = Y_{\mathcal{R}}$. The range equation is $\mathcal{F}_{\mathcal{R}}(z(t,y),t) = y$. If $A = D_z\mathcal{F}_{\mathcal{R}}(0)$, then

$$D_t z(0,0) = 0, \qquad D_y z(0,0) = A^{-1}.$$

The cokernel component $G(t,y) = \mathcal{F}_{\mathcal{C}}(z(t,y),t)$ satisfies $G(0,0) = 0$ and $DG(0,0) = 0$, since $D\mathcal{F}_{\mathcal{C}}(0) = 0$. The chain rule also gives

$$\tfrac{1}{2}D^2_{tt}G(0,0)[t,t] = Q(t).$$

All terms containing a second derivative of z are multiplied by $D\mathcal{F}_{\mathcal{C}}(0) = 0$. The bilinear ty term is bounded by $C|t||y|$, the yy term by $C|y|^2$, and the third-order remainder by $C(|t| + |y|)^3$; on a sufficiently small product ball the latter is bounded by $C'(|t|^3 + |t||y| + |y|^2)$. Thus

$$G(t, Y_{\mathcal{R}}) = Q(t) + O(|t|^3 + |t|\,|Y_{\mathcal{R}}| + |Y_{\mathcal{R}}|^2). \tag{8.13}$$

Set $t = \varepsilon w$ and $Y = \varepsilon^2 A$. The equation is

$$\varepsilon^{-2}G(\varepsilon w, \varepsilon^2 A_{\mathcal{R}}) = A_{\mathcal{C}}. \tag{8.14}$$

To check smoothness at $\varepsilon = 0$, write $y = A_{\mathcal{R}}$ and apply Taylor's formula at $(0,0)$:

$$\varepsilon^{-2}G(\varepsilon w, \varepsilon^2 y) = \int_0^1 (1-s)D^2G(s\varepsilon w, s\varepsilon^2 y)[(w,\varepsilon y),(w,\varepsilon y)]\,ds.$$

The integrand is smooth in all variables on a fixed compact s interval. Its value at $\varepsilon = 0$ integrates to $Q(w)$. Choose the d-dimensional subspace $W \subset V$ furnished by Theorem 7.1, so that $DQ(z_+)|_W : W \to \mathcal{C}$ is an isomorphism. Restrict w to $z_+ + W$ and keep its coordinates in a complementary subspace fixed. Subtract $A_{\mathcal{C}}$ from the left side of (8.14). The resulting smooth map vanishes at $(\varepsilon, w, A) = (0, z_+, 0)$ and has derivative $DQ(z_+)|_W$ in the variable $w - z_+$. Lemma 3.2, after translation and multiplication by this derivative's inverse, gives a unique smooth $w(\varepsilon, A) \in z_+ + W$ for (ε, A) in a product neighborhood of zero. Restricting to the closure of a smaller product neighborhood bounds w and all its fixed-order derivatives uniformly.

For $Y \ne 0$, choose $\varepsilon = L\sqrt{\|Y\|}$ with a fixed L large enough that $A = Y/\varepsilon^2$ lies in the permitted target ball. For small Y, (8.14) gives $|t| = O(\sqrt{\|Y\|})$ and $|z| = O(\|Y\|)$. Smoothness of the finite family gives

$$\mathsf{L}(g_Y) = \lambda I_m + Y, \qquad \|g_Y - g\|_{C^k} \le C_k\,\|Y\|^{1/2}. \tag{8.15}$$

For $Y = 0$ take $g_Y = g$. This choice is distinct from the nonconstant exact curves in (8.9). Fix the right-inverse coordinates in (8.14) once and for all. The parameter-dependent inverse then selects the realizations consistently for all nonzero targets in a small ball. If $\mathcal{C} = 0$, Lemma 3.2 gives the stronger bound $C_k\,\|Y\|$.

8.4. **A Hölder local section.**

Proposition 8.1. *The realizations in* (8.15) *may be selected by one map* $Y \mapsto \mathscr{S}(Y)$*, with* $\mathscr{S}(0) = g$*, which is smooth for* $Y \neq 0$ *and satisfies*

$$\left\|\mathscr{S}(Y) - \mathscr{S}(Y')\right\|_{C^k} \leq C_k \left\|Y - Y'\right\|^{1/2} \tag{8.16}$$

for all targets in a sufficiently small ball. For every target, $\operatorname{supp}(\mathscr{S}(Y) - g)$ *is contained in the union of the supports of the fixed finite family of parameter functions.*

Proof. The solution of (8.14) gives a smooth metric-valued map $\Psi(\varepsilon, A)$ on a product neighborhood of $(0, 0)$ such that

$$\Psi(0, A) = g, \qquad \mathsf{L}(\Psi(\varepsilon, A)) = \lambda I_m + \varepsilon^2 A. \tag{8.17}$$

Indeed, its kernel parameter is $t = \varepsilon w(\varepsilon, A)$ and its range parameter is $z(t, \varepsilon^2 A_{\mathcal{R}})$; both vanish at $\varepsilon = 0$. The functions in the metric family are fixed and smooth. On a smaller compact parameter product, for each k,

$$\|\partial_\varepsilon \Psi\|_{C^k} \leq C_k, \qquad \|D_A \Psi\|_{\mathcal{L}(\mathrm{Sym}_m, C^k)} \leq C_k |\varepsilon|. \tag{8.18}$$

For the second inequality, differentiate $\Psi(0, A) = g$ in A and integrate $\partial_\varepsilon D_A \Psi$ from zero to ε. The product neighborhood is fixed by the finite-dimensional inverse equations. Its closure is compact, and Ψ is a finite linear combination of fixed smooth functions followed by the smooth conformal chart. Thus the same neighborhood is valid for every k, with constants C_k depending on k.

Use the Frobenius norm on Sym_m. Choose $L > 0$ with L^{-2} smaller than the allowed A radius. For $Y \neq 0$, put $r = \|Y\|$ and

$$\mathscr{S}(Y) = \Psi\left(L\sqrt{r}, \frac{Y}{L^2 r}\right), \qquad \mathscr{S}(0) = g. \tag{8.19}$$

The target identity in (8.17) proves $\mathsf{L}(\mathscr{S}(Y)) = \lambda I_m + Y$. The first estimate in (8.18) gives $\|\mathscr{S}(Y) - g\|_{C^k} \leq C_k L\sqrt{r}$, proving continuity at zero. Both arguments in (8.19) are smooth for $Y \neq 0$.

For $H \in \mathrm{Sym}_m$ and $Y \neq 0$, the Frobenius norm gives

$$Dr[H] = \frac{\operatorname{tr}(YH)}{r}, \qquad D(L\sqrt{r})[H] = \frac{L\operatorname{tr}(YH)}{2r^{3/2}},$$

and

$$D\left(\frac{Y}{L^2 r}\right)[H] = \frac{H}{L^2 r} - \frac{Y\operatorname{tr}(YH)}{L^2 r^3}.$$

Since $|\operatorname{tr}(YH)| \leq r\|H\|$, the norms of the last two derivatives are at most $L\|H\|/(2\sqrt{r})$ and $2\|H\|/(L^2 r)$, respectively. The chain rule and (8.18) imply

$$\|D\mathscr{S}(Y)[H]\|_{C^k} \leq C_k r^{-1/2} \|H\| \qquad (Y \neq 0). \tag{8.20}$$

The constants are uniform in the direction Y/r, which lies in a compact sphere inside the allowed auxiliary target ball.

Let $H = Y' - Y \neq 0$ and $Y_t = Y + tH$, $0 \leq t \leq 1$. Put $t_0 = -\langle Y, H\rangle / \|H\|^2$. Orthogonal projection onto the line spanned by H gives

$$\|Y_t\| \geq \|H\|\, |t - t_0|, \qquad \int_0^1 |t - t_0|^{-1/2}\, dt \leq 4.$$

Integrating (8.20) on the segment gives

$$\left\|\mathscr{S}(Y') - \mathscr{S}(Y)\right\|_{C^k} \leq 4C_k \|H\|^{1/2}.$$

If the segment meets zero, integrate on the two truncated segments and use continuity at zero to pass to the limit. If $H = 0$, both sides of the desired inequality are zero. This proves (8.16). Every selected metric uses the fixed parameter functions, so its support has the stated property. If $C = 0$, the smooth inverse already constructed gives a locally Lipschitz section instead. ☐

Suppose $\mathcal{C} \neq 0$ and choose $C \in \mathcal{C}$ with $\|C\| = 1$. The scalar function $\langle C, \mathsf{L}(g_v) - \lambda I_m\rangle$ has zero value and first derivative at zero. Boundedness of its second derivative on a sufficiently small C^k ball gives

$$|\langle C, \mathsf{L}(g_v) - \lambda I_m\rangle| \leq C_0 \|v\|_{C^k}^2 \leq C_1 \|g_v - g\|_{C^k}^2 . \tag{8.21}$$

The last inequality follows from the smooth pointwise inverse of the conformal chart near its positive reference factor. If $\mathsf{L}(g_v) = \lambda I_m + tC$, $t > 0$, then $\|g_v - g\|_{C^k} \geq c\sqrt{t}$. Therefore no exponent strictly larger than $1/2$ can replace the exponent at zero in (8.16) uniformly over all matrix directions at an unstable conformal reference.

The section, approximation, and degree arguments prove Theorem B. A differentiable right inverse at an unstable reference would give $D\mathsf{L}(g)D\mathscr{S}(0) = I$, contradicting failure of surjectivity.

8.5. **Nonsmoothness of the multiplicity locus.** Use the conformal parameter chart $v \mapsto g_v$ throughout this paragraph. Its derivative at zero is the injective map

$$\iota(a) = \begin{cases} \beta a g, & n \geq 3, \\ a g, & n = 2. \end{cases}$$

The two exact curves in (8.9) have parameter velocities $x + y$ and $x - y$, because $Dz(0) = 0$ in (8.4); their metric velocities are consequently $\iota(x + y)$ and $\iota(x - y)$. Define the map on the conformal parameter space by

$$\mathcal{H}(v) = \mathrm{proj}_{\mathcal{C}}\big(\mathsf{L}(g_v) - \lambda I_m\big).$$

Its derivative vanishes on the entire conformal parameter space. In particular,

$$D^2\mathcal{H}(0)[a, a] = 2Q(a) \qquad (a \in \mathcal{K}), \qquad Q(2y) = 4Q(y) \neq 0. \tag{8.22}$$

No vanishing of the unrestricted metric derivative against the conformal cokernel is used.

Every $C \in \mathcal{C}$ is trace-free. Thus $\mathcal{H}(v) = 0$ whenever $\mathsf{L}(g_v)$ is scalar, including when the common eigenvalue differs from λ. Let a C^1 curve in the conformal multiplicity locus have parameter expression $v(t) = ta + r(t)$, where $\|r(t)\|_{C^k} = o(|t|)$. Since $\mathcal{H}(0) = D\mathcal{H}(0) = 0$, Taylor's formula in this parameter chart gives

$$0 = \mathcal{H}(v(t)) = \tfrac{1}{2}t^2 D^2\mathcal{H}(0)[a, a] + o(t^2).$$

The mixed bilinear term is $O(|t|\,\|r(t)\|_{C^k}) = o(t^2)$, and the quadratic term in $r(t)$ is $o(t^2)$. Consequently

$$D^2\mathcal{H}(0)[a, a] = 0. \tag{8.23}$$

Only the first derivative of the curve is required.

If the conformal multiplicity locus were a C^1 embedded submanifold, its inverse image in the conformal parameter chart would have a linear tangent space. That space contains $x + y$ and $x - y$, hence $2y$. Every vector of that tangent space is the velocity of a C^1 curve in the submanifold. Applying (8.23) to that curve gives $2Q(2y) = 0$, contrary to (8.22). This proves the unstable conformal case of Theorem C.

On a closed manifold, put

$$\mathcal{R}_{\mathrm{all}} = \mathrm{im}\, D\mathsf{L}(g), \qquad \mathcal{C}_{\mathrm{all}} = \mathcal{R}_{\mathrm{all}}^{\perp},$$

where this derivative is taken on all symmetric metric variations. Assume $\mathcal{C}_{\mathrm{all}} \neq 0$. Proposition 2.1 gives $\mathcal{C}_{\mathrm{all}} \subset \mathcal{C}$ and $\mathrm{tr}\, C = 0$ for every $C \in \mathcal{C}_{\mathrm{all}}$. On the conformal kernel $\mathcal{K}$, define

$$Q_{\mathrm{all}}(a) = \tfrac{1}{2}\, \mathrm{proj}_{\mathcal{C}_{\mathrm{all}}} D^2\big(\mathsf{L}(g_v)\big)(0)[a, a].$$

Every nonzero scalar component has infinite negative index by Theorem 6.1. Theorem 7.1 therefore provides $x, y \in \mathcal{K}$ for which $x + y$ and $x - y$ are regular zeros and $Q_{\mathrm{all}}(y) \neq 0$.

Choose smooth symmetric tensors $h_1, \dots, h_q$ whose derivative images form a basis of $\mathcal{R}_{\mathrm{all}}$. Put $d_{\mathrm{all}} = \dim \mathcal{C}_{\mathrm{all}}$. Let $V \subset \mathcal{K}$ be a finite-dimensional subspace containing x, y and vectors $w_1, \dots, w_{d_{\mathrm{all}}}$ on whose span both derivatives $DQ_{\mathrm{all}}(x \pm y)$ are isomorphisms onto $\mathcal{C}_{\mathrm{all}}$. Consider the metric family

$$\Theta(s, a) = g_a + \sum_{j=1}^{q} s_j h_j, \qquad (s, a) \in \mathbb{R}^q \times V.$$

It is positive definite for small parameters. Its derivative in s maps isomorphically onto $\mathcal{R}_{\mathrm{all}}$, and its derivative in a has image $\iota(V) \subset \ker D\mathsf{L}(g)$. Projecting its second derivative onto $\mathcal{C}_{\mathrm{all}}$ removes the acceleration term, since $\mathrm{proj}_{\mathcal{C}_{\mathrm{all}}} D\mathsf{L}(g) = 0$. The range equation has a solution $s = s(a)$ with $s(0) = Ds(0) = 0$ by Lemma 3.2. The reduced map is consequently $Q_{\mathrm{all}}(a) + O(|a|^3)$. Equations (8.7)–(8.9), applied to these explicitly chosen finite spaces, produce two curves of metrics with $\mathsf{L} = \lambda I_m$ and velocities $\iota(x + y)$ and $\iota(x - y)$.

Now define on the full C^k metric chart

$$\mathcal{H}_{\mathrm{all}}(g') = \mathrm{proj}_{\mathcal{C}_{\mathrm{all}}}(\mathsf{L}(g') - \lambda I_m).$$

This map and its derivative vanish at g, and it vanishes on every nearby scalar cluster matrix because its target consists of trace-free matrices. If the full multiplicity locus were a C^1 embedded submanifold, its tangent space would contain the difference $\iota(2y)$ of the two velocities above. A C^1 curve with that velocity would force

$$D^2\mathcal{H}_{\mathrm{all}}(g)[\iota(2y), \iota(2y)] = 0$$

by the Taylor argument of (8.23). On the other hand, the chain rule and $D\mathcal{H}_{\mathrm{all}}(g) = 0$ give

$$D^2\mathcal{H}_{\mathrm{all}}(g)[\iota(2y), \iota(2y)] = 2Q_{\mathrm{all}}(2y) = 8Q_{\mathrm{all}}(y) \neq 0.$$

The contradiction proves the unrestricted case without identifying its cokernel with the larger conformal cokernel.

At a strongly stable point the derivative of Γ is onto Sym^0_m. A finite right inverse splits the tangent space into its kernel and a finite-dimensional complement. The split-kernel argument in Section 3, applied to Γ, identifies $\Gamma^{-1}(0)$ locally with a graph over that kernel. Its codimension is $\dim \mathrm{Sym}^0_m = m(m + 1)/2 - 1$. This proves the other implication of Theorem C.

9. Perturbations supported in the interior

Theorem 9.1. *Let (M^n, g) be compact, connected, and smooth, $n \geq 2$, and let $\lambda > 0$ have multiplicity m for the closed, Dirichlet, or Neumann Laplacian. Let $Z \subset M$ be closed with $Z \cap M^\circ$ of empty interior. There is a compact set $K \Subset M^\circ \setminus Z$ such that the following assertions hold.*

1. *There is a local section $Y \mapsto g_Y$, continuous at zero, smooth off zero, and taking values in the smooth conformal metrics equal to g on $M \setminus K$, such that $g_0 = g$, $\mathsf{L}(g_Y) = \lambda I_m + Y$, and $\|g_Y - g_{Y'}\|_{C^k} \leq C_k \|Y - Y'\|^{1/2}$ for every fixed $k \geq 4$.*
2. *There are smooth conformal metrics $g_j \to g$ in C^∞, equal to g on $M \setminus K$, with $\mathsf{L}(g_j) = \lambda I_m$, at which the cluster derivative is onto using conformal variations supported in K.*
3. *The Weak Arnold Hypothesis holds relative to these interior-supported variations.*

Proof. The cokernel calculation (5.9) and the oscillatory construction use $\mathcal{V} = C^\infty_c(M^\circ \setminus Z)$. The quadratic theorem and the range correction select finitely many members of this space. Their supports have compact union contained in $M^\circ \setminus Z$. Enlarge this union slightly to a compact set K still contained in that open set. Every metric used in (8.3), including the correction for the prescribed matrix, equals g outside K.

The arguments of Section 8, including Proposition 8.1, take place in this fixed finite family. They prove the exact target identity and the pairwise estimate. At a realized parameter v_*, a parameter variation a has metric derivative

$$\frac{d}{dt} g_{v_*+ta}\Big|_{t=0} = \begin{cases} \dfrac{\beta a}{1+v_*}\, g_{v_*}, & n \geq 3, \\ \dfrac{a}{1+v_*}\, g_{v_*}, & n = 2. \end{cases}$$

It is conformal and supported in K. Thus the surjective finite-family derivative in Section 8 is also surjective among the stated metric variations at g_j, not only at the original parameter. The local section used in the degree argument has this same support. This proves all three assertions. Since K has positive distance from ∂M and from Z, the metric equals g on neighborhoods of both sets. Thus the boundary conditions used in Section 5 remain fixed throughout the construction. The width of the fixed boundary collar is a consequence of the chosen K. □

The theorem does not claim that one may prescribe K in advance. In particular, it does not give support in any arbitrarily chosen ball, nor does it permit replacing the hypothesis on Z by a set with nonempty interior. The proof uses the density of $M^\circ \setminus Z$ in deriving the global equation for F_C. It supplies no replacement for that step when an open region is excluded.

Lemma 6.2 gives a finite upper bound for the order ℓ at each fixed reference pair. The proof of the quadratic realization does not require a numerical value of that bound. For each fixed C, Theorem 6.1 first supplies a finite negative-definite subspace. Continuity of its finite matrix in C then supplies a neighborhood of that dual direction. Only after these choices is compactness used in Section 7. No interchange of those quantifiers is required.

Corollary 9.2. *If* $\mathrm{Ric}_g > cg$ *on a compact neighborhood of* K*, the metrics in Theorem 9.1 may be chosen to retain this inequality there. Any strict* C^k*-open condition on the reference metric is retained after decreasing the parameter range.*

Proof. Let K_1 be the indicated compact neighborhood. The minimum δ of $(\mathrm{Ric}_g - cg)(X, X)$ over $x \in K_1$ and $|X|_g = 1$ is positive by compactness. The estimates (3.9)–(3.10) apply to $h = g_Y - g$ and to $h = g_j - g$. Their C^2 norms tend to zero, so these differences have norm less than $\min\{\epsilon, \delta/(2(C+|c|))\}$ after restricting the parameters. Equation (3.10) proves the asserted strict inequality. Outside K the metric is unchanged. For a specified C^k-open condition, choose a C^k neighborhood on which it holds; convergence places the constructed metrics inside it. □

The conformal volume factors are $H^{2n/(n-2)}$ for $n \geq 3$ and $1 + v$ for $n = 2$. Thus nontrivial parameters in this theorem do not preserve the pointwise volume form. The fixed-density result in the next sections uses trace-free tensor variations and a separate submersion theorem. Neither construction preserves an Einstein equation merely by C^2 closeness. No constants uniform in a degenerating family are claimed.

10. Grassmannian retractions and spectral bundles

For the topology of finite-dimensional matrix families, see [2, 7]. The use of a nontrivial spectral bundle to obstruct extension through a gap region occurs for Dirac families in [23, Lemma 1.6 and Section 4]; the statement numbers here refer to its arXiv version. The proposition below concerns a section of the scalar Laplace matrix map and identifies the pullback bundle explicitly. It does not determine the homotopy type of the entire gap space.

Let $\mathcal{M}$ be a metric parameter space on which a spectral-cluster frame is fixed on a neighborhood $\mathcal{U}$ of g_0. Suppose that a continuous map

$$s : B_\delta(0) \subset \mathrm{Sym}^0_m \longrightarrow \mathcal{U}, \qquad s(0) = g_0, \quad \Gamma(s(A)) = A \tag{10.1}$$

is given and is smooth on the punctured matrix ball. A centered submersion gives a section smooth also at zero. At an arbitrary reference of Theorem B, Proposition 8.1 gives (10.1) by restricting the full matrix section to trace-free targets. The latter section need not be differentiable at zero.

Fix $1 \le r < m$. Let $\mathcal{U}_r \subset \mathcal{U}$ be the metrics for which the r-th and $(r+1)$-st ordered eigenvalues of L differ. Define

$$p_r : \mathcal{U}_r \longrightarrow \mathrm{Gr}_r(\mathbb{R}^m) \tag{10.2}$$

by the lower rank-r eigenspace of L. The Grassmannian is identified with real orthogonal rank-r projections. The finite-dimensional Riesz formula [19, Chapter II, Section 1.4] makes the map smooth. At any metric in $\mathcal{U}_r$, choose a contour containing precisely the lower r matrix eigenvalues. The strict internal gap keeps that contour disjoint from the nearby matrix spectra; the resolvent series of Section 2 differentiates its integral.

For a rank-r orthogonal projection P, put

$$A_\varepsilon(P) = \varepsilon(rI_m - mP), \qquad \sigma_\varepsilon(P) = s(A_\varepsilon(P)). \tag{10.3}$$

This matrix is trace-free. Its eigenvalues are $-(m-r)\varepsilon$ on $\operatorname{im} P$ and $r\varepsilon$ on $\ker P$. Its Frobenius norm is independent of P:

$$\|A_\varepsilon(P)\|^2 = \varepsilon^2\big(r(m-r)^2 + (m-r)r^2\big) = \varepsilon^2 mr(m-r) > 0. \tag{10.4}$$

For a fixed sufficiently small $\varepsilon > 0$, every target therefore belongs to the punctured ball $B_\delta(0) \setminus \{0\}$. The restriction of s to this family is smooth.

Proposition 10.1. *The family* (10.3) *lies in* $\mathcal{U}_r$ *and satisfies*

$$p_r \circ \sigma_\varepsilon = \mathrm{Id}_{\mathrm{Gr}_r(\mathbb{R}^m)}\,. \tag{10.5}$$

Its image is a smooth embedded copy of the Grassmannian, and $\sigma_\varepsilon \circ p_r$ is a retraction onto that image. The lower spectral bundle restricts to the tautological bundle.

Proof. The full matrix $\mathsf{L}(s(A))$ differs from A by a scalar multiple of I_m, which does not change its eigenspaces. The eigenvalue difference in (10.3) is $m\varepsilon > 0$. This proves (10.5). It also gives injectivity and injectivity of the derivative of σ_ε. The Grassmannian is compact and the metric space is Hausdorff, so the injective continuous map is a homeomorphism onto its image. Differentiation of $p_r\sigma_\varepsilon = \mathrm{Id}$ gives a left inverse for its tangent map. In a C^k metric chart, the finite-dimensional tangent image is complemented: compose that left inverse with $D\sigma_\varepsilon$ to obtain a bounded projection onto the image. The local inverse argument in Section 3 therefore expresses the image as a smooth graph over that finite-dimensional subspace. This proves the asserted smooth embedding. The composite $\sigma_\varepsilon p_r$ has image in that embedded copy and is the identity there; it is a retraction.

Let γ_r be the tautological bundle. For $g' \in \mathcal{U}_r$, use the physical frame $V_{g'} = J_{g'}^{-1}U_{g'}$ of (2.10). The map

$$(g', v) \longmapsto (g', V_{g'}v), \qquad v \in \operatorname{im} p_r(g'),$$

is a smooth fibrewise isometry from $p_r^*\gamma_r$ onto the lower spectral bundle of $\Delta_{g'}\ \mathcal{E}_r$. Its inverse is the adjoint of $V_{g'}$ restricted to that bundle. Thus this identification is valid even when the Riemannian density varies. Pulling back by σ_ε gives

$$\sigma_\varepsilon^*\mathcal{E}_r \cong \gamma_r. \tag{10.6}$$

□

With compatible basepoints, (10.5) gives split injections on homotopy groups and split injections $p_r^* : H^*(\mathrm{Gr}_r(\mathbb{R}^m); A) \to H^*(\mathcal{U}_r; A)$ for every abelian coefficient group A. A deformation retraction is not asserted. Nor does the proposition determine the connected components or the entire homotopy type of $\mathcal{U}_r$.

Corollary 10.2. *Let B be a compact smooth manifold, let $m \ge 2$ and $1 \le r < m$, and let $E \subset B \times \mathbb{R}^m$ be a smooth rank-r subbundle. Its orthogonal projections P_b are smooth, and the family $b \mapsto \sigma_\varepsilon(P_b)$ has lower spectral bundle isomorphic to E.*

Proof. In a local smooth frame represented by a full-rank matrix $V(b)$, the orthogonal projection is $V(V^TV)^{-1}V^T$, hence is smooth. These local formulas agree. The bundle E is the pullback of γ_r by $b \mapsto P_b$. Apply (10.6). □

Corollary 10.3. *Fix a smooth reference metric with a positive eigenvalue λ of multiplicity $m \geq 2$, and let $1 \leq r < m$. In every sufficiently small neighborhood with its whole-cluster frame, the internal gap space contains the Grassmannian sections of Proposition 10.1, within the reference conformal class. They can be chosen with*

$$\mathsf{L}(\sigma_\varepsilon(P)) = \lambda I_m + \varepsilon(rI_m - mP), \qquad \|\sigma_\varepsilon(P) - g_0\|_{C^k} \leq C_k\sqrt{\varepsilon}. \tag{10.7}$$

Proof. Use the full matrix section $\mathscr{S}$ of Proposition 8.1 and set $\sigma_\varepsilon(P) = \mathscr{S}(\varepsilon(rI_m - mP))$. The target matrix has nonzero constant Frobenius norm for fixed $\varepsilon > 0$, so the section is smooth along the entire Grassmannian. The target identity gives the first equality in (10.7), and the Hölder estimate at zero gives the second, uniformly over the compact set of projections. For small ε the image lies in the prescribed neighborhood. Proposition 10.1 applies with the original cluster frame. □

At an unstable reference the full section is continuous at $A = 0$ and smooth on $A \neq 0$; only nonzero targets occur in the Grassmannian family. No smooth right inverse through that unstable reference is asserted.

11. Localized spectral bundles on spheres

11.1. **A centered submersion with fixed volume form.** Let $n \geq 2$, $m = n + 1$, and let γ be the unit-round metric on $S^n \subset \mathbb{R}^m$. Fix $a > 0$ and put $g_* = a\gamma$. Its first positive eigenvalue is $\lambda_* = n/a$, with eigenspace spanned by

$$u_i = \left(\frac{m}{\mathrm{Vol}(S^n, g_*)}\right)^{1/2} x_i, \qquad 1 \leq i \leq m. \tag{11.1}$$

The next eigenvalue is $2(n+1)/a$. For a homogeneous harmonic polynomial $r^\ell Y(\omega)$, the separation formula $\sum_{j=1}^{n+1} \partial_{x_j}^2 = \partial_r^2 + nr^{-1}\partial_r - r^{-2}\Delta_\gamma$ gives $\Delta_\gamma Y = \ell(\ell + n - 1)Y$. The Euclidean operator on the left is the sum of second partial derivatives; it is the negative of the positive Euclidean Laplacian.

These eigenfunctions span a dense subspace of $L^2(S^n)$. If P is homogeneous of degree ℓ, let $D = \sum_j \partial_{x_j}^2$, $r^2 = \sum_j x_j^2$, and set

$$H = \sum_{j=0}^{\lfloor \ell/2 \rfloor} a_j r^{2j} D^j P, \quad a_0 = 1, \quad a_j = -\frac{a_{j-1}}{2j(2\ell + n - 1 - 2j)}. \tag{11.2}$$

The identity

$$D(r^{2j}D^jP) = 2j(2\ell + n - 1 - 2j)r^{2j-2}D^jP + r^{2j}D^{j+1}P$$

shows $DH = 0$ by cancellation of consecutive terms. The difference $P - H$ is divisible by r^2; induction on ℓ decomposes the restriction of P into restrictions of harmonic homogeneous polynomials. Polynomial restrictions are dense in $C(S^n)$ by Stone–Weierstrass, and hence in L^2. There are no other eigenspaces. Indeed, an eigenfunction with eigenvalue ν different from every $\ell(\ell + n - 1)$ is orthogonal, by self-adjointness, to every restricted harmonic polynomial. Density would make it zero. If $\nu = \ell(\ell + n - 1)$, subtract its orthogonal projection onto that finite-dimensional polynomial space. The remainder is still a ν-eigenfunction, is orthogonal to that space by construction and to every other polynomial eigenspace by self-adjointness, and is consequently zero. Thus these are the complete eigenspaces. The values $\ell(\ell + n - 1)$ increase strictly for integers $\ell \geq 0$. Scaling the metric by a divides them by a, proving both asserted positive levels and their multiplicities. Reflection in a coordinate hyperplane gives $\int x_i x_j = 0$ for $i \neq j$. Coordinate permutations and $\sum_i x_i^2 = 1$ give $\int x_i^2 = \mathrm{Vol}(S^n, g_*)/m$, which verifies (11.1).

For $h \in C_c^\infty(O; S_0^2 T^* S^n)$, where $O \subset S^n$ is nonempty and open, (2.21) becomes

$$D\mathsf{L}(g_*)[h]_{ij} = -\int_{S^n} h(\nabla u_i, \nabla u_j)\, dv_{g_*}. \tag{11.3}$$

Moreover, $\sum_i du_i \otimes du_i = m\gamma / \operatorname{Vol}(S^n, g_*)$ is a constant multiple of g_*. Therefore $\operatorname{tr} D\mathsf{L}(g_*)[h] = 0$, and $D\Gamma(g_*)[h] = D\mathsf{L}(g_*)[h]$.

Lemma 11.1. *The map $D\Gamma(g_*) : C_c^\infty(O; S_0^2 T^* S^n) \to \mathrm{Sym}_m^0$ is onto for every nonempty open O.*

Proof. If $C \in \mathrm{Sym}_m^0$ annihilates the map, set $T_C = \sum C_{ij} du_i \otimes du_j$. Testing (11.3) gives $T_C^{\mathrm{TF}} = 0$ on O. Up to the fixed normalization in (11.1), this says that the restriction of the ambient form C to $x^\perp$ is scalar for every $x \in O$.

Let $P_x = I - x \otimes x$. The tensor condition is

$$P_x C P_x - \frac{\operatorname{tr} C - x^T C x}{n} P_x = 0. \tag{11.4}$$

Its entries are polynomial restrictions to S^n. Fix a point of O. Along a great circle through that point each entry is a trigonometric polynomial, vanishes on an interval, and thus vanishes on the whole circle. Every point of S^n lies on a great circle through the fixed point. Hence (11.4) holds on all of S^n.

For distinct i, j, choose $k \neq i, j$; this is possible because $m \geq 3$. At $x = e_k$, the vectors e_i, e_j are tangent. Equation (11.4) gives $C_{ij} = 0$ and $C_{ii} = C_{jj}$. Thus $C = cI_m$. Its trace is zero, so $C = 0$. A map to a finite-dimensional space with trivial annihilator is onto. □

Choose $\chi \in C_c^\infty(O)$, $\chi \geq 0$, positive on some nonempty open subset. On Sym_m^0 define the operator G by

$$\langle GC, D \rangle = \int_{S^n} \chi \left\langle T_C^{\mathrm{TF}}, T_D^{\mathrm{TF}} \right\rangle_{g_*} dv_{g_*}. \tag{11.5}$$

If its value on (C, C) is zero, T_C^{TF} vanishes on a nonempty open set and the preceding proof gives $C = 0$. Thus $G > 0$. Put

$$R(C) = -\chi T_{G^{-1}C}^{\mathrm{TF}}. \tag{11.6}$$

For every $D \in \mathrm{Sym}_m^0$, contraction with (11.3) gives $\langle D\Gamma(g_*) R(C), D \rangle = \langle C, D \rangle$; hence $D\Gamma(g_*) R(C) = C$.

For a small $B \in \mathrm{Sym}_m^0$, define

$$g(B)(X, Y) = g_*\big(e^{g_*^{-1} R(B)} X, Y\big). \tag{11.7}$$

The exponent is self-adjoint and trace-free. Its exponential is positive and self-adjoint, and $\det e^{g_*^{-1} R(B)} = e^{\operatorname{tr}(g_*^{-1} R(B))} = 1$. Consequently $dv_{g(B)} = dv_{g_*}$. Outside $\operatorname{supp} \chi$ the exponent is zero and $g(B) = g_*$. The smoothness, support, and derivative assertions follow from Proposition 3.1.

The map $B \mapsto \Gamma(g(B))$ has derivative I at zero. Lemma 3.2 gives a smooth local inverse and hence a family

$$A \longmapsto \widehat{g}(A), \qquad \Gamma(\widehat{g}(A)) = A, \qquad dv_{\widehat{g}(A)} = dv_{g_*}, \qquad \widehat{g}(A) = g_* \text{ off } \operatorname{supp} \chi. \tag{11.8}$$

It is defined on a full ball around zero in Sym_m^0. Its image converges to g_* in every C^k norm as $A \to 0$.

Theorem 11.2. *Let B be compact and smooth and $E \subset B \times \mathbb{R}^{n+1}$ a smooth rank-r subbundle, $1 \leq r \leq n$. For every sufficiently small fixed $\varepsilon > 0$, the metrics*

$$g_b = \widehat{g}\big(\varepsilon(rI_m - mP_b)\big) \tag{11.9}$$

have a rank-r spectral bundle isomorphic to E, corresponding to the first r positive eigenvalues counted with multiplicity, where P_b is the orthogonal projection onto E_b. They have the fixed volume form and support properties of (11.8).

Proof. The cluster matrix is
$$\mathsf{L}(g_b) = c(b)I_m + \varepsilon(rI_m - mP_b). \tag{11.10}$$
Its eigenvalues are $c(b)-(m-r)\varepsilon$ with multiplicity r and $c(b)+r\varepsilon$ with multiplicity $m-r$. Compactness of the set of rank-r projections gives a common parameter range. The original cluster is the first m positive eigenvalues and is separated from zero and from $2(n+1)/a$. For sufficiently small parameters it retains those properties.

Choose $\eta > 0$ smaller than half of both n/a and $(n+2)/a$. Use two rectangular contours with real intersections $(-1, n/a - \eta)$ and $(-1, n/a + \eta)$. Their reference ranks are 1 and $m + 1$. The resolvent construction in Section 2 retains these ranks for small parameters. The Laplacian is nonnegative, and
$$\Delta_{g_b} u = 0 \implies \int_{S^n} |du|^2_{g_b}\, dv_{g_b} = 0 \implies u \text{ is constant.}$$
Thus precisely one eigenvalue, the simple zero eigenvalue, lies below the chosen cluster. The cluster still occupies indices $1, \dots, m$. The frame (2.10) identifies its lower eigenspaces with $\operatorname{im} P_b = E_b$, smoothly over B. $\square$

Every smooth real vector bundle of positive finite rank over a compact smooth manifold embeds in a finite-dimensional trivial bundle. To construct an embedding, choose finitely many local trivializations and smooth functions ϕ_i supported in their domains, with $\sum_i \phi_i^2 = 1$. Such functions are obtained by normalizing a finite family of subordinate bump functions with positive sum of squares. Send a vector to the tuple of its local coordinates multiplied by ϕ_i, extending each component by zero. At least one multiplier is nonzero at every point, so this map is fibrewise injective. Adding zero coordinates if necessary gives ambient dimension at least three and strictly larger than the rank. Theorem 11.2 therefore realizes any such bundle after increasing the sphere dimension.

11.2. **Equivariance.** Let $G_0 \subset O(n+1)$ be a compact group preserving χ. For $Q \in G_0$, use the metric action $g \mapsto (Q^{-1})^* g$ and the function action $\mathcal{T}_Q f = f \circ Q^{-1}$. If $U_* v = \big(m/\operatorname{Vol}(S^n, g_*)\big)^{1/2}\langle v, x\rangle$, then
$$(\mathcal{T}_Q U_* v)(x) = \left(\frac{m}{\operatorname{Vol}(S^n, g_*)}\right)^{1/2} \langle v, Q^{-1}x\rangle = (U_* Q v)(x).$$
Thus $\mathcal{T}_Q U_* = U_* Q$ as operators from $\mathbb{R}^m$ to $L^2(S^n, dv_{g_*})$.

For the metric action $(Q^{-1})^*$, the identity $\mathcal{T}_Q U_* = U_* Q$ gives $(Q^{-1})^* T_C = T_{QCQ^{-1}}$. It commutes with the trace-free projection because Q is an isometry of g_*. Invariance of χ and the measure in (11.5) gives $G(QCQ^{-1}) = QG(C)Q^{-1}$. Its inverse has the same property. Hence $R(QCQ^{-1}) = (Q^{-1})^* R(C)$, and the exponential family (11.7) is equivariant.

Volume forms are fixed in this family, so all functions belong to the same L^2. The Laplacian and its projection transform by $\mathcal{T}_Q$. Substitution in (2.8), using $\mathcal{T}_Q U_* = U_* Q$, gives
$$U_{(Q^{-1})^* g} = \mathcal{T}_Q U_g Q^{-1}, \qquad \mathsf{L}((Q^{-1})^* g) = Q\mathsf{L}(g)Q^{-1}. \tag{11.11}$$
For the Gram square root this uses $(QBQ^{-1})^{-1/2} = QB^{-1/2}Q^{-1}$ for positive B, which follows by diagonalizing B.

Choose invariant matrix balls on which $B \mapsto \Gamma(g(B))$ is injective and its image contains a smaller invariant ball.

Such balls exist because the derivative at zero is the identity and the Frobenius norm is invariant under orthogonal conjugation. First choose a Frobenius ball within the inverse domain, then choose a smaller target ball contained in its open image. Every conjugate of a source point in the first ball remains in that ball. If B is the preimage of A, equivariance makes QBQ^{-1} the preimage of QAQ^{-1}. Uniqueness gives
$$\widehat{g}(QAQ^{-1}) = (Q^{-1})^* \widehat{g}(A). \tag{11.12}$$
If $\chi \equiv 1$, the group can be $O(n+1)$. Rank-r projections form one orbit. Hence in that case the family realizing any embedded bundle is pairwise isometric and has constant complete spectrum. No exterior is then prescribed.

11.3. **An isometry family with fixed exterior on the four-sphere.** Write $S^4 \subset \mathbb{C}^2 \oplus \mathbb{R}$, with coordinates (z, s), and put $g_* = \frac{1}{2}\gamma$. Then

$$\operatorname{Ric}_{g_*} = 6g_*, \qquad \lambda_1(g_*) = \cdots = \lambda_5(g_*) = 8, \qquad \lambda_6(g_*) = 20. \tag{11.13}$$

Fix any polar cap $O = \{s > a\}$, $-1 < a < 1$. Choose a nonnegative nonzero cutoff $\chi(s)$ with compact support in O. The action of $U(2)$ on z, fixing s, preserves χ and g_*. Its real representation is contained in $SO(5)$.

For $\ell \in \mathbb{CP}^1$, let $p_\ell : \mathbb{C}^2 \to \ell$ be Hermitian orthogonal projection. Regard it as a real rank-two projection and put

$$P_\ell = p_\ell^{\mathbb{R}} \oplus I_{\mathbb{R}}, \qquad A_{\varepsilon,\ell} = \varepsilon(3I_5 - 5P_\ell), \qquad g_{\varepsilon,\ell} = \widehat{g}(A_{\varepsilon,\ell}). \tag{11.14}$$

The target has trace zero, eigenvalue -2ε on $\operatorname{im} P_\ell$, and eigenvalue 3ε on its orthogonal complement. The family is smooth in ℓ, since the projection in homogeneous coordinates is $zz^*/|z|^2$.

The group $U(2)$ acts transitively on $\mathbb{CP}^1$ and satisfies $P_{Q\ell} = QP_\ell Q^{-1}$. Equation (11.12) gives $g_{\varepsilon,Q\ell} = (Q^{-1})^* g_{\varepsilon,\ell}$. Thus the metrics for a fixed ε are pairwise isometric. In particular every scalar eigenvalue, not just the first five, is independent of ℓ. These isometries need not fix the common exterior pointwise.

The scalar part $c(\varepsilon) = \operatorname{tr} \mathsf{L}(g_{\varepsilon,\ell})/5$ is independent of ℓ by conjugacy. Since its first derivative vanishes for trace-free metric variations by (11.3),

$$\begin{aligned} c(\varepsilon) &= 8 + O(\varepsilon^2), \\ \lambda_1 = \lambda_2 = \lambda_3 &= c(\varepsilon) - 2\varepsilon, \\ \lambda_4 = \lambda_5 &= c(\varepsilon) + 3\varepsilon. \end{aligned} \tag{11.15}$$

Choose contours with real intersections $(-1, 4)$ and $(-1, 16)$, avoiding the reference spectrum. Their ranks at g_* are 1 and 6, respectively: the reference eigenvalues below 16 are 0 and 8, with multiplicities 1 and 5. The uniform resolvent construction of Section 2 retains these ranks and excludes spectrum on the endpoints for all sufficiently small parameters. On a closed connected manifold the Laplacian is nonnegative and its zero eigenspace consists exactly of the constants, since $\langle \Delta u, u\rangle = \|du\|_2^2$. Thus there is no additional positive eigenvalue below the first-five cluster, and the next eigenvalue satisfies $\lambda_6 > 16$. Compactness of the projection orbit makes one choice of $\varepsilon_0 > 0$ valid for every ℓ.

Let $K = \operatorname{supp} \chi \Subset O$. Equation (11.8) gives

$$g_{\varepsilon,\ell} = g_* \text{ on } S^4 \setminus K, \qquad dv_{g_{\varepsilon,\ell}} = dv_{g_*}. \tag{11.16}$$

By (11.13), $\operatorname{Ric}_{g_*} - 3g_* = 3g_*$. The finite family satisfies

$$\sup_{\ell \in \mathbb{CP}^1} \|g_{\varepsilon,\ell} - g_*\|_{C^2} \le C_2|\varepsilon|.$$

Apply Lemma 3.5 with $c = 3$ and $\delta = 3$. If $C_2|\varepsilon|$ is smaller than both its admissible radius and $3/(2(C+3))$, then

$$\operatorname{Ric}_{g_{\varepsilon,\ell}} - 3g_{\varepsilon,\ell} \ge \tfrac{3}{2} g_* > 0$$

for every ℓ.

The frame $U_{g_{\varepsilon,\ell}}$ is a smooth isometry from the trivial five-plane bundle onto the sum of the first five positive eigenspaces. Equation (11.14) identifies the lower eigenspace with $\operatorname{im} P_\ell$. Therefore the first-three eigenspace bundle is

$$\mathcal{E}_\varepsilon \cong L_{\mathbb{R}} \oplus \underline{\mathbb{R}}, \tag{11.17}$$

where L is the tautological complex line. The full first-five bundle remains trivial; (11.17) concerns a separated subbundle of it.

Choose unit frames of L on the two coordinate disks of $\mathbb{CP}^1$. On their equator the transition can be chosen $e^{-i\theta}$. The bundle in (11.17) has clutching loop $\operatorname{diag}(\operatorname{Rot}_{-\theta}, 1) \in SO(3)$. Its lift through the double covering $\mathrm{Spin}(3) \to SO(3)$ starts at 1 and ends at -1, since a one-plane rotation of angle 2π lifts to a half-angle path. Thus it does not lift to a closed loop. The obstruction is

$$\left\langle w_2(\mathcal{E}_\varepsilon), [S^2] \right\rangle = 1. \tag{11.18}$$

Equivalently, $w_2(L_{\mathbb{R}}) = c_1(L) \bmod 2$ and $c_1(L) = -1$ on the chosen orientation; the sign does not affect (11.18). These clutching descriptions of w_2 are also given in [22, Sections 4 and 14].

Equations (11.15)–(11.18), Proposition 10.1, and Theorem 11.2 prove Theorem D.

11.4. **Topology of the gap space.** Let $\mathcal{G}_{\mathrm{gap}}$ consist of smooth metrics on this fixed S^4 with $\mathrm{Ric} > 3g$, $dv_g = dv_{g_*}$, $g = g_*$ off K, and $\lambda_3(g) < \lambda_4(g)$. For fixed $0 < \varepsilon < \varepsilon_0$, (11.14) defines a map $f_\varepsilon : S^2 \to \mathcal{G}_{\mathrm{gap}}$.

Corollary 11.3. *The map f_ε is not nullhomotopic. In particular it represents a nonzero element of the based π_2 of its component of $\mathcal{G}_{\mathrm{gap}}$.*

Proof. A nullhomotopy extends the family to a continuous family over D^3. We use the spectral-bundle nonextension argument of [23, Lemma 1.6], with a two-dimensional parameter boundary; the characteristic class is calculated in (11.18). The first-three projections are locally continuous by resolvent continuity and the gap, and define a rank-three bundle. Compactness of the parameter ball gives a uniform positive minimum for the third-to-fourth gap; one may also construct the bundle with local contours. Homotopy invariance of pullback bundles [22, Section 3], applied to the contraction of D^3 to a point, makes this bundle trivial. Naturality of Stiefel–Whitney classes [22, Section 4] then gives zero w_2 on its restriction to S^2, contrary to (11.18). Choosing a basepoint on the original sphere family gives the based conclusion. □

Corollary 11.4. *Fix $0 < \varepsilon < \varepsilon_0$. Let $g_{t,b}$, $(t, b) \in [0, 1] \times S^2$, be a continuous family of smooth metrics, with the C^∞ topology, such that $g_{0,b} = g_{\varepsilon,b}$ and*

$$\lambda_3(g_{t,b}) < \lambda_4(g_{t,b}) \qquad \text{for all } (t, b).$$

Then for some $b \in S^2$,

$$\lambda_1(g_{1,b}) = \lambda_2(g_{1,b}) \quad \text{or} \quad \lambda_2(g_{1,b}) = \lambda_3(g_{1,b}).$$

Proof. Local Riesz projections define a bundle on $S^2 \times [0, 1]$. The two endpoint inclusions are homotopic, so their pullback bundles are isomorphic by [22, Section 3]. Suppose the final three eigenvalues are distinct at every parameter. Their increasing order defines three continuous rank-one Riesz projections, and hence a splitting into real line bundles. Real line bundles are classified by their first Stiefel–Whitney class [22, Sections 4–5]; each is trivial because $H^1(S^2; \mathbb{Z}/2) = 0$. The Whitney product formula [22, Section 4] gives zero second Stiefel–Whitney class for their sum, contradicting (11.18). □

The family is constant after passage to isometry classes, by (11.12). Its nontriviality lies in the marked metric gap space and the associated eigenbundle. The corollary asserts the existence of a crossing, not a formula equating an unweighted crossing count with w_2. The gap equals 5ε within each fixed family and closes as $\varepsilon \downarrow 0$; no uniform gap through that limit is asserted.

12. Potentials supported in a prescribed open set

Let (M^n, g) be compact, connected, and smooth, with $n \geq 2$. If $\partial M \neq \emptyset$, fix either the Dirichlet or the Neumann realization throughout this section. Let $V \in C^\infty(M; \mathbb{R})$ and

$$P = \Delta_g + V.$$

Fix an eigenvalue $\lambda \in \mathbb{R}$ of multiplicity m, an isolating interval I, and a nonempty open set $O \subset M^\circ$. The metric, the measure, and the boundary realization will not vary.

Multiplication by the real smooth V is bounded and self-adjoint on $L^2(M, dv_g)$. On the fixed domain of the Laplacian, P is self-adjoint by bounded perturbation; its resolvent is compact. Indeed, for a real $z < -\|V\|_\infty - 1$, the inverse $(z - \Delta_g)^{-1}$ is compact and $\|M_V(z - \Delta_g)^{-1}\| < 1$. The identity

$$(z - P)^{-1} = (z - \Delta_g)^{-1}[I - M_V(z - \Delta_g)^{-1}]^{-1}$$

expresses one resolvent as a compact operator times a bounded operator. The resolvent identity gives compactness at every point of the resolvent set. It is bounded below by $-\|V\|_\infty$. For an eigenfunction, $\Delta_g u = (\lambda - V)u$. Multiplication by the smooth function $\lambda - V$ preserves every H^s. Starting with $u \in H^2$, equation (2.2) gives $u \in H^{s+2}$ whenever $u \in H^s$; hence $u \in C^\infty(M)$, up to the boundary. In particular, an arbitrary real eigenvalue λ has finite multiplicity and positive distance from the excluded spectrum. For a sufficiently small real potential q, let $\mathsf{L}(q)$ be the symmetric matrix of the I-cluster of $P + q$ in its Riesz–Gram frame, based at a fixed real orthonormal basis of $\ker(P - \lambda)$. Thus $\mathsf{L}(0) = \lambda I_m$.

Theorem 12.1. *There are a compact set $K \Subset O$, a finite-dimensional subspace $W \subset C_c^\infty(O; \mathbb{R})$ whose members are supported in K, a number $\delta > 0$, and a map*

$$\mathscr{S} \,:\, B_\delta(0) \subset \operatorname{Sym}_m(\mathbb{R}) \longrightarrow W$$

such that

$$\mathscr{S}(0) = 0, \qquad \mathsf{L}(\mathscr{S}(Y)) = \lambda I_m + Y. \tag{12.1}$$

The map is smooth on $B_\delta(0) \setminus \{0\}$, and, for every integer $k \geq 0$,

$$\|\mathscr{S}(Y) - \mathscr{S}(Y')\|_{C^k(M)} \leq C_k \|Y - Y'\|^{1/2}. \tag{12.2}$$

There exist $q_j \in W$ tending to zero in C^∞ for which

$$\mathsf{L}(q_j) = \lambda I_m, \qquad D\mathsf{L}(q_j)|_W \,:\, W \longrightarrow \operatorname{Sym}_m(\mathbb{R}) \quad \textit{is surjective.}$$

If $D\mathsf{L}(0)$ is surjective on $C_c^\infty(O)$, the section may be chosen smooth at zero with a linear estimate in place of (12.2). *Otherwise, no realization estimate at zero with an exponent greater than* $1/2$ *can hold in every target direction.*

The assertion concerns one isolated cluster. It imposes no conditions on the other eigenvalues. In particular, λ need not be positive. The choices are ordered as follows: first P, λ, and O; then the finite collection of variations and its compact support K.

12.1. **The localized annihilator.** Let $E = \ker(P - \lambda)$, let $u_1, \dots, u_m$ be its real orthonormal basis, and let $U : \mathbb{R}^m \to E$ be the associated isometry. Denote the orthogonal projection by Π. For z on a fixed contour enclosing the cluster, the identity

$$(z - P - q)^{-1} = (z - P)^{-1}[I - M_q(z - P)^{-1}]^{-1}$$

and the norm-convergent Neumann series apply whenever $\|q\|_\infty \sup_z \|(z - P)^{-1}\| < 1$. Both the Riesz projection $\Pi(q)$ and the bounded spectral operator

$$A(q) = \frac{1}{2\pi i} \int_\gamma z(z - P - M_q)^{-1}\, dz = (P + M_q)\Pi(q)$$

are analytic in the L^∞ norm of q. If U_q is the Gram-normalized frame, then $\mathsf{L}(q) = U_q^* A(q) U_q$ is analytic as well. The operator domain and boundary conditions are unchanged. These assertions use the fixed-domain perturbation theorem, not a variation of the Riemannian density [19].

Differentiating at $q = 0$ gives

$$D\mathsf{L}(0)[f]_{ij} = \int_M f u_i u_j \, dv_g. \tag{12.3}$$

Hence the annihilator and kernel are, respectively,

$$\mathcal{C}_O = \left\{ C \in \mathrm{Sym}_m \,:\, F_C := \sum_{i,j} C_{ij} u_i u_j = 0 \text{ on } O \right\}, \tag{12.4}$$

$$\mathcal{K}_O = \left\{ f \in C_c^\infty(O) \,:\, \Pi M_f \Pi = 0 \right\}. \tag{12.5}$$

Indeed, the vanishing of $\int fF_C$ for every $f \in C_c^\infty(O)$ is equivalent to $F_C|_O = 0$ by the defining property of distributions, and F_C is smooth. No global identity for F_C is asserted. In particular, a member of $\mathcal{C}_O$ is not assumed trace-free.

The range $\mathcal{R}_O$ and this annihilator are orthogonal complements in the entire space Sym_m, with its Frobenius product. Therefore a basis of $\mathcal{R}_O$ has preimages in $C_c^\infty(O)$ even when $\mathcal{C}_O$ contains matrices of nonzero trace. The scalar matrix direction is obtained from that full decomposition and the subsequent quadratic correction; constant potentials, which need not be supported in O, are not inserted.

Set

$$R = (P - \lambda)^{-1}|_{E^\perp}, \qquad R|_E = 0, \qquad B_C = \sum_{i,j} C_{ij} M_{u_i} R M_{u_j}.$$

The operator B_C is self-adjoint because $R^* = R$ and $C^T = C$.

Moreover,

$$\|B_C\|_{L^2\to L^2} \le \|R\|_{L^2\to L^2} \sum_{i,j} |C_{ij}|\, \|u_i\|_\infty \|u_j\|_\infty < \infty.$$

This global bound is used only for the finite constraint corrections. The asymptotic sign calculation below is localized inside O and does not assert cancellation of lower-order symbols outside that set.

Lemma 12.2. *For $C \in \mathcal{C}_O$ and $f \in \mathcal{K}_O$,*

$$\langle C, D^2\mathsf{L}(0)[f,f]\rangle = -2\langle f, B_C f\rangle_{L^2(M)}. \tag{12.6}$$

Proof. Write $(P + tM_f)U(t) = U(t)\mathsf{L}(t)$, where $U(t)^*U(t) = I_m$. Equation (12.3) gives $\mathsf{L}'(0) = 0$. The first differentiated equation, projected onto $E^\perp$, gives

$$(I - \Pi)U'(0) = -RM_f U.$$

Project the second differentiated equation onto E. The terms containing PU'' and $\lambda U''$ cancel, and the term $2U'\mathsf{L}'$ is zero. Thus

$$\mathsf{L}''(0) = 2U^* M_f U'(0) = -2U^* M_f R M_f U.$$

The E-component of $U'(0)$ contributes zero because $U^* M_f U = 0$. Contracting with C proves (12.6). □

12.2. **The first nonzero diagonal coefficient.** For $C \in \mathcal{C}_O$ define

$$b_C(x,y) = \sum_{i,j} C_{ij} u_i(x) u_j(y).$$

Then $b_C(x,y) = b_C(y,x)$ and $(P_y - \lambda)b_C(x,y) = 0$. Suppose that every transverse derivative of b_C vanishes at every (x,x) with $x \in O$. For each such x, strong unique continuation for $\Delta_g + V - \lambda$ gives $b_C(x,\cdot) = 0$ on the connected manifold. Taking its inner product with u_j gives

$$\sum_i C_{ij} u_i = 0 \quad \text{on } O.$$

Unique continuation applied once more to this eigenfunction gives the same identity on M. Orthonormality implies $C = 0$. All uses of continuation are interior uses for a scalar elliptic operator with smooth coefficients and smooth real zeroth-order term [5].

For $C \neq 0$, there is therefore a least finite integer ℓ such that a transverse derivative of order ℓ is nonzero at some point of O. In a coordinate neighborhood compactly contained in O,

$$b_C(x, x+z) = P_x(z) + O(|z|^{\ell+1}), \tag{12.7}$$

where P_x is homogeneous of degree ℓ. All lower coefficients vanish identically on O.

The integral remainder (6.4) applies on each compact product chart in O. In particular its derivatives of z-order $0, 1, 2$ are $O(|z|^{\ell+1})$, $O(|z|^{\ell})$, and $O(|z|^{\ell-1})$, respectively, uniformly for x in a smaller compact set. This justifies taking the lowest homogeneous degree after applying $P_y - \lambda$. The equality $b_C(x, x+z) = b_C(x+z, x)$ implies $P_x(z) = P_x(-z)$ at the first nonzero order. Since $F_C|_O = 0$, one has $\ell \geq 2$, and ℓ is even. The term of degree $\ell - 2$ in $(P_y - \lambda)b_C = 0$ is

$$g^{ab}(x)\partial_{z_a}\partial_{z_b}P_x(z) = 0. \tag{12.8}$$

The first-order coefficients of Δ_g multiply derivatives vanishing to order at least $\ell - 1$, and $V - \lambda$ multiplies a function vanishing to order ℓ; neither contributes at degree $\ell - 2$. At a point where $P_x \neq 0$, its restriction to the unit sphere for the frozen metric has mean zero and takes both signs. Here the assumption $n \geq 2$ is used.

12.3. **Interior symbol and infinite inertia.** After conjugating the local Riemannian measure to Lebesgue measure, the localized reduced inverse has principal symbol $|\xi|_g^{-2}$. To see that this applies also to either boundary realization, insert two cutoffs supported in the interior. An elliptic parametrix is two-sided modulo smooth kernels there; the equations for the reduced inverse differ from the inverse equations by the finite-rank projection Π, which has a smooth kernel. For the difference kernel, apply the sum of the two localized operators, acting in its separate variables. Its principal symbol is $|\xi|_g^2 + |\eta|_g^2$, which is positive for $(\xi, \eta) \neq 0$. The kernel argument in Section 6, based on interior elliptic regularity [12, Theorem 15.1], therefore gives a smooth difference kernel. The smooth zeroth-order coefficient $V - \lambda$ changes neither ellipticity nor the order -2 principal symbol. Thus the localization of either boundary realization has that same symbol.

The symbol composition formula gives

$$\sigma(B_C) \sim \sum_{\alpha} \frac{i^{-|\alpha|}}{\alpha!}\left(\sum_{i,j} C_{ij}u_i\partial^\alpha u_j\right)\partial_\xi^\alpha \sigma(R).$$

The coefficients with $|\alpha| < \ell$ vanish by the definition of ℓ. In coordinates orthonormal at the point, the first nonzero homogeneous symbol is

$$i^{-\ell}P_x(\partial_\xi)|\xi|^{-2}.$$

Represent P_x by a symmetric trace-free coefficient tensor. In the ℓ-fold derivative of $|\xi|^{-2}$, every term containing a contraction of two polynomial indices vanishes against that tensor. The remaining coefficient is $(-2)(-4)\cdots(-2\ell)$. Hence

$$\sigma_{-\ell-2}(B_C)(x, \xi) = (-1)^{\ell/2}2^\ell \ell!\,\frac{P_x(\xi)}{|\xi|^{2\ell+2}}. \tag{12.9}$$

The conjugating density factors commute with multiplication by the u_i; thus they do not change the leading coefficient in (12.9).

Lemma 12.3. *For every nonzero $C \in \mathcal{C}_O$, the form*

$$f \longmapsto \langle C, D^2\mathsf{L}(0)[f, f]\rangle$$

has infinite positive and infinite negative index on $\mathcal{K}_O$.

Proof. Choose a nonzero coordinate covector ξ_0 and a ball compactly contained in O on which one sign of (12.9) is strict. Given N, choose real nonzero amplitudes $a_1, \ldots, a_N$ with pairwise disjoint compact supports in that ball, and put

$$f_{j,T}(x) = a_j(x)\cos(T\xi_0 \cdot x).$$

The localized symbol expansion, applied separately to the phases $\pm\xi_0$, gives

$$T^{\ell+2}\langle B_C f_{j,T}, f_{j,T}\rangle \longrightarrow \frac{1}{2}\int_M a_j^2\, \sigma_{-\ell-2}(B_C)(x,\xi_0)\, dv_g.$$

The terms containing twice the phase are $O(T^{-L})$ for every L by integration by parts. For $i \neq j$, the kernel is smooth on the product of the two separated supports, so the same integrations in the two variables give

$$\langle B_C f_{i,T}, f_{j,T}\rangle = O(T^{-L}) \quad \text{for every } L.$$

Apply the oscillatory estimate proved in Section 6 with order $-\ell-2$ to the localized operator B_C. The off-diagonal integrations use its smooth kernel between separated supports. These uses of the symbol calculus have the same hypotheses as [18, Section 18.1]: the symbols are classical, the inserted cutoffs have compact interior support, and the chosen linear phase has nonzero differential. The amplitudes are fixed after N is chosen.

Let $\mathcal{R}_O = \operatorname{im} D\mathsf{L}(0)$. Choose a linear right inverse

$$Q_O : \mathcal{R}_O \longrightarrow C_c^\infty(O)$$

with finite-dimensional image by choosing one smooth preimage of each vector in a basis of $\mathcal{R}_O$. Every entry of $D\mathsf{L}(0)[f_{j,T}]$ is $O(T^{-L})$ for all L, since it is an integral of a fixed smooth compactly supported amplitude against a nonstationary linear phase. Define

$$\widehat{f}_{j,T} = f_{j,T} - Q_O(D\mathsf{L}(0)[f_{j,T}]).$$

Then $\widehat{f}_{j,T} \in \mathcal{K}_O$, and the correction is $O(T^{-L})$ in every fixed C^k norm for every L. Let $e_{j,T} = \widehat{f}_{j,T} - f_{j,T}$. Since R is bounded and the u_i are smooth on the compact manifold, $\|B_C\|_{L^2\to L^2} < \infty$. Thus

$$\big|\langle B_C \widehat{f}_{i,T}, \widehat{f}_{j,T}\rangle - \langle B_C f_{i,T}, f_{j,T}\rangle\big| \le \|B_C\|\big(\|e_{i,T}\|_2\|f_{j,T}\|_2 + \|f_{i,T}\|_2\|e_{j,T}\|_2 + \|e_{i,T}\|_2\|e_{j,T}\|_2\big).$$

Choose the decay power larger than $\ell+2$. After multiplication by $T^{\ell+2}$, the pairing matrix converges entrywise to a diagonal matrix with all diagonal entries of the same strict sign. In finite dimension, entrywise convergence implies operator-norm convergence. If the absolute values of those diagonal entries have minimum $c > 0$, the matrix difference has norm less than $c/2$ for large T. The resulting quadratic form is definite. A linear relation among the $\widehat{f}_{j,T}$ would give a nonzero null vector of this matrix, so they are independent and their span has dimension N. The opposite sign region gives the other index. Finally use (12.6).

For each chosen N the amplitudes and the finite right inverse Q_O are fixed before T tends to infinity. The corrected span has dimension N and consists of functions in $C_c^\infty(O)$. Thus taking the supremum over integers N proves infinite index in that algebraic space. No single fixed-frequency family or infinite sum of controls is required. □

12.4. **Finite-dimensional correction and support.** The finite-dimensional implication used below concerns subspaces of the space of compactly supported variations.

Lemma 12.4. *Let $d \geq 1$, let E be a real vector space, and let $Q : E \to \mathbb{R}^d$ have quadratic-form coordinates, with bilinear polarization. Suppose every nonzero scalar form $c \cdot Q$ has infinite negative index. Then Q has a nonzero regular zero. The zero lies in a finite-dimensional subspace $E_0 \subset E$ on which the restricted derivative is surjective.*

Proof. Apply Theorem 7.1. It gives $x, y \in E$ and $w_1, \dots, w_d \in E$ with

$$Q(x+y) = 0, \qquad Q(x) = -Q(y) \neq 0, \qquad DQ(x+y)|_{\operatorname{span}\{w_1,\dots,w_d\}} \text{ an isomorphism.}$$

If $x + y = 0$, then $Q(y) = Q(-x) = Q(x)$, contradicting $Q(y) = -Q(x) \neq 0$. Thus $z_0 = x + y \neq 0$. The restriction of Q to $E_0 = \operatorname{span}\{x, y, w_1, \dots, w_d\}$ has a regular zero at z_0. This is a finite-dimensional restriction of the original vector space; no closure or enlargement of its elements is used. □

Proof of Theorem 12.1. If $C_O = 0$, choose finitely many preimages of a basis of Sym_m under (12.3). Lemma 3.2, restricted to those preimages, gives a smooth local section. On a smaller convex target ball its derivative is bounded in every C^k, which gives the linear estimate.

Assume $C_O \neq 0$. On $\mathcal{K}_O$ set

$$Q(f) = \tfrac{1}{2}\,\mathrm{proj}_{C_O} D^2\mathsf{L}(0)[f, f].$$

Lemma 12.3 and Lemma 12.4 give a nonzero regular zero t_0 and vectors on which the derivative is surjective. Add finitely many functions mapping onto $\mathcal{R}_O$ under $D\mathsf{L}(0)$. All these functions are compactly supported in O; their supports have a compact union $K \Subset O$. Let W be their span. Write the parameters as (z, t), where the z-directions map isomorphically onto $\mathcal{R}_O$ and the t-directions belong to $\mathcal{K}_O$. The range equation

$$\mathrm{proj}_{\mathcal{R}_O}(\mathsf{L}(z, t) - \lambda I_m) = Y_{\mathcal{R}_O}$$

has a smooth solution $z = z(t, Y_{\mathcal{R}_O})$ with

$$z(0, 0) = 0, \qquad D_t z(0, 0) = 0, \qquad z = O(|t|^2 + |Y_{\mathcal{R}_O}|).$$

Since the projection onto the cokernel annihilates the first derivative, the remaining map satisfies

$$G(t, Y_{\mathcal{R}_O}) = Q(t) + O\big(|t|^3 + |t||Y_{\mathcal{R}_O}| + |Y_{\mathcal{R}_O}|^2\big).$$

Set $t = \varepsilon w$ and $Y = \varepsilon^2 A$. Since $G(0, 0) = DG(0, 0) = 0$,

$$\varepsilon^{-2} G(\varepsilon w, \varepsilon^2 A_{\mathcal{R}_O}) = \int_0^1 (1 - s) D^2 G(s\varepsilon w, s\varepsilon^2 A_{\mathcal{R}_O})\big[(w, \varepsilon A_{\mathcal{R}_O}), (w, \varepsilon A_{\mathcal{R}_O})\big]\, ds.$$

This defines a smooth extension at $\varepsilon = 0$, with value $Q(w)$. Put $d = \dim C_O$ and choose a d-dimensional subspace E_1 of the kernel-parameter space such that $DQ(t_0)|_{E_1}$ is an isomorphism onto C_O, and write $w = t_0 + e$, $e \in E_1$. At $(\varepsilon, e, A) = (0, 0, 0)$, the derivative in e of the extended expression minus A_{C_O} is this isomorphism. Lemma 3.2 gives a smooth solution $e = e(\varepsilon, A)$. Set $t = \varepsilon(t_0 + e(\varepsilon, A))$ and substitute it into the solved range equation. The resulting W-valued map $\Psi(\varepsilon, A)$ satisfies

$$\Psi(0, A) = 0, \qquad \mathsf{L}(\Psi(\varepsilon, A)) = \lambda I_m + \varepsilon^2 A.$$

On a smaller product neighborhood its estimates are

$$\|\Psi(\varepsilon, A)\|_{C^k} \le C_k|\varepsilon|, \quad \|\partial_\varepsilon \Psi\|_{C^k} \le C_k, \quad \|D_A\Psi\|_{C^k} \le C_k|\varepsilon|.$$

The last follows from $\Psi(0, A) = 0$ and the integral formula for $D_A\Psi$ in ε.

The same parameter neighborhood serves every fixed k: choose it by the finite-dimensional inverse equations and by compactness in the finite parameter space, then estimate its finitely many smooth coefficient functions in C^k. The constants may depend on k; no norm bound uniform in all derivatives is claimed. Fix L sufficiently large and, for $Y \neq 0$, set

$$\mathscr{S}(Y) = \Psi\left(L\sqrt{\|Y\|}, \frac{Y}{L^2\|Y\|}\right).$$

Take the matrix norm from its Euclidean inner product. The auxiliary target has norm L^{-2}, so one choice of L works in every direction; a sufficiently small target radius keeps the first argument in the same product neighborhood. Differentiation gives

$$\|D\mathscr{S}(Y)\|_{\mathrm{Sym}_m \to C^k} \le C_k\|Y\|^{-1/2}.$$

To obtain the pairwise estimate, write $\rho = \|Y - Y'\|$. If $\rho \ge \frac{1}{2}\max(\|Y\|, \|Y'\|)$, use the bound at zero and the triangle inequality. Otherwise, after interchanging endpoints if necessary, $\|Y\| = \max(\|Y\|, \|Y'\|) > 2\rho$. Every point on the segment has norm at least $\|Y\|/2 > \rho$. Integrating the derivative bound along that segment gives $C_k\rho/\sqrt{\|Y\|} \le C_k\sqrt{\rho}$. This proves (12.2).

With $A = 0$ and $\varepsilon \neq 0$, the reduced derivative is $\varepsilon DQ(t_0) + O(\varepsilon^2)$ and is onto for sufficiently small ε. The range block remains invertible. Taking a sequence $\varepsilon_j \to 0$ gives the stated q_j.

To check the full derivative, let A_0 be its range-to-range block and let B_0, C_0, D_0 be the other three blocks. Differentiation of the solved range equation gives the reduced block $D_0 - C_0A_0^{-1}B_0$. It is onto by the displayed expansion on the fixed derivative complement at t_0. Given target components (a, b), solve

$$(D_0 - C_0A_0^{-1}B_0)t = b - C_0A_0^{-1}a, \qquad z = A_0^{-1}(a - B_0t).$$

Then the unreduced derivative sends (z, t) to (a, b). All these directions belong to the fixed space W; its support is the same compact K.

Finally, fix a unit nonzero $C \in \mathcal{C}_O$. The Riesz matrix is analytic as a function of a sufficiently small bounded potential. Its second derivative is bounded on a smaller L^∞ ball. For q supported in O, (12.4) makes the first derivative vanish against C. Taylor's formula on this subspace gives

$$|\langle C, \mathsf{L}(q) - \lambda I_m\rangle| \leq C_0\|q\|_\infty^2.$$

A realization of $Y = tC$, $t > 0$, must therefore satisfy $\|q\|_\infty \geq C_0^{-1/2}\sqrt{t}$. This proves optimality of the exponent. □

Corollary 12.5. *Every eigenvalue of $\Delta_g + V$ satisfies the Weak Arnold Hypothesis for scalar potentials supported in the prescribed set O.*

Proof. Let $\mathcal{U}$ be a neighborhood of zero in the allowed potential space. Choose $q_j \in \mathcal{U}$ as in Theorem 12.1. Restrict $D\mathsf{L}(q_j)|_W$ to a subspace on which it is invertible. The local inverse gives, for some $\rho > 0$,

$$\sigma : \overline{B}_\rho(\lambda I_m) \to \mathcal{U} \cap W, \qquad \mathsf{L} \circ \sigma = \mathrm{Id}.$$

If $\widetilde{\mathsf{L}}$ is continuous and $\sup_{\mathcal{U}} \|\widetilde{\mathsf{L}} - \mathsf{L}\| < \rho/2$, define

$$H_t(A) = A - \lambda I_m + t\big(\widetilde{\mathsf{L}}(\sigma(A)) - A\big).$$

For $\|A - \lambda I_m\| = \rho$, one has $\|H_t(A)\| \geq \rho/2$. Hence $\deg(H_1, B_\rho(\lambda I_m), 0) = 1$, and $\widetilde{\mathsf{L}}$ attains λI_m on $\mathcal{U} \cap W$. □

13. Spectral bundles on a fixed round two-sphere

The round-sphere product isomorphism below is the algebraic input in Colin de Verdière's conformal stability theorem [11, p. 186]; see also [14, Proposition 4.6 and its proof]. We include a proof to specify the real multiplication map. The subsequent compactly supported Gram construction gives a potential section smooth at zero with a linear estimate. This construction, and hence the fixed-surface bundle realization, does not use Lemma 12.3 or the quadratic correction theorem.

Let γ be the unit-round metric on S^2. Denote by $\mathcal{H}_d$ the real homogeneous harmonic polynomials of degree d in $\mathbb{R}^3$, restricted to S^2. Their eigenvalue is $d(d + 1)$ and their dimension is $2d + 1$.

Lemma 13.1 (Round-sphere multiplication; [11, p. 186], [14, Proposition 4.6]). *For every $\ell \geq 0$, multiplication induces an isomorphism*

$$\mathrm{Sym}^2\,\mathcal{H}_\ell \longrightarrow \bigoplus_{j=0}^{\ell} \mathcal{H}_{2j}. \tag{13.1}$$

Here multiplication sends $u \odot v$ to uv, where $u \odot v = (u \otimes v + v \otimes u)/2$. Consequently, a symmetric matrix C in a basis $u_1, \dots, u_{2\ell+1}$ of $\mathcal{H}_\ell$ for which $\sum C_{ij}u_iu_j$ vanishes on a nonempty open subset of S^2 is zero.

Proof. Write $\mathscr{D} = \partial_{X_1}^2 + \partial_{X_2}^2 + \partial_{X_3}^2$ for the Euclidean trace operator and $\rho^2 = X_1^2 + X_2^2 + X_3^2$. Every homogeneous polynomial of degree d has the unique decomposition

$$p = \sum_{a=0}^{\lfloor d/2 \rfloor} \rho^{2a} h_{d-2a}, \qquad \mathscr{D} h_{d-2a} = 0.$$

Existence follows from the polynomial projection formula (11.2), specialized to $n = 2$, and induction on d. For uniqueness, suppose a sum of the displayed form is zero, and choose the largest index a with nonzero h_{d-2a}. Applying $\mathscr{D}^a$ annihilates all terms of smaller index and gives a nonzero scalar multiple of h_{d-2a}, a contradiction. The scalar is nonzero because

$$\mathscr{D}(\rho^{2a} h_e) = 2a(2e + 2a + 1)\rho^{2a-2} h_e, \qquad a > 0.$$

It follows that the image in (13.1) is contained in the stated direct sum.

The decomposition also determines dimensions. Let $\mathcal{P}_d$ be the homogeneous polynomials of degree d in three variables, with $\mathcal{P}_d = 0$ for $d < 0$. Uniqueness gives

$$\mathcal{P}_d = \mathcal{H}_d^{\mathrm{hom}} \oplus \rho^2 \mathcal{P}_{d-2}.$$

Restriction to the sphere is injective on homogeneous harmonic polynomials by homogeneity. Therefore

$$\dim \mathcal{H}_d = \binom{d+2}{2} - \binom{d}{2} = 2d + 1,$$

where the second binomial term is zero for $d = 0, 1$.

Put $z = X_1 + iX_2$. Both $\operatorname{Re} z^\ell$ and $\operatorname{Im} z^\ell$ are harmonic. Their sum of squares is

$$p_\ell = (X_1^2 + X_2^2)^\ell,$$

so $p_\ell|_{S^2}$ belongs to the real multiplication image. Write its harmonic decomposition as

$$p_\ell = \sum_{j=0}^{\ell} \rho^{2(\ell-j)} h_{2j}.$$

For each j,

$$\mathscr{D}^{\ell-j} p_\ell = 4^{\ell-j} \left(\frac{\ell!}{j!} \right)^2 (X_1^2 + X_2^2)^j.$$

The top homogeneous harmonic component of the left side is a nonzero scalar multiple of h_{2j}: terms indexed above j are annihilated, and terms indexed below j retain a positive power of ρ^2. The top harmonic component of $(X_1^2 + X_2^2)^j$ is nonzero. Otherwise that polynomial would be divisible by ρ^2; evaluation at $(X_1, X_2, X_3) = (1, 0, i)$ over $\mathbb{C}$ gives the contradiction $1 = 0$. Thus every h_{2j} is nonzero.

Let $\mathcal{I}$ be the multiplication image on S^2. It is invariant under rotations. For $L_{ij} = X_i \partial_{X_j} - X_j \partial_{X_i}$, differentiation of the rotation action preserves $\mathcal{I}$ because this finite-dimensional subspace is closed. Direct differentiation gives, with $\mathcal{E} = \sum_i X_i \partial_{X_i}$,

$$\sum_{i<j} L_{ij}^2 = \rho^2 \mathscr{D} - \mathcal{E}^2 - \mathcal{E}.$$

On the restriction of a degree-d homogeneous harmonic polynomial, the right-hand side equals $-d(d+1)$. The sphere decomposition thus identifies $-\sum_{i<j} L_{ij}^2$ with Δ_Y on the finite sum containing $\mathcal{I}$. Hence $\mathcal{I}$ is invariant under Δ_Y. Put $\mu_j = 2j(2j+1)$. The projection onto $\mathcal{H}_{2j}$ on the displayed direct sum is

$$\mathcal{P}_j = \prod_{\substack{0 \le k \le \ell \\ k \ne j}} \frac{\Delta_Y - \mu_k I}{\mu_j - \mu_k}.$$

The denominators are nonzero. Since $\mathcal{I}$ is invariant under Δ_γ and contains $p_\ell|_{S^2}$, it contains $\mathcal{P}_j(p_\ell|_{S^2}) = h_{2j}|_{S^2}$.

Each h_{2j} is invariant under the rotations fixing the X_3-axis, by uniqueness of harmonic decomposition. An invariant homogeneous polynomial has the form

$$h = \sum_{a=0}^{\lfloor d/2 \rfloor} c_a (X_1^2 + X_2^2)^a X_3^{d-2a}.$$

The coefficients of $\mathscr{D}h = 0$ satisfy

$$4(a+1)^2 c_{a+1} + (d-2a)(d-2a-1)c_a = 0 \qquad \left(0 \le a < \left\lfloor \frac{d}{2} \right\rfloor\right).$$

At the largest index the remaining X_3-second derivative is zero because its exponent is 0 or 1. Every coefficient is consequently determined by c_0, and every choice of c_0 produces a harmonic polynomial. Thus this invariant subspace is exactly one-dimensional. For an orthonormal basis $Y_1, \dots, Y_{2d+1}$ of $\mathcal{H}_d$, the evaluation kernel

$$K_d(\omega, \nu) = \sum_i Y_i(\omega) Y_i(\nu)$$

is rotation invariant, reproduces every member of $\mathcal{H}_d$, and is nonzero at (ν, ν). The last assertion follows because $K_d(\nu, \nu)$ is rotation invariant and its integral in ν equals $2d + 1$. Thus a nonzero axis-invariant harmonic is a scalar multiple of $K_d(\cdot, e_3)$. Its rotations span $\mathcal{H}_d$: a function orthogonal to all those rotations has zero evaluation at every point. Since $\mathcal{I}$ is rotation invariant and contains h_{2j}, it contains $\mathcal{H}_{2j}$ for every j. This proves surjectivity. Finally,

$$\dim \operatorname{Sym}^2 \mathcal{H}_\ell = (\ell+1)(2\ell+1) = \sum_{j=0}^{\ell} (4j+1),$$

so the map is an isomorphism. The restriction of a polynomial to S^2 is real analytic. Its vanishing on a nonempty open set implies its vanishing on the connected sphere. Injectivity now proves the last assertion. ☐

Proposition 13.2. *Fix $\ell \ge 1$, $m = 2\ell + 1$, $\lambda = \ell(\ell+1)$, and a nonempty open set $O \subset S^2$. There are $K \Subset O$, a finite-dimensional $W \subset \{q \in C^\infty(S^2; \mathbb{R}) : \operatorname{supp} q \subset K\}$, and a smooth local section $\mathscr{S}_\ell$ at zero such that*

$$\mathsf{L}(\mathscr{S}_\ell(Y)) = \lambda I_m + Y, \qquad \mathscr{S}_\ell(0) = 0,$$

and, on a sufficiently small target ball,

$$\|\mathscr{S}_\ell(Y) - \mathscr{S}_\ell(Y')\|_{C^k} \le C_{k,\ell} \|Y - Y'\| \qquad (k \ge 0). \tag{13.2}$$

Proof. Choose $\chi \in C_c^\infty(O)$ nonnegative and positive on a nonempty open set. In an orthonormal eigenbasis put $F_C = \sum C_{ij} u_i u_j$ and define the matrix-space Gram operator G by

$$\langle GC, D \rangle = \int_{S^2} \chi F_C F_D \, dv_\gamma.$$

Lemma 13.1 shows that G is positive definite. Define

$$R(C) = \chi F_{G^{-1}C}.$$

Equation (12.3) gives $D\mathsf{L}(0)R(C) = C$. Hence $C \mapsto \mathsf{L}(R(C)) - \lambda I_m$ has derivative the identity. Its finite-dimensional local inverse defines $\mathscr{S}_\ell$ with values in $\operatorname{im} R$ and support in $\operatorname{supp} \chi$. On a smaller convex ball the derivatives into every C^k space are bounded, since the image is finite-dimensional and consists of fixed smooth functions. Integration along segments proves (13.2). ☐

Theorem 13.3. *Let B be a compact smooth manifold and let $E \to B$ be a smooth real vector bundle of rank $r > 0$. Fix a nonempty open disk $O \subset S^2$. There are an integer $\ell \geq 1$, $m = 2\ell + 1 > r$, a compact set $K \Subset O$, and a smooth family of real potentials*

$$q_{\varepsilon,b} \in C^\infty(S^2;\mathbb{R}), \quad \operatorname{supp} q_{\varepsilon,b} \subset K, \qquad b \in B, \quad |\varepsilon| < \varepsilon_0,$$

with $q_{0,b} = 0$, for which the $\ell(\ell+1)$-cluster of $\Delta_\gamma + q_{\varepsilon,b}$ has, for $0 < \varepsilon < \varepsilon_0$, exactly the eigenvalues

$$\ell(\ell+1) - (m-r)\varepsilon \quad \text{with multiplicity } r, \qquad \ell(\ell+1) + r\varepsilon \quad \text{with multiplicity } m - r. \tag{13.3}$$

The bundle of eigenspaces for the first value is isomorphic to E. For every fixed k, the family is $O(\varepsilon)$ in $C^k(S^2)$, uniformly in b.

Proof. Choose finitely many trivializing neighborhoods for E and smooth functions ϕ_i with supports in them and $\sum_i \phi_i^2 = 1$. The map sending a vector to the direct sum of its local coordinates multiplied by the ϕ_i is a smooth fibrewise injection into a finite trivial bundle. The terms extend by zero because the supports lie inside the trivializing neighborhoods. At each point at least one ϕ_i is nonzero, which proves injectivity. If the finite trivial bundle has rank N, choose an odd $m = 2\ell + 1 \geq N$ with $m > r$, and compose the injection with the coordinate inclusion into $B \times \mathbb{R}^m$. Let P_b be the Euclidean orthogonal projection onto the image of E_b. Define

$$Y_{\varepsilon,b} = \varepsilon(rI_m - mP_b), \qquad q_{\varepsilon,b} = \mathscr{S}_\ell(Y_{\varepsilon,b}).$$

Compactness of B permits one ε_0. Proposition 13.2 gives a family smooth also at $\varepsilon = 0$ and the stated estimates. The matrix $Y_{\varepsilon,b}$ acts by $-(m-r)\varepsilon$ on $\operatorname{im} P_b$ and by $r\varepsilon$ on its orthogonal complement. The exact section identity therefore proves (13.3). Let $U_q : \mathbb{R}^m \to L^2(S^2, dv_\gamma)$ be the Riesz–Gram frame. The map

$$(b, v) \longmapsto U_{q_{\varepsilon,b}} v, \qquad v \in \operatorname{im} P_b,$$

is a smooth fibrewise isomorphism onto the required eigenspace bundle. The density is fixed, so no Hilbert-space conjugation is needed here. Spectral isolation from the other round clusters persists for small potentials.

For an explicit isolation bound, let d_ℓ be the distance from $\ell(\ell+1)$ to the excluded round spectrum and choose $0 < \eta < d_\ell/3$. The variational characterization of the ordered eigenvalues gives

$$|\lambda_j(\Delta_\gamma + q) - \lambda_j(\Delta_\gamma)| \leq \|q\|_\infty \quad \text{for every } j \ :$$

the Rayleigh quotients differ by at most $\|q\|_\infty$, and taking the infimum and supremum preserves the bound. Take the parameter range so that $\|q_{\varepsilon,b}\|_\infty < \eta$ and $m|\varepsilon| < \eta$ uniformly in b. The displayed two eigenvalues then remain in the isolated cluster, and no other eigenvalue has either value. ☐

This theorem concerns a finite spectral cluster, in general above other positive eigenvalues. It does not assert that E is a bundle of lowest eigenfunctions or that the entire spectra are independent of b. The metric remains the unit-round metric, and all perturbations occur in the fixed potential support K.

Corollary 13.4. *There is a smooth family of arbitrarily small potentials on the fixed round S^2, supported in any prescribed nonempty open disk and parametrized by $\mathbb{CP}^1$, for which the degree-two cluster has the two constant eigenvalues*

$$6 - 3\varepsilon \quad \text{of multiplicity } 2, \qquad 6 + 2\varepsilon \quad \text{of multiplicity } 3.$$

The lower rank-two eigenbundle has Euler number -1 for the complex orientation. Fix a sufficiently small $\varepsilon > 0$. Let $q_{t,b}$, $(t,b) \in [0,1] \times \mathbb{CP}^1$, be a continuous map into $C^\infty(S^2;\mathbb{R})$ beginning at this family. Order the complete spectrum as $\nu_0(t,b) \leq \nu_1(t,b) \leq \cdots$. If

$$\nu_3(t,b) < \nu_4(t,b), \qquad \nu_5(t,b) < \nu_6(t,b) \quad \text{for all } (t,b),$$

then $\nu_4(1,b) = \nu_5(1,b)$ for some $b \in \mathbb{CP}^1$. The two inequalities are the gaps immediately below and above the selected rank-two subcluster.

Proof. Apply Theorem 13.3 to the real plane bundle underlying the tautological complex line over $\mathbb{CP}^1$, embedded in $\mathbb{C}^2 \subset \mathbb{R}^5$, and take $\ell = 2$ and $r = 2$. Its clutching function is $e^{-i\theta}$, so its Euler number is -1. At the unperturbed sphere, the levels below 6 have total multiplicity $1 + 3 = 4$. The isolation estimate in Theorem 13.3 therefore places the selected rank-two eigenspace at indices $4, 5$. Under the two gap inequalities in the statement, local Riesz projections onto these two eigenvalues agree on overlaps, and define a rank-two bundle over $\mathbb{CP}^1 \times [0, 1]$. Its endpoint restrictions are isomorphic. If the two eigenvalues were simple at every parameter, the endpoint bundle would split into two real line bundles. Each real line bundle over S^2 is trivial because $H^1(S^2; \mathbb{Z}/2) = 0$. Their sum has Euler class zero, a contradiction. This uses the homotopy invariance and line-bundle classification cited in the preceding spectral-bundle section. □

Appendix A. Flat tori and finite-order differential variations

A.1. **All first-order obstructions on a planar frequency shell.** Let $T^2 = \mathbb{R}^2/\Lambda$ be flat and let $\Lambda^* = \{k : k \cdot \gamma \in 2\pi\mathbb{Z} \text{ for all } \gamma \in \Lambda\}$. For a positive eigenvalue, write its complete shell as

$$\{k \in \Lambda^* : |k|^2 = \lambda\} = \{\pm k_1, \dots, \pm k_r\}, \qquad m = 2r. \tag{A.1}$$

The real orthonormal basis consists of $\sqrt{2/\operatorname{Vol}(T^2)}\cos(k_j \cdot x)$ and the corresponding sine. Put $C_a = \operatorname{diag}(a_1 I_2, \dots, a_r I_2)$.

If $s = k + \ell \neq 0$ and $|k| = |\ell| = \sqrt{\lambda}$, then $k \cdot s = |s|^2/2$. This line meets the circle in at most two points, exchanged by $k \leftrightarrow \ell$. Therefore s determines the unordered pair. Expanding a real quadratic relation in the complex exponentials shows that all coefficients with nonzero sum frequency vanish separately. The zero-sum pairs are $k_j, -k_j$. In the real basis they give exactly $a_j I_2$ blocks, and their constant terms cancel precisely when $\sum_j a_j = 0$.

Thus

$$\begin{aligned} \mathcal{C}_{\mathrm{conf}} &= \{C_a : \sum_j a_j = 0\}, \\ \mathcal{C}_{\mathrm{all}} &= \{C_a : \sum_j a_j k_j \otimes k_j = 0\}. \end{aligned} \tag{A.2}$$

The trace of the second relation implies the first because $|k_j|^2 = \lambda > 0$. The span of the matrices $k_j \otimes k_j$ has dimension $\min\{r, 3\}$: up to a fixed invertible change of coordinates their entries are $(1, \cos 2\theta_j, \sin 2\theta_j)$, with distinct θ_j modulo π. Any three distinct points of a circle are not collinear. Hence

$$\dim \mathcal{C}_{\mathrm{conf}} = r - 1, \qquad \dim \mathcal{C}_{\mathrm{all}} = \max\{r - 3, 0\}. \tag{A.3}$$

Together with Theorem C, this gives smooth full-metric multiplicity loci for $m = 2, 4, 6$ and nonsmooth loci for $m \geq 8$; within the conformal class, only $m = 2$ is strongly stable.

A.2. **Differential order.** Let $\mathscr{D}_{2s}$ be the real formally self-adjoint scalar differential operators of order at most $2s$ on T^2, where $s \geq 0$. Consider

$$\mathscr{D}_{2s} \longrightarrow \operatorname{Sym}(E_\lambda), \qquad D \longmapsto P_\lambda D P_\lambda. \tag{A.4}$$

Proposition A.1. *The cokernel dimension of* (A.4) *is*

$$\max\{r - (2s + 1), 0\}. \tag{A.5}$$

The least even differential order for which the compression map is surjective is $2\lfloor r/2 \rfloor$.

Proof. Multiplication by every real smooth function belongs to $\mathscr{D}_{2s}$. The preceding product calculation forces every annihilator to be C_a. Testing real differential operators is equivalent to testing their self-adjoint parts: for symmetric real C_a, transposition gives $\operatorname{tr}(C_a PDP) = \operatorname{tr}(C_a P(D + D^*)P/2)$.

For a multi-index α, direct trigonometric differentiation gives

$$\cos(k\cdot x)\,\partial^\alpha\cos(k\cdot x)+\sin(k\cdot x)\,\partial^\alpha\sin(k\cdot x)=\begin{cases}0, & |\alpha| \text{ odd},\\ (-1)^{|\alpha|/2}k^\alpha, & |\alpha| \text{ even}.\end{cases} \tag{A.6}$$

Testing each coefficient of a differential operator therefore gives precisely $\sum_j a_j k_j^\alpha = 0$ for even $|\alpha| \le 2s$.

Contracting two indices in $k_j^{\otimes 2s}$ gives $\lambda k_j^{\otimes(2s-2)}$. Since $\lambda > 0$, all lower moment equations follow from

$$\sum_j a_j k_j^{\otimes 2s} = 0. \tag{A.7}$$

Conversely, the order-$2s$ equation is among the moment equations derived from (A.6).

The space of homogeneous polynomials of degree $2s$ in two variables has dimension $2s+1$. Any at most $2s+1$ distinct projective directions give independent evaluation functionals. To isolate a fixed direction in such a collection, choose one linear factor vanishing on each other direction but not on the fixed direction. Their product has degree at most $2s$. Multiply it by a power of another linear form nonzero on the fixed direction to reach degree $2s$. Its evaluations vanish at the other directions and not at the fixed one. Dualizing proves that the map in (A.7) has rank $\min\{r, 2s+1\}$. This gives (A.5). The smallest s satisfying $r \le 2s+1$ is $\lfloor r/2 \rfloor$. ☐

On the square torus the shell $|k|^2 = 5^s$ has $4(s+1)$ vectors. Indeed, in the Gaussian integers, $5 = (2+i)(2-i)$ with two nonassociate primes; a Gaussian integer of norm 5^s is a unit times $(2+i)^a(2-i)^{s-a}$, $0 \le a \le s$. For $z, w \in \mathbb{Z}[i]$, $w \ne 0$, choose a Gaussian integer q by rounding the two coordinates of z/w. Then $|z-qw|^2 \le |w|^2/2 < |w|^2$. The Euclidean algorithm therefore gives, for any $a, b \in \mathbb{Z}[i]$, a greatest common divisor d and Gaussian integers u, v with $d = ua + vb$. If an irreducible π does not divide a, this identity for π, a gives $1 = u\pi + va$ after multiplication by a unit. Consequently $\pi \mid ab$ implies $\pi \mid b$; every irreducible is prime. Induction on the positive integer norm factors every nonzero nonunit into irreducibles, since a proper nonunit factor has strictly smaller norm. Primality then proves uniqueness by cancellation, up to units and order. The two factors $2+i$ and $2-i$ have prime norm five and are not associates.

If $z\bar{z} = 5^s$, every prime factor of z divides $(2+i)^s(2-i)^s$ and is therefore associated to one of these two primes. Taking norms in $z = u(2+i)^a(2-i)^b$, $u \in \{1, -1, i, -i\}$, gives $a+b = s$. These are exactly the asserted choices. Unique factorization makes the displayed $4(s+1)$ choices distinct. Thus $r = 2s+2$, and (A.5) is one. This shows that no fixed finite even order removes every obstruction. For $s \ge 1$ and $D \in \mathscr{D}_{2s}$, use the cluster of Δ^s at λ^s. The function $x \mapsto x^s$ is strictly increasing on $[0,\infty)$, so its eigenspace is E_λ. The principal symbol of $\Delta^s + tD$ is $|\xi|^{2s} + t\sigma_{2s}(D)(x,\xi)$. Boundedness of $\sigma_{2s}(D)$ on the compact unit cosphere makes it positive for sufficiently small $|t|$. This formally self-adjoint elliptic family has fixed domain H^{2s}, and its cluster derivative is $P_\lambda D P_\lambda$. No perturbation changing differential order at the Laplacian is asserted to have a fixed domain.

A.3. **Second variations at the eigenvalue five.** Let $\ell \in \Lambda^*$ satisfy $|\ell| > 2\sqrt{\lambda}$, and put $f_\ell(x) = \cos(\ell\cdot x)$. This function is well-defined on $\mathbb{R}^2/\Lambda$. If $|k|^2 = \lambda$, then $|k \pm \ell| \ge |\ell| - |k| > \sqrt{\lambda}$. Thus multiplication by f_ℓ sends each shell exponential into the orthogonal complement of E_λ, and $PM_{f_\ell}P = 0$. For the conformal curve $(1+tf_\ell)g$, put $q_a(\ell) = \langle C_a, D^2\mathsf{L}(0)[f_\ell, f_\ell]\rangle$. The dimension-two second variation gives

$$q_a(\ell) = -\lambda^2 \sum_{j=1}^r a_j \left(\frac{1}{|\ell|^2 + 2k_j\cdot\ell} + \frac{1}{|\ell|^2 - 2k_j\cdot\ell}\right). \tag{A.8}$$

For the normalized exponential $e_k = \mathrm{Vol}(T^2)^{-1/2}e^{ik\cdot x}$,

$$M_{f_\ell}e_k = \tfrac{1}{2}(e_{k+\ell} + e_{k-\ell}), \qquad \langle M_{f_\ell}RM_{f_\ell}e_k, e_k\rangle = \tfrac{1}{4}\sum_{\epsilon=\pm1}\frac{1}{|k+\epsilon\ell|^2 - \lambda}.$$

The same sum is obtained for $-k$. The scalar block $a_j I_2$ is unchanged on complexifying the corresponding sine–cosine plane. Its trace contribution is therefore twice this diagonal entry times a_j. In dimension two, $A_0'' = B_0'' = 0$. Since $PM_{f_\ell}P = 0$, equation (5.16) gives the uncontracted identity

$$D^2\mathsf{L}(0)[f_\ell, f_\ell] = -2\lambda^2 U^* M_{f_\ell} R M_{f_\ell} U.$$

Its contraction with any C_a gives (A.8); no additional condition on the weights a_j is required. All denominators are positive because $|\ell| > 2\sqrt{\lambda}$.

For the shell in Example 4.2, with $a = (1, 1, -1, -1)$, the second moment is zero and the fourth is

$$\sum_{j=1}^{4} a_j (k_j \cdot \xi)^4 = -48\xi_1\xi_2(\xi_1^2 - \xi_2^2). \tag{A.9}$$

Equation (A.8) gives

$$q_a((5,1)) = \frac{845}{352}, \qquad q_a((5,-1)) = -\frac{845}{352}. \tag{A.10}$$

For example the bracketed sum for $(5, 1)$ is

$$\frac{1}{40} + \frac{1}{12} + \frac{1}{44} + \frac{1}{8} - \frac{1}{32} - \frac{1}{20} - \frac{1}{48} - \frac{1}{4},$$

and multiplication by -25 gives the first value.

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