HOMOLOGICAL OBSTRUCTIONS TO LOCAL CONTRACTIBILITY IN NONCOLLAPSED RICCI LIMITS

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ABSTRACT. For every integer $N \geq 4$, we construct a compact noncollapsed Ricci limit X of closed N-manifolds with nonnegative Ricci curvature such that $0 < \mathcal{H}^{N-2}(S_{\text{top}}(X)) < \infty$, where $S_{\text{top}}(X)$ is the nonmanifold locus. There is a compact subset $Z \subset S_{\text{top}}(X)$ of positive $(N-2)$-dimensional Hausdorff measure such that, for every subset $E \subset X$ with $\dim_{\mathcal{H}} E < N-2$, every relative open neighborhood U in $X \setminus E$ of a point of $Z \setminus E$ satisfies $\operatorname{rank} \operatorname{im}[H_2(U;\mathbb{Z}) \to H_2(X;\mathbb{Z})] = \infty$. Thus no such E has manifold complement, without any closedness or measurability assumption on E. These examples attain the codimension-two upper bound for the nonmanifold locus and contradict the Hausdorff-codimension-three manifold conjecture. The realizing sequences have uniformly bounded total scalar curvature and total Ricci norm; in dimensions four and five, their Ricci curvature has a uniform strictly positive lower bound. A separate compact four-dimensional example has exactly one nonmanifold point, with unique tangent cone $\mathbb{R}^2 \times \mathsf{C}(S^1_{2\pi q})$ for some $q \in (0,1)$, and an empty Cheeger–Colding zero-stratum.

CONTENTS

1. Main results and conventions

1.1. **Noncollapsed limits and manifold complements.** Let $N \geq 2$ be an integer. A noncollapsed Ricci limit of dimension N is a pointed Gromov–Hausdorff limit

$$(M_j^N, d_{g_j}, p_j) \longrightarrow (X, d, p)$$

of complete connected smooth Riemannian N-manifolds without boundary such that

$$\mathrm{Ric}_{g_j} \geq K g_j, \qquad \mathrm{Vol}_{g_j} B_1(p_j) \geq v > 0, \tag{1.1}$$

where $K \in \mathbb{R}$ and v are independent of j. Here d_{g_j} is the Riemannian length distance; no factor $N-1$ is included in K. Hausdorff measure is normalized to agree with Lebesgue measure on Euclidean space. The volume-convergence theorem identifies the limiting volume measure with $\mathcal{H}^N$ and gives $\dim_{\mathcal{H}} X = N$ [4, Theorem 5.9, p. 434].

A topological manifold is Hausdorff, second countable, and without boundary. Define

$$\mathcal{R}_{\mathrm{top}}(X) = \{x \in X \,:\, x \text{ has an open neighborhood homeomorphic to } \mathbb{R}^N\}.$$

Put

$$S_{\mathrm{top}}(X) = X \setminus \mathcal{R}_{\mathrm{top}}(X).$$

The set $\mathcal{R}_{\mathrm{top}}(X)$ is open: every point of a chart domain has a smaller Euclidean-ball chart. Thus $S_{\mathrm{top}}(X)$ is closed. Cheeger–Jiang–Naber's Theorem 1.14 and its proof supply ambient manifold charts outside an $(N-2)$-rectifiable exceptional set of locally finite $\mathcal{H}^{N-2}$ measure [5]. In particular, $\dim_{\mathcal{H}} S_{\mathrm{top}}(X) \leq N-2$. The finite-measure consequence needed for the compact spaces constructed here is proved in Lemma 11.2.

Naber's codimension-three manifold conjecture [13, Conjecture 2.4], with codimension understood in the Hausdorff sense, asks for a subset $E \subset X$ such that

$$\dim_{\mathcal{H}} E \leq N-3, \qquad X \setminus E \text{ is a topological manifold.} \tag{1.2}$$

The complement has its subspace topology. We do not impose closedness on E in (1.2). A stronger closed-exceptional-set formulation, recorded in [2, Introduction, p. 3], asks, for $N \geq 4$, for a closed $E \subset X$ such that

$$\dim_{\mathcal{H}} E \leq N-4, \qquad X \setminus E \text{ is a topological } N\text{-manifold.} \tag{1.3}$$

We call (1.3) the generalized topological codimension-four conjecture.

For a closed E, the subspace $X \setminus E$ is open in X, so its manifold charts are ambient charts and $S_{\mathrm{top}}(X) \subset E$. Conversely, $X \setminus S_{\mathrm{top}}(X)$ is an open locally Euclidean subspace of a separable metric space and is therefore a topological N-manifold. Consequently (1.3) is equivalent to

$$\dim_{\mathcal{H}} S_{\mathrm{top}}(X) \leq N-4. \tag{1.4}$$

This argument does not apply to an arbitrary nonclosed E. The deletion statements below concern relative neighborhoods and do not use that implication. Cheeger–Colding's Conjecture 0.7 specifies a tangent-cone stratum instead of allowing the exceptional set to vary; its formulation is given in Section 1.3.

1.2. **Neighborhood homology and exceptional sets.** For an abelian group A, put $\operatorname{rank} A = \dim_{\mathbb{Q}}(A \otimes_{\mathbb{Z}} \mathbb{Q})$. The homology maps displayed in Theorems 1.1 and 1.3 and Corollary 1.2 are induced by inclusions. Subsets of Euclidean space carry the restricted Euclidean distance. We write $\mathrm{Scal}_g = \mathrm{tr}_g \mathrm{Ric}_g$ and use the Hilbert–Schmidt norm of the Ricci tensor. If $\mathrm{Ric}_g \geq 0$ and $\lambda_1, \dots, \lambda_N$ are its eigenvalues, then

$$|\mathrm{Ric}_g|_g = \left(\sum_{a=1}^N \lambda_a^2\right)^{1/2} \leq \sum_{a=1}^N \lambda_a = \mathrm{Scal}_g\,. \tag{1.5}$$

Thus a bound on the total Ricci norm follows from the scalar-curvature bound in this setting.

Theorem 1.1. *Let $n \in \{4, 5\}$, and let $F \subset \{z \in \mathbb{R}^{n-2} : |z| < 1/40\}$ be nonempty, compact, and have empty interior. There are closed connected smooth n-manifolds (M_j, g_j, p_j), constants $\kappa, v, \Lambda > 0$, a compact noncollapsed Ricci limit (X, d, p), and a bi-Lipschitz embedding $\iota : F \hookrightarrow X$ such that*

$$\operatorname{Ric}_{g_j} \geq \kappa g_j, \qquad \operatorname{Vol}_{g_j} B_1(p_j) \geq v, \qquad (M_j, d_{g_j}, p_j) \longrightarrow (X, d, p), \tag{1.6}$$

$$\int_{M_j} \left(\operatorname{Scal}_{g_j} + |\operatorname{Ric}_{g_j}|_{g_j}\right) dv_{g_j} \leq \Lambda. \tag{1.7}$$

For every subset $E \subset X$ with $\dim_{\mathcal{H}} E < n-2$, every $x \in \iota(F) \setminus E$, and every relative open neighborhood U of x in $X \setminus E$,

$$\operatorname{rank} \operatorname{im}\left[H_2(U; \mathbb{Z}) \longrightarrow H_2(X; \mathbb{Z})\right] = \infty. \tag{1.8}$$

In particular, $\iota(F) \subset S_{\mathrm{top}}(X)$, and $X \setminus E$ is not locally contractible at any point of $\iota(F) \setminus E$.

The bounds in (1.6)–(1.7) hold for every j. The constants and the realizing sequence may depend on F. No closedness or measurability assumption is made on E; Hausdorff dimension is defined by Hausdorff outer measures. The image in (1.8) is taken in the homology of X, not merely in that of $X \setminus E$.

Corollary 1.2. *For every integer $N \geq 4$ there are a compact noncollapsed Ricci limit X_N and a compact subset $Z_N \subset S_{\mathrm{top}}(X_N)$ such that*

$$0 < \mathcal{H}^{N-2}(Z_N) \leq \mathcal{H}^{N-2}(S_{\mathrm{top}}(X_N)) < \infty, \qquad \dim_{\mathcal{H}} S_{\mathrm{top}}(X_N) = N-2. \tag{1.9}$$

For every subset $E \subset X_N$ with $\dim_{\mathcal{H}} E < N-2$, every $z \in Z_N \setminus E$, and every relative open neighborhood U of z in $X_N \setminus E$,

$$\operatorname{rank} \operatorname{im}\left[H_2(U; \mathbb{Z}) \longrightarrow H_2(X_N; \mathbb{Z})\right] = \infty. \tag{1.10}$$

Consequently $X_N \setminus E$ is not a topological manifold. The realizing sequence consists of closed connected smooth N-manifolds with nonnegative Ricci curvature, a uniform positive lower bound for based unit-ball volumes, and a uniform bound for the integral in (1.7)*. For $N = 4, 5$, the Ricci curvature has a uniform strictly positive lower bound.*

Indeed, $\dim_{\mathcal{H}} E < N-2$ implies $\mathcal{H}^{N-2}(E) = 0$, whereas $\mathcal{H}^{N-2}(Z_N) > 0$. Thus $Z_N \setminus E \neq \emptyset$. A manifold chart at one of these points would contain a relative open Euclidean ball U, for which $H_2(U; \mathbb{Z}) = 0$, contrary to (1.10). Since $N-3 < N-2$, Corollary 1.2 contradicts (1.2); it also contradicts (1.3). The upper bound in (1.9) is a consequence of [5, Theorem 1.14]. The construction gives the positive-measure lower bound and the deletion property.

1.3. **An isolated nonmanifold point in dimension four.** For a pointed proper metric space (Z, d, z), let $\operatorname{Tan}(Z, z)$ be the set of pointed-isometry classes of pointed limits of $(Z, r_i^{-1} d, z)$ as $r_i \downarrow 0$. The circle of circumference $2\pi q$ is denoted by $S^1_{2\pi q}$. Its metric cone $\mathsf{C}(S^1_{2\pi q})$ has length tensor $dt^2 + q^2 t^2 d\theta^2$, where $\theta \in \mathbb{R}/(2\pi\mathbb{Z})$ and the circle at $t = 0$ is collapsed to the vertex. For an N-dimensional noncollapsed Ricci limit, define

$$\mathcal{S}_k(Z) = \{z \in Z : \text{no tangent cone at } z \text{ splits } \mathbb{R}^{k+1}\}, \qquad 0 \leq k \leq N-1.$$

A splitting is a pointed isometry with a metric product. Cheeger–Colding's definitions give $D_k = \mathcal{S}_k$ and $\mathcal{S}_k^{\mathrm{CC}} = D_k \setminus \mathcal{R}$, where $\mathcal{R}$ is their regular set [4, Definitions 0.1, 0.3–0.4, pp. 408–410]. In particular, $\mathcal{S}_k = \emptyset$ implies $\mathcal{S}_k^{\mathrm{CC}} = \emptyset$, without any further comparison of regular-set conventions. For $N \geq 4$, their Conjecture 0.7 asserts that

$$\operatorname{Int}_Z\left(Z \setminus \mathcal{S}^{\mathrm{CC}}_{N-4}(Z)\right) \text{ is a topological manifold} \tag{1.11}$$

[4, p. 413].

Theorem 1.3. *There are $q \in (0,1)$, constants $\kappa, v, \Lambda > 0$, closed connected smooth four-manifolds (M_j, g_j, p_j), and a compact noncollapsed limit (Y, d, p) such that*

$$\mathrm{Ric}_{g_j} \geq \kappa g_j, \qquad \mathrm{Vol}_{g_j} B_1(p_j) \geq v, \qquad (M_j, d_{g_j}, p_j) \longrightarrow (Y, d, p),$$

$$\mathcal{S}_{\mathrm{top}}(Y) = \{p\}, \qquad \mathrm{Tan}(Y, p) = \{\mathbb{R}^2 \times \mathsf{C}(S^1_{2\pi q})\}, \qquad \mathcal{S}_1(Y) = \varnothing. \tag{1.12}$$

The realizing sequence satisfies (1.7). *The space $Y \setminus \{p\}$ is locally isometric to a smooth Riemannian four-manifold. For every open neighborhood U of p,*

$$\operatorname{rank} \operatorname{im}\big[H_2(U;\mathbb{Z}) \longrightarrow H_2(Y;\mathbb{Z})\big] = \infty.$$

The space Y is not locally contractible at p, and no open neighborhood of p is homeomorphic to an open topological cone over a compact space. Consequently $\mathcal{S}_0^{\mathrm{CC}}(Y) = \varnothing$, although Y is not a topological manifold.

The inclusions $\mathcal{S}_0(Y) \subset \mathcal{S}_1(Y)$ imply that the interior in (1.11) is all of Y when $N = 4$. Thus Theorem 1.3 contradicts the prescribed-stratum conjecture in that dimension. It does not contradict (1.3): removing the closed zero-dimensional set $\{p\}$ leaves a manifold. The four-dimensional instance of Corollary 1.2 gives the obstruction to an arbitrary exceptional set of the stated dimension.

1.4. **Comparison with earlier results.** Menguy's Theorem 0.6 constructs a noncollapsed limit $(X_{\mathrm{M}}, p_{\mathrm{M}})$ of complete four-manifolds with positive Ricci curvature and locally infinite topology [11, p. 601]. The discussion following that theorem states that $X_{\mathrm{M}} \setminus \{p_{\mathrm{M}}\}$ is a topological manifold and that no neighborhood of p_{M} has trivial homology. Remark 0.14 of the same paper attributes the infinite second Betti number to infinitely many inserted copies of $\mathbb{CP}^2$ [11, p. 603]. Thus neither a single nonmanifold accumulation point nor accumulation of second homology is new. The tangent cones are topologically Euclidean; see [2, Section 1.2].

Menguy also states that his example satisfies the prescribed-stratum conjecture [11, p. 601]. In contrast, Theorem 1.3 gives the specified metric tangent $\mathbb{R}^2 \times \mathsf{C}(S^1_{2\pi q})$ and an empty prescribed zero-stratum, together with a compact realization and the bounds (1.7). Tangent uniqueness alone is not used to distinguish the two constructions.

The direct products of Menguy's example do not have the deletion property of Corollary 1.2. For a flat torus T^k, with T^0 a point,

$$(X_{\mathrm{M}} \times T^k) \setminus (\{p_{\mathrm{M}}\} \times T^k) = (X_{\mathrm{M}} \setminus \{p_{\mathrm{M}}\}) \times T^k \tag{1.13}$$

is a manifold. The deleted set is isometric to T^k, so its Hausdorff dimension is $k = N - 4 < N - 2$, where $N = 4 + k$. This argument uses the exhibited removable set; it does not require any assertion about the exact nonmanifold locus of a product.

Hupp–Naber–Wang constructed four-rectifiable limits of six-dimensional manifolds with uniform lower Ricci bounds for which every nonempty open subset has infinitely generated second homology [9, Theorem 1.1]. Their proof also detects independent second-homology classes in the whole limit [9, Section 2.0.1, pp. 6–8]. Thus accumulation of homology, including its detection in the ambient limit, already occurs in their construction. The assertions of Theorem 1.1 and Corollary 1.2 combine same-dimensional noncollapse, a positive-measure codimension-two nonmanifold locus, and persistence under every deletion of Hausdorff dimension less than $N - 2$.

Positive-measure Cantor sets also occur as metric singular strata. Li–Naber construct noncollapsed limits of three-manifolds with nonnegative sectional curvature whose quantitative one-stratum is a prescribed compact subset of a bi-Lipschitz embedded circle [10, Theorem 1.7]; see also [13, Example 2.14]. This concerns the induced topology of a metric stratum, not the existence of ambient manifold charts. In Corollary 1.2, the positive-measure set consists of points where the homology of ambient neighborhoods, and of the indicated relative neighborhoods, obstructs local contractibility.

Under a two-sided Ricci bound and noncollapse, Cheeger–Naber prove the codimension-four estimate for the metric singular set [6, Theorem 1.4]. Under a global metric-measure splitting

$X = \mathbb{R}^{N-3} \times Z^3$, Bruè–Pigati–Semola prove that Z^3 is a topological three-manifold [2, Theorem 1.7]. In dimension four, their Theorem 1.1 shows that the cross-sections of all tangent cones at a fixed point are homeomorphic to one spherical space form. For the cone in Theorem 1.3, the cross-section is $S^1 * S^1_{2\pi q}$, homeomorphic to S^3; the obstruction concerns neighborhoods in Y.

The author's preprint [14, Theorem 3.2, Corollary 3.4 and Remark 3.5] constructs, for $N \geq 5$, a single nonmanifold point with empty prescribed codimension-four stratum and a unique product-cone tangent. Its sufficiently small neighborhoods are open topological cones over the compact manifold $M_Q = Q\#(-Q)$. They are contractible. More precisely, put

$$C_{\mathrm{top},a}(M_Q) = ([0,a) \times M_Q)/(\{0\} \times M_Q), \qquad a > 0.$$

The map

$$H([r,z],s) = [(1-s)r, z], \qquad 0 \leq s \leq 1, \tag{1.14}$$

contracts this cone to its vertex. To check continuity at the vertex, let V be a neighborhood of it. The inverse image of V contains $\{0\} \times M_Q$. Compactness of M_Q gives $b > 0$ such that $[0,b) \times M_Q$ is contained in that inverse image. Since $(1-s)r \leq r$, the contraction preserves the corresponding smaller cone for every s. Away from the vertex it is the quotient of the displayed continuous formula; as $s \to 1$, its radial coordinate tends to zero locally uniformly. The same compactness argument shows that the smaller cones form a neighborhood basis. The distinguished point in [14] is therefore locally contractible, and these neighborhoods have zero absolute second homology. Its nonmanifold obstruction is instead expressed by relative local homology.

For the point in Theorem 1.3, the inclusion-induced image of second homology has infinite rank in every neighborhood. If $p \in V \subset U$ are open and $V \hookrightarrow U$ is null-homotopic, then

$$H_2(V;\mathbb{Z}) \longrightarrow H_2(U;\mathbb{Z}) \longrightarrow H_2(Y;\mathbb{Z})$$

is zero, contrary to that assertion for V. Thus this point is not locally contractible, and no open neighborhood of it is homeomorphic to an open topological cone over a compact space. No relative local-homology formula from [14] is used. The examples of Corollary 1.2 concern the distinct question of the least possible Hausdorff dimension of an arbitrary exceptional set.

1.5. **Construction and dependencies.** The two compact manifolds with boundary used in the construction are

$$P_4 = \mathbb{CP}^2 \setminus \operatorname{Int} D^4, \qquad P_5 = (S^2 \times S^3) \setminus \operatorname{Int} D^5.$$

The positive-Ricci metric on P_4 with round strictly convex boundary goes back to Perelman; we use the formulation in [3, Theorem C and Section 3.3]. For P_5, we use [15, Theorem 5.1 and Lemma 2.18]. The normalization and the homology isomorphism induced by boundary collapse are proved in Lemma 2.1.

On $S^n = S^{n-2} * S^1$, with $n = 4, 5$, the circle regularization has a parameter depending on S^{n-2} and vanishing on F. Proposition 5.2 uses the inequality

$$A \geq M\left(|\operatorname{Hess}\delta| + \frac{|d\delta|^2}{\delta}\right)$$

to control the spatial and mixed Ricci terms. Its constants do not depend on the positive minimum of δ. This is the uniform estimate used for degeneration along F; parameter-dependent smoothing and control of mixed Ricci terms already occur in [11, Sections 1.3 and 1.5]. The geometric construction preceding the dimension calculation requires only that F be compact with empty interior. Choosing a positive-measure Cantor subset of $\mathbb{R}^{n-2}$ gives the lower measure bound in Corollary 1.2. The passage to all $N \geq 4$ uses products with flat tori.

The annuli, their boundary identifications, and the collar smoothing are developed in Sections 3, 6 and 8. The logarithmic interpolation uses Colding–Naber's Lemma 2.1 and the functions in equations (9)–(10) of [7, Section 2]. With $r_0 = e^{-T_0}$, their quantity $-\log(r_0 r)$ equals $T_0 - \log r$; the functions in Lemma 6.2 use this substitution and a slow reparametrization of the metric path. The common-volume-form normalization is Moser's construction [12]. The smoothing follows the Ricci gluing argument of Reiser–Wraith [16, Theorem A and Corollary B]. The collar interpolation, spherical deformation, and regularized-link calculation are adapted from [14, Sections 5.2, 6 and 7.1]. All statements needed from those sections are proved below. The main proofs do not invoke any theorem of that preprint.

For each domain W_i, the induced metric is independent of j for $j \geq i$. The limiting inclusions and boundary-collapse maps satisfy

$$f_i : W_i \longrightarrow X, \qquad \pi_i : X \longrightarrow W_i/\partial W_i, \qquad \pi_i f_i = q_i, \qquad \pi_i f_k \text{ constant for } k \neq i.$$

If c_i generates $H_2(W_i; \mathbb{Z})$ and $a_i = (f_i)_* c_i$, then

$$(\pi_i)_* a_i = (q_i)_* c_i \neq 0, \qquad (\pi_i)_* a_k = 0 \quad (k \neq i).$$

The nonzero class is a generator of an infinite cyclic group. Applying each $(\pi_i)_*$ to a finite relation proves independence of the a_i. The local isometry assertion of Lemma 2.5 and the perturbation lemma 2.8 then give the deletion statement in Proposition 2.9. No continuity of singular homology under Gromov–Hausdorff convergence is used.

Section 10 bounds the scalar integrals by boundary-flux identities, integration by parts, and summation of bounds of the form CR_i^{n-2} for the inserted regions. These estimates concern the constructed sequences. They are not a universal scalar-curvature bound under a lower Ricci bound; compare [13, Conjecture 2.18]. For Theorem 1.3, Section 12 proves convergence under every rescaling at the marked point, Section 13 bounds the replacement error by $O(r^3)$ on balls of radius r, and Section 14 identifies the smooth complement. Dividing that error by r gives the same tangent cone as the background.

1.6. **Geometric and topological conventions.** A space Y is locally contractible at y if, for every open neighborhood U of y, there is an open neighborhood V with $y \in V \subset U$ such that $V \hookrightarrow U$ is null-homotopic. We use the curvature convention

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \qquad \sec(X \wedge Y) = \langle R(X, Y)Y, X \rangle$$

for orthonormal X, Y. For functions, $\Delta_h = \operatorname{div}_h \nabla_h = \operatorname{tr}_h \operatorname{Hess}_h$. For a symmetric covariant two-tensor S, $(\operatorname{div}_h S)_i = \nabla^j S_{ji}$. A smooth function is flat on a closed set if all of its derivatives, including the function itself, vanish there.

All homology groups are singular homology groups with the indicated coefficients; reduced groups are denoted by $\widetilde{H}_r$. For compact boundary pairs, the relative-to-quotient identification is induced by the quotient map [8, Proposition 2.22, p. 124]. The symbols Int and ∂ denote the manifold interior and manifold boundary when applied to a smooth domain. All metric balls are open unless a closure is displayed. An intrinsic distance on a domain is the infimum of lengths of curves contained in that domain; an ambient distance is the restriction of the distance of the containing space.

The notation g_{S^m} denotes the unit round metric, $\sigma_m = \operatorname{Vol}(S^m, g_{S^m})$, and $\omega_N = \operatorname{Vol}(B_1(0) \subset \mathbb{R}^N)$; thus $\sigma_{N-1} = N\omega_N$. A closed manifold is compact and has empty boundary. Gromov–Hausdorff convergence of compact spaces is understood in the unpointed sense unless marked points are included.

For a boundary component with outward unit normal ν, set

$$\mathrm{II}(V, W) = g(\nabla_V \nu, W).$$

For $g = ds^2 + f(s)^2 g_{S^m}$ and a boundary slice with induced metric h, the outer and inner second fundamental forms are respectively $(f'/f)h$ and $-(f'/f)h$. Under $\widehat{g} = c^2 g$, with $c > 0$ constant,

$$d_{\widehat{g}} = c\, d_g, \qquad \operatorname{Vol}_{\widehat{g}} = c^{\dim M} \operatorname{Vol}_g, \qquad \operatorname{Ric}_{\widehat{g}} = \operatorname{Ric}_g, \qquad \mathrm{II}_{\widehat{g}} = c\, \mathrm{II}_g.$$

The Ricci identity is an identity of covariant two-tensors.

Inequalities between symmetric two-tensors mean pointwise quadratic-form inequalities. Norms of covariant tensors are the Hilbert–Schmidt norms induced by the specified metric. For a differential between tangent spaces, $\|dF\|_{\mathrm{op}}$ is the operator norm; it is this norm that bounds the Lipschitz constant of a smooth map. For functions on a fixed compact Riemannian manifold,

$$\|f\|_{C^2} = \max_{0\leq r\leq 2} \sup |\nabla^r f|.$$

A constant in a uniform estimate may depend on the data fixed before that estimate, but not on a positive lower bound for a regularization parameter. The order of the geometric parameter choices is fixed in Section 7. For [2], theorem and section numbers refer to the accepted manuscript identified in the bibliography; for [9, 10], they refer to the stated arXiv versions. References to [15, 16] use the versions specified there.

2. Quotient maps and second homology under metric convergence

In Sections 2–11, fix $n \in \{4, 5\}$ and a nonempty compact set $F \subset \{|z| < 1/40\} \subset \mathbb{R}^{n-2}$ with empty interior. The parameters may depend on this F, but not on the index j of the realizing sequence.

Lemma 2.1. *For $n = 4, 5$, the manifold $P = P_n$ has a smooth metric k, a boundary parametrization $\beta_0 : S^{n-1} \to \partial P$, and constants $\mu, \alpha > 0$ such that*

$$\beta_0^*(k|_{\partial P}) = g_{S^{n-1}}, \qquad \mathrm{Ric}_k \geq \mu k, \qquad \beta_0^*\mathrm{II}_{\partial P} \geq \alpha g_{S^{n-1}}.$$

The manifold P is connected. The quotient map induces

$$H_2(P;\mathbb{Z}) \cong \mathbb{Z}, \qquad H_2(P;\mathbb{Z}) \xrightarrow{\cong} \widetilde{H}_2(P/\partial P;\mathbb{Z}). \tag{2.1}$$

Boundary tensors below are written in this parametrization.

Proof. For $n = 4$, set $Q = \mathbb{CP}^2$. By [3, Theorem C], $Q \setminus \mathrm{Int}\, D^4$ has a positive-Ricci metric with round strictly convex boundary, with second fundamental form defined using the outward normal. For $n = 5$, set $Q = S^3 \times S^2$, the trivial linear S^2-bundle over S^3. The round S^3 contains an isometrically embedded unit hemisphere and hence has a core metric in the sense of [15, Definition 1.2]. Theorem 5.1 and Lemma 2.18 of that paper give an embedded disc $D^5 \subset Q$ and a positive-Ricci metric on $Q \setminus \mathrm{Int}\, D^5$ with round strictly convex boundary.

In either case let $\bar{k}$ be the resulting metric, and choose $R > 0$ and β_0 with $\beta_0^*(\bar{k}|_{\partial P}) = R^2 g_{S^{n-1}}$. Set $k = R^{-2}\bar{k}$. Constant scaling gives

$$\mathrm{Ric}_k = \mathrm{Ric}_{\bar{k}}, \qquad \mathrm{II}_k = R^{-1}\mathrm{II}_{\bar{k}}, \qquad \beta_0^*(k|_{\partial P}) = g_{S^{n-1}}.$$

The least eigenvalues of $k^{-1}\,\mathrm{Ric}_k$ on P and of $(g_{S^{n-1}})^{-1}\beta_0^*\mathrm{II}_k$ on S^{n-1} have positive minima, denoted by μ and α. The connected boundary lies in one component of P; any other component would be open and closed in the connected manifold Q. Thus P is connected.

A collar of ∂P in Q gives a neighborhood of P that deformation retracts onto P: in the added collar the normal coordinate is multiplied by $1 - t$, with $0 \leq t \leq 1$, and P is fixed. Thus (Q, P) is a good pair. Collapsing P identifies Q/P with D^n/S^{n-1}. The relative-to-quotient theorem [8, Proposition 2.22] gives

$$H_r(Q, P;\mathbb{Z}) \cong \widetilde{H}_r(D^n/S^{n-1};\mathbb{Z}) \cong \widetilde{H}_r(S^n;\mathbb{Z}) = 0 \quad (r = 2, 3).$$

The exact sequence of (Q, P) therefore identifies $H_2(P;\mathbb{Z})$ with $H_2(Q;\mathbb{Z})$. For $Q = \mathbb{CP}^2$, its CW decomposition has one cell in each of dimensions $0, 2, 4$; the cellular boundary maps entering and leaving degree two vanish. For $Q = S^3\times S^2$, the integral Künneth formula gives $H_2(Q;\mathbb{Z}) = H_0(S^3;\mathbb{Z})\otimes H_2(S^2;\mathbb{Z}) \cong \mathbb{Z}$. In the exact sequence

$$H_2(S^{n-1};\mathbb{Z}) \longrightarrow H_2(P;\mathbb{Z}) \longrightarrow H_2(P, \partial P;\mathbb{Z}) \longrightarrow H_1(S^{n-1};\mathbb{Z}),$$

both end groups vanish. The collar makes $(P, \partial P)$ a good pair. The quotient identification [8, Proposition 2.22] is induced by $P \to P/\partial P$ and proves (2.1). □

Lemma 2.2. *Let (W,h) be a compact connected Riemannian manifold with nonempty boundary, and let d_W be its intrinsic length distance. The quotient $Q_W = W/\partial W$ is a compact metric space for*

$$d_{Q_W}([x],[y]) = \min\{d_W(x,y),\ d_W(x,\partial W) + d_W(y,\partial W)\}. \tag{2.2}$$

Suppose W is a smooth domain in a connected Riemannian manifold M and the metric on W is h. The map

$$\pi_W : M \longrightarrow Q_W, \qquad \pi_W|_W = q_W, \quad \pi_W|_{M\setminus \mathrm{Int}\, W} = [\partial W]$$

is 1-Lipschitz for the ambient distance of M.

Proof. Put $A = \partial W$ and $r(x) = d_W(x,A)$. The function r is 1-Lipschitz. Define

$$D(x,y) = \min\{d_W(x,y), r(x) + r(y)\}.$$

If both minima in $D(x,y) + D(y,z)$ use intrinsic distances, the triangle inequality for d_W gives $D(x,z) \le D(x,y) + D(y,z)$. If both use boundary distances, then

$$D(x,z) \le r(x) + r(z) \le r(x) + 2r(y) + r(z).$$

If the first uses $d_W(x,y)$ and the second uses $r(y) + r(z)$, then

$$D(x,z) \le r(x) + r(z) \le d_W(x,y) + r(y) + r(z).$$

The remaining case follows by interchanging x, z. Hence D is a pseudometric. Its zero classes are the single boundary class and the individual interior points. If $x \in A$, then $D(x,y) = r(y)$, independent of the representative x. Thus D induces (2.2) on Q_W.

In an interior coordinate ball, or a boundary half-ball, choose nested convex coordinate neighborhoods with compact closures. On the larger one there are $m, M > 0$ such that $m|v|^2 \le h(v,v) \le M|v|^2$. A coordinate segment in the smaller neighborhood has length at most $\sqrt{M}$ times its Euclidean length. Every curve that leaves the larger neighborhood from a fixed point of the smaller one has length at least $\sqrt{m}$ times the positive coordinate distance to its relative frontier. These two inequalities show that d_W induces the manifold topology. Connectedness gives finite intrinsic distances, by joining a finite chain of such neighborhoods. Since $D \le d_W$, the quotient map $q_W : W \to Q_W$ is continuous. It factors as a continuous bijection from the compact topological quotient onto the metric quotient. That bijection is a homeomorphism; in particular Q_W is compact.

Let γ be a piecewise smooth curve in M with endpoints x, y. If $x, y \in W$ and γ stays in W, then $L(\gamma) \ge d_W(x,y) \ge D(x,y)$. If $x, y \in W$ and γ leaves W, its initial segment up to the first exit and its final segment after the last entrance give

$$L(\gamma) \ge r(x) + r(y) \ge D(x,y).$$

For $x \in W$ and $y \notin \mathrm{Int}\, W$, the first boundary encounter gives $L(\gamma) \ge r(x)$. If both endpoints are outside $\mathrm{Int}\, W$, their images under π_W agree. Taking the infimum over curves proves

$$d_{Q_W}(\pi_W(x), \pi_W(y)) \le d_M(x,y).$$

□

Proposition 2.3. *Let compact connected Riemannian manifolds (M_j, g_j) converge in Gromov–Hausdorff distance to a compact metric space X. Suppose that for each i a fixed compact connected Riemannian manifold (W_i, h_i) with $\partial W_i \ne \emptyset$ occurs as a smooth domain in M_j for every $j \ge i$, with unchanged Riemannian metric. Suppose also that these domains are pairwise disjoint at every index j. Then, after a subsequence, there are continuous maps*

$$f_i : W_i \longrightarrow X, \qquad \pi_i : X \longrightarrow Q_{W_i}$$

satisfying

$$\pi_i \circ f_i = q_{W_i}, \qquad \pi_i \circ f_k \text{ is constant if } i \ne k. \tag{2.3}$$

The maps f_i and π_i are 1-Lipschitz when W_i has its intrinsic distance.

Proof. For each j, choose an admissible metric on $M_j \sqcup X$ for which $d_H(M_j, X) < \epsilon_j$, where $\epsilon_j \downarrow 0$; see [1, Section 7.3]. Amalgamate these metric spaces along their common copy of X. Write d_j for the chosen metric on $M_j \sqcup X$. For $u \in M_i$ and $v \in M_j$, $i \neq j$, define

$$D(u,v) = \inf_{x\in X}\big(d_i(u,x) + d_j(x,v)\big),$$

and retain d_j on $M_j \sqcup X$. For $x, y \in X$ and $v \in M_j$,

$$d_j(x,v) + d_j(v,y) \geq d_X(x,y).$$

Consequently, for $u \in M_i$ and $w \in M_k$, a sum through v is bounded below by

$$d_i(u,x) + d_X(x,y) + d_k(y,w) \geq d_i(u,y) + d_k(y,w).$$

Taking infima proves the triangle inequality when $i \neq k$; when $i = k$ the last expression is at least $d_i(u,w)$. If two adjacent points lie in the same $M_i \sqcup X$, use $d_i(u,v)+d_i(v,x) \geq d_i(u,x)$ before taking the infimum. This gives the remaining triangle inequalities, including points in X. Distinct components have positive distance because each M_j and X are disjoint compact subsets of $(M_j \sqcup X, d_j)$. Thus D is a metric extending every d_j. The completion of the amalgam is a compact metric space Z: for every $r > 0$, all sufficiently late M_j lie within $r/3$ of X, and a finite $r/3$-net in X, together with finite r-nets in the remaining finitely many M_j, is an r-net in the amalgam. Completeness then gives compactness. Write $e_{i,j} : W_i \to M_j \subset Z$ for the inclusions. Since

$$d_Z(e_{i,j}(u), e_{i,j}(v)) \leq d_{W_i}(u,v),$$

Arzelà–Ascoli and a countable diagonal subsequence give uniform limits $f_i : W_i \to Z$. Hausdorff convergence implies $f_i(W_i) \subset X$.

Let $\pi_{i,j} : M_j \to Q_{W_i}$ be the maps of Lemma 2.2, and let $\Gamma_{i,j} \subset Z \times Q_{W_i}$ be their graphs. For any compact metric space T, its nonempty compact subsets form a compact metric space for Hausdorff distance. To prove this, let $A_h \subset T$ be nonempty and compact. The functions $x \mapsto d_T(x, A_h)$ are 1-Lipschitz and bounded by diam T. Arzelà–Ascoli gives a subsequence converging uniformly to a function f. Choosing $a_h \in A_h$ and passing to a convergent subsequence proves that $A := f^{-1}(0)$ is nonempty. It is compact. For fixed $x \in T$, choose $a_h(x) \in A_h$ with $d_T(x, a_h(x)) = d_T(x, A_h)$. A convergent subsequence has limit $a(x) \in A$ and $d_T(x, a(x)) = f(x)$, so $d_T(x, A) \leq f(x)$. Conversely, $f(x) \leq d_T(x, a)$ for $a \in A$, by the Lipschitz inequality for f. Thus $f = d_T(\cdot, A)$. The identity

$$d_H(C,D) = \|d_T(\cdot, C) - d_T(\cdot, D)\|_{C^0(T)}$$

for nonempty compact $C, D \subset T$ follows by evaluating on $C \cup D$ and using the triangle inequality. Hence $A_h \to A$ in Hausdorff distance. Apply this result with $T = Z \times Q_{W_i}$, for each i, and take a diagonal subsequence:

$$\Gamma_{i,j} \longrightarrow \Gamma_i \quad \text{in Hausdorff distance, for every fixed } i.$$

For $x \in X$, choose $x_j \in M_j$ with $x_j \to x$. Compactness of Q_{W_i} gives a convergent subsequence of $\pi_{i,j}(x_j)$. Hence the first projection of Γ_i is X.

For $(x,u), (y,v) \in \Gamma_i$, Hausdorff convergence gives, along the same indices, graph points $(x_j, u_j), (y_j, v_j)$ tending to these two points. Therefore

$$d_{Q_{W_i}}(u,v) = \lim_j d_{Q_{W_i}}(u_j, v_j) \leq \lim_j d_Z(x_j, y_j) = d_X(x,y).$$

Taking $y = x$ proves single-valuedness. Thus Γ_i is the graph of a 1-Lipschitz map $\pi_i : X \to Q_{W_i}$.

For $j \geq \max\{i,k\}$ and $w \in W_k$, the graph $\Gamma_{i,j}$ contains

$$(e_{k,j}(w), \pi_{i,j}(e_{k,j}(w))) = \begin{cases} (e_{i,j}(w), q_{W_i}(w)), & k = i, \\ (e_{k,j}(w), *), & k \neq i. \end{cases}$$

Pass to the limit, using uniform convergence of $e_{k,j}$ and closedness of Γ_i. This gives (2.3). Passing to the limit in $d_Z(e_{i,j}(u), e_{i,j}(v)) \leq d_{W_i}(u,v)$ proves that f_i is 1-Lipschitz. □

Proposition 2.4. *In the situation of Proposition 2.3, suppose*

$$W_i \cong P_n, \qquad \partial W_i \cong S^{n-1}.$$

Suppose there are embeddings $\iota_j : F \to M_j$ with common bi-Lipschitz constants, points $z_i \in F$, and numbers $\eta_i \downarrow 0$ such that for every $z \in F$ there is a strictly increasing sequence (i_ℓ) with $z_{i_\ell} \to z$, and

$$\sup_{w \in W_i} d_{M_j}(w, \iota_j(z_i)) \le \eta_i \qquad (j \ge i). \tag{2.4}$$

Then a subsequence has a bi-Lipschitz limit embedding $\iota : F \to X$, and (1.8) *holds with $E = \emptyset$ at every $x \in \iota(F)$.*

Proof. In the compact realization used for Proposition 2.3, Arzelà–Ascoli gives a uniform subsequential limit $\iota : F \to X$. The two bi-Lipschitz inequalities pass to the limit, so ι is an embedding.

Let $q_i : W_i \to Q_{W_i}$ be the quotient map. By Lemma 2.1 and invariance under a diffeomorphism of boundary pairs,

$$(q_i)_* : H_2(W_i; \mathbb{Z}) \xrightarrow{\cong} \widetilde{H}_2(Q_{W_i}; \mathbb{Z}) \cong \mathbb{Z}.$$

Choose a generator $c_i \in H_2(W_i; \mathbb{Z})$, and put $a_i = (f_i)_* c_i$. Functoriality and (2.3) give

$$(\pi_i)_* a_i = (q_i)_* c_i, \qquad (\pi_i)_* a_k = 0 \quad (k \ne i).$$

If $\sum_{i \in I} n_i a_i = 0$ for a finite index set I, application of $(\pi_k)_*$ gives $n_k (q_k)_* c_k = 0$ in an infinite cyclic group. Thus $n_k = 0$ for every $k \in I$. The same homomorphisms also prove independence after tensoring with $\mathbb{Q}$. Indeed, if $r_i \in \mathbb{Q}$ and $\sum_{i \in I} r_i (a_i \otimes 1) = 0$, then application of $(\pi_k)_* \otimes \mathrm{Id}_{\mathbb{Q}}$ gives

$$r_k\big((q_k)_* c_k \otimes 1\big) = 0 \quad \text{in } \widetilde{H}_2(Q_{W_k}; \mathbb{Z}) \otimes \mathbb{Q} \cong \mathbb{Q}. \tag{2.5}$$

The displayed generator is nonzero, so every r_k is zero. This calculation applies also to any subgroup of $H_2(X; \mathbb{Z})$ containing the classes under consideration.

Fix $z \in F$, put $x = \iota(z)$, and let $U \subset X$ be an open neighborhood of x. Choose $r > 0$ with $B_r(x) \subset U$. There is a strictly increasing sequence i_ℓ with $z_{i_\ell} \to z$. Passing (2.4) to the limit yields

$$\sup_{w \in W_{i_\ell}} d_X(f_{i_\ell}(w), x) \le \eta_{i_\ell} + d_X(\iota(z_{i_\ell}), \iota(z)) \longrightarrow 0.$$

For all large ℓ, f_{i_ℓ} factors through U. Hence the image of $H_2(U; \mathbb{Z}) \to H_2(X; \mathbb{Z})$ contains the independent classes a_{i_ℓ} and has infinite rank.

If $V \subset U \subset X$ are open neighborhoods of x and $V \hookrightarrow U$ is null-homotopic, then the induced map

$$H_2(V; \mathbb{Z}) \longrightarrow H_2(U; \mathbb{Z}) \longrightarrow H_2(X; \mathbb{Z})$$

is zero, contrary to the infinite-rank assertion for V. Thus local contractibility fails at x. A manifold chart would contain a neighborhood homeomorphic to a Euclidean ball, whose second homology is zero. Therefore $x \in S_{\mathrm{top}}(X)$. □

Lemma 2.5. *In Proposition 2.3, each f_i is injective on* $\operatorname{Int} W_i$ *and is an isometry on a neighborhood of each point of* $\operatorname{Int} W_i$*, with the restricted ambient distances. If $A \subset X$ is arbitrary, then*

$$\dim_{\mathcal{H}}\big(f_i^{-1}(A) \cap \operatorname{Int} W_i\big) \le \dim_{\mathcal{H}} A. \tag{2.6}$$

Proof. By (2.3) and the Lipschitz bounds,

$$\min\{d_{W_i}(u,v), d_{W_i}(u, \partial W_i) + d_{W_i}(v, \partial W_i)\} \le d_X(f_i(u), f_i(v)) \le d_{W_i}(u,v).$$

For distinct interior points the minimum is positive. Fix $u_0 \in \operatorname{Int} W_i$ and put $r_0 = d_{W_i}(u_0, \partial W_i) > 0$. If $u, v \in B^{W_i}_{r_0/4}(u_0)$, then $d_{W_i}(u,v) < r_0/2$, whereas the sum of the boundary distances is greater than $3r_0/2$. The two bounds therefore agree. A countable family of these balls covers $\operatorname{Int} W_i$. On each ball the inverse of f_i on its image is an isometry; no openness of this image in X is asserted or needed here. For the countable cover (B_a) just chosen,

$$f_i^{-1}(A) \cap \operatorname{Int} W_i = \bigcup_a (f_i|_{B_a})^{-1}\big(A \cap f_i(B_a)\big).$$

Each inverse is 1-Lipschitz. Applying the covering definition of Hausdorff dimension on each set and then countable-union stability proves (2.6). ☐

Lemma 2.6. *For $n = 4, 5$, every generator of $H_2(P_n; \mathbb{Z})$ is represented by a smooth map $\sigma : S^2 \to \operatorname{Int} P_n$.*

Proof. Write $Q = \mathbb{CP}^2$ or $S^3 \times S^2$ and $P = Q \setminus \operatorname{Int} D^n$. The CW decomposition of $\mathbb{CP}^2$ has one zero-cell and no one-cells; its two-cell is attached to the zero-cell, and attaching the four-cell does not change the fundamental group. Thus $\pi_1(\mathbb{CP}^2) = 0$. Both factors of $S^3 \times S^2$ are simply connected, so $\pi_1(S^3 \times S^2) = 0$. Use a bicollar of ∂D^n to cover Q by an open neighborhood of P and an open ball. The first retracts onto P, the second is contractible, and their intersection retracts onto S^{n-1}. Since $n - 1 \geq 3$, the intersection is simply connected. The van Kampen theorem gives $\pi_1(P) = \pi_1(Q) = 0$.

Choose a collar $\partial P \times [0, a)$ and $0 < 3\epsilon < a$. The formula $s \mapsto s + t\epsilon\zeta(s)$ on this collar, extended by the identity, defines a homotopy, where $\zeta = 1$ near zero, $\zeta = 0$ for $s \geq 2\epsilon$, and $0 \leq \zeta \leq 1$. Its terminal map takes P into $\operatorname{Int} P$, and its restriction to $\operatorname{Int} P$ stays in $\operatorname{Int} P$. Hence $\operatorname{Int} P \hookrightarrow P$ is a homotopy equivalence. The degree-two Hurewicz theorem [8, Theorem 4.32] gives $\pi_2(\operatorname{Int} P) \cong H_2(\operatorname{Int} P; \mathbb{Z}) \cong \mathbb{Z}$. A chosen generator is therefore the image of $[S^2]$ under a continuous map $\sigma_0 : S^2 \to \operatorname{Int} P$.

Cover S^2 by finitely many compact sets K_b whose interiors cover S^2, with $K_b \subset O_b$ open and $\sigma_0(\overline{O}_b)$ contained in a coordinate ball of $\operatorname{Int} P$. Shrink the O_b so that their target coordinate images lie in smaller convex balls. Choose smooth χ_b supported in O_b, equal to one near K_b. There is a positive uniform tolerance for perturbing σ_0 while retaining all these coordinate containments, because the cover is finite and the relevant images are compact. At step b, write the current map in the target coordinates on O_b. On $\operatorname{supp} \chi_b$ approximate its coordinate functions uniformly by smooth functions. Such approximations are obtained by multiplying by a second cutoff supported in O_b, extending the resulting continuous functions by zero on S^2, extending radially to an annulus in $\mathbb{R}^3$, and convolving with a smooth kernel of sufficiently small support. Replace the coordinate map v by $(1 - \chi_b)v + \chi_b v_{\mathrm{sm}}$. Convex interpolation gives a homotopy in the target ball. The new map is smooth near K_b and remains smooth wherever the preceding map was smooth. Choose the finitely many errors with sum smaller than the uniform tolerance. After the final step the map is smooth on S^2, remains in $\operatorname{Int} P$, and is homotopic to σ_0. Its homology class is unchanged. ☐

Lemma 2.7. *If $A \subset \mathbb{R}^a$ is arbitrary and $b \geq 0$ is an integer, then* $\dim_{\mathcal{H}}(A \times \mathbb{R}^b) \leq \dim_{\mathcal{H}} A + b$.

Proof. If $A = \emptyset$, the assertion holds because the product is empty. Suppose $A \neq \emptyset$. Restrict both factors to bounded boxes, and take $s > \dim_{\mathcal{H}} A$. For every $\epsilon > 0$, cover the first factor by balls of radii $0 < r_\ell < \epsilon$ with $\sum_\ell r_\ell^s < \epsilon$. For each ℓ, the bounded second box is covered by at most $C_b r_\ell^{-b}$ cubes of diameter at most $C_b r_\ell$. The resulting product cover has diameters at most $C_b' r_\ell$ and the sum of their $(s + b)$-th powers is at most $C_{b,s}\epsilon$. Thus its $(s + b)$-dimensional Hausdorff outer measure is zero. Let $s \downarrow \dim_{\mathcal{H}} A$ and exhaust both factors by countably many bounded boxes. If $\dim_{\mathcal{H}} A = a$, the ambient dimension bound gives the same result. ☐

Lemma 2.8. *Let V be a smooth Riemannian d-manifold without boundary, $A \subset V$ an arbitrary subset with $\dim_{\mathcal{H}} A < d - 2$, and $\sigma : S^2 \to V$ a smooth map. For every open neighborhood Ω of $\sigma(S^2)$ there is a smooth isotopy $\Phi_s : V \to V$, $0 \leq s \leq 1$, equal to the identity outside a compact subset of Ω, with $\Phi_0 = \mathrm{Id}$ and $\Phi_1(\sigma(S^2)) \cap A = \emptyset$. The isotopy can be chosen arbitrarily close to the identity on compact sets.*

Proof. If $A = \emptyset$, take $\Phi_s = \mathrm{Id}$ for every s. Suppose $A \neq \emptyset$. Then $\dim_{\mathcal{H}} A \geq 0$, and the hypothesis implies $d \geq 3$. Take finitely many relatively compact coordinate balls in Ω and smooth cutoffs whose value is one on smaller balls covering $\sigma(S^2)$. Multiply every coordinate vector field by the corresponding cutoff and extend by zero. The resulting fields $Y_1, \dots, Y_L$ have compact support in Ω, and their values span TV on a neighborhood of $\sigma(S^2)$. Their flows ϕ_t^ℓ exist for all real times: outside a fixed compact set each field vanishes, and on that compact set its smooth coefficients are bounded in finitely many charts. For $t = (t_1, \dots, t_L)$ in a small open cube B about zero, define

$$\Phi_t = \phi_{t_L}^L \circ \cdots \circ \phi_{t_1}^1, \qquad \mathcal{E} : S^2 \times B \to V, \quad \mathcal{E}(z,t) = \Phi_t(\sigma(z)).$$

At $t = 0$, the parameter differential is

$$D_t\mathcal{E}(z,0)(v) = \sum_{\ell=1}^{L} v_\ell Y_\ell(\sigma(z)).$$

It is surjective. Equip the domain with its Euclidean metric and the target with the fixed Riemannian metric. The least eigenvalue of $D_t\mathcal{E}(z,0)D_t\mathcal{E}(z,0)^*$ is a positive continuous function of $z \in S^2$; let $c > 0$ be its minimum. Smooth dependence and compactness give a cube $B = (-a,a)^L$ on which that eigenvalue is at least $c/2$. Thus $\mathcal{E}$ is a submersion on $S^2 \times B$.

Submersion coordinates identify $\mathcal{E}$ locally with the projection $\mathbb{R}^d \times \mathbb{R}^{L+2-d} \to \mathbb{R}^d$. On relatively compact coordinate boxes, all coordinate changes are bi-Lipschitz, after further subdivision if necessary. A countable collection of such boxes covers $S^2 \times B$. By Lemma 2.7,

$$\dim_{\mathcal{H}} \mathcal{E}^{-1}(A) \le \dim_{\mathcal{H}} A + L + 2 - d < L. \tag{2.7}$$

Projection to $B \subset \mathbb{R}^L$ is Lipschitz. The set of parameters t for which $\Phi_t(\sigma(S^2))$ meets A consequently has Hausdorff dimension less than L. It cannot contain the open cube B, whose Hausdorff dimension is L. Choose $t \in B$ outside this set and define $\Phi_s = \Phi_{st}$ for $0 \le s \le 1$. Then $\Phi_0 = \mathrm{Id}$ and $\Phi_1(\sigma(S^2)) \cap A = \emptyset$. If $K = \bigcup_{\ell=1}^L \operatorname{supp} Y_\ell \Subset \Omega$, every factor, and hence Φ_s, fixes $V \setminus K$. Both Φ_s and its inverse fix $V \setminus \Omega$; injectivity in both directions gives $\Phi_s(\Omega) = \Omega$. For each compact $K' \subset V$, each finite derivative order, and each prescribed positive error, smooth dependence on t makes the corresponding difference from Id smaller than that error on K', after decreasing a. The set $\mathrm{pr}_B(\mathcal{E}^{-1}(A))$ still has dimension less than L in every such cube. All dimension estimates use Hausdorff outer covers and require no measurability of A. □

Proposition 2.9. *Assume the hypotheses of Proposition 2.4, with $n = 4$ or $n = 5$. Let T^k be a flat torus, with T^0 a point, and put $Z = \iota(F) \times T^k$. If $E \subset X \times T^k$ has $\dim_{\mathcal{H}} E < n + k - 2$, then for every $(x,y) \in Z \setminus E$ and every relative open neighborhood U of (x,y) in $(X \times T^k) \setminus E$,*

$$\operatorname{rank} \operatorname{im}\big[H_2(U;\mathbb{Z}) \longrightarrow H_2(X \times T^k;\mathbb{Z})\big] = \infty.$$

Proof. Write $U = O \cap ((X \times T^k) \setminus E)$ with O open in $X \times T^k$. Choose open neighborhoods $O_x \subset X$ and $B \subset T^k$ of x, y with $O_x \times B \subset O$. For $k > 0$ take B smaller than the injectivity radius of the flat torus; for $k = 0$ take $B = T^0$. By (2.4) and its limit, there are infinitely many indices i with $f_i(W_i) \subset O_x$. Fix these indices. The map

$$F_i = f_i \times \mathrm{Id} : \operatorname{Int} W_i \times B \longrightarrow X \times T^k$$

restricts to an isometry on sufficiently small neighborhoods: apply Lemma 2.5 and the product-distance formula. A countable cover by such neighborhoods gives

$$\dim_{\mathcal{H}} A_i < n + k - 2, \qquad A_i := F_i^{-1}(E).$$

By Lemma 2.6, the generator c_i used in Proposition 2.4 is represented by a smooth map $\sigma_i : S^2 \to \operatorname{Int} W_i$. Apply Lemma 2.8 in the smooth $(n+k)$-manifold $\operatorname{Int} W_i \times B$ to $z \mapsto (\sigma_i(z), y)$. The resulting homotopic map σ_i' avoids A_i. Hence $F_i\sigma_i'$ maps S^2 into U. Let $b_i = (F_i\sigma_i')_*[S^2] \in H_2(X \times T^k;\mathbb{Z})$. Homotopy invariance and projection onto X give

$$(\mathrm{pr}_X)_* b_i = (f_i\sigma_i)_*[S^2] = a_i.$$

The a_i are independent by Proposition 2.4; applying this projection to any finite relation proves that the b_i are independent. Each b_i comes from $H_2(U;\mathbb{Z})$. □

3. Gluing along isometric boundaries with a lower Ricci bound

Lemma 3.1. *Let $n \geq 2$, and let two compact Riemannian n-manifolds have isometric boundary components (Σ, h) and Ricci tensors strictly greater than λ times their metrics near those components. Suppose*

$$\mathrm{II}_- + \mathrm{II}_+ > 0.$$

For every $\lambda' < \lambda$ and every sufficiently small prescribed neighborhood of the identified boundary, there is a smooth glued metric with $\mathrm{Ric} > \lambda' g$ on the modified region, equal to the given metrics outside that neighborhood. The modified collar may be required to be 2-bi-Lipschitz to $ds^2 + h$. Identical parametrized collar metrics admit identical choices of the smoothing.

Proof. In signed unit-normal collars write the two metrics as $ds^2 + H_-(s)$, $s \leq 0$, and $ds^2 + H_+(s)$, $s \geq 0$. Then

$$H_-(0) = H_+(0) = h, \qquad J := H_-'(0) - H_+'(0) = 2(\mathrm{II}_- + \mathrm{II}_+) > 0.$$

Fix $\lambda' < \lambda'' < \lambda$. For $d > 0$ put $A_- = H_-(-d)$, $A_+ = H_+(d)$ and $K_\pm = H_\pm'(\pm d)$. On $[-d, d]$ set

$$\begin{aligned} H_d(s) = \frac{d-s}{2d}A_- + \frac{d+s}{2d}A_+ + \frac{(s-d)^2(s+d)}{4d^2}\left(K_- - \frac{A_+ - A_-}{2d}\right) \\ + \frac{(s+d)^2(s-d)}{4d^2}\left(K_+ - \frac{A_+ - A_-}{2d}\right). \end{aligned} \tag{3.1}$$

Thus $H_d(\pm d) = A_\pm$ and $H_d'(\pm d) = K_\pm$. Taylor expansion of the four endpoint tensors, in fixed tangential charts, gives

$$H_d(s) = h - \frac{d}{4}J + \frac{s}{2}(H_-'(0) + H_+'(0)) - \frac{s^2}{4d}J + R_d(s),$$

$$\sup_{|s|\leq d} \|\partial_s^r R_d(s)\|_{C^2(\Sigma)} \leq Cd^{2-r} \quad (0 \leq r \leq 2).$$

Consequently

$$H_d = h + O_{C^2}(d), \qquad H_d' = O_{C^2}(1), \qquad H_d'' = -J/(2d) + O_{C^2}(1).$$

For $g = ds^2 + H(s)$, contraction of the Christoffel identities

$$\Gamma^s_{ij} = -\tfrac{1}{2}H'_{ij}, \qquad \Gamma^i_{sj} = \tfrac{1}{2}H^{ik}H'_{kj}, \qquad \Gamma^k_{ij} = \Gamma(H)^k_{ij}$$

gives

$$\begin{aligned} \mathrm{Ric}_{ss} &= -\tfrac{1}{2}\operatorname{tr}_H H'' + \tfrac{1}{4}\operatorname{tr}(H^{-1}H'H^{-1}H'), \\ \mathrm{Ric}_{si} &= \tfrac{1}{2}(\nabla^j H'_{ij} - \nabla_i \operatorname{tr}_H H'), \\ \mathrm{Ric}_{ij} &= (\mathrm{Ric}_H)_{ij} - \tfrac{1}{2}H''_{ij} - \tfrac{1}{4}(\operatorname{tr}_H H')H'_{ij} + \tfrac{1}{2}(H'H^{-1}H')_{ij}. \end{aligned} \tag{3.2}$$

All inverses of H_d are uniformly bounded for small d. Therefore

$$\mathrm{Ric}_{ss} = \frac{\operatorname{tr}_h J}{4d} + O(1), \quad \mathrm{Ric}_{si} = O(1), \quad \mathrm{Ric}_{ij} = \frac{J_{ij}}{4d} + O(1).$$

Let $j_0 > 0$ be the minimum on Σ of the least eigenvalue of $h^{-1}J$. Since $n \geq 2$,

$$J \geq j_0 h, \qquad \operatorname{tr}_h J \geq (n-1)j_0 \geq j_0.$$

The three remainder tensors have uniformly bounded operator norm with respect to $ds^2 + h$. Thus, for some $C > 0$ independent of sufficiently small d,

$$\mathrm{Ric}_{ds^2+H_d} \geq \left(\frac{j_0}{4d} - C\right)(ds^2 + h).$$

Put $c = j_0/4$. Choose d so small that $(c/d - C)(ds^2 + h) > \lambda''(ds^2 + H_d)$ and that the normal collar lies in the prescribed neighborhood. The unchanged portions also satisfy the strict λ'' inequality. Compactness gives a positive margin on each one-sided closure at $\pm d$.

Extend H_d by H_- for $s < -d$ and by H_+ for $s > d$. The first derivative of the extended tensor H_d is continuous. Fix

$$\varrho \in C_c^\infty((-1,1)), \qquad \varrho \geq 0, \qquad \varrho(-s) = \varrho(s), \qquad \int_{\mathbb{R}} \varrho(s)\, ds = 1,$$

and put $\varrho_\sigma(s) = \sigma^{-1}\varrho(s/\sigma)$ for $\sigma > 0$. Distributional differentiation gives

$$\partial_s^2(\varrho_{\nu^2} * H_d) = \varrho_{\nu^2} * (\partial_s^2 H_d).$$

Keep d fixed and take $0 < \nu < \min\{d/8, 1/4\}$. Use $\varrho_{\nu^2} * H_d$ on $|s \mp d| \leq \nu/2$, and multiply $(\varrho_{\nu^2} * H_d) - H_d$ by a cutoff supported in $|s \mp d| < \nu$. On each interval $|s \mp d| \leq \nu/2$, write $B_\nu = \varrho_{\nu^2} * H_d$ and $P_0 = ds^2 + h$. For fixed d, the one-sided second normal derivatives are bounded. Equation (3.2) has the form

$$\mathrm{Ric}_{ds^2+H} = \mathcal{L}(H)[H''] + \mathcal{R}(H, H', \nabla_x H, \nabla_x H', \nabla_x^2 H),$$

where $\mathcal{L}$ and $\mathcal{R}$ are smooth on the bounded set of positive metric jets under consideration. All entries of $\mathcal{R}$ are continuous across each join and converge uniformly under convolution. The identity for B_ν'' therefore gives

$$\left\| \mathrm{Ric}_{ds^2+B_\nu}(s) - \int \varrho_{\nu^2}(u)\, \mathrm{Ric}_{ds^2+H_d}(s-u)\, du \right\|_{P_0} \longrightarrow 0.$$

For the $\mathcal{L}$ term, this follows by bounding

$$\sup_{|u| \leq \nu^2} \|\mathcal{L}(B_\nu(s)) - \mathcal{L}(H_d(s-u))\| \cdot \sup \|H_d''\|;$$

the first factor tends to zero and the second is finite for fixed d. Choose $m_d > 0$ such that, on the two one-sided collars,

$$\mathrm{Ric}_{ds^2+H_d} - \lambda''(ds^2 + H_d) \geq m_d P_0.$$

Nonnegativity of the kernel and its unit integral give

$$\int \varrho_{\nu^2}(u) \big[\mathrm{Ric}_{ds^2+H_d} - \lambda''(ds^2 + H_d)\big](s-u)\, du \geq m_d P_0.$$

The convolution of $ds^2 + H_d$ is $ds^2 + B_\nu$. Taking the commutator error smaller than $m_d/2$ yields

$$\mathrm{Ric}_{ds^2+B_\nu} - \lambda''(ds^2 + B_\nu) \geq (m_d/2) P_0$$

for $|s \mp d| \leq \nu/2$. On the cutoff transition intervals $\nu/2 < |s \mp d| < \nu$, the integration interval $[s - \nu^2, s + \nu^2]$ lies on one smooth side of the interface, since $\nu < 1/4$. Evenness of the kernel and Taylor's formula give

$$\|g_{d,\nu} - (ds^2 + H_d)\|_{C^2} \leq C_d \nu^4 (1 + \nu^{-1} + \nu^{-2}) \longrightarrow 0.$$

Thus $\mathrm{Ric}_{g_{d,\nu}} > \lambda' g_{d,\nu}$ on the entire modified collar. The same choices give

$$\tfrac{1}{4}(ds^2 + h) \leq g_{d,\nu} \leq 4(ds^2 + h).$$

Outside the collar the metric is unchanged. For repeated parametrized collar data, retain d, ν, the kernels, and the cutoffs; the resulting tensor and signed-collar smooth structure then agree identically. This proves the required relative assertion. The same local smoothing is used in [16, Theorem A and Corollary B]. $\square$

Lemma 3.2. *In Lemma 3.1, put $J = 2(\mathrm{II}_- + \mathrm{II}_+)$ in the common boundary parametrization. For every $\epsilon > 0$, the smoothing can also be chosen so that, on its modified collar C,*

$$\int_C \mathrm{Scal}_g \, dv_g \le \int_\Sigma \mathrm{tr}_h J \, dv_h + \epsilon. \tag{3.3}$$

This choice is compatible with every support, comparison, and Ricci inequality in Lemma 3.1.

Proof. Tracing (3.2) gives, for $g = ds^2 + H(s)$,

$$\mathrm{Scal}_g = \mathrm{Scal}_H - \mathrm{tr}_H H'' + \tfrac{3}{4}\,\mathrm{tr}((H^{-1}H')^2) - \tfrac{1}{4}(\mathrm{tr}_H H')^2. \tag{3.4}$$

For the interpolation tensor in (3.1), uniformly on $|s| \le d$,

$$H_d = h + O_{C^2}(d), \quad H_d' = O_{C^2}(1), \quad H_d'' = -J/(2d) + O_{C^2}(1), \quad dv_{H_d} = (1 + O(d))dv_h.$$

The constants are allowed to depend on this fixed pair of collars. All terms in (3.4) except $-\mathrm{tr}_{H_d} H_d''$ are bounded independently of d. Thus

$$\int_{[-d,d]\times\Sigma} \mathrm{Scal}_{ds^2+H_d} \, dv_{ds^2+H_d} = \int_\Sigma \mathrm{tr}_h J \, dv_h + O(d). \tag{3.5}$$

Choose d small enough to make the error less than $\epsilon/2$ and to satisfy the requirements in Lemma 3.1.

Keep d fixed. The extended H_d is C^1, is smooth on each side of the two interfaces, and has bounded one-sided second normal derivatives. The mollification used in that lemma is supported on intervals of total length at most 4ν about the interfaces. On their inner halves the second normal derivative is a nonnegative-kernel average of these bounded derivatives; all lower and tangential derivatives remain bounded. On the cutoff transition intervals, the C^2 difference is $O_d(\nu^2)$, as proved there. The inverses and volume densities stay bounded on both kinds of interval. Equation (3.4) therefore bounds the absolute scalar integral over the affected intervals by $C_d\nu$, for both the interpolated and mollified tensors. The extra unchanged intervals outside $[-d, d]$ have scalar integral tending to zero as $\nu \to 0$. Take ν small enough that the total integral error is less than $\epsilon/2$, while retaining the strict Ricci and product-comparison inequalities. Equation (3.3) follows from (3.5). □

4. Regularization of the circle warping function

Fix $q \in (0, 1)$ and put $c = -\log q$. For $\delta > 0$ define

$$e_\delta = e^{-c/\delta}, \qquad b_\delta(t) = \sin t \left(1 + \frac{\sin^2 t}{e_\delta^2}\right)^{-\delta/2}. \tag{4.1}$$

Fix preliminary bounds $0 < \bar{t} < 1/10$ and $0 < \bar{\delta} < \min\{1/4, c/4\}$ with $\sin^2 \bar{t} + e_{\bar{\delta}}^2 < e^{-2}$. All estimates in this section hold on

$$\mathcal{D}_q = \{(t, \delta) : 0 < t \le \bar{t},\ 0 < \delta \le \bar{\delta}\}.$$

Every constant in this section is fixed on $\mathcal{D}_q$ before a smaller radial interval is selected. Put

$$Z = \frac{\sin^2 t}{\sin^2 t + e_\delta^2}, \quad J = 1 - Z, \quad L = 1 - \tfrac{1}{2}\log(\sin^2 t + e_\delta^2), \quad P = \log \frac{b_\delta}{\sin t}. \tag{4.2}$$

Then $L \ge 2$, $0 < Z < 1$, and $0 < J < 1$.

Lemma 4.1. *The identities*

$$\frac{b_t}{b} = (1 - \delta Z)\cot t, \tag{4.3}$$

$$-\frac{b_{tt}}{b} = 1 - \delta Z + \delta Z[3 - (2+\delta)Z]\cot^2 t, \tag{4.4}$$

$$P_\delta = -\tfrac{1}{2}\log(1 + \sin^2 t/e_\delta^2) + \frac{c}{\delta}Z, \tag{4.5}$$

$$P_{\delta\delta} = -\frac{2c^2}{\delta^3}ZJ, \tag{4.6}$$

$$P_{\delta t} = Z\left(-1 + \frac{2c}{\delta}J\right)\cot t \tag{4.7}$$

hold. Moreover,

$$0 \le P_\delta \le ZL, \qquad |P_{\delta t}| \le 3ZL\cot t, \qquad P_{\delta\delta} \le 0. \tag{4.8}$$

Proof. Since $\partial_t Z = 2ZJ\cot t$ and $\partial_\delta Z = -2c\delta^{-2}ZJ$, direct differentiation of (4.1) gives (4.3) and (4.5)–(4.7). Differentiating (4.3) and adding its square gives (4.4).

The relations $\log J = -2c/\delta + 2(L-1)$ and (4.5) give

$$\frac{P_\delta}{Z} = L - 1 + \frac{J}{2Z}\log J.$$

The function $x\log x - x + 1$ has derivative $\log x \le 0$ on $(0,1]$ and value zero at $x = 1$. Thus $J\log J \ge J - 1 = -Z$, whereas $J\log J \le 0$. It follows that $J\log J/(2Z) \in [-1/2, 0]$, and therefore $0 \le P_\delta \le ZL$. The derivative of $-x\log x$ vanishes only at $x = e^{-1}$; its sign changes there from positive to negative. Hence $-J\log J \le e^{-1}$. Consequently

$$\frac{c}{\delta}J = J(L-1) - \tfrac{1}{2}J\log J \le L - 1 + \frac{1}{2e}.$$

Substitution in (4.7), together with $L \ge 2$, proves the second bound in (4.8). The inequality $P_{\delta\delta} \le 0$ follows from (4.6). □

Lemma 4.2. *Define*

$$\Psi_\delta(t) = \int_0^t (t-u)L_\delta(u)^2\,du. \tag{4.9}$$

There is a constant C_q such that

$$tL_\delta(t)^2 \le \Psi'_\delta(t) \le C_q tL_\delta(t)^2, \qquad 0 \le \Psi_\delta(t) \le C_q t^2 L_\delta(t)^2. \tag{4.10}$$

The functions Ψ_δ, Ψ'_δ and their first two δ-derivatives are uniformly bounded on $\mathcal{D}_q$. Both suprema below are taken over $(t,\delta) \in \mathcal{D}_q$:

$$\sup_{\delta,t} L_\delta(t)|\partial_\delta\Psi_\delta(t)| < \infty, \qquad \sup_{\delta,t} t^2 L_\delta(t)^r < \infty \quad (r = 2,3,4). \tag{4.11}$$

Proof. The function L_δ decreases in t. For $0 < u \le t \le \bar{t}$,

$$0 \le L_\delta(u) - L_\delta(t) = \tfrac{1}{2}\log\frac{\sin^2 t + e_\delta^2}{\sin^2 u + e_\delta^2} \le \log 2 + \log(t/u),$$

where $u/2 \le \sin u$ and $\sin t \le t$ were used. Set $\ell = L_\delta(t) \ge 2$. Substitution $u = tv$ and integration give

$$\begin{aligned}\Psi'_\delta(t) &\le t\int_0^1 (\ell + \log 2 + \log(1/v))^2\,dv \\ &= t\big((\ell+\log 2)^2 + 2(\ell + \log 2) + 2\big) \le 4t\ell^2.\end{aligned}$$

Here $\int_0^1 \log(1/v)\,dv = 1$ and $\int_0^1 \log^2(1/v)\,dv = 2$; the last inequality follows from $\log 2 < 3/4$ and $\ell \geq 2$. Monotonicity gives $\Psi'_\delta(t) \geq t\ell^2$, and $0 \leq \Psi_\delta(t) \leq t\Psi'_\delta(t)$. This proves (4.10) with any $C_q \geq 4$.

Write $W = L^2$. One has

$$\partial_\delta L = -c\delta^{-2}J, \qquad \partial_\delta^2 L = 2c\delta^{-3}J - 2c^2\delta^{-4}ZJ.$$

Since $L \leq 1 + c/\delta$ and $\delta \leq 1$,

$$|W_\delta| \leq C_q\delta^{-3}J, \qquad |W_{\delta\delta}| \leq C_q\delta^{-5}J.$$

Furthermore,

$$\int_0^{\bar t} J\,du \leq \int_0^{\bar t} \frac{e_\delta^2}{u^2/4 + e_\delta^2}\,du \leq \pi e_\delta.$$

Thus, for $r = 1, 2$ and $N_1 = 3, N_2 = 5$,

$$|\partial_\delta^r \Psi'_\delta(t)| \leq C_q e^{-c/\delta}\delta^{-N_r}, \qquad |\partial_\delta^r \Psi_\delta(t)| \leq C_q t e^{-c/\delta}\delta^{-N_r}.$$

Every function $\delta^{-a}e^{-c/\delta}$, with fixed $a \geq 0$, is bounded on $(0, \bar\delta]$: substitution $v = c/\delta$ reduces this assertion to boundedness of $v^a e^{-v}$ on $[c/\bar\delta, \infty)$. Moreover, $L \leq 1 + c/\delta$ gives

$$L_\delta(t)|\partial_\delta \Psi_\delta(t)| \leq C_q t e^{-c/\delta}(\delta^{-3} + c\delta^{-4}) \leq C'_q. \tag{4.12}$$

Finally $L_\delta(t) \leq 1 + \log 2 + |\log t|$. For every fixed integer $r \geq 0$, $t^2(1 + \log 2 + |\log t|)^r$ is bounded on $(0, \bar t]$; the substitution $v = -\log t$ gives an exponentially decreasing factor times a polynomial. These estimates and (4.10) prove all the asserted bounds. □

5. Ricci bounds with variable regularization parameters

Let $\gamma = g_{S^{n-2}}$, and give $S^1 = \mathbb{R}/(2\pi\mathbb{Z})$ the coordinate θ. The map

$$(t, x, \theta) \longmapsto (\cos t\, x, \sin t \cos\theta, \sin t \sin\theta) \in S^n \subset \mathbb{R}^{n-1} \oplus \mathbb{R}^2$$

identifies $(0, \pi/2) \times S^{n-2} \times S^1$ with $S^n \setminus (\Sigma_0 \cup \Sigma_1)$, where

$$\Sigma_0 = S^n \cap (\mathbb{R}^{n-1} \oplus \{0\}) \cong S^{n-2}, \qquad \Sigma_1 = S^n \cap (\{0\} \oplus \mathbb{R}^2) \cong S^1.$$

For $(v, w) \in S^n$ with $v \neq 0$ and $w \neq 0$, the inverse is $t = \arcsin|w|$, $x = v/|v|$, and $(\cos\theta, \sin\theta) = w/|w|$. For $Y \in T_x S^{n-2}$, differentiation and $\langle x, Y\rangle = 0$ give the round norm

$$|T\partial_t + Y + \vartheta\partial_\theta|^2_{g_{S^n}} = T^2 + \cos^2 t\,|Y|^2_\gamma + \sin^2 t\,\vartheta^2.$$

At $t = 0$ the circle is collapsed; at $t = \pi/2$ the S^{n-2} factor is collapsed. We identify Σ_0 with S^{n-2}. Norms and covariant derivatives of functions on S^{n-2} in this section are taken with respect to γ. Choose $K \geq 1$, depending only on the preliminary domain $\mathcal{D}_q$, so that

$$tL^2 \leq \Psi_t \leq KtL^2, \qquad 0 \leq \Psi \leq Kt^2L^2, \tag{5.1}$$

$$|\Psi| + |\Psi_t| + |\partial_\delta\Psi| + |\partial_\delta\Psi_t| + |\partial_\delta^2\Psi| + L|\partial_\delta\Psi| \leq K, \tag{5.2}$$

$$t^2L^r \leq K \quad (r = 2, 3, 4). \tag{5.3}$$

The finiteness of K follows from Lemma 4.2. It is independent of the subsequently chosen functions δ, A and of every positive lower bound for δ.

Lemma 5.1. *Put $m = n - 2 \in \{2, 3\}$. Let $a, b > 0$ be smooth on $I \times S^m$, independent of θ, and set*

$$G = dt^2 + h(t) + b(t,x)^2 d\theta^2, \qquad h = a^2\gamma, \qquad k = a_t/a.$$

The Ricci components, with the circle direction normalized, are

$$D := \mathrm{Ric}_G(\partial_t, \partial_t) = -m a_{tt}/a - b_{tt}/b, \tag{5.4}$$
$$B_i := \mathrm{Ric}_G(\partial_t, \partial_i) = -(m-1)\partial_i k - (b_{ti} - k b_i)/b, \tag{5.5}$$
$$T := \mathrm{Ric}_G|_{TS^m \times TS^m} = \mathrm{Ric}_h - (a_{tt}/a + (m-1)k^2 + k b_t/b)h - b^{-1}\operatorname{Hess}_h b, \tag{5.6}$$
$$V := \mathrm{Ric}_G(b^{-1}\partial_\theta, b^{-1}\partial_\theta) = -b_{tt}/b - mkb_t/b - b^{-1}\Delta_h b. \tag{5.7}$$

The other components involving the circle direction vanish. For $v = \log a$*,*

$$\mathrm{Ric}_h = (m-1)\gamma - (m-2)(\operatorname{Hess}_\gamma v - dv \otimes dv) - (\Delta_\gamma v + (m-2)|dv|^2_\gamma)\gamma. \tag{5.8}$$

In particular, for $m = 2$ *and* $v = \log\cos t - U$, $\mathrm{Ric}_h = (1 + \Delta_\gamma U)\gamma$*.*

Proof. Write $\bar{G} = dt^2 + h$. The identities $h_t = 2kh$ and $h_{tt} = 2(k_t + 2k^2)h$ give

$$\tfrac{1}{2}h^{-1}h_t = k\,\mathrm{Id}, \qquad \mathrm{tr}_h\, h_t = 2mk.$$

Substitution in (3.2) yields

$$\mathrm{Ric}_{\bar{G},tt} = -m(k_t + k^2), \quad \mathrm{Ric}_{\bar{G},ti} = -(m-1)k_i, \quad \mathrm{Ric}_{\bar{G}}|_{TS^m} = \mathrm{Ric}_h - (k_t + mk^2)h.$$

The same Christoffel symbols give

$$(\operatorname{Hess}_{\bar{G}} b)_{tt} = b_{tt}, \quad (\operatorname{Hess}_{\bar{G}} b)_{ti} = b_{ti} - kb_i, \quad \operatorname{Hess}_{\bar{G}} b|_{TS^m} = \operatorname{Hess}_h b + kb_t h,$$

$$\Delta_{\bar{G}} b = b_{tt} + mkb_t + \Delta_h b.$$

For base indices α, λ, the additional symbols are $\Gamma^\theta_{\alpha\theta} = b^{-1}b_\alpha$ and $\Gamma^\alpha_{\theta\theta} = -b\bar{G}^{\alpha\lambda}b_\lambda$. Their contraction gives

$$\mathrm{Ric}_G|_{T\bar{G}} = \mathrm{Ric}_{\bar{G}} - b^{-1}\operatorname{Hess}_{\bar{G}} b, \qquad \mathrm{Ric}_G(\partial_\theta, \partial_\theta) = -b\Delta_{\bar{G}} b, \qquad \mathrm{Ric}_G(\partial_\theta, T\bar{G}) = 0.$$

These formulas prove (5.4)–(5.7). For $h = e^{2v}\gamma$, the difference of the two connections is $C^k_{ij} = \delta^k_i v_j + \delta^k_j v_i - \gamma_{ij}v^k$. Contraction gives

$$\nabla_k C^k_{ij} - \nabla_j C^k_{ik} = -(m-2)v_{ij} - (\Delta_\gamma v)\gamma_{ij},$$

$$C^k_{k\ell}C^\ell_{ij} - C^k_{j\ell}C^\ell_{ik} = (m-2)(v_i v_j - |dv|^2_\gamma \gamma_{ij}).$$

Adding $\mathrm{Ric}_\gamma = (m-1)\gamma$ proves (5.8). □

Proposition 5.2. *For fixed* $q \in (0,1)$ *there are* $M > 1$, $\eta, t_0, \delta_0 > 0$ *such that the following holds for* $m = n - 2 \in \{2, 3\}$*. Suppose* $\delta, A \in C^\infty(S^m)$ *satisfy*

$$0 < \delta \le \delta_0, \qquad A \ge M\left(|\operatorname{Hess}\delta| + \frac{|d\delta|^2}{\delta}\right), \qquad \|A\|_{C^2} + \|\delta\|_{C^2} \le \eta. \tag{5.9}$$

For $0 < t \le t_0$*, define*

$$U = A(x)\Psi_{\delta(x)}(t), \qquad a = \cos t\, e^{-U}, \qquad G = dt^2 + a^2\gamma + b_{\delta(x)}(t)^2 d\theta^2. \tag{5.10}$$

Then $\mathrm{Ric}_G \ge G$*. The constants do not depend on* $\min\delta$*.*

Proof. Use K from (5.1)–(5.3). Set

$$M = 9216, \quad C_E = 6(K^2 + K), \quad C_R = 40K + 8K^2, \quad \delta_0 = \bar{\delta}.$$

Choose $0 < t_0 < \bar{t}/4$ so that, for $0 < t \le t_0$,

$$6Kt\tan t \le \tfrac{1}{4}, \qquad (1 - \delta_0)\cot t \ge 6\tan t, \qquad 2(1 - \delta_0)t\cot t \ge 1. \tag{5.11}$$

Such a choice exists because $t\tan t \to 0$, $t\cot t \to 1$, and $\delta_0 \le 1/4$. Choose $\eta > 0$ such that

$$\begin{gathered}\eta \le 1, \quad K\eta \le \tfrac{1}{10}, \quad 3K^3\eta \le \tfrac14, \quad 2K^2\eta \le 1,\\ C_R\eta + C_E\eta^2 \le \tfrac14, \qquad 3C_E\eta^2 \le \tfrac14, \qquad 64K^2\eta^2 \le \tfrac14.\end{gathered} \tag{5.12}$$

There are finitely many inequalities and each left side tends to zero with η.

Abbreviate $L = L_{\delta(x)}(t)$, $Z = Z(t,\delta(x))$, $Q = AL^2$, $X = \delta Z\cot^2 t$, $H = |\operatorname{Hess}\delta|$, and $p_0 = |d\delta|$. Put $\Psi^{[r]} = \partial_\delta^r \Psi_\delta|_{\delta=\delta(x)}$ for $r = 0,1,2$. Since $0 \le U \le K\eta \le 1/10$ and $t < 1/10$,

$$\sec^2 t \le a^{-2} = e^{2U}\sec^2 t < 2. \tag{5.13}$$

The spatial chain rules are

$$\begin{aligned} dU &= \Psi\, dA + A\Psi^{[1]} d\delta,\\ \operatorname{Hess} U &= \Psi \operatorname{Hess} A + \Psi^{[1]}(dA\otimes d\delta + d\delta\otimes dA) + A\Psi^{[1]}\operatorname{Hess}\delta + A\Psi^{[2]} d\delta\otimes d\delta,\\ dU_t &= \Psi_t\, dA + A\Psi_t^{[1]} d\delta. \end{aligned}$$

Thus

$$|dU| \le 2K\eta, \qquad |\operatorname{Hess} U| \le 5K\eta, \qquad |dU_t| \le 2K\eta. \tag{5.14}$$

Also, since $0 \le P_\delta \le ZL$,

$$|dU| P_\delta p_0 \le K\eta^2 t^2 L^3 + \eta^3 L |\Psi^{[1]}| \le (K^2+K)\eta^2. \tag{5.15}$$

The spatial Hessian of $b = b_{\delta(x)}(t)$ satisfies

$$b^{-1}\operatorname{Hess}_\gamma b = P_\delta \operatorname{Hess}\delta + (P_{\delta\delta} + P_\delta^2) d\delta\otimes d\delta \le (ZLH + Z^2L^2p_0^2)\gamma.$$

The connection change $d\log a = -dU$ gives

$$b^{-1}\operatorname{Hess}_h b = b^{-1}\operatorname{Hess}_\gamma b + dU\otimes(b^{-1}db) + (b^{-1}db)\otimes dU - \langle dU, b^{-1}db\rangle_\gamma \gamma.$$

The last three terms have γ-operator norm at most $3|dU|P_\delta p_0$. Set

$$E = 2ZLH + 2Z^2L^2p_0^2 + C_E\eta^2.$$

Equations (5.13) and (5.15) give

$$b^{-1}\operatorname{Hess}_h b \le Eh, \qquad b^{-1}\Delta_h b \le mE, \qquad E \le \frac{2}{M}Q + C_E\eta^2. \tag{5.16}$$

For the last inequality, the hypotheses give

$$2ZLH \le \frac{2Q}{ML} \le \frac{Q}{M}, \qquad 2Z^2L^2p_0^2 \le \frac{2\delta Q}{M} \le \frac{Q}{2M}.$$

The radial derivatives are

$$k = -\tan t - U_t, \qquad -a_{tt}/a = 1 + Q - 2\tan t\, U_t - U_t^2, \qquad tQ \le U_t \le KtQ. \tag{5.17}$$

The choices above imply

$$2m\tan t\, U_t \le \tfrac14 Q, \qquad mU_t^2 \le 3K^2A(t^2L^2)Q \le \tfrac14 Q, \qquad U_t\tan t \le 2K^2\eta \le 1.$$

In particular $U_t \le \cot t$. Equation (4.4) gives $-b_{tt}/b \ge 1 - \delta + 3X/4$. Therefore

$$D \ge m + 1 - \delta + mQ + \tfrac34 X - 2m\tan t\, U_t - mU_t^2 \ge \tfrac52 + Q + \tfrac12 X. \tag{5.18}$$

From (5.8), with $v = \log\cos t - U$ and $m \le 3$, we have

$$\operatorname{Ric}_h \ge \big((m-1)a^{-2} - 8|\operatorname{Hess} U| - 2|dU|^2\big)h \ge \big((m-1)\sec^2 t - C_R\eta\big)h.$$

For the least h-eigenvalue λ_T of T, substitution gives

$$\begin{aligned}\lambda_T &\geq m+1-\delta Z+Q+[(1-\delta Z)\cot t-2m\tan t]U_t-mU_t^2-C_R\eta-E\\ &\geq 3-\delta_0-C_R\eta-C_E\eta^2+(\tfrac{3}{4}-\tfrac{2}{M})Q.\end{aligned}$$

Hence

$$\lambda_T \geq \tfrac{5}{2}+\tfrac{1}{2}Q. \tag{5.19}$$

For the normalized circle component,

$$\begin{aligned}V &= (m+1)(1-\delta Z)+\delta Z[3-(2+\delta)Z]\cot^2 t+mU_t(1-\delta Z)\cot t-b^{-1}\Delta_h b\\ &\geq 3(1-\delta_0)+\tfrac{3}{4}X+(1-\tfrac{6}{M})Q-3C_E\eta^2\geq 2.\end{aligned}$$

The Q term in this inequality follows from $mU_t(1-\delta Z)\cot t \geq 2(1-\delta_0)t\cot t\, Q \geq Q$.

The mixed covector is

$$B=(m-1)dU_t-[P_{\delta t}+((1-\delta Z)\cot t+\tan t+U_t)P_\delta]d\delta.$$

The bracket has absolute value at most $6ZL\cot t$. Since $m-1\leq 2$ and $a^{-2}<2$,

$$\begin{aligned}|B|_h^2 &\leq 16|dU_t|^2+144p_0^2Z^2L^2\cot^2 t\\ &\leq 64K^2\eta^2+\frac{144}{M}QX\leq \tfrac{1}{4}+\tfrac{1}{64}QX.\end{aligned} \tag{5.20}$$

Here $p_0^2\leq \delta A/M$ and $Z^2\leq Z$ were used. Equations (5.18), (5.19), and (5.20) give

$$\begin{aligned}(D-1)(\lambda_T-1)-|B|_h^2 &\geq (\tfrac{3}{2}+Q+\tfrac{1}{2}X)(\tfrac{3}{2}+\tfrac{1}{2}Q)-\tfrac{1}{4}-\tfrac{1}{64}QX\\ &= 2+\tfrac{9}{4}Q+\tfrac{1}{2}Q^2+\tfrac{3}{4}X+\tfrac{15}{64}QX>0.\end{aligned}$$

For $s\in\mathbb{R}$ and $Y\in TS^m$, completing the square gives

$$(D-1)\left(s+\frac{B(Y)}{D-1}\right)^2+(T-h)(Y,Y)-\frac{B(Y)^2}{D-1}.$$

The last two terms are at least $(\lambda_T-1-|B|_h^2/(D-1))|Y|_h^2$. This coefficient is positive. The circle component of $\mathrm{Ric}_G - G$ is at least one and has no mixed terms. Thus $\mathrm{Ric}_G\geq G$. ☐

Lemma 5.3. *After decreasing η and δ_0 in Proposition 5.2, while keeping M and t_0 fixed, every pair (δ, A) satisfying* (5.9) *with these decreased constants defines a smooth metric $G[\delta, A]$ on S^n with*

$$\mathrm{Ric}_{G[\delta,A]}\geq G[\delta,A],\qquad c_0^2G_q\leq G[\delta,A]\leq g_{S^n},\qquad c_0>9/10, \tag{5.21}$$

where

$$G_q=dt^2+\cos^2 t\,\gamma+q^2\sin^2 t\,d\theta^2$$

is interpreted on the regular join cylinder. Every metric $G[\delta,A]$ equals G_q for $t\geq 3t_0$ and induces γ on Σ_0. Two such metrics agree wherever their respective parameter pairs (δ, A) agree.

Proof. Choose fixed smooth cutoffs $0\leq \chi_a,\chi_b\leq 1$ with

$$\chi_a=1\ (t\leq t_0),\quad \chi_a=0\ (t\geq 2t_0),\qquad \chi_b=1\ (t\leq 2t_0),\quad \chi_b=0\ (t\geq 3t_0).$$

On the regular join cylinder define

$$\begin{aligned}U&=\chi_a A\Psi_{\delta(x)},\qquad a=\cos t\, e^{-U},\\ b&=\sin t\exp\{\chi_b P(t,\delta(x))+(1-\chi_b)\log q\},\\ G[\delta,A]&=dt^2+a^2\gamma+b^2d\theta^2.\end{aligned}$$

The interval $[0,3t_0]$ lies inside the preliminary kernel interval. On $t\leq t_0$, Proposition 5.2 applies.

On the fixed strip $\mathcal{S} = [t_0, 3t_0] \times S^{n-2} \times S^1$,

$$P + c = -\frac{\delta}{2} \log(\sin^2 t + e_\delta^2).$$

The function $P + c$ and its first two radial derivatives are bounded in absolute value by $C\delta$. The quantities P_δ, $P_{\delta t}$ and $P_{\delta\delta}$ are uniformly bounded on $\mathcal{S}$; this follows from (4.5)–(4.7) and the boundedness of $e^{-c/\delta}\delta^{-r}$ for every fixed r. The chain rules

$$d_x P = P_\delta d\delta, \qquad \operatorname{Hess}_x P = P_\delta \operatorname{Hess} \delta + P_{\delta\delta} d\delta \otimes d\delta, \qquad d_x P_t = P_{\delta t} d\delta$$

and Lemma 4.2 therefore give, in fixed finite coordinate atlases,

$$\|G[\delta, A] - G_q\|_{C^2(\mathcal{S})} \le C_{q,t_0,\chi_a,\chi_b}(\|\delta\|_{C^2} + \|A\|_{C^2}).$$

The constant C_{q,t_0,χ_a,χ_b} does not depend on $\min \delta$. The tensor G_q is positive definite on $\mathcal{S}$. Substitution of $a = \cos t$ and $b = q \sin t$ in (5.4)–(5.7) gives $\operatorname{Ric}_{G_q} = (n-1)G_q$. In the fixed finite charts, the inverse matrices of metrics sufficiently close to G_q are uniformly bounded. The coordinate Ricci formula is a smooth function of these inverses and of the first and second metric derivatives. On a fixed bounded neighborhood of the metric jets of G_q, its derivative is bounded. The mean-value inequality therefore bounds the change of Ricci by a constant times the preceding C^2 error. Choose η so that, pointwise on the strip, $|G[\delta, A] - G_q|_{G_q} \le 1/4$ and $|\operatorname{Ric}_{G[\delta,A]} - (n-1)G_q|_{G_q} \le 1$. Since $n - 1 \ge 3$, these choices give

$$\tfrac{3}{4}G_q \le G[\delta, A] \le \tfrac{5}{4}G_q, \qquad \operatorname{Ric}_{G[\delta,A]} \ge 2G_q \ge G[\delta, A] \quad \text{on } \mathcal{S}.$$

For $t \ge 3t_0$ equality with G_q is exact.

For each positive smooth $\delta(x)$, the function $\mathcal{W}(s, x) = (1 - \frac{1}{2} \log(\sin^2 \sqrt{s} + e_{\delta(x)}^2))^2$ is smooth for s near zero, since $\sin^2 \sqrt{s}$ is given there by a convergent power series. The identity

$$\Psi_{\delta(x)}(t) = t^2 \int_0^1 (1 - v)\mathcal{W}(t^2 v^2, x)\, dv$$

expresses the kernel as a smooth function of t^2 and x, with zero value and first t-derivative at zero. Near $t = 0$, a^2 is smooth and even, while b is smooth and odd with $b_t(0, x) = 1$. In normal Cartesian coordinates $y_1 = t \cos \theta$, $y_2 = t \sin \theta$, write

$$b(t, x) = t + t^3 B(t^2, x), \qquad a(t, x)^2 = A_0(t^2, x).$$

Then

$$dt^2 + b^2 d\theta^2 = dy_1^2 + dy_2^2 + [2B(t^2, x) + t^2 B(t^2, x)^2](y_1 dy_2 - y_2 dy_1)^2.$$

This tensor and $A_0(t^2, x)\gamma$ are smooth at $y = 0$. Also $A_0(0, x) = 1$, so the induced metric on Σ_0 is γ. At $t = \pi/2$, put $r = \pi/2 - t$; the metric is

$$dr^2 + \sin^2 r\, \gamma + q^2 \cos^2 r\, d\theta^2.$$

Set $y = rx \in \mathbb{R}^{n-1}$, with $|x| = 1$, and put $s(r) = \sin r/r$, extended by $s(0) = 1$. Then

$$dr^2 + \sin^2 r\, \gamma = s(r)^2 \sum_{a=1}^{n-1} dy_a^2 + \frac{1 - s(r)^2}{r^2} \left(\sum_{a=1}^{n-1} y_a\, dy_a \right)^2.$$

Both $s(r)^2$ and $(1 - s(r)^2)/r^2$ are smooth functions of r^2 at zero; the same holds for $\cos^2 r$. The displayed tensor therefore extends smoothly across the collapsed S^{n-2}. The tensor $\operatorname{Ric}_{G[\delta,A]} - G[\delta, A]$ is continuous on S^n and nonnegative on the dense open set $(0, \pi/2) \times S^{n-2} \times S^1$. It is therefore nonnegative on Σ_0 and Σ_1.

The preliminary parameter bounds give $\sin^2(3t_0) + e_{\delta_0}^2 < 1$. Thus on the cutoff support

$$-c \le P = -c - \tfrac{\delta}{2} \log(\sin^2 t + e_\delta^2) \le 0, \qquad q \sin t \le b \le \sin t.$$

Decrease η so that $U \le -\log(19/20)$, and take $c_0 = 19/20$. Then

$$c_0 \cos t \le a \le \cos t, \qquad c_0^2 G_q \le G[\delta, A] \le g_{S^n}$$

on the regular cylinder. The endpoint strata have the induced metrics γ and $q^2 d\theta^2$. The length distances induced by G_q, $G[\delta, A]$, and g_{S^n} are uniformly comparable to the round distance: the displayed coefficients give comparison constants bounded below by $\min\{q, c_0\} > 0$ on the regular cylinder, with the same bound on the endpoint strata. Let c be an absolutely continuous round curve. In a chart adapted to either endpoint stratum, write its normal coordinate functions as $y_1, \dots, y_a$. For a real absolutely continuous function y, its derivative is zero at every differentiability point of $y^{-1}(0)$ that is an accumulation point of this zero set: use a sequence of distinct zeros in the difference quotient. Isolated zeros form a countable set, and nondifferentiability points form a null set. Applying this assertion to each y_b, in a countable family of adapted chart intervals, proves that c' is tangent to the stratum at almost every parameter for which c lies in it. Integrating the coefficient inequalities on the regular pieces and the induced inequalities on the endpoint strata therefore compares its lengths. Taking infima gives the comparison on the join completion. The defining formulas also show equality of the metrics wherever both parameter functions agree. □

6. Ricci-positive annuli with prescribed boundary tensors

Throughout this section $n \in \{4, 5\}$. A metric path on S^{n-1} is understood to be stationary near its endpoints and extended constantly outside its parameter interval.

Lemma 6.1. *Let $(\widehat{h}_u)_{-2 \le u \le 2}$ be a smooth path on the oriented S^{n-1} with constant total volume. There is a smooth isotopy ϕ_u, with $\phi_{-2} = \mathrm{Id}$, such that $h_u = \phi_u^* \widehat{h}_u$ has volume form independent of u. If the metric path is constant on a parameter interval adjacent to either endpoint, then the isotopy is constant on that interval. Pullback preserves every Ricci lower bound of the path.*

Proof. Put $\omega_u = dv_{\widehat{h}_u}$ and $\alpha_u = \partial_u \omega_u$. Differentiation under the integral gives $\int_{S^{n-1}} \alpha_u = 0$. Fix a smooth background metric h_{ref} and let $\mathcal{H}, \mathcal{G}, d^*$ be its harmonic projection, Hodge Green operator, and codifferential. A top-degree form harmonic for h_{ref} is a constant multiple of $dv_{h_{\mathrm{ref}}}$, since S^{n-1} is connected and oriented. Thus $\mathcal{H}\alpha_u = 0$, while $d\mathcal{G}\alpha_u = 0$ by degree. The identity

$$(dd^* + d^*d)\mathcal{G} = \mathrm{Id} - \mathcal{H}$$

gives $d\eta_u = \alpha_u$ for $\eta_u = d^*\mathcal{G}\alpha_u$.

Contraction with the positive form ω_u is an isomorphism from vector fields to $(n-2)$-forms. There is therefore a unique smooth vector field Y_u with $\iota_{Y_u}\omega_u = -\eta_u$. On the compact manifold S^{n-1} its time-dependent flow exists for the full parameter interval; smooth dependence for the flow equation gives a smooth isotopy ϕ_u with $\phi_{-2} = \mathrm{Id}$. Since $d\omega_u = 0$, Cartan's identity yields

$$\partial_u(\phi_u^* \omega_u) = \phi_u^*(\alpha_u + d\iota_{Y_u}\omega_u) = 0.$$

Thus $dv_{\phi_u^*\widehat{h}_u} = \omega_{-2}$. On every parameter interval where $\widehat{h}_u$ is constant, α_u vanishes identically on S^{n-1}; hence $\eta_u = Y_u = 0$ on S^{n-1}, and ϕ_u is constant on that parameter interval. Finally,

$$\mathrm{Ric}_{\phi_u^*\widehat{h}_u} = \phi_u^* \,\mathrm{Ric}_{\widehat{h}_u};$$

pullback preserves each of the asserted tensor inequalities. □

Lemma 6.2. *Let h_u be a smooth path on S^{n-1}, constant for $u \le -2$ and $u \ge 2$, with*

$$dv_{h_u} = dv_{h_{-2}}, \qquad \mathrm{Ric}_{h_u} \ge (n-2)h_u.$$

For each $0 < \theta < 1/4$ there are smooth functions $\chi : (0, 1] \to (0, 1)$ and $\psi : (0, 1] \to [-2, 2]$ such that

$$G_0 = dr^2 + (r\chi(r))^2 h_{\psi(r)}$$

has positive Ricci curvature. They satisfy

$$\psi = -2 \text{ near } 0, \qquad \psi = 2 \text{ near } 1, \tag{6.1}$$

$$\chi \to 1, \quad r\chi' \to 0 \quad (r \downarrow 0), \tag{6.2}$$

$$|1-\chi| + r|\chi'/\chi| + r|g'|_g + r^2|g''|_g \le \theta, \qquad g(r) = h_{\psi(r)}, \tag{6.3}$$

$$r^2|\operatorname{Ric}_{G_0}(\partial_r, (r\chi)^{-1}e)| \le \theta \quad (|e|_g = 1), \tag{6.4}$$

$$|1-\chi(1)| + |\chi'(1)| \le \theta. \tag{6.5}$$

Proof. Set $b_0(t) = 0$ for $t \le 0$ and $b_0(t) = e^{-1/t}$ for $t > 0$, and put

$$H_0(t) = \frac{b_0(t)}{b_0(t) + b_0(1-t)}, \qquad \rho_0(v) = -2 + 4H_0((v+1)/2).$$

Each derivative of b_0 on $t > 0$ is $e^{-1/t}$ times a polynomial in t^{-1} and tends to zero at zero. The denominator defining H_0 is positive on $\mathbb{R}$. For $0 < t < 1$,

$$H_0'(t) = H_0(t)(1 - H_0(t))\big(t^{-2} + (1-t)^{-2}\big) > 0.$$

Outside $[0,1]$, the function H_0 is constant. Hence ρ_0 is smooth and nondecreasing, equals -2 on $(-\infty,-1]$, and equals 2 on $[1,\infty)$. Choose $A_{\text{path}} \ge 1$ such that

$$\sup_{(u,x)\in[-2,2]\times S^{n-1}} \max\big\{ |\partial_u h_u|_{h_u},\ |\partial_u^2 h_u|_{h_u},\ |\nabla^{h_u}\partial_u h_u|_{h_u} \big\} \le A_{\text{path}}.$$

These tensors are smooth on the compact parameter–space product and vanish where the path is stationary. Put $a_1 = \|\rho_0'\|_\infty$, $a_2 = \|\rho_0''\|_\infty$, and choose

$$L_* = 2 + \max\{4a_1 A_{\text{path}}, \sqrt{2(a_2 + a_1^2)A_{\text{path}}}\}.$$

For $k_v = h_{\rho_0(v/L_*)}$, the chain rule gives

$$|\dot k_v|_{k_v} + |\ddot k_v|_{k_v} + |\nabla^{k_v}\dot k_v|_{k_v} \le \frac{2a_1 A_{\text{path}}}{L_*} + \frac{(a_2 + a_1^2)A_{\text{path}}}{L_*^2} \le 1.$$

Set

$$E = \theta/8, \quad F_0 = \theta/320, \quad L_0 = 320\theta^{-2}, \quad T_0 = e^{L_0},$$

$$T(r) = T_0 - \log r, \quad L(r) = \log T(r), \quad \chi(r) = 1 - E/L(r),$$

$$f(r) = -F_0 \log L(r) + L_* + 1 + F_0 \log L_0, \qquad \psi(r) = \rho_0(f(r)/L_*).$$

Then $f' = F_0/(rTL) > 0$, $f(1) = L_* + 1$, and $f(r) \to -\infty$ as $r \downarrow 0$. Thus (6.1) holds. Put $w = r\chi$ and $g(r) = k_{f(r)}$.

The common volume form gives $\operatorname{tr}_g g' = 0$. Differentiating $g^{ab}g'_{ab} = 0$ and using $(g^{ab})' = -g^{ac}g'_{cd}g^{db}$ yields $\operatorname{tr}_g g'' = |g'|_g^2$. Put $p = w'/w$ and $H = w^2 g$. Then

$$H' = w^2(2pg + g'), \quad H'' = w^2\big(2(p^2 + w''/w)g + 4pg' + g''\big),$$

$$\operatorname{tr}_H H' = 2(n-1)p, \quad |H'|_H^2 = 4(n-1)p^2 + |g'|_g^2, \quad \operatorname{tr}_H H'' = 2(n-1)(p^2 + w''/w) + |g'|_g^2.$$

In the tangential formula of (3.2), the coefficients of pg' are $-2 + 2 - (n-1)/2 = -(n-1)/2$. Since $\operatorname{Ric}_{w^2 g} = \operatorname{Ric}_g$ on each slice, substitution gives, for $|e|_g = 1$,

$$\operatorname{Ric}_{G_0,rr} = -(n-1)w''/w - \tfrac14|g'|_g^2, \tag{6.6}$$

$$\operatorname{Ric}_{G_0}(\partial_r, w^{-1}e) = (2w)^{-1}(\operatorname{div}_g g')(e), \tag{6.7}$$

$$\begin{aligned}\operatorname{Ric}_{G_0}(w^{-1}e, w^{-1}e) = {} & w^{-2}\operatorname{Ric}_g(e,e) - (n-2)(w'/w)^2 - w''/w \\ & - \tfrac{n-1}{2}(w'/w)g'(e,e) - \tfrac12 g''(e,e) + \tfrac12|g'(e,\cdot)|_g^2.\end{aligned} \tag{6.8}$$

Differentiation yields

$$\chi' = -\frac{E}{rTL^2}, \quad w'' = -\frac{E}{rTL^2}\left(1+\frac{1}{T}+\frac{2}{TL}\right), \quad f'' = -\frac{F_0}{r^2TL}\left(1-\frac{1}{T}-\frac{1}{TL}\right), \qquad |f''| \le \frac{F_0}{r^2TL}.$$

Also

$$|g'| \le \frac{F_0}{rTL}, \qquad |g''| \le \frac{F_0}{r^2TL} + \frac{F_0^2}{r^2T^2L^2}, \qquad |\nabla g'| \le \frac{F_0}{rTL}.$$

The inequalities $E/L \le 1/2$, $T/L > 4$, $E \ge 4F_0$, and $E \ge F_0^2$ hold. If $d = E/(\chi TL^2)$, then $0 < d < 1$ and $w'/w = (1-d)/r$. Since $\chi^{-2} - 1 \ge 2E/L$, (6.6)–(6.8) imply

$$\mathrm{Ric}_{G_0,rr} \ge \frac{E}{2r^2TL^2}, \qquad \mathrm{Ric}_{G_0}(w^{-1}e, w^{-1}e) \ge \frac{E}{r^2L},$$

$$|\operatorname{Ric}_{G_0}(\partial_r, w^{-1}e)| \le \frac{2F_0}{r^2TL}.$$

For the tangential inequality, after multiplication by r^2L the lower bound is

$$2(n-2)E - \frac{nF_0}{2T} - \frac{F_0^2}{2T^2L} \ge 4E - \frac{5E}{8T} - \frac{E}{2T^2L} > E.$$

Since the slices have dimension $n-1 \le 4$, $|\operatorname{div}_g g'|_g \le 2|\nabla^g g'|_g$. The product of the radial and tangential lower bounds minus the square of the mixed-covector norm bound is

$$\frac{E^2}{2r^4TL^3}\left(1 - \frac{8F_0^2L}{E^2T}\right) > \frac{799}{800}\frac{E^2}{2r^4TL^3} > 0.$$

Indeed, $F_0/E = 1/40$ and $L/T < 1/4$ give $8F_0^2L/(E^2T) < 1/800$. The Schur complement proves $\mathrm{Ric}_{G_0} > 0$.

Finally, the left side of (6.3) is bounded by

$$\frac{E}{L_0} + \frac{2E}{T_0L_0^2} + \frac{2F_0}{T_0L_0} + \frac{F_0^2}{T_0^2L_0^2} < \theta.$$

The bound for $\mathrm{Ric}_{G_0}(\partial_r, w^{-1}e)$ gives (6.4). The formulas $\chi = 1 - E/L$ and $\chi' = -E/(rTL^2)$ give (6.2) and (6.5). □

Proposition 6.3. *There are constants $\varepsilon_* > 0$ and $C < \infty$, independent of h_u, with the following property. Let h_u satisfy the hypotheses of Lemma 6.2, with*

$$h_{-2} = \tau^2 g_{S^{n-1}}, \qquad h_2 = h, \qquad 0 < \tau \le 1.$$

For every $0 < \varepsilon < \varepsilon_*$ there is a Ricci-positive metric G_ε on $(0,1] \times S^{n-1}$ satisfying

$$\mathrm{Ric}_{G_\varepsilon} \ge (n-1-C\varepsilon^2)G_\varepsilon, \tag{6.9}$$

$$G_\varepsilon|_{\{1\}\times S^{n-1}} = \sin^2\varepsilon\, h, \qquad \mathrm{II}_{\mathrm{out}} \ge \left(\cot\varepsilon + \frac{2\varepsilon^4}{\sin\varepsilon}\right)\sin^2\varepsilon\, h. \tag{6.10}$$

Near $r = 0$, the metric G_ε is a round warped product $ds^2 + F(s)^2 g_{S^{n-1}}$ with $F(s) \to 0$ and $F'(s) \to \tau$. Every sufficiently small inner truncation has intrinsic diameter at most $C\varepsilon$ and volume at most $C\varepsilon^n$.

Proof. Apply Lemma 6.2 with $\theta = \varepsilon^4$ and write $g(r) = h_{\psi(r)}$, $b = r\chi$. For $0 < k < 1/2$, put

$$s_k(r) = \frac{\sin(kr)}{k}, \qquad a = s_k\chi, \qquad G_k = dr^2 + a(r)^2 g(r).$$

Let $\alpha = \chi'/\chi$ and $d_k = k\cot(kr) - r^{-1}$. For $0 < x \le 1/2$, define

$$F_1(x) = \frac{x\cot x - 1 + x^2/3}{x^4}, \qquad F_2(x) = \frac{(x/\sin x)^2 - 1 - x^2/3}{x^4}, \qquad F_3(x) = \frac{x/\sin x - 1}{x^2}.$$

The sine and cosine Taylor series give the continuous extensions

$$F_1(0) = -1/45, \qquad F_2(0) = 1/15, \qquad F_3(0) = 1/6.$$

Fix $C_0 \geq 1 + \max_{j=1,2,3} \|F_j\|_{C^0([0,1/2])}$. Substitution of $x = kr$ gives

$$|d_k + k^2 r/3| \leq C_0 k^4 r^3, \quad |s_k^{-2} - r^{-2} - k^2/3| \leq C_0 k^4 r^2,$$

$$0 \leq b/a - 1 \leq C_0 k^2 r^2, \qquad a''/a - b''/b = -k^2 + 2d_k \alpha.$$

Subtract (6.6) and (6.8) for G_k and G_0, using their respective orthonormal frames. The radial difference is $(n-1)k^2 - 2(n-1)d_k\alpha$. The tangential difference is

$$\begin{aligned} D_T = (a^{-2} - b^{-2}) \operatorname{Ric}_g(e,e) + k^2 - 2(n-2)d_k/r \\ - 2(n-1)d_k\alpha - (n-2)d_k^2 - \tfrac{n-1}{2} d_k g'(e,e). \end{aligned}$$

Here $a^{-2} - b^{-2} = \chi^{-2}(s_k^{-2} - r^{-2}) \geq 0$. Since $\chi \leq 1$ and $\operatorname{Ric}_g \geq (n-2)g$,

$$(a^{-2} - b^{-2}) \operatorname{Ric}_g(e,e) \geq \tfrac{n-2}{3} k^2 - (n-2)C_0 k^4 r^2,$$

$$-2(n-2)d_k/r \geq \tfrac{2(n-2)}{3} k^2 - 2(n-2)C_0 k^4 r^2.$$

Set $C_1 = 40(1 + C_0)^2$ and $C_2 = C_1 + C_0$. Also $|d_k| \leq (1/3 + C_0)k^2 r$, while (6.3) gives

$$|d_k\alpha| + |d_k g'(e,e)| \leq C\theta k^2, \qquad d_k^2 \leq Ck^4.$$

The sum of the leading coefficients is $(n-2)/3 + 1 + 2(n-2)/3 = n-1$. Thus

$$D_T \geq (n - 1 - C_1\theta - C_1 k^2)k^2,$$

and the same lower bound holds for the radial difference. Formula (6.7), applied with $w = a$ and $w = b$, gives the mixed-component difference

$$\operatorname{Ric}_{G_k}(\partial_r, a^{-1}e) - \operatorname{Ric}_{G_0}(\partial_r, b^{-1}e) = (b/a - 1) \operatorname{Ric}_{G_0}(\partial_r, b^{-1}e).$$

The absolute value of this mixed-component difference is at most $C_0\theta k^2$ by (6.4). Evaluation on an arbitrary unit vector $x\partial_r + ya^{-1}e$, with $x^2 + y^2 = 1$, and $2|xy| \leq 1$ therefore gives

$$\operatorname{Ric}_{G_k} \geq (n - 1 - C_2\theta - C_2 k^2)k^2 G_k. \tag{6.11}$$

Initially require $0 < \varepsilon < 1/100$. The identity

$$\sin\varepsilon = \varepsilon - \int_0^\varepsilon (\varepsilon - t)\sin t \, dt$$

and $0 \leq \sin t \leq t$ for $0 \leq t \leq \varepsilon$ give

$$0 < \varepsilon - \frac{\varepsilon^3}{6} \leq \sin\varepsilon \leq \varepsilon.$$

Both sides of the middle inequality are positive. Squaring gives

$$\begin{aligned} \sin^2\varepsilon - 8\varepsilon^4 &\geq \left(\varepsilon - \frac{\varepsilon^3}{6}\right)^2 - 8\varepsilon^4 \\ &= \varepsilon^2 \left(1 - \frac{25}{3}\varepsilon^2 + \frac{\varepsilon^4}{36}\right) \\ &\geq \varepsilon^2 \left(1 - \frac{25}{3}\varepsilon^2\right) > \frac{1199}{1200}\varepsilon^2 > \frac{\varepsilon^2}{4}. \end{aligned} \tag{6.12}$$

Thus the positive square root

$$\lambda_\varepsilon = (\sin^2\varepsilon - 8\varepsilon^4)^{1/2} \quad \text{satisfies} \quad \varepsilon/2 < \lambda_\varepsilon \leq \sin\varepsilon \leq \varepsilon.$$

The equation $\chi(1)\sin k = \lambda_\varepsilon$ has a unique solution $k = k_\varepsilon \in (0, 1/2)$ and

$$\varepsilon/2 \le k_\varepsilon \le 4\varepsilon, \qquad \varepsilon/2 \le \lambda_\varepsilon \le \varepsilon.$$

Indeed $1 - \varepsilon^4 \le \chi(1) \le 1$ and $\lambda_\varepsilon/\chi(1) \le 2\varepsilon$; use $\sin k \le k \le 2\sin k$ on this interval. Put

$$c_\varepsilon = \frac{\sin\varepsilon\, k_\varepsilon}{\lambda_\varepsilon}, \qquad G_\varepsilon = c_\varepsilon^2 G_{k_\varepsilon}.$$

Then $\varepsilon/4 \le c_\varepsilon \le 8\varepsilon$ and $c_\varepsilon a(1) = \sin\varepsilon$.

Since

$$\chi(1)^2 - \lambda_\varepsilon^2 - \cos^2\varepsilon = \chi(1)^2 - 1 + 8\varepsilon^4 \ge 6\varepsilon^4,$$

and the nonnegative quantities $\chi(1)\cos k_\varepsilon$ and $\cos\varepsilon$ are at most one,

$$\chi(1)\cos k_\varepsilon - \cos\varepsilon \ge 3\varepsilon^4.$$

Equation (6.5) gives

$$a'(1) = \chi(1)\cos k_\varepsilon + s_{k_\varepsilon}(1)\chi'(1) \ge \cos\varepsilon + 2\varepsilon^4.$$

The slice path is stationary near $r = 1$, so the outward second fundamental form after scaling is $(a'(1)/\sin\varepsilon)$ times the boundary metric. This proves (6.10).

Constant scaling leaves the covariant Ricci tensor unchanged, and

$$k_\varepsilon^2/c_\varepsilon^2 = \lambda_\varepsilon^2/\sin^2\varepsilon = 1 - 8\varepsilon^4/\sin^2\varepsilon \ge 1 - 32\varepsilon^2.$$

Since $k_\varepsilon^2 \le 16\varepsilon^2$ and $\theta = \varepsilon^4 \le \varepsilon^2$, the two lower factors are $n - 1 - 17C_2\varepsilon^2$ and $1 - 32\varepsilon^2$. Where they are positive, their product is at least $n - 1 - (17C_2 + 128)\varepsilon^2$. Choose

$$C = 1 + \max\{17C_2 + 128, 16 + \pi, 8^4\sigma_3, 8^5\sigma_4\}, \qquad \varepsilon_* = \min\{1/100, (2C+2)^{-1/2}\}.$$

These choices prove (6.9), positivity of the required factors, and the size bounds below, for both $n = 4$ and $n = 5$.

Near zero $g(r) = \tau^2 g_{S^{n-1}}$. In the coordinate $s = c_\varepsilon r$, the warping is $F(s) = c_\varepsilon \tau a(r)$. Hence $F \to 0$ and $F' \to \tau$ by (6.2). For each r, Bonnet–Myers gives $\operatorname{diam}(S^{n-1}, g(r)) \le \pi$. Radial segments have length at most $c_\varepsilon \le 8\varepsilon$, and the outer slice has diameter at most $\pi\sin\varepsilon$. Thus the intrinsic diameter of every truncated annulus is at most $(16 + \pi)\varepsilon$. The volume of the truncated annulus is

$$c_\varepsilon^n \tau^{n-1}\sigma_{n-1}\int a(r)^{n-1}\,dr \le 8^n\sigma_{n-1}\varepsilon^n,$$

since $a(r) \le r \le 1$ and $\tau \le 1$. □

7. Smooth parameters with prescribed constant values and vanishing jets

Fix (P, k) as in Lemma 2.1. Choose

$$0 < \beta < \min\{\alpha/4, 1/8\}, \qquad 0 < q < \beta^{n-1}/1000. \tag{7.1}$$

Use this q in Sections 4 and 5. Complete the choices in Lemma 5.3, and let $C_{\rm ann}$ and ε_* be the constants in Proposition 6.3. Fix, before choosing any centers or radii,

$$0 < R_* < \min\left\{10^{-9}, \frac{t_0}{100}, \frac{\varepsilon_*}{4}, \frac{1}{4\sqrt{1 + C_{\rm ann}}}\right\}. \tag{7.2}$$

Then $20R_* < t_0$, $2R_* < \varepsilon_*$, and $(n-1) - C_{\rm ann}(2R_*)^2 > 2$. Every enlarged round ball below has radius less than 10^{-3}. No later choice changes R_*.

Identify a small neighborhood in S^{n-2} with $\{z \in \mathbb{R}^{n-2} : |z| < 1/10\}$ by

$$z \longmapsto (\sqrt{1 - |z|^2}, z) \in S^{n-2} \subset \mathbb{R}^{n-1}.$$

Use the compact set F in $|z| < 1/40$. Choose all centers below in $|z| < 1/20$. Their enlarged balls then lie in the displayed chart.

Lemma 7.1. *There are $x_i \in S^{n-2} \setminus F$ in the same fixed coordinate chart, $z_i \in F$, and radii $R_i > 0$ such that*

$$d_\gamma(x_i, z_i) \to 0, \quad \text{every point of } F \text{ is approached by a subsequence of } (z_i), \tag{7.3}$$

$$\overline{B}^\gamma_{10^6 R_i}(x_i) \cap F = \emptyset, \quad \overline{B}^\gamma_{10^6 R_i}(x_i) \text{ are pairwise disjoint,} \tag{7.4}$$

$$R_i < 2^{-i} \min\{R_*, d_\gamma(x_i, F)^3\}, \tag{7.5}$$

where $R_* > 0$ is any prescribed sufficiently small constant.

Proof. For each integer $h \geq 1$, fix a finite $1/h$-net $E_h \subset F$. Enumerate the finite lists $E_1, E_2, \ldots$, allowing repetitions, as (z_i); write $h(i)$ for the level of z_i. Then $h(i) \to \infty$.

Suppose the first $i - 1$ closed enlarged balls have been chosen, and let K_{i-1} be their union. It is a compact set disjoint from F. Since F has empty interior, there exists

$$x_i \in \{|z| < 1/20\} \setminus (F \cup K_{i-1}), \qquad d_\gamma(x_i, z_i) < 1/h(i).$$

When $i > 1$, require in addition $d_\gamma(x_i, z_i) < d_\gamma(z_i, K_{i-1})/4$. This is possible because the latter distance is positive. Choose $R_i > 0$ satisfying

$$R_i < 2^{-i} \min\{R_*, d_\gamma(x_i, F)^3\}, \qquad 10^6 R_i < \tfrac{1}{2} d_\gamma(x_i, F),$$

$$10^6 R_i < \tfrac{1}{2} d_\gamma(x_i, K_{i-1}) \quad (i > 1).$$

These inequalities make the new closed enlarged ball disjoint from F and from K_{i-1}. They also give (7.5).

For $z \in F$, choose $z_{i_h} \in E_h$ with $d_\gamma(z, z_{i_h}) \leq 1/h$. The indices i_h are strictly increasing and $z_{i_h} \to z$. Also $d_\gamma(x_i, z_i) \leq 1/h(i) \to 0$. These are the assertions in (7.3). □

Lemma 7.2. *The radii in Lemma 7.1 admit a function $r \in C^\infty(S^{n-2} \setminus F)$ such that*

$$0 < r(x) \leq C d_\gamma(x, F), \qquad r = R_i \text{ on } B^\gamma_{60R_i}(x_i),$$

$$|\nabla^m r| \leq C_m r^{1-m} \qquad (m \geq 1).$$

The function $f = e^{-1/r^2}$ on $S^{n-2} \setminus F$, extended by zero on F, is smooth and flat on F.

Proof. The value of R_* has already been fixed by (7.2). Define

$$r_0(x) = \min\left\{ R_*, 10^{-3} d_\gamma(x, F), \inf_i \big(R_i + 10^{-2} d_\gamma(x, \overline{B}_{100R_i}(x_i))\big) \right\}.$$

Each entry is 10^{-2}-Lipschitz; the infimum is finite and has the same Lipschitz bound. Thus r_0 is 10^{-2}-Lipschitz. It vanishes on F. Let $K \subset S^{n-2} \setminus F$ be nonempty and compact, and put $d = d_\gamma(K, F) > 0$. Choose i_0 such that $d_\gamma(x_i, F) + 100R_i < d/2$ for $i \geq i_0$. For $x \in K$ and $z \in \overline{B}_{100R_i}(x_i)$, the triangle inequality gives

$$d_\gamma(x, z) \geq d_\gamma(x, F) - d_\gamma(x_i, F) - 100R_i > d/2.$$

Thus the entries indexed by $i \geq i_0$ are at least $10^{-2} d/2$ on K. The finitely many earlier entries are at least $\min_{i<i_0} R_i > 0$ when present; the other two entries are R_* and at least $10^{-3} d$. Their minimum is positive. Hence $r_0 > 0$ on $S^{n-2} \setminus F$.

If $x \in B_{100R_i}(x_i)$, the i-th entry is R_i. The entry R_* and the distance-to-F entry exceed R_i. An entry with $R_k \geq R_i$ is at least R_i. For $R_k < R_i$, disjointness of the enlarged balls implies

$$d_\gamma(x_i, x_k) > 10^6 (R_i + R_k).$$

Indeed, otherwise a minimizing geodesic between the centers would contain a point in both closed balls. Consequently

$$d_\gamma(x, \overline{B}_{100R_k}(x_k)) > (10^6 - 100)(R_i + R_k) > 100R_i.$$

The k-th entry is then larger than R_i. Thus

$$r_0 = R_i \quad \text{on } B_{100R_i}(x_i). \tag{7.6}$$

Set $s(x) = r_0(x)/100$ and $b = R_*/100$. For $h \geq 0$, put

$$\mathcal{B}_h = \{B_{s(x)}(x) \,:\, 2^{-h-1}b < s(x) \leq 2^{-h}b\}.$$

Choose inductively a disjoint subfamily $\mathcal{A}_h \subset \mathcal{B}_h$ of maximum cardinality among those disjoint from all balls in $\bigcup_{k<h} \mathcal{A}_k$. Such a maximum exists: the round polar volume formula gives $\mathrm{Vol}_\gamma B_\rho \geq 2^{-(n-2)}\omega_{n-2}\rho^{n-2}$ for the radii in question, so the cardinalities of disjoint subfamilies of $\mathcal{B}_h$ are bounded integers. Enumerate $\bigcup_{h\geq 0} \mathcal{A}_h$ as $\{B_{s(x_a)}(x_a)\}_a$. A ball in $\mathcal{B}_h$ either meets a previously chosen ball or meets a member of $\mathcal{A}_h$, by maximality. In either case it meets a selected ball whose radius is greater than half its own radius. If $B_{s(x)}(x)$ and $B_{s(y)}(y)$ meet, then

$$|s(x) - s(y)| \leq 10^{-4}d_\gamma(x, y) < 10^{-4}(s(x) + s(y)), \qquad \frac{1 - 10^{-4}}{1 + 10^{-4}} < \frac{s(x)}{s(y)} < \frac{1 + 10^{-4}}{1 - 10^{-4}}.$$

Thus $d_\gamma(x, y) < 3s(y)$, and the balls $B_{5s(x_a)}(x_a)$ cover $S^{n-2} \setminus F$. For $x \in B_{10s(x_a)}(x_a)$, $|s(x) - s(x_a)| \leq 10^{-3}s(x_a)$; hence

$$\tfrac{1}{2}s(x) \leq s(x_a) \leq 2s(x).$$

Every selected ball contributing at x is contained in $B_{22s(x)}(x)$ and has radius at least $s(x)/2$. Write $m = n - 2$. For $0 < \rho < 10^{-2}$, integration of the round polar volume form, using $\rho/2 \leq \sin\rho \leq \rho$, gives

$$2^{-m}\omega_m\rho^m \leq \mathrm{Vol}_\gamma(B_\rho) \leq \omega_m\rho^m.$$

The disjointness of the selected balls therefore bounds the number contributing at x by 88^m. These radii lie in the stated range because $s \leq R_*/100$. On a compact subset disjoint from F, s has a positive minimum. Every ball whose tenfold enlargement meets that compact set has radius bounded below by half that minimum; the same packing estimate makes their number finite. Thus the enlarged cover is locally finite off F. At each dyadic size only finitely many disjoint balls can be selected, by the same volume bound. The descending maximal selection is consequently countable.

Choose smooth functions θ_a supported in $B_{10s(x_a)}(x_a)$, equal to one on $B_{5s(x_a)}(x_a)$, with

$$0 \leq \theta_a \leq 1, \qquad |\nabla^m\theta_a| \leq C_m s(x_a)^{-m} \quad (m \geq 1).$$

Such functions are radial cutoffs in the fixed round normal charts. Since $1 \leq \sum_a \theta_a \leq C$, the functions

$$\varphi_a = \theta_a \Big/ \sum_b \theta_b$$

form a smooth partition of unity. To bound their derivatives, put $S = \sum_b \theta_b$. The overlap bound gives $1 \leq S \leq 88^{n-2}$ and $|\nabla^h S| \leq C_h r_0^{-h}$. Apply the iterated covariant Leibniz rule to $SS^{-1} = 1$. Since $S \geq 1$,

$$|\nabla^h(S^{-1})| \leq S^{-1}\sum_{b=1}^{h}\binom{h}{b}|\nabla^b S|\,|\nabla^{h-b}(S^{-1})|.$$

This estimate uses only the product rule and the invariance of the tensor norm under permutations of factors; it does not commute covariant derivatives. Induction gives $|\nabla^h(S^{-1})| \leq C'_h r_0^{-h}$. The Leibniz rule applied to $\varphi_a = \theta_a S^{-1}$ then gives $|\nabla^h\varphi_a| \leq C''_h r_0^{-h}$ on each support. Define

$$r_s = \sum_a r_0(x_a)\varphi_a.$$

Bounded overlap and comparability give

$$cr_0 \leq r_s \leq Cr_0, \qquad |\nabla^m r_s| \leq C_m r_0^{1-m}.$$

For disjoint cutoffs ζ_i, equal to one on $B_{60R_i}(x_i)$ and zero off $B_{80R_i}(x_i)$, set

$$r = r_s + \sum_i \zeta_i(R_i - r_s).$$

On supp $\zeta_i \subset \overline{B}_{80R_i}(x_i)$, (7.6) and the derivative bounds for ζ_i prove

$$cr_0 \le r \le Cr_0, \qquad |\nabla^m r| \le C_m r^{1-m}, \qquad r = R_i \text{ on } B_{60R_i}(x_i).$$

In particular $r \le C d_\gamma(\cdot, F)$.

Let $f = e^{-1/r^2}$ off F. Repeated chain rules give

$$|\nabla^m f| \le C_m r^{-N_m} e^{-1/r^2} \quad \text{for some } N_m \ge 0.$$

For every m, ℓ, the right side is at most $C_{m,\ell} r^\ell$, and hence at most $C'_{m,\ell} d_\gamma(\cdot, F)^\ell$. On a relatively compact coordinate chart, every ordinary partial derivative $\partial^\alpha f$ is a finite sum of components of $\nabla^b f$, $0 \le b \le |\alpha|$, with smooth bounded coefficients. Therefore

$$|\partial^\alpha f(x)| \le C_{\alpha,\ell} d_\gamma(x, F)^\ell \qquad (x \notin F)$$

on each smaller chart. Let F_α be its extension by zero on F. These functions are continuous. If $x \in F$ and e_j is a coordinate basis vector, local comparison with Euclidean distance and the bound with $\ell = 2$ give

$$\left| \frac{F_\alpha(x + he_j) - F_\alpha(x)}{h} \right| \le C_\alpha |h| \longrightarrow 0.$$

Off F one has $\partial_j F_\alpha = F_{\alpha+e_j}$; at F both sides are zero by the preceding limit. Thus this equality holds everywhere and its right side is continuous. Induction proves that F_0 is smooth and that all its derivatives vanish on F. □

Lemma 7.3. *There are a number $\lambda > 0$ and smooth nonnegative functions δ, A on S^{n-2}, flat on F, with $\delta > 0$ on $S^{n-2} \setminus F$, such that*

$$\delta = \delta_i := \lambda e^{-2/R_i^2}, \quad A = 0 \quad \text{on } B^\gamma_{20R_i}(x_i), \tag{7.7}$$

$$\delta_i \le R_i^4/100, \qquad \delta_i |\log \sin R_i| \le \log 2. \tag{7.8}$$

There are $\varepsilon_{\rm adm} > 0$ and positive smooth functions δ_ε, $0 < \varepsilon < \varepsilon_{\rm adm}$, such that $\delta_\varepsilon \to \delta$ pointwise, $\delta_\varepsilon = \delta$ wherever $\delta \ge 2\varepsilon$, and every pair (δ_ε, A) satisfies (5.9).

Proof. Choose $\xi \in C^\infty(S^{n-2} \setminus F, [0,1])$ equal to zero on $B_{20R_i}(x_i)$, equal to one outside $\bigcup_i B_{40R_i}(x_i)$, and with transitions in $B_{40R_i}(x_i) \setminus \overline{B}_{20R_i}(x_i)$, where $f = e^{-1/R_i^2}$ by Lemma 7.2. The derivatives of a transition cutoff satisfy $|\nabla^m \xi| \le C_m R_i^{-m}$. On such a transition, the Leibniz rule gives

$$|\nabla^m(\xi f^2)| \le C_m R_i^{-m} e^{-2/R_i^2} \le C_{m,\ell} R_i^\ell \le C'_{m,\ell} d_\gamma(\,\cdot\,, F)^\ell \quad (\ell \ge 1).$$

The last inequality uses $d_\gamma(x_i, F) > 10^6 R_i$ and $d_\gamma(x, x_i) \le 40R_i$ on the transition. Outside the transitions, ξf^2 is either zero or f^2. The difference-quotient argument of Lemma 7.2 therefore extends ξf^2 smoothly and flatly by zero on F.

For a constant M_1 chosen below put

$$\delta = \lambda f^2, \qquad A = M_1 \lambda \big(\xi f^2 + |\operatorname{Hess} f|^2 + 6|df|^2 \big).$$

Off F, differentiation gives

$$\operatorname{Hess} \delta = 2\lambda f \operatorname{Hess} f + 2\lambda\, df \otimes df, \qquad \frac{|d\delta|^2}{\delta} = 4\lambda |df|^2. \tag{7.9}$$

The inequality $2f|\operatorname{Hess} f| \le f^2 + |\operatorname{Hess} f|^2$ and $|df \otimes df| = |df|^2$ imply

$$|\operatorname{Hess} \delta| + \frac{|d\delta|^2}{\delta} \le \lambda(f^2 + |\operatorname{Hess} f|^2 + 6|df|^2).$$

Where df or Hess f is nonzero, $\xi = 1$. Where $\xi \neq 1$, the derivatives of f vanish and the left side is zero. The quotient extends smoothly across F as $4\lambda|df|^2$. On $B_{20R_i}(x_i)$, $f = e^{-1/R_i^2}$ and $\xi = df = \operatorname{Hess} f = 0$; hence (7.7) holds.

Let H_0 be the smooth nondecreasing function used in the proof of Lemma 6.2; it equals zero on $(-\infty, 0]$ and one on $[1, \infty)$. Define, for $u \geq 0$,

$$\vartheta(u) = 1 + (u-1)H_0(u-1), \qquad \delta_\varepsilon = \varepsilon\vartheta(\delta/\varepsilon). \tag{7.10}$$

Then $\vartheta = 1$ on $[0,1]$, $\vartheta(u) = u$ on $[2,\infty)$, and $1 \leq \vartheta(u) \leq u$ on $[1,2]$. In particular, $\frac{1}{2}\max\{1,u\} \leq \vartheta(u) \leq \max\{1,u\}$. All derivatives of ϑ of positive order are bounded; those of order at least two are supported in $[1,2]$. On the transition set $\varepsilon \leq \delta \leq 2\varepsilon$,

$$d\delta_\varepsilon = \vartheta'(\delta/\varepsilon)d\delta, \quad \operatorname{Hess}\delta_\varepsilon = \vartheta'(\delta/\varepsilon)\operatorname{Hess}\delta + \varepsilon^{-1}\vartheta''(\delta/\varepsilon)d\delta \otimes d\delta.$$

Put $b_1 = \|\vartheta'\|_\infty$, $b_2 = \|\vartheta''\|_\infty$, and $C_\vartheta = 1 + b_1 + 2b_2 + 2b_1^2$. On the transition, $\varepsilon^{-1} \leq 2/\delta$ and $\delta_\varepsilon \geq \delta/2$. The displayed derivative identities give

$$|\operatorname{Hess}\delta_\varepsilon| + \frac{|d\delta_\varepsilon|^2}{\delta_\varepsilon} \leq b_1|\operatorname{Hess}\delta| + (2b_2 + 2b_1^2)\frac{|d\delta|^2}{\delta} \leq C_\vartheta\left(|\operatorname{Hess}\delta| + \frac{|d\delta|^2}{\delta}\right).$$

On $\{\delta \leq \varepsilon\}$ the left side vanishes; on $\{\delta \geq 2\varepsilon\}$ the regularization equals δ. Thus the last bound holds on the entire sphere. The quotient on the right is extended as $4\lambda|df|^2$ across F. Take $M_1 \geq MC_\vartheta$.

The functions $f, \xi f^2$ and their derivatives are fixed and bounded. The displayed estimates give constants $C_1, C_2 > 0$, independent of λ and ε, such that

$$\|A\|_{C^2} \leq C_1\lambda, \qquad \|\delta_\varepsilon\|_{C^2} \leq C_2(\lambda + \varepsilon).$$

Choose $\lambda > 0$ so small that $(C_1 + C_2)\lambda < \eta/2$ and $C_2\lambda < \delta_0/2$. Then choose $\varepsilon_{\mathrm{adm}} > 0$ with

$$C_2\varepsilon_{\mathrm{adm}} < \min\{\eta/2, \delta_0/2\}.$$

This gives the C^2 and value bounds in (5.9) for $0 < \varepsilon < \varepsilon_{\mathrm{adm}}$. The functions

$$r \longmapsto e^{-2/r^2}r^{-4}, \qquad r \longmapsto e^{-2/r^2}|\log\sin r|$$

extend continuously by zero at $r = 0$ and are bounded on $[0, R_*]$. A further decrease of λ gives (7.8) simultaneously for all i and preserves the preceding bounds with the same $\varepsilon_{\mathrm{adm}}$. □

8. Annular replacement with a round inner boundary

Choose ν_j recursively so that

$$0 < \nu_j < \min\left\{\varepsilon_{\mathrm{adm}},\, 2^{-j},\, \frac{1}{2}\min_{1\leq i\leq j}\delta_i,\, \frac{1}{2}\nu_{j-1}\right\}, \tag{8.1}$$

omitting the last entry when $j = 1$. Each minimum is positive. Hence $\nu_j \downarrow 0$, each regularization is admissible, and $\delta_{\nu_j} = \delta$ on $\bigcup_{i=1}^j B^\gamma_{20R_i}(x_i)$ by (7.7). Let $\mathcal{G}_j = G[\delta_{\nu_j}, A]$ on S^n. Then $\operatorname{Ric}_{\mathcal{G}_j} \geq \mathcal{G}_j$. On the region $x \in B_{20R_i}(x_i)$, $t \leq t_0$, and $j \geq i$, the metric is exactly

$$dt^2 + \cos^2 t\,\gamma + b_{\delta_i}(t)^2 d\theta^2. \tag{8.2}$$

The previously imposed bounds on R_* give $20R_i < t_0$ and place the annulus parameters below in the range of Proposition 6.3.

Lemma 8.1. *The radius-R_i sphere about $x_i \in \Sigma_0$ in* (8.2) *has induced metric*

$$\sin^2 R_i\,\widetilde{h}_i, \tag{8.3}$$

where, on the join $S^{n-1} = S^{n-3} * S^1$,

$$\widetilde{h}_i = du^2 + \cos^2 u\, g_{S^{n-3}} + \sin^2 u\left(1 + \frac{\sin^2 u}{e_i^2}\right)^{-\delta_i} d\theta^2, \qquad e_i = \frac{e_{\delta_i}}{\sin R_i}. \tag{8.4}$$

The metric $\widetilde{h}_i$ is smooth and satisfies

$$\sec(\widetilde{h}_i) \geq 1 - \delta_i, \qquad \mathrm{Vol}(S^{n-1}, \widetilde{h}_i) \leq 10q\sigma_{n-1}. \tag{8.5}$$

The complement of the open radius-R_i ball has outward second fundamental form at least $-\cot R_i$ *times the boundary metric* (8.3).

Proof. The base metric $dt^2 + \cos^2 t\gamma$ is a round hemisphere. In its polar coordinates about x_i,

$$\sin t = \sin r \sin u, \qquad dt^2 + \cos^2 t\gamma = dr^2 + \sin^2 r(du^2 + \cos^2 u g_{S^{n-3}}).$$

Substitution of (4.1) gives (8.4) at $r = R_i$. The same substitution gives $g_{rr} = 1$ and $g_{rA} = 0$ for every angular index A. Hence $\Gamma^\alpha_{rr} = 0$ in these coordinates, and the radial curves are unit-speed geodesics. The projection

$$\varpi([t, x, \theta]) = (\cos t\, x, \sin t) \in S^{n-1}_+$$

is defined on the join quotient. Its differential satisfies

$$|d\varpi(T\partial_t + Y + \vartheta\partial_\theta)|^2 = T^2 + \cos^2 t\,|Y|^2_\gamma \leq c_0^{-2}\big(T^2 + a^2|Y|^2_\gamma + b^2\vartheta^2\big).$$

Integration along curves, with the induced metrics on the collapsed strata, makes ϖ globally c_0^{-1}-Lipschitz. In polar coordinates about x_i, $t \leq r$ and $d_\gamma(x, x_i) \leq r$. Thus the base ball $r \leq 10R_i$ lies where $\delta_{v_j} = \delta_i$ and $A = 0$. Any curve from x_i exiting this ball has length at least $10c_0R_i > R_i$. Inside it, $|dr| = 1$, while each radial segment has length r. It follows that the full ambient distance from x_i equals r for $r \leq R_i$. The sphere used below therefore has precisely the stated induced metric and normal.

For the metric in (8.4), set $a = \cos u$ and let b be its circle warping. Then

$$b'/b = (1 + z)\cot u, \quad z = -\delta_i \frac{\sin^2 u}{e_i^2 + \sin^2 u}, \quad -\delta_i \leq z \leq 0, \quad z' \leq 0.$$

Choose a geodesic orthonormal frame $e_1, \dots, e_{n-3}$ on the unit S^{n-3} at the point under consideration, and put $E_0 = \partial_u$, $E_a = a^{-1}e_a$, and $E_\theta = b^{-1}\partial_\theta$. Koszul's formula gives, at that point,

$$\nabla_{E_0}E_a = \nabla_{E_0}E_\theta = 0, \qquad \nabla_{E_a}E_0 = \frac{a'}{a}E_a, \qquad \nabla_{E_\theta}E_0 = \frac{b'}{b}E_\theta,$$

$$\nabla_{E_a}E_b = -\delta_{ab}\frac{a'}{a}E_0, \qquad \nabla_{E_\theta}E_\theta = -\frac{b'}{b}E_0, \qquad \nabla_{E_a}E_\theta = \nabla_{E_\theta}E_a = 0.$$

When $n = 5$, also use $R^{S^2}(e_1, e_2)e_2 = e_1$. The resulting curvature operator is diagonal on

$$E_0 \wedge TS^{n-3}, \quad \Lambda^2 TS^{n-3}, \quad TS^{n-3} \wedge \mathbb{R}E_\theta, \quad \mathbb{R}(E_0 \wedge E_\theta),$$

with respective eigenvalues

$$-\frac{a''}{a} = 1, \qquad \frac{1 - (a')^2}{a^2} = 1, \qquad -\frac{a'b'}{ab} = 1 + z, \qquad -\frac{b''}{b} = 1 + z - z'\cot u - z(1 + z)\cot^2 u.$$

For $n = 4$, $\Lambda^2 TS^1 = 0$, so the second block is absent. Every off-diagonal entry vanishes. Since $z' \leq 0$, $-\delta_i \leq z \leq 0$, and $1+z > 0$, the last two correction terms $-z'\cot u$ and $-z(1+z)\cot^2 u$ are nonnegative. All the displayed eigenvalues that occur are at least $1 - \delta_i$. If ω is the unit simple two-vector of a plane and $\omega = \sum_\ell c_\ell\omega_\ell$ in an orthonormal eigenbasis, then

$$\sec(\omega) = \sum_\ell \lambda_\ell c_\ell^2 \geq (1 - \delta_i)\sum_\ell c_\ell^2 = 1 - \delta_i.$$

Near $u = 0$, $b(u) = u + O(u^3)$ is odd, and a is even. Near $u = \pi/2$, put $v = \pi/2 - u$. Then $a = \sin v$ and $b(\pi/2 - v)$ is a positive smooth function of v^2. In Cartesian coordinates $y = v\xi \in \mathbb{R}^{n-2}$, $|\xi| = 1$,

$$dv^2 + \sin^2 v\, g_{S^{n-3}} = \left(\frac{\sin v}{v}\right)^2 \sum_{a=1}^{n-2} dy_a^2 + \frac{1 - (\sin v/v)^2}{v^2}\left(\sum_{a=1}^{n-2} y_a\, dy_a\right)^2.$$

The two scalar coefficients are smooth functions of v^2. At $u = 0$, the expression $b(u) = u + u^3 B(u^2)$ gives the smooth normal two-plane tensor as in Lemma 5.3. Hence the metric is smooth at both collapsed strata.

The outward principal curvatures of the removed ball are $\cot R_i$ in the base directions and $(1 - \delta_i Z)\cot R_i$ in the circle direction. Reversing the normal proves the complement estimate.

Finally, for $0 < u \le \pi/2$,

$$(1 + \sin^2 u/e_i^2)^{-\delta_i/2} \le e_i^{\delta_i} \sin^{-\delta_i} u = q(\sin R_i)^{-\delta_i} \sin^{-\delta_i} u.$$

The round join volume is $\sigma_{n-1} = 2\pi\sigma_{n-3}/(n-2)$. Thus

$$\begin{aligned}\frac{\mathrm{Vol}(\widetilde{h}_i)}{\sigma_{n-1}} &\le (n-2)q(\sin R_i)^{-\delta_i} \int_0^{\pi/2} \cos^{n-3} u \, \sin^{1-\delta_i} u \, du \\ &\le 3\pi q < 10q.\end{aligned}$$

Here $(\sin R_i)^{-\delta_i} \le 2$, $1 - \delta_i > 0$, and both trigonometric factors in the integrand are at most one. For $n = 4$ one may instead integrate explicitly to obtain

$$\frac{\mathrm{Vol}(\widetilde{h}_i)}{\sigma_3} \le \frac{2q(\sin R_i)^{-\delta_i}}{2 - \delta_i} \le \frac{16q}{7}.$$

This proves (8.5). □

Lemma 8.2. *Put $h_i = (1 - \delta_i)\widetilde{h}_i$. There is a smooth Ricci-positive path from $\tau_i^2 g_{S^{n-1}}$ to h_i with constant total volume and* $\mathrm{Ric} \ge (n-2)g$ *along the path, where*

$$\tau_i^{n-1} = \frac{\mathrm{Vol}(h_i)}{\sigma_{n-1}}, \qquad 0 < \tau_i < \beta/2.$$

Proof. In (8.4), keep e_i fixed and replace the exponent δ_i by $p \in [0, \delta_i]$. Denote the resulting metric by $h(p)$. The curvature calculation in Lemma 8.1 gives $\mathrm{Ric}_{h(p)} \ge (n-2)(1-\delta_i)h(p)$. The volume $\mathrm{Vol}(S^{n-1}, h(p))$ is nonincreasing in p. Let $V_i = (1-\delta_i)^{(n-1)/2}\,\mathrm{Vol}(h(\delta_i))$ and set

$$\widehat{h}(p) = \left(\frac{V_i}{\mathrm{Vol}(h(p))}\right)^{2/(n-1)} h(p).$$

The metric scaling factor is at most $1 - \delta_i$, so $\mathrm{Ric}_{\widehat{h}(p)} \ge (n-2)\widehat{h}(p)$. For every $p \in [0, \delta_i]$, $\mathrm{Vol}(S^{n-1}, \widehat{h}(p)) = V_i$; the endpoint metrics are $\widehat{h}(\delta_i) = h_i$ and $\widehat{h}(0) = \tau_i^2 g_{S^{n-1}}$. Equations (7.1) and (8.5) give

$$\tau_i^{n-1} \le 10q < \beta^{n-1}/100 < (\beta/2)^{n-1}.$$

A smooth monotone reparametrization constant near the endpoints gives the required path. □

Proposition 8.3. *For every i the metric* (8.2) *on the radius-R_i ball can be replaced by a smooth annulus with one inner round boundary of radius ρ_i, such that*

$$\mathrm{II}_{\mathrm{in}} \ge -\frac{\beta}{\rho_i}\rho_i^2 g_{S^{n-1}}, \qquad 0 < \rho_i < \min\{2^{-i}R_i, \rho_{i-1}/2, 1\}, \tag{8.6}$$

$$\mathrm{Ric} \ge \tfrac{1}{2} g \text{ on the annulus and its outer smoothing,} \tag{8.7}$$

$$\mathrm{diam}_{\mathrm{int}}(\text{annulus}) \le CR_i, \qquad \mathrm{Vol}(\text{annulus}) \le CR_i^n. \tag{8.8}$$

For $i = 1$, omit the condition involving ρ_0. The replacement and its outer smoothing are contained inside the radius-$2R_i$ sphere and can be fixed once for all $j \ge i$.

Proof. Apply Lemma 6.1 to the metric path supplied by Lemma 8.2. The resulting path has final metric $\phi_i^* h_i$, where $\phi_i : S^{n-1} \to S^{n-1}$ is the final diffeomorphism of the isotopy supplied by Lemma 6.1. The attaching map from the annular boundary to the boundary of the removed ball is ϕ_i.

Choose

$$\varepsilon_i = \arcsin\left(\frac{\sin R_i}{\sqrt{1-\delta_i}}\right) \in (0, \pi/2).$$

Since $R_i < 1/10$ and $\delta_i \leq R_i^4/100 < 1/100$, one has

$$4(1-\delta_i)\cos^2 R_i \geq 4(99/100)(1-R_i^2) > 1.$$

Consequently the inverse-sine argument is in $(0, 1)$ and

$$\frac{\sin R_i}{\sqrt{1-\delta_i}} \leq \sin(2R_i), \qquad \sin^2\varepsilon_i = \frac{\sin^2 R_i}{1-\delta_i}.$$

For the chosen R_*, $R_i \leq \varepsilon_i \leq 2R_i < \varepsilon_*$. Since $h_i = (1-\delta_i)\widetilde{h}_i$,

$$\sin^2\varepsilon_i\, \phi_i^* h_i = \phi_i^*\left(\sin^2 R_i\, \widetilde{h}_i\right).$$

Thus the induced metrics agree under the attaching map ϕ_i. Write $h_\partial = \sin^2\varepsilon_i\, \phi_i^* h_i$ for the common boundary metric expressed on the annular boundary. By Proposition 6.3, the annular second fundamental form is at least

$$\left(\cot\varepsilon_i + \frac{2\varepsilon_i^4}{\sin\varepsilon_i}\right) h_\partial.$$

On the other hand,

$$\begin{aligned}\cot R_i - \cot\varepsilon_i &= \frac{\cos R_i - \sqrt{\cos^2 R_i - \delta_i}}{\sin R_i} \\ &= \frac{\delta_i}{\sin R_i(\cos R_i + \sqrt{\cos^2 R_i - \delta_i})} \leq \frac{4\delta_i}{R_i} \leq \frac{R_i^3}{25}.\end{aligned}$$

Since $2\varepsilon_i^4/\sin\varepsilon_i \geq 2R_i^3$, the two outward second fundamental forms in the common boundary parametrization satisfy

$$\mathrm{II}_{\mathrm{ann,out}} + \phi_i^*\mathrm{II}_{\mathrm{ext}} \geq \left(2 - \frac{1}{25}\right) R_i^3 h_\partial = \frac{49}{25} R_i^3 h_\partial > 0. \tag{8.9}$$

The inequality $(n-1) - C_{\mathrm{ann}}(2R_*)^2 > 2$ from (7.2) gives the annular Ricci bound $\mathrm{Ric} \geq 2g$. The metric on the original sphere satisfies $\mathrm{Ric} \geq g$. Apply Lemma 3.1, with $\lambda = 3/4$ and $\lambda' = 1/2$, to compact collars of the two outer boundary components, and extend by the unchanged metrics outside those collars. Choose the annulus-side collar disjoint from a fixed inner round subcylinder; on the original sphere, require the smoothing support to lie in $r < 3R_i/2$. The resulting metric satisfies (8.7); its inner round subcylinder is unchanged. Use Lemma 3.2 at this outer seam with error $R_i^{n-2}/2$; this only decreases the admissible smoothing widths.

In the inner round region, $F'(s) \to \tau_i < \beta/2$ and $F(s) \to 0$. Choose $s_i > 0$ sufficiently small that $0 < F'(s_i) < \beta$ and all radius inequalities in (8.6) hold for $\rho_i = F(s_i)$. The outward second fundamental form of the retained annulus at $s = s_i$ is $-F'(s_i)\rho_i^{-1}(\rho_i^2 g_{S^{n-1}})$, which gives the boundary-form inequality in (8.6).

The intrinsic estimates before smoothing follow from Proposition 6.3 and $\varepsilon_i \leq 2R_i$. Take both smoothing widths smaller than $R_i/100$, and impose the 2-bi-Lipschitz collar bound. The common boundary tensor is $h_\partial = \sin^2\varepsilon_i\, \phi_i^* h_i$. Consequently

$$\mathrm{Vol}_{n-1}(h_\partial) = \sin^{n-1} R_i\ \mathrm{Vol}(\widetilde{h}_i) \leq 10 q \sigma_{n-1} R_i^{n-1}. \tag{8.10}$$

Since $\mathrm{Ric}_{h_i} \geq (n-2)h_i$, Bonnet–Myers gives $\mathrm{diam}(h_\partial) \leq \pi \sin\varepsilon_i \leq 2\pi R_i$. If a collar has total normal width $\ell_i \leq 2R_i/100$, its product metric has volume $\ell_i \mathrm{Vol}_{n-1}(h_\partial)$. The quadratic-form upper bound $g \leq 4(ds^2 + h_\partial)$ therefore gives

$$\mathrm{Vol}_n(\mathrm{collar}) \leq 2^n \ell_i \mathrm{Vol}_{n-1}(h_\partial) \leq C R_i^n.$$

Normal segments and curves in one boundary slice give intrinsic diameter at most $2\ell_i + 2\,\mathrm{diam}(h_\partial) \le CR_i$ for the smoothed collar. Joining the two sides through this collar proves (8.8). All data used depend on i, not on $j \ge i$. □

9. Closed realizations and uniform metric estimates

For each $j \ge 1$, remove the first j balls from $(S^n, \mathcal{G}_j)$ and apply Proposition 8.3. Attach $(P, \rho_i^2 k)$ to the i-th round inner boundary using the fixed round boundary parametrization of P from Lemma 2.1. The boundary tensors of $(P, \rho_i^2 k)$ are

$$(\rho_i^2 k)|_{\partial P} = \rho_i^2 g_{S^{n-1}}, \qquad \mathrm{II}_{\partial P, \rho_i^2 k} \ge \frac{\alpha}{\rho_i} \rho_i^2 g_{S^{n-1}}, \qquad \mathrm{Ric}_{\rho_i^2 k} \ge \frac{\mu}{\rho_i^2}(\rho_i^2 k).$$

The boundary-form sum is at least

$$\frac{\alpha - \beta}{\rho_i} \rho_i^2 g_{S^{n-1}} > 0. \tag{9.1}$$

Choose the smoothing along the inner gluing hypersurface once for each i, with both normal support widths less than $\min\{R_i, \rho_i\}/100$, disjoint from the outer smoothing collar, and with the product comparison of Lemma 3.1. Such widths exist because the two gluing hypersurfaces are disjoint and compact with positive normal collar radii. At this inner seam also impose (3.3) with error $R_i^{n-2}/2$. Since $\rho_i < 1$, Lemma 3.1 retains a common lower bound

$$\mathrm{Ric}_{g_j} \ge \kappa g_j, \qquad \kappa = \tfrac{1}{8} \min\{1, \mu\} > 0 \tag{9.2}$$

on the resulting closed connected smooth manifold M_j. Indeed, the annulus and the scaled copy of P have Ricci lower coefficients at least $1/2$ and μ, respectively, and both exceed 2κ; apply Lemma 3.1 with $\lambda = 2\kappa$ and $\lambda' = \kappa$. The operations for distinct i have disjoint supports. For each i, fix the boundary parametrization, the two Gaussian collar maps, the interpolation and mollification widths, and the cutoff functions when that piece is first attached. Equation (8.1) makes all these collar tensors identical for $j \ge i$. Their signed-collar charts therefore give the same smooth structure and the same smoothed tensor on the completed piece. No earlier piece is smoothed again.

Let $W_i \subset M_j$, $j \ge i$, consist of the attached copy of P, its annulus, and the unchanged outer annular region up to the coordinate sphere $r_i = 2R_i$. Then W_i is a fixed smooth domain, with one fixed metric h_i^W, independent of $j \ge i$. It is obtained from P by attaching a collar, so

$$W_i \cong P, \qquad \partial W_i \cong S^{n-1}, \qquad \mathrm{diam}_{\mathrm{int}}(W_i) \le C_P R_i. \tag{9.3}$$

The diameter bound follows from (8.8), $\rho_i < R_i$, the fixed intrinsic diameter of P, and the collar product estimates in Lemma 3.1. More explicitly, a point in the unchanged part of the scaled copy of P can be joined to the inner gluing collar along an initial segment of an intrinsic path in P of scaled length at most $\rho_i\,\mathrm{diam}(P, k) + \varepsilon$. The remaining collar and annular portions have lengths at most CR_i. Let $\varepsilon \downarrow 0$, and join two such paths through the common outer slice. All these curves lie in W_i; their lengths therefore bound its intrinsic diameter.

The domains W_i are pairwise disjoint. A cylinder attached to P by a boundary diffeomorphism is diffeomorphic to P with an extended collar: use the given diffeomorphism on the cylinder and the collar coordinate on P. Thus $W_i \cong P$ independently of the boundary parametrization.

Lemma 9.1. *There are smooth maps $\Phi_j : M_j \to \mathbb{R}^{n-2}$ such that*

$$\mathrm{Lip}(\Phi_j) \le 32, \qquad \Phi_j|_F = z, \tag{9.4}$$

and Φ_j is constant on each W_i, $i \le j$.

Proof. On the join cylinder set $\Pi(t, x, \theta) = \cos t\, z(x)$, where $x = (x_0, z) \in S^{n-2} \subset \mathbb{R}^{n-1}$. This is the restriction of a linear coordinate projection of the unit sphere $S^n \subset \mathbb{R}^{n+1}$, and is therefore smooth across the join collapses. For a tangent vector $T\partial_t + Y + \vartheta\partial_\theta$,

$$d\Pi(T\partial_t + Y + \vartheta\partial_\theta) = -T \sin t\, z + \cos t\, dz(Y).$$

The coordinate projection on S^{n-2} satisfies $|z| \leq 1$ and $\|dz\|_{\mathrm{op},\gamma} \leq 1$. Cauchy–Schwarz and $a \geq c_0 \cos t$ therefore give

$$|d\Pi(T\partial_t + Y + \vartheta\partial_\theta)| \leq \sqrt{\sin^2 t\,|z|^2 + \frac{\cos^2 t}{a^2}}\sqrt{T^2 + a^2|Y|^2_\gamma + b^2\vartheta^2},$$

$$\|d\Pi\|_{\mathrm{op},\mathcal{G}_j} \leq \sqrt{1 + c_0^{-2}} < 2.$$

On $2R_i \leq r_i \leq 4R_i$, put

$$\Phi_j = \Pi(x_i) + \zeta(r_i)\big(\Pi - \Pi(x_i)\big),$$

where $\zeta = 0$ near $2R_i$, $\zeta = 1$ near $4R_i$, $0 \leq \zeta \leq 1$, and $|\zeta'| \leq 2/R_i$. Extend it by $\Pi(x_i)$ on W_i and by Π off these annuli. The round radial coordinate satisfies $|dr_i|_{\mathcal{G}_j} \leq c_0^{-1}$ and $|\Pi - \Pi(x_i)| \leq 4R_i$. Thus

$$\|d\Phi_j\|_{\mathrm{op}} \leq 2 + 8c_0^{-1} < 32.$$

The transition annuli lie outside all smoothing supports and are pairwise disjoint. Hence the map is smooth globally and integration along curves proves its Lipschitz bound. Every point of F lies outside these annuli, and at $t = 0$ the map Π equals z. □

Lemma 9.2. *For the coordinate inclusion $\iota_j : F \hookrightarrow M_j$,*

$$\frac{1}{32}|z - z'| \leq d_{M_j}(\iota_j(z), \iota_j(z')) \leq 2|z - z'| \qquad (z, z' \in F). \tag{9.5}$$

Moreover,

$$\sup_{w \in W_i} d_{M_j}(w, \iota_j(z_i)) \leq C_P R_i + 2d_\gamma(x_i, z_i) \qquad (j \geq i). \tag{9.6}$$

Proof. For $z, z' \in F$, Lemma 9.1 gives

$$|z - z'| = |\Phi_j(\iota_j(z)) - \Phi_j(\iota_j(z'))| \leq 32 d_{M_j}(\iota_j(z), \iota_j(z')).$$

This is the lower bound in (9.5).

The unchanged part of Σ_0 has induced metric γ. In the chart $z \mapsto (\sqrt{1 - |z|^2}, z)$,

$$\gamma_z(v, v) = |v|^2 + \frac{\langle z, v\rangle^2}{1 - |z|^2} \leq (1 - 10^{-2})^{-1}|v|^2 \qquad (|z| < 1/10).$$

The coordinate segment from z to z' lies in this chart. Its round length is at most $(1 - 10^{-2})^{-1/2}|z - z'|$. The shorter round geodesic has no greater length. For each of the finitely many γ-balls in Σ_0 $B_{2R_i}(x_i)$, $i \leq j$, which it enters, replace the interior segment by a shortest arc in that boundary sphere. A geodesic ball of radius $r < \pi/2$ is geodesically convex, so there is at most one such segment per ball.

If two boundary points have angular separation $\vartheta \leq \pi$ as seen from the center and ambient spherical distance ℓ, then

$$\sin(\ell/2) = \sin r \sin(\vartheta/2), \qquad \sin r\, \vartheta \leq \frac{\pi}{2}\ell.$$

The second inequality follows from $\sin(\vartheta/2) \geq \vartheta/\pi$ and $\sin(\ell/2) \leq \ell/2$. Thus the modified curve has length at most $\pi/2$ times the original length. Disjointness of the enlarged balls ensures that every replacement arc avoids the other modified regions. Since $(\pi/2)(1 - 10^{-2})^{-1/2} < 2$, the upper bound follows.

For (9.6), join z_i to x_i by its shorter round geodesic, and stop at the first point on $\partial B_{2R_i}(x_i)$. Replace each portion inside another deleted γ-ball by the boundary arc just constructed. This gives a path from z_i to a point of ∂W_i with length at most $2d_\gamma(x_i, z_i)$. The diameter estimate in (9.3) completes the proof. All centers lie in the fixed coordinate chart chosen in Lemma 7.1. □

Lemma 9.3. *There are points $p_j \in M_j$ and $v > 0$ such that $\mathrm{Vol}_{g_j}(B_1(p_j)) \geq v$ for every j. The sequence has a subsequence converging to a compact noncollapsed n-dimensional Ricci limit.*

Proof. Choose a point p in the fixed region $3t_0 < t < \pi/2$ and a sufficiently small round coordinate ball B around it, with radial paths of length less than 1. The metric on this region is G_q for every j, and the region is disjoint from every annular replacement. The same coordinate point defines p_j, and $B \subset B_1^{M_j}(p_j)$. Its volume $v = \mathrm{Vol}_{G_q}(B) > 0$ is independent of j.

Bonnet–Myers applied to (9.2) gives

$$\operatorname{diam}(M_j, g_j) \le D := \pi\sqrt{(n-1)/\kappa}.$$

Since $\mathrm{Ric} \ge 0$ and $B_{D+1}(y) = M_j$, Bishop–Gromov gives, for $0 < r < 1$,

$$\mathrm{Vol}(B_r(y)) \ge \frac{r^n}{(D+1)^n}\mathrm{Vol}(M_j) \ge \frac{vr^n}{(D+1)^n}, \qquad \mathrm{Vol}(M_j) \le \omega_n(D+1)^n.$$

If $y_1, \dots, y_m$ is an r-separated set, its open $r/2$-balls are pairwise disjoint. Therefore

$$m\frac{v(r/2)^n}{(D+1)^n} \le \sum_{a=1}^{m} \mathrm{Vol}(B_{r/2}(y_a)) \le \omega_n(D+1)^n, \qquad m \le \frac{2^n\omega_n(D+1)^{2n}}{vr^n}.$$

A maximal such set is an r-net. The uniform diameter and net bounds imply Gromov–Hausdorff precompactness [1, Section 7.4]; mark the points p_j and pass to a convergent subsequence. Its limit X is compact. The approximants are closed, smooth, complete, and connected, with the bounds (1.1). Thus X is a noncollapsed Ricci limit. Theorem 5.9 of [4, p. 434] gives $\dim_{\mathcal{H}} X = n$ and identifies the limiting volume with $\mathcal{H}^n$. □

10. Total scalar and Ricci curvature

The scalar curvature convention in this section is $\mathrm{Scal}_g = \mathrm{tr}_g \mathrm{Ric}_g$. In particular, the unit round d-sphere has scalar curvature $d(d-1)$. All constants in this section are independent of the indices j and i, the inner truncation radius of an annulus, and the derivatives of the individual boundary path, unless a fixed collar is explicitly under consideration.

Lemma 10.1. *For the backgrounds $\mathcal{G}_j$ in Section 8,*

$$0 \le \int_{S^n} \mathrm{Scal}_{\mathcal{G}_j}\, dv_{\mathcal{G}_j} \le C_0.$$

The assertion holds for every compact empty-interior set F allowed in Theorem 1.1.

Proof. Put $m = n-2$. On $0 < t < T := t_0/2$, write

$$G = \bar{G} + b^2 d\theta^2, \qquad \bar{G} = dt^2 + a^2\gamma, \qquad a = \cos t\, e^{-U}, \qquad U = A\Psi_{\delta_{v_j}}, \qquad b = b_{\delta_{v_j}}.$$

Tracing the circle-warping formulas in Lemma 5.1 gives

$$\mathrm{Scal}_G = \mathrm{Scal}_{\bar{G}} - 2b^{-1}\Delta_{\bar{G}} b, \qquad dv_G = b\,a^m\, dt\, dv_\gamma\, d\theta. \tag{10.1}$$

Tracing the Gaussian and conformal formulas gives

$$\begin{aligned}
\mathrm{Scal}_{\bar{G}} &= \mathrm{Scal}_{a^2\gamma} - 2m\frac{a_{tt}}{a} - m(m-1)\left(\frac{a_t}{a}\right)^2,\\
\mathrm{Scal}_{a^2\gamma} &= a^{-2}\big(m(m-1) + 2(m-1)\Delta_\gamma U - (m-1)(m-2)|d_x U|_\gamma^2\big).
\end{aligned} \tag{10.2}$$

By (5.14), the spatial derivatives in this expression are bounded uniformly in j. The functions U, U_t are uniformly bounded by (5.2). Also $U_{tt} = AL_{\delta_{v_j}}^2$ and $L_\delta(t) \le 1 + \log 2 + |\log t|$ by Lemma 4.2. The identities

$$\frac{a_t}{a} = -\tan t - U_t, \qquad \frac{a_{tt}}{a} = -1 - U_{tt} + 2\tan t\, U_t + U_t^2$$

therefore imply

$$|\mathrm{Scal}_{\bar{G}}(t,x)| \le C(1 + |\log t|^2). \tag{10.3}$$

The function $b = b_{\delta_{v_j}}$ satisfies $0 < b \le \sin t$, $0 \le b_t \le 1$, and $b_t(0,x) = 1$. Indeed (4.3) gives $b_t = (1-\delta_{v_j}Z)\cos t\,(1+\sin^2 t/e^2_{\delta_{v_j}})^{-\delta_{v_j}/2}$, whose factors lie in $[0,1]$. Moreover

$$\Delta_{\bar G} b = b_{tt} + m(a_t/a)b_t + \Delta_{a^2\gamma} b, \qquad \int_{S^m} \Delta_{a^2\gamma} b \, dv_{a^2\gamma} = 0.$$

The last identity follows by integrating the divergence on the closed sphere. Integrate (10.1) first over $\epsilon < t < T$. For fixed j, $a(0,x) = 1$ and $b_t(0,x) = 1$ uniformly on the compact sphere. Letting $\epsilon \downarrow 0$ gives

$$\begin{aligned}\int_{\{t<T\}} \mathrm{Scal}_G \, dv_G = 2\pi \int_0^T \int_{S^m} b \, \mathrm{Scal}_{\bar G} \, a^m \, dv_\gamma \, dt \\ - 4\pi \int_{S^m} a(T,x)^m b_t(T,x) \, dv_\gamma + 4\pi\sigma_m.\end{aligned} \tag{10.4}$$

The contribution from the upper boundary is nonpositive. Since $a \le 1$ and $b \le t$,

$$\int_{\{t<T\}} \mathrm{Scal}_G \, dv_G \le C \int_0^T t(1+|\log t|^2)\, dt + 4\pi\sigma_m < \infty,$$

with a bound independent of j. On $T \le t \le 3t_0$, $\sin t \ge \sin T > 0$ and $\cos t \ge \cos(3t_0) > 0$. The formulas in Lemmas 4.1 and 4.2 bound all coefficient derivatives through order two there independently of j; factors $\delta^{-a}e^{-2c/\delta}$ remain bounded as $\delta \downarrow 0$. The fixed cutoffs preserve these bounds. The lower tensor comparison in Lemma 5.3 then bounds the inverse matrices in a fixed finite atlas. Thus the metrics, their inverses, and their derivatives through order two are uniformly bounded on this entire strip, including $[T, t_0]$. For $t \ge 3t_0$ they equal the fixed smooth outer metric. These regions have uniformly bounded scalar integrals. Finally $\mathrm{Ric}_{\mathcal{G}_j} \ge \mathcal{G}_j$, so scalar curvature is nonnegative everywhere. This proves both inequalities. ☐

Lemma 10.2. *Let $d = n-1$. Every common-volume-form path used in Proposition 8.3 satisfies*

$$0 \le \int_{S^d} \mathrm{Scal}_{h_u} \, dv_{h_u} \le C_d := (d-1)(d-2)\sigma_d + 4\pi\sigma_{d-2}. \tag{10.5}$$

Proof. Before volume normalization the path is

$$h(p) = du^2 + \cos^2 u \, g_{S^{d-2}} + b_p(u)^2 d\theta^2, \qquad b_p(u) = \sin u(1+\sin^2 u/e_i^2)^{-p/2}, \quad 0 \le p \le \delta_i,$$

where e_i is fixed along this path. Its base $du^2 + \cos^2 u \, g_{S^{d-2}}$ is the unit round hemisphere of dimension $d-1$ and has scalar curvature $(d-1)(d-2)$. Apply (10.1) on $\epsilon < u < \pi/2 - \epsilon$ and integrate the base Laplacian. At $u = 0$, $b_p'(0) = 1$. At $u = \pi/2$, b_p' is bounded and $\cos^{d-2} u \to 0$, since $d \ge 3$. The two boundary limits give the exact formula

$$\int_{S^d} \mathrm{Scal}_{h(p)} \, dv_{h(p)} = (d-1)(d-2)\,\mathrm{Vol}(h(p)) + 4\pi\sigma_{d-2}. \tag{10.6}$$

Since $b_p \le \sin u$, $\mathrm{Vol}(h(p)) \le \sigma_d$. The curvature calculation in Lemma 8.1 also gives $\mathrm{Scal}_{h(p)} > 0$. In Lemma 8.2, the metric is multiplied by a positive number $\lambda(p) \le 1$. Under this change the scalar integral is multiplied by $\lambda(p)^{(d-2)/2} \le 1$. A diffeomorphism pullback leaves the integral unchanged, by the change-of-variables formula. Thus volume normalization, reparametrization, and Moser pullback preserve the bound (10.5). ☐

Lemma 10.3. *For each annulus in Proposition 8.3, before either boundary smoothing,*

$$0 \le \int_{\text{annulus}} \mathrm{Scal} \, dv \le CR_i^{n-2}.$$

The bound is uniform in the inner truncation radius.

Proof. Use the unscaled tensor $G_k = dr^2 + a(r)^2 g(r)$ from Proposition 6.3 and put $d = n - 1$. Write $dv_{g(r)} = \nu$, $V_i = \int_{S^d} \nu$, and $S(r) = \int_{S^d} \mathrm{Scal}_{g(r)}\ \nu$. By Lemma 10.2, $0 \le S(r) \le C_d$; also $V_i \le \sigma_d$. Tracing (6.6)–(6.8) and using $\mathrm{tr}_g\, g' = 0$, $\mathrm{tr}_g\, g'' = |g'|_g^2$ gives

$$\mathrm{Scal}_{G_k} = a^{-2}\,\mathrm{Scal}_g - d(d-1)(a'/a)^2 - 2da''/a - \tfrac{1}{4}|g'|_g^2. \tag{10.7}$$

For any retained interval $[r_0, 1]$, integration by parts yields

$$\begin{aligned}\int_{[r_0,1]\times S^d} \mathrm{Scal}_{G_k}\ dv_{G_k} &= \int_{r_0}^1 a^{d-2} S(r)\, dr + d(d-1)V_i \int_{r_0}^1 a^{d-2}(a')^2\, dr \\ &\quad - 2dV_i[a^{d-1}a']_{r_0}^1 - \tfrac{1}{4}\int_{r_0}^1 a^d \int_{S^d} |g'|_g^2\ \nu\ dr. \end{aligned} \tag{10.8}$$

Indeed $(a^{d-1}a')' = (d-1)a^{d-2}(a')^2 + a^{d-1}a''$. The final integral in (10.8) is nonpositive. The explicit functions give

$$0 < a = s_k\chi \le r, \qquad |a'| = |\cos(kr)\chi + s_k\chi'| \le 1 + r|\chi'| \le 2.$$

Consequently the first two integrals and the boundary term together are at most $C_d/(d-1) + 12d\sigma_d$. This bound does not involve r_0 or derivatives of the path. The actual annulus tensor is $c_{\varepsilon_i}^2 G_{k_{\varepsilon_i}}$ and $c_{\varepsilon_i} \le 8\varepsilon_i \le 16R_i$. In dimension $n = d + 1$, a scalar integral is multiplied by $c_{\varepsilon_i}^{n-2}$. The claimed upper bound follows. The Ricci lower bound in Proposition 6.3 gives nonnegativity of the scalar curvature on the retained annulus. □

Proposition 10.4. *The smoothing choices in Sections 8 and 9 can be made once for each i so that all their stated conclusions hold and*

$$\int_{M_j} \mathrm{Scal}_{g_j}\ dv_{g_j} \le C_0 + C\sum_{i=1}^{j} R_i^{n-2} \le C_0 + \frac{CR_*^{n-2}}{2^{n-2}-1}. \tag{10.9}$$

Moreover $|\mathrm{Ric}_{g_j}|_{g_j} \le \mathrm{Scal}_{g_j}$ *pointwise. In particular,* (1.7) *holds.*

Proof. At the outer seam the common boundary area is at most CR_i^{n-1} by (8.10). The original boundary principal curvatures have absolute value at most $\cot R_i \le R_i^{-1}$. On the annular side the slice path is constant near its endpoint, so every principal curvature equals $a'(1)/\sin\varepsilon_i$. The bound $|a'| \le 2$ and $\sin\varepsilon_i \ge R_i/2$ give absolute value at most $4/R_i$. Thus, for the positive tensor $J_i = 2(\mathrm{II}_- + \mathrm{II}_+)$ at that seam,

$$0 < \int \mathrm{tr}\, J_i\, dv \le CR_i^{n-2}. \tag{10.10}$$

At the inner seam the area is $\sigma_{n-1}\rho_i^{n-1}$. The annular shape operator is $-F'(s_i)/\rho_i$ times the identity, with $0 < F'(s_i) < \beta$. The core shape operator has norm at most C_k/ρ_i, where C_k is the maximum norm of the fixed core boundary shape operator. Consequently

$$0 < \int \mathrm{tr}\, J_i^{\mathrm{in}}\, dv \le C\rho_i^{n-2} \le CR_i^{n-2}. \tag{10.11}$$

Apply Lemma 3.2 at each seam with error $R_i^{n-2}/2$. Both errors and all width restrictions are imposed when the i-th domain is constructed. The two seams have disjoint neighborhoods; the second choice leaves the first unchanged. For $j \ge i$ the collar tensors are identical, so the same choices apply.

The scaled core has scalar integral

$$\int_{(P,\rho_i^2 k)} \mathrm{Scal}\ dv = \rho_i^{n-2}\int_P \mathrm{Scal}_k\ dv_k \le CR_i^{n-2}.$$

By Lemmas 10.1 and 10.3 and (10.10)–(10.11), the background, the retained annuli, the cores, and the two smoothing collars have the bounds in (10.9). All these scalar curvatures are nonnegative; restricting a region decreases its integral of nonnegative scalar curvature. Finally $R_i < 2^{-i}R_*$, so $\sum_{i\ge1} R_i^{n-2} < R_*^{n-2}/(2^{n-2}-1)$. If $\lambda_1, \dots, \lambda_n \ge 0$ are the eigenvalues of $g_j^{-1}\,\mathrm{Ric}_{g_j}$, then

$$|\mathrm{Ric}_{g_j}|_{g_j} = (\textstyle\sum_a \lambda_a^2)^{1/2} \le \sum_a \lambda_a = \mathrm{Scal}_{g_j}.$$

This proves the remaining assertion. □

11. Proofs of the exceptional-set results

Proof of Theorem 1.1. Fix $n = 4$ or $n = 5$ and the stated compact set F. The selection of centers in Lemma 7.1 uses compactness to choose finite nets and empty interior to choose their approximating centers in the complement. The construction of Lemma 7.2 uses only the separation and proximity properties of those centers. The remaining parameter estimates require only the derivative bounds for that scale function. They impose no restriction on the Hausdorff dimension or Lebesgue measure of F. Thus Sections 7–9 apply to this F. At each seam make the scalar-integral choice of Lemma 3.2, as specified in Proposition 10.4. No previously fixed domain is changed.

By Lemma 9.3, a subsequence converges to a compact noncollapsed limit X. The fixed domains satisfy Proposition 2.3, and the markings satisfy Lemma 9.2. Put

$$e_i = C_P R_i + 2d_Y(x_i, z_i), \qquad \eta_i = \sup_{k\geq i} e_k.$$

Equations (7.3) and (7.5) give $e_i \to 0$. Hence $\eta_i < \infty$, $\eta_i \downarrow 0$, and (9.6) implies (2.4). Proposition 2.4 gives the bi-Lipschitz embedding and independent classes represented in every neighborhood of each point of $\iota(F)$. Proposition 2.9, with $k = 0$, proves (1.8) for the specified E and relative U. For $E = \emptyset$, a Euclidean-ball neighborhood would have zero second homology; thus $\iota(F) \subset S_{\mathrm{top}}(X)$. If a smaller relative neighborhood V of x contracted in a relative neighborhood U, the map $H_2(V;\mathbb{Z}) \to H_2(U;\mathbb{Z}) \to H_2(X;\mathbb{Z})$ would be zero, contrary to (1.8). This proves the asserted failure of local contractibility. The curvature and volume bounds are (9.2) and Lemma 9.3; Proposition 10.4 proves (1.7). □

Lemma 11.1. *There is a compact perfect nowhere-dense set $K \subset [0,1]$ with Lebesgue measure $1/2$. For every positive integer m and every similarity $T : \mathbb{R}^m \to \mathbb{R}^m$ of ratio $\rho > 0$, the set $F = T(K^m)$ is compact, has empty interior, and satisfies*

$$\mathcal{H}^m(F) = \rho^m 2^{-m} > 0. \tag{11.1}$$

Proof. Set $K_0 = [0,1]$. For $h \geq 1$, define K_h by removing from the center of each of the 2^{h-1} component intervals of K_{h-1} an open interval of length 4^{-h}. All component intervals of K_h have length

$$\ell_h = 2^{-h-1} + 2^{-2h-1}.$$

Indeed $\ell_0 = 1$ and $\ell_h = (\ell_{h-1} - 4^{-h})/2$; substitution gives the displayed formula, and $\ell_{h-1} > 4^{-h}$ verifies that every removal is possible. The decreasing intersection $K = \bigcap_h K_h$ is compact. Since

$$\mathcal{L}^1(K_h) = 2^h \ell_h = \tfrac{1}{2} + 2^{-h-1},$$

continuity of Lebesgue measure on decreasing compact sets gives $\mathcal{L}^1(K) = 1/2$. An interval in K would lie in a component of every K_h and have length at most $\ell_h \to 0$; hence K has empty interior. Every endpoint of a component of K_h belongs to K. For any $x \in K$, choose an endpoint of its level-h interval different from x. Its distance from x is at most ℓ_h. Thus K has no isolated points.

The Cartesian product K^m is compact. If it contained an open ball, its projection onto the first coordinate would contain an interval in K, which is impossible. The product formula for Lebesgue measure gives $\mathcal{L}^m(K^m) = 2^{-m}$. With the normalization of Hausdorff measure used here, $\mathcal{H}^m = \mathcal{L}^m$ on $\mathbb{R}^m$. A similarity multiplies both measures by ρ^m, proving (11.1). □

Lemma 11.2. *Let $N \geq 4$, and let X be a compact N-dimensional noncollapsed Ricci limit realized by closed manifolds with nonnegative Ricci curvature and a common finite diameter bound. Then $\mathcal{H}^{N-2}(S_{\mathrm{top}}(X)) < \infty$.*

Proof. Let D bound the diameters and let v be the based unit-ball volume bound. For every point y_j in a realizing manifold, Bishop–Gromov gives

$$\operatorname{Vol} B_1(y_j) \geq (D+1)^{-N} \operatorname{Vol} M_j \geq (D+1)^{-N} v.$$

Any $y \in X$ is the limit of points y_j in a common compact realization; the sequence rebased at y_j therefore has this same noncollapse bound. By [5, Theorem 1.14 and its proof], there are a set $S_y \subset B_1(y)$ and a finite constant C_y such that

$$\mathcal{H}^{N-2}(S_y) \leq C_y, \qquad x \in B_1(y) \setminus S_y \implies \exists\, U_x \subset X \text{ open}, \quad x \in U_x \cong \mathbb{R}^N.$$

Here the ambient neighborhoods U_x are those obtained by the Reifenberg theorem in that proof; a statement about the relative topology of $X \setminus S_y$ alone would not suffice. By the definition of $S_{\mathrm{top}}(X)$, $S_{\mathrm{top}}(X) \cap B_1(y) \subset S_y$. Choose $y_1, \dots, y_m$ such that $X = \bigcup_{a=1}^m B_1(y_a)$, using compactness. Countable subadditivity of Hausdorff outer measure gives

$$\mathcal{H}^{N-2}(S_{\mathrm{top}}(X)) \leq \sum_{a=1}^{m} \mathcal{H}^{N-2}(S_{\mathrm{top}}(X) \cap B_1(y_a)) \leq \sum_{a=1}^{m} C_{y_a} < \infty.$$

The set $S_{\mathrm{top}}(X)$ is closed, so it is Borel and Hausdorff measurable. □

Proof of Corollary 1.2. For $N = 4$ put $n = 4$ and $k = 0$. For $N \geq 5$ put $n = 5$ and $k = N - 5$. Choose a similarity image F of K^{n-2} inside the chart of Theorem 1.1, and let $(M_j, g_j, p_j) \to X$ and $\iota : F \to X$ be given by that theorem. Let T^k be a fixed flat torus, with T^0 a point, and set

$$X_N = X \times T^k, \qquad Z_N = \iota(F) \times T^k.$$

The products $(M_j \times T^k, g_j + g_{T^k})$ are closed smooth N-manifolds with nonnegative Ricci curvature and a common diameter bound. A product of radius-$1/2$ balls is contained in the based unit ball. Since $\operatorname{Vol} B_{1/2}(p_j) \geq 2^{-n}v$, these products are noncollapsed. A correspondence between M_j and X, paired with the identity of T^k, has no larger distortion, by

$$\left| \sqrt{a^2 + b^2} - \sqrt{a'^2 + b^2} \right| \leq |a - a'|.$$

It follows that the product sequence converges to X_N. The product scalar curvature and Ricci norm equal those of g_j at the first coordinate. Their integrals are multiplied by the fixed factor $\operatorname{Vol}(T^k)$. For $k = 0$ this factor is one. This proves all bounds for the realizing sequence, including the strictly positive Ricci bound when $N = 4, 5$.

For $k > 0$, write $T^k = \mathbb{R}^k/\Lambda$, where Λ is a full lattice, and put $\ell = \min\{|v| : v \in \Lambda \setminus \{0\}\} > 0$. Choose a closed Euclidean cube $\widehat{Q}$ with nonempty interior and diameter less than $\ell/2$, and denote its image in T^k by Q. For $x, y \in \widehat{Q}$ and $v \in \Lambda \setminus \{0\}$,

$$|x - y + v| \geq \ell - |x - y| > |x - y|.$$

The quotient distance therefore equals $|x - y|$ on $\widehat{Q}$. Thus $\widehat{Q} \to Q$ is an isometry for the restricted ambient distance, and $\iota \times \mathrm{Id} : F \times Q \to \iota(F) \times Q$ is bi-Lipschitz. By Lemma 11.1 and the Euclidean product formula,

$$0 < \mathcal{H}^{n-2+k}(F \times Q) < \infty.$$

Finitely many such cubes cover the torus. The covering definition of Hausdorff measure under the two Lipschitz inequalities gives $0 < \mathcal{H}^{N-2}(Z_N) < \infty$. For $k = 0$ use F alone.

Proposition 2.9 gives (1.10). Taking $E = \varnothing$ shows $Z_N \subset S_{\mathrm{top}}(X_N)$. Lemma 11.2 proves the finite upper measure bound in (1.9). Positive finite $(N-2)$-measure implies $\dim_{\mathcal{H}} S_{\mathrm{top}}(X_N) = N - 2$. If $\dim_{\mathcal{H}} E < N - 2$, then $\mathcal{H}^{N-2}(E) = 0$, so $Z_N \setminus E$ is nonempty. A manifold chart at any of these points would contain a relative Euclidean-ball neighborhood, whose second homology is zero. This contradicts (1.10). □

11.1. **The middle-third choices.** Let C be the middle-third Cantor set, take $a_4 = 1$, $a_5 = 3$, and embed C^{a_n} by a similarity in $\mathbb{R}^{n-2}$, using $C \times \{0\}$ when $n = 4$. Put $s_n = a_n \log 2/\log 3$. If $a = a_n$ and $s = s_n$, the level-h cover has 2^{ah} cubes of diameter $\sqrt{a}\,3^{-h}$. For $d > s$,

$$2^{ah}(\sqrt{a}\,3^{-h})^d = a^{d/2}(2^a 3^{-d})^h \longrightarrow 0.$$

Give each level-h cube mass 2^{-ah}, using the Bernoulli measure on its ternary digits. If $3^{-h-1} < r \le 3^{-h}$, each coordinate projection of a radius-r ball meets at most four level-h intervals. Hence

$$\nu(B_r(z)) \le 4^a 2^{-ah} \le 4^a 3^s r^s. \tag{11.2}$$

For every sufficiently fine countable cover $\{U_b\}$, choose $z_b \in U_b \cap C^a$ when this intersection is nonempty. If $\operatorname{diam} U_b > 0$, then $U_b \cap C^a \subset B_{2 \operatorname{diam} U_b}(z_b)$. Zero-diameter sets have zero mass by the ball bound. Thus

$$1 \le \sum_b \nu^*(U_b \cap C^a) \le 4^a 6^s \sum_b (\operatorname{diam} U_b)^s.$$

This proves $\dim_{\mathcal{H}} C^a = s$. The resulting subsets $\iota(F)$ have dimensions $\log 2/\log 3$ in dimension four and $\log 8/\log 3$ in dimension five. Products of the latter subsets with a flat k-torus have dimension $k + \log 8/\log 3$: the product of the measure above with Lebesgue measure has ball bound Cr^{s+k}, while the level covers use at most $C2^{ah}3^{hk}$ sets of diameter $C3^{-h}$. The same upper and lower covering estimates give the stated dimension. These choices are distinct from the positive-measure choices in Corollary 1.2.

For the exceptional-set examples, no equality $S_{\mathrm{top}}(X) = \iota(F)$ and no uniqueness of tangent cones along $\iota(F)$ are asserted. The next three sections concern the singleton accumulation set in Theorem 1.3.

12. The background at an isolated zero of the parameter

Throughout Sections 12–14, take $n = 4$. Let $x_* \in \Sigma_0 = S^2$. Use the construction of Sections 7–9 with $F = \{x_*\}$. In particular, $\delta^{-1}(0) = \{x_*\}$, while $A \ge 0$ is flat at x_* and vanishes on the prescribed constant-parameter balls. No assertion that A is positive away from x_* is imposed. Write $p = (x_*, 0)$ for the corresponding point of S^4. Let $\widehat{d}_j$ be the distance of the unmodified background $\mathcal{G}_j$.

Lemma 12.1. *For the parameters of Lemma 7.3, $\delta_\varepsilon \to \delta$ in $C^\infty(S^{n-2})$. On $0 < t \le \bar{t}$, the functions $b_\delta(t)$ and $\Psi_\delta(t)$ extend smoothly to $\delta = 0$, with*

$$b_0(t) = q \sin t, \qquad \Psi_0(t) = \int_0^t (t-u)(1 - \log \sin u)^2 \, du.$$

Consequently the background metrics converge in C^∞ on compact subsets of $S^4 \setminus \{p\}$ to a smooth metric g_0.

Proof. The derivative bounds for the function r in Lemma 7.2, applied to $\delta = \lambda e^{-2/r^2}$, give for each integer $h \ge 1$ constants C_h, N_h with

$$|\nabla^h \delta| \le C_h \delta r^{-N_h} \quad \text{off } F. \tag{12.1}$$

Indeed each chain-rule term is $\lambda e^{-2/r^2}$ times a finite product of powers of r^{-1} and derivatives of r; the latter satisfy $|\nabla^a r| \le C_a r^{1-a}$. Where $\delta \ge 2\varepsilon$, the difference $\delta_\varepsilon - \delta$ vanishes. Where $\delta \le \varepsilon$, its derivatives of positive order are $-\nabla^h \delta$. For sufficiently small ε, the condition $\delta \le 2\varepsilon$ implies $r^{-2} \ge \frac{1}{2}\log(\lambda/(2\varepsilon))$. Exponential decay and (12.1) give

$$\sup_{\{\delta \le \varepsilon\}} |\nabla^h \delta| \le C_h \varepsilon (1 + |\log \varepsilon|)^{N_h'}.$$

On $\varepsilon \le \delta \le 2\varepsilon$, a derivative of order h of $\varepsilon\vartheta(\delta/\varepsilon)$ is a sum of terms

$$\varepsilon^{1-a} \vartheta^{(a)}(\delta/\varepsilon)(\nabla^{h_1}\delta) \cdots (\nabla^{h_a}\delta), \qquad h_1 + \cdots + h_a = h,$$

with contractions and fixed integer coefficients. Here $h_b \ge 1$ and $r^{-2} = \frac{1}{2}\log(\lambda/\delta)$. Every term is bounded by $C_h \varepsilon (1 + |\log \varepsilon|)^{N_h''}$. The same estimate holds for the difference and extends across F by its vanishing derivatives. The zeroth-order difference is bounded by $C\varepsilon$. This proves C^∞ convergence.

For $\delta > 0$,

$$b_\delta(t) = q \sin t \, \exp\{-\tfrac{\delta}{2} \log(\sin^2 t + e^{-2c/\delta})\}. \tag{12.2}$$

The function $e^{-2c/\delta}$, extended by zero, is smooth and flat at $\delta = 0$. On every compact subinterval of $(0, \bar{t}]$, (12.2) is therefore smooth through $\delta = 0$.

For the integral kernel, put $W = L^2$. The identities $\partial_\delta Z = -2c\delta^{-2}ZJ$, $\partial_\delta J = 2c\delta^{-2}ZJ$, and $\partial_\delta L = -c\delta^{-2}J$ show by induction that, for each $a \geq 1$,

$$|\partial_\delta^a W(t,\delta)| \leq C_a \delta^{-N_a} J(t,\delta).$$

To see that the factor J persists, differentiate any product of powers of δ^{-1}, L, Z, and J arising after the first derivative; differentiating L, Z, or J introduces another factor containing J, and differentiating δ^{-1} does not remove it. The bound $L \leq 1 + c/\delta$ absorbs the remaining powers of L. Since $\int_0^{\bar{t}} J(t,\delta)\,dt \leq \pi e^{-c/\delta}$, integration gives

$$|\partial_\delta^a \Psi_\delta(t)| + |\partial_\delta^a \partial_t \Psi_\delta(t)| \leq C_a e^{-c/\delta} \delta^{-N_a}.$$

For two or more t-derivatives on a compact subinterval of $(0, \bar{t}]$, use $\partial_t^2 \Psi_\delta = L_\delta^2$. The derivatives of J on that interval have the same exponential decay, with additional fixed powers of δ^{-1}. The bound $L_\delta(u) \leq 1 - \log \sin u$ gives dominated convergence in the defining integral. It identifies its limit with Ψ_0 and, together with the preceding derivative estimates, proves smooth extension. The integral is finite since $\int_0^t |\log u|^2 du < \infty$.

On $0 < t \leq 3t_0$, all coefficient functions thus converge smoothly after composition with δ_{ν_j}. For $t \geq 3t_0$ the cutoffs give the fixed tensor G_q. Near $t = 0$ with $x \neq x_*$, the parameter δ has a positive lower bound on a smaller coordinate neighborhood; for large j, $\delta_{\nu_j} = \delta$ there. The Cartesian expressions in Lemma 5.3 give smooth convergence across that stratum. Near $t = \pi/2$ the metrics are all G_q. This proves the final assertion. □

Lemma 12.2. *The completion of $(S^4 \setminus \{p\}, g_0)$ is a compact length space (Y_0, d_0) whose underlying topological space is S^4. On that fixed sphere, $\widehat{d}_j \to d_0$ uniformly. Moreover,*

$$(Y_0, r^{-1}d_0, p) \longrightarrow \big(\mathbb{R}^2 \times \mathrm{C}(S^1_{2\pi q}), (0, o)\big) \quad (r \downarrow 0). \tag{12.3}$$

Proof. Let d_S be the round distance. The coefficient comparison in Lemma 5.3 gives

$$c_0 q\, d_S \leq \widehat{d}_j \leq d_S. \tag{12.4}$$

The same comparisons hold for the completion distance of g_0. For the upper comparison, a round geodesic through p can be replaced, inside a round ball of radius ρ, by an arc avoiding p of length at most $\pi\rho$; the additional length tends to zero. The coefficient inequality $g_0 \leq g_{S^4}$ bounds its g_0-length. The lower comparison follows by integrating $g_0 \geq c_0^2 q^2 g_{S^4}$. These comparisons identify the completion with S^4 and prove compactness. Its distance is a length distance, as is seen by approximating the endpoints by points of the punctured sphere and concatenating curves whose lengths approach the infimum. It is geodesic: for $x, y \in Y_0$, choose curves of lengths tending to $d_0(x, y)$, parametrized on $[0, 1]$ with constant speed. Their speeds are bounded. Compactness and Arzelà–Ascoli give a uniformly convergent subsequence. For each finite partition of $[0, 1]$, the sum of successive distances converges; taking the supremum over partitions proves that the limit curve has length at most $d_0(x, y)$. Its endpoints force the reverse inequality. A constant-speed parametrization of this curve is a minimizing geodesic.

Fix a small round closed ball K_ρ about p. On its complement, Lemma 12.1 gives, for numbers $\eta_j(\rho) \to 0$,

$$(1 - \eta_j(\rho))g_0 \leq \mathcal{G}_j \leq (1 + \eta_j(\rho))g_0.$$

The intrinsic diameter of K_ρ for either metric is at most 2ρ: round geodesic chords stay in K_ρ. A chord through p can be changed inside $B_\sigma(p)$, where $0 < \sigma < \rho$, by an arc of $\partial B_\sigma(p)$ of length at most $\pi \sin\sigma$. The resulting curve avoids p and has length at most $2\rho + \pi\sigma$; let $\sigma \downarrow 0$. In a curve joining two given points, replace the portion between the first and last visits to K_ρ by such a chord. The remaining portions avoid K_ρ. Comparing their lengths gives

$$\sup_{u,v \in S^4} |\widehat{d}_j(u,v) - d_0(u,v)| \leq C\eta_j(\rho) + 4\rho$$

for large j; the constant uses the diameter bound π in (12.4). Endpoints in K_ρ are included in the same first- and last-visit construction. First let $j \to \infty$, then $\rho \to 0$. This proves uniform convergence.

Take normal coordinates y on (S^2, γ) centered at x_*. For a fixed $B > 0$, write

$$\delta_r = \sup_{|y| \le Br} \delta(y), \qquad A_r = \sup_{|y| \le Br} A(y).$$

Flatness gives $\delta_r = O_h(r^h)$ and $A_r = O_h(r^h)$ for every h. On the rescaled coordinate region $y = rY$, $t = rT$, with $|Y|, T \le B$, the cutoffs equal one for small r. Since $\gamma_{ab}(rY) = \delta_{ab} + O_B(r^2)$ and $0 \le U(rT, rY) \le KA_r$, the longitudinal coefficient converges uniformly to the Euclidean one.

For $0 < t \le t_0$ and $0 \le \delta \le \delta_0$, the logarithmic formula gives

$$q \sin t \le b_\delta(t) \le q(\sin t)^{1-\delta}.$$

The lower bound uses $\sin^2 t + e^{-2c/\delta} < 1$; the value at $\delta = 0$ is understood by continuity. The mean-value theorem for $u \mapsto x^{-u}$ yields

$$0 \le b_\delta(t) - q \sin t \le q\delta |\log \sin t| (\sin t)^{1-\delta}. \tag{12.5}$$

For $0 < T \le B$ and small r, use $\sin(rT) \ge rT/2$, $|\log \sin(rT)| \le |\log r| + |\log T| + \log 2$, and $(\sin(rT))^{1-\delta} \le (rT)^{1-\delta}$. For $0 < T \le 1$ and $0 \le \delta \le 1/4$,

$$T^{1-\delta} \le 1, \qquad T^{1-\delta} |\log T| \le T^{3/4} |\log T| \le \frac{4}{3e}.$$

For $1 \le T \le B$, the two quantities are bounded by B and $B \log B$, respectively, when this interval is nonempty. Consequently

$$e_B(r) := \sup_{|Y|, T \le B} \left| r^{-1} b_{\delta(rY)}(rT) - qT \right| \le C_B \left(r^2 + \delta_r (1 + |\log r|) e^{\delta_r |\log r|} \right) \longrightarrow 0. \tag{12.6}$$

The formula at $T = 0$ follows by continuity.

Let $g_* = |dY|^2 + dT^2 + q^2 T^2 d\theta^2$ denote the product-cone tensor. On the regular part of each rescaled coordinate region, the lower bounds just proved give $g_r \ge (1 - \omega_B(r)) g_*$, with $\omega_B(r) \to 0$. The same length comparison holds for curves in $T = 0$, where the induced metric is the rescaled γ. For two points in a bounded product-cone ball, choose their product minimizing curve. In the cone factor, choose lifts whose angular difference is at most π. After multiplying the angle by q, their difference is less than π; the straight segment in the corresponding planar sector has monotone angle and total θ-variation at most π. A radial segment is used when one endpoint is the vertex. The Euclidean factor is a straight segment. Both factors stay in a bounded coordinate region. The upper coefficient bounds and (12.6) give

$$L_{g_r}(\gamma) \le (1 + \omega_B(r)) L_{g_*}(\gamma) + \pi e_B(r).$$

The same inequality is obtained by approximation if the curve meets $T = 0$.

For the reverse distance estimate, a curve of rescaled length at most $3B + 1$ with endpoints in a fixed bounded coordinate region cannot leave a larger such region: its round length is at most $(c_0 q)^{-1}(3B+1)r$ by (12.4). Choose the larger region before taking $r \to 0$ and integrate its lower tensor comparison. Thus coordinate correspondences have distortion tending to zero on each bounded ball. Distances from the basepoint have the same error, so every ball corresponds to the other ball with a radius error tending to zero. Moving a point by that error along a minimizing curve from the basepoint gives correspondences on closed balls of the same radius. This proves (12.3) for the full limit $r \downarrow 0$. □

13. Distance comparison for small replacement domains

Every domain in this section is equipped with its intrinsic length distance unless an ambient restricted distance is specified.

Lemma 13.1. *Let B and M be compact connected Riemannian manifolds. Suppose $K_1, \dots, K_j \subset B$ and $W_1, \dots, W_j \subset M$ are pairwise disjoint compact connected smooth domains. Suppose that their complements, including their boundaries, are identified with one compact connected Riemannian manifold E, and the metrics agree on E. Put*

$$d_i^* = \max\{\operatorname{diam}_{\mathrm{int}} K_i, \operatorname{diam}_{\mathrm{int}} W_i\}.$$

Suppose $L > 0$ and there are nonnegative L-Lipschitz functions λ_B, λ_M, equal on E and constant with the common value $t_i > 0$ on K_i, W_i. Let $p \in E$ have value zero. There is a relation $\mathcal{R} \subset M \times B$, surjective onto both factors and containing (p, p), such that for every $R > 0$, whenever $(u, v), (u', v') \in \mathcal{R}$ and $d_M(p, u), d_M(p, u') \leq R$,

$$|d_M(u, u') - d_B(v, v')| \leq 4 \sum_{\{i\,:\, t_i \leq 4LR\}} d_i^*. \tag{13.1}$$

The same assertion holds with B and M interchanged.

Proof. Let g_M be the metric of M. Write $C_i = \partial K_i = \partial W_i \subset E$ and let d_E be the intrinsic length distance of E. This distance is finite and induces the manifold topology, by the coordinate argument in Lemma 2.2; hence (E, d_E) is compact. The C_i are pairwise disjoint compact sets. They are nonempty: if ∂W_i were empty, W_i would be open and closed in connected M, hence equal to M, contrary to $\lambda_M(p) = 0 < t_i$. Let Q be the set of classes for the equivalence relation

$$x \sim y \quad \Longleftrightarrow \quad x = y \ \text{ or } \ x, y \in C_i \text{ for some } i,$$

and let $q : E \to Q$ be the quotient map. A class is regarded below as its subset of E.

For $A, A' \in Q$, define

$$d_Q(A, A') = \inf \left\{ \sum_{\ell=1}^{s} d_E(x_\ell, y_\ell) \ : \ \begin{array}{l} s \geq 1, \quad x_\ell, y_\ell \in E, \quad x_1 \in A, \quad y_s \in A', \\ y_\ell \sim x_{\ell+1} \quad (1 \leq \ell < s) \end{array} \right\}. \tag{13.2}$$

A chain with one pair shows that this number is finite. Reversal of chains proves symmetry, and concatenation proves the triangle inequality. A constant pair gives $d_Q(A, A) = 0$.

For nonempty compact subsets $C, D \subset E$, put

$$\Delta(C, D) = \min\{d_E(x, y) \ : \ x \in C, \ y \in D\}.$$

For distinct classes A, A', let $\mathcal{P}(A, A')$ be the finite collection of lists $(D_0, \dots, D_{r+1})$, with $0 \leq r \leq j$, such that $D_0 = A$, $D_{r+1} = A'$, all entries are distinct, and $D_1, \dots, D_r$ belong to $\{C_1, \dots, C_j\}$. The list (A, A') is allowed. Then

$$d_Q(A, A') = \min_{(D_0, \dots, D_{r+1}) \in \mathcal{P}(A, A')} \sum_{\ell=0}^{r} \Delta(D_\ell, D_{\ell+1}). \tag{13.3}$$

Indeed, if an intermediate equivalence class in a chain is a singleton, $y_\ell = x_{\ell+1}$ and

$$d_E(x_\ell, y_{\ell+1}) \leq d_E(x_\ell, y_\ell) + d_E(x_{\ell+1}, y_{\ell+1}).$$

The two pairs can therefore be replaced by one. If $x_a, y_b \in C_i$ for $a \leq b$, omit the pairs indexed by $a, \dots, b$; the adjacent representatives remain equivalent. The initial or terminal representative, when removed, is replaced by another point of its class. For $A \neq A'$ this operation cannot remove the entire chain. Each operation strictly decreases the number of pairs. After finitely many operations, no class C_i is repeated, including an endpoint class. The sum of the successive set distances does not exceed the original chain length. This proves the lower inequality in (13.3). Conversely, for each successive pair of classes in a list, choose points attaining Δ. The choices made for two adjacent pairs need not agree, but they belong to the same intermediate C_i and are equivalent. They form a chain of exactly the stated sum. This proves the reverse inequality.

Every two distinct classes are disjoint compact subsets of (E, d_E), so each $\Delta(D_\ell, D_{\ell+1})$ in (13.3) is positive. There are finitely many lists. Their minimum is therefore positive, and d_Q is a metric. Moreover

$$d_Q(q(x), q(y)) \le d_E(x, y) \qquad (x, y \in E).$$

Thus q is continuous and Q is compact. The induced bijection from the compact topological quotient to (Q, d_Q) is continuous, hence a homeomorphism. For every $\epsilon > 0$, each pair in a chain is joined in E by a curve whose length exceeds its d_E-distance by less than ϵ/s. Their images under q concatenate, because adjacent endpoints are equivalent. Consequently their total d_Q-length is at most the chain length plus ϵ. Taking the infima in (13.2) proves that Q is a length space.

Define $\pi_M : M \to Q$ by $\pi_M = q$ on E and $\pi_M(W_i) = C_i$; define π_B in the same way with K_i in place of W_i. Both definitions agree on the boundaries. We prove

$$d_Q(\pi_M(u), \pi_M(u')) \le d_M(u, u'), \qquad d_Q(\pi_B(v), \pi_B(v')) \le d_B(v, v'). \tag{13.4}$$

Let $\gamma : [0,1] \to M$ be a piecewise smooth curve, and put $F_i = \gamma^{-1}(W_i)$. These are pairwise disjoint compact subsets of $[0,1]$. Set $b_0 = 0$. Suppose $i_1, \dots, i_{\ell-1}$ and $b_{\ell-1}$ have been defined. If

$$J_\ell = \{i \notin \{i_1, \dots, i_{\ell-1}\} : F_i \cap [b_{\ell-1}, 1] \ne \emptyset\}$$

is nonempty, define

$$a_\ell = \min \bigcup_{i \in J_\ell} (F_i \cap [b_{\ell-1}, 1]), \qquad a_\ell \in F_{i_\ell}, \qquad b_\ell = \max F_{i_\ell}.$$

The index i_ℓ is unique. No F_{i_h} with $h \le \ell$ meets $(b_\ell, 1]$. The recursion thus terminates after at most j intervals. The curve is in E outside their interiors, except possibly at an endpoint of $[0,1]$ lying in a W_i; such an endpoint already represents the class C_i. Each nondegenerate complementary interval has image in E, and each deleted interval has endpoints with the same image under π_M. If no complementary interval is nondegenerate, the two endpoints have the same quotient class; use a constant pair in (13.2). Otherwise the complementary portions give a chain in that definition. In either case,

$$d_Q(\pi_M(\gamma(0)), \pi_M(\gamma(1))) \le \sum L_{d_E}(\text{complementary portions}) \le L_{g_M}(\gamma).$$

Infimizing over γ proves the first inequality in (13.4); the proof for B is identical. In particular, the two maps are 1-Lipschitz. This also identifies Q with the metric quotient obtained by collapsing the domains in either manifold: any pseudometric on that quotient for which the projection is 1-Lipschitz is bounded by every chain length in (13.2), and hence by d_Q.

The common restriction of λ_M and λ_B to E is constant on C_i, with value t_i, and satisfies

$$|\lambda_M(x) - \lambda_M(y)| \le L d_E(x, y) \qquad (x, y \in E).$$

Summing this inequality over chains shows that its induced function $\bar\lambda : Q \to [0, \infty)$ is L-Lipschitz. Set

$$\mathcal{R} = \{(u, v) \in M \times B : \pi_M(u) = \pi_B(v)\}.$$

Both projections are surjective, and $(p, p) \in \mathcal{R}$. Suppose $(u, v), (u', v') \in \mathcal{R}$ and $d_M(p, u), d_M(p, u') \le R$. Put $A = \pi_M(u)$ and $A' = \pi_M(u')$. By (13.4),

$$d_Q(\pi_M(p), A) \le R, \qquad d_Q(A, A') \le 2R, \qquad \bar\lambda(A) \le LR.$$

Choose a minimizing list in (13.3) when $A \ne A'$. For any of its classes D_h,

$$\bar\lambda(D_h) \le \bar\lambda(A) + L \sum_{\ell < h} \Delta(D_\ell, D_{\ell+1}) \le 3LR.$$

Thus every class C_i in the list, including either endpoint class, has $t_i \le 3LR < 4LR$. For $A = A'$, the only possible nontrivial lift is within their common domain; its value satisfies $t_i \le LR$.

Choose points realizing every set distance in the list and join them in E by curves with total excess length less than $\epsilon/2$. In M, join the two boundary representatives belonging to each C_i through W_i; if an endpoint belongs to W_i, join that endpoint to the corresponding boundary representative. The analogous curves in B use K_i. The sum of all additional lengths in either manifold is at most $2\sum_{t_i\le 4LR} d_i^* + \epsilon/2$: an intermediate class occurs once, and at most two endpoint portions can belong to the same class. It follows, after $\epsilon\downarrow 0$, that both $d_M(u,u')$ and $d_B(v,v')$ belong to the interval

$$\left[d_Q(A,A'),\ d_Q(A,A')+2\sum_{t_i\le 4LR} d_i^*\right].$$

Their difference is at most $2\sum_{t_i\le 4LR} d_i^*$, which implies (13.1). Interchanging M and B proves the symmetric assertion. □

Lemma 13.2. *Suppose the centers lie on a short geodesic issuing from $p\in\Sigma_0$, and write $t_i = d_S(p,x_i)$. Assume $10^6 R_i < t_i/2$ and the separation conditions of Lemma 7.1. For each j, let*

$$B_j = (S^4, \mathcal{G}_j), \qquad K_i = \overline{B}^{d_S}_{2R_i}(x_i), \quad i\le j,$$

and let M_j, W_i be the corresponding closed realization and its fixed domains. There are functions λ_j on M_j and $\widehat{\lambda}_j$ on B_j, equal on the common exterior, with

$$\mathrm{Lip}(\lambda_j), \mathrm{Lip}(\widehat{\lambda}_j)\le 32, \quad \lambda_j(p)=\widehat{\lambda}_j(p)=0, \quad \lambda_j|_{W_i}=\widehat{\lambda}_j|_{K_i}=t_i. \tag{13.5}$$

For a constant C_* independent of j,

$$\lambda_j(z)\le 32 d_{M_j}(p,z), \qquad d_{M_j}(p,z)\le C_*\lambda_j(z). \tag{13.6}$$

On the common exterior, λ_j is between one half and twice $\varrho(z) := d_S(p,z)$. The analogous two-sided bounds hold on B_j. Moreover $\mathrm{diam}_{\mathrm{int}}\, K_i \le 4R_i$ *and* $\mathrm{diam}_{\mathrm{int}}\, W_i\le C_P R_i$.

Proof. The function ϱ is independent of θ. It is the distance from the corresponding boundary point in the round hemisphere with metric $dt^2+\cos^2 t\,\gamma$. The projection to that hemisphere is c_0^{-1}-Lipschitz by Lemma 8.1. Thus ϱ is c_0^{-1}-Lipschitz for every background metric. On the unchanged annulus $2R_i\le r_i\le 4R_i$, choose the cutoff of Lemma 9.1 and set

$$\lambda_j = t_i + \zeta(r_i)(\varrho - t_i).$$

Use the constant t_i on W_i, and use ϱ outside these annuli. Define $\widehat{\lambda}_j$ by the same prescription with K_i in place of W_i. On a transition annulus, $|\varrho - t_i|\le 4R_i$, $|\zeta'|\le 2/R_i$, and $|dr_i|\le c_0^{-1}$. The local Lipschitz constant is at most $9c_0^{-1} < 32$. The annuli are disjoint and the definitions agree on their boundaries; integration along curves proves the global Lipschitz assertion. The value at p is zero because p is in none of the closed enlarged balls.

For a point in a transition annulus, both ϱ and λ_j belong to $[t_i-4R_i, t_i+4R_i]$. The assumption $4R_i < t_i/100$ gives $\varrho/2\le\lambda_j\le 2\varrho$. The same comparison is exact outside the annuli. For a common-exterior point z, take a shorter round geodesic from p to z. Its intersection with any open K_i is an interval, because the round balls have radius less than $\pi/2$ and are geodesically convex. Replace each such interval by a shortest arc in ∂K_i. The calculation in Lemma 9.2, now on S^4, bounds the new length by $\pi/2$ times the old one. Separation ensures that each replacement boundary arc avoids every other K_k. On the exterior the metric is at most the round metric. Consequently $d_{M_j}(p,z)\le 2\varrho(z)\le 4\lambda_j(z)$. For $z\in W_i$, restrict the round geodesic from p to x_i at its first intersection with ∂K_i, and replace its portions in the other domains by boundary arcs as above. Join its endpoint to z inside W_i. By (9.3),

$$d_{M_j}(p,z)\le 2t_i + C_P R_i \le (2+C_P)\lambda_j(z).$$

This proves (13.6); the first inequality follows directly from the Lipschitz bound. In B_j, a round chord gives $\widehat{d}_j(p,z)\le\varrho(z)$; within K_i, $\varrho(z)\le t_i+2R_i\le 2t_i$, which proves its bounds. Round geodesic chords in K_i have length at most $4R_i$ and the background metric is at most round. The final diameter assertion follows from (9.3). □

Proposition 13.3. *Under the hypotheses of Lemma 13.2, suppose*

$$\sum_{\{i\,:\,t_i\le u\}} R_i \le C_0 u^3 \qquad (0<u<u_0). \tag{13.7}$$

All closed balls below carry the restrictions of their ambient distances. For every $A>0$ there is $C_A<\infty$, independent of j, such that for all sufficiently small $r>0$,

$$d_{\mathrm{GH}}\big(\overline{B}^{M_j}_{Ar}(p), \overline{B}^{B_j}_{Ar}(p)\big) \le C_A r^3. \tag{13.8}$$

The corresponding closed balls admit surjective relations containing (p,p) whose distortions are at most $2C_A r^3$. If a subsequence $M_j \to (Y,d,p)$ is chosen, both assertions hold with M_j, B_j replaced by Y, Y_0.

Proof. For $i\le j$, the spaces M_j and B_j have the same exterior of W_i and K_i. This exterior is connected. Indeed, join two exterior points by a shorter round geodesic and replace each portion in a deleted ball by a path in its boundary S^3. Only finitely many balls occur, their boundaries are connected, and separation keeps these boundary paths outside every other deleted ball. The resulting path lies in the exterior. Increase C_0 if necessary so that $C_0>0$, and put

$$c_P=\max\{4,C_P\},\qquad K_A=4c_PC_0(256A)^3,\qquad C_A=\frac{5}{2}K_A.$$

Apply Lemma 13.1 with $L=32$ and $d_i^*\le c_PR_i$. On points based within $2Ar$ in either factor, the distortion of the common quotient relation is at most

$$D_A(r) := 4c_P\sum_{t_i\le 256Ar} R_i \le K_A r^3 \qquad (256Ar<u_0).$$

Require throughout this proof that

$$0<r<\min\left\{\frac{u_0}{256A}, \sqrt{\frac{A}{2K_A}}\right\}. \tag{13.9}$$

Then $D_A(r)<Ar/2$, independently of j. Because (p,p) belongs to the relation, it also gives

$$|d_{M_j}(p,u)-d_{B_j}(p,v)|\le D_A(r)$$

whenever either endpoint of (u,v) lies in the closed Ar-ball. Put $D=D_A(r)<Ar/2$. For either of the geodesic spaces $T=M_j,B_j$, define

$$P_T(x)=\{x_0\in T\,:\,d_T(p,x_0)=\min\{Ar,d_T(p,x)\},\quad d_T(x,x_0)=(d_T(p,x)-Ar)_+\}.$$

This set is nonempty: restrict a minimizing geodesic from p to x to the indicated radial parameter. It equals $\{x\}$ for $x\in\overline{B}^T_{Ar}(p)$. If $\mathcal{R}$ is the relation in Lemma 13.1, set

$$\begin{aligned}
C_1&=\{(u,v_0)\,:\,(u,v)\in\mathcal{R},\ d_{M_j}(p,u)\le Ar,\ v_0\in P_{B_j}(v)\},\\
C_2&=\{(u_0,v)\,:\,(u,v)\in\mathcal{R},\ d_{B_j}(p,v)\le Ar,\ u_0\in P_{M_j}(u)\},\\
C&=C_1\cup C_2.
\end{aligned}$$

The first projection of C_1 is $\overline{B}^{M_j}_{Ar}(p)$ and the second projection of C_2 is $\overline{B}^{B_j}_{Ar}(p)$. All pairs belong to the product of those two balls, and $(p,p)\in C$. For each $(x,y)\in C$ there is $(u,v)\in\mathcal{R}$ with

$$d_{M_j}(x,u)\le D,\qquad d_{B_j}(y,v)\le D,\qquad d_{M_j}(p,u),d_{B_j}(p,v)\le Ar+D<2Ar.$$

For two such pairs, the triangle inequality and the distortion bound on $\mathcal{R}$ give

$$|d_{M_j}(x,x')-d_{B_j}(y,y')|\le d_{M_j}(x,u)+d_{M_j}(x',u')+D+d_{B_j}(y,v)+d_{B_j}(y',v')\le 5D.$$

Thus $\operatorname{dis}C\le 5D$ and $d_{\mathrm{GH}}\le\frac12\operatorname{dis}C$, with the distinguished pair retained. This proves (13.8) and its pointed assertion.

The limit Y is geodesic. To prove this, use the common compact realization of Proposition 2.3, take $x_j \to x$ and $y_j \to y$, and choose constant-speed minimizing geodesics $\gamma_j : [0,1] \to M_j$ between them. They have uniformly bounded Lipschitz constants. A uniformly convergent subsequence has image in Y and satisfies

$$d_Y(\gamma(s), \gamma(t)) = |s-t| d_Y(x,y), \qquad 0 \le s, t \le 1.$$

Thus it is a minimizing geodesic. The corresponding assertion for Y_0 was proved in Lemma 12.2. Suppose compact geodesic spaces (T_j, p_j) and (T, p) are isometrically embedded in a compact metric space Z, with $\epsilon_j > d_H^Z(T_j, T) + d_Z(p_j, p)$ and $\epsilon_j \to 0$. For $x \in \overline{B}_R^T(p)$ choose $x_j \in T_j$ with $d_Z(x, x_j) < \epsilon_j$. Then $d_{T_j}(p_j, x_j) < R + 2\epsilon_j$. A minimizing geodesic from p_j to x_j contains a point x_j' with $d_{T_j}(p_j, x_j') \le R$ and $d_{T_j}(x_j, x_j') \le 2\epsilon_j$. Therefore $d_Z(x, x_j') < 3\epsilon_j$. Interchanging T_j and T gives the other inclusion, and hence

$$d_H^Z(\overline{B}_R^{T_j}(p_j), \overline{B}_R^T(p)) \le 3\epsilon_j.$$

Apply this estimate to $M_j \to Y$ and to $B_j \to Y_0$ from Lemma 12.2, for the fixed radius $R = Ar$. For a fixed radius, take compact common realizations of the two sequences of closed balls and replace the relations for the approximating manifolds by their closures. A Hausdorff-convergent subsequence of these compact relations has surjective projections onto the two limiting balls: for either limiting point, choose approximating points and associated points in the other factor, then use compactness of that factor. The limit contains (p, p). Distances converge in the common realizations, so its distortion is no greater than $5D_A(r) \le 5K_A r^3 = 2C_A r^3$. The same C_A therefore gives both assertions, with the distinguished points retained. □

14. Proof of the isolated-point theorem

Choose x_* at the center of the chart in Section 7, and let e be a unit vector in $T_{x_*}S^2$. Fix $0 < t_* < 1/100$ and define

$$t_i = t_* 2^{-i}, \qquad x_i = \exp_{x_*}(t_i e), \qquad z_i = x_*.$$

Choose

$$0 < r_* < \min\{R_*/4, t_*^3/4, t_*/10^8\}, \qquad R_i = r_* 2^{-4i}. \tag{14.1}$$

For $k > i$, $d_Y(x_i, x_k) \ge t_i/2$, while $10^6(R_i + R_k) \le 2 \cdot 10^6 r_* 2^{-4i} < t_i/2$. Also $10^6 R_i < t_i/2$ and $R_i < 2^{-i}\min\{R_*, t_i^3\}$. These inequalities verify (7.3)–(7.5) for $F = \{x_*\}$. Moreover,

$$\sum_{k \ge i} R_k = \frac{16}{15} r_* 2^{-4i} \le t_i^3. \tag{14.2}$$

If $t_i \le u < t_{i-1}$, (14.2) gives $\sum_{t_k \le u} R_k \le t_i^3 \le u^3$. Thus (13.7) holds.

Use these data in Lemmas 7.2 and 7.3, then perform the replacements of Sections 8 and 9. All proofs there use only (7.3)–(7.5) and the empty-interior condition on F. The resulting closed manifolds satisfy (9.2), and every W_i is fixed with $\operatorname{diam}_{\mathrm{int}} W_i \le C_P R_i$. Take $p_j = p = x_*$, which is outside all replacements. The unchanged coordinate region in Lemma 9.3 has a positive volume v_0. The diameter bound $D = \pi\sqrt{3/\kappa}$ and Bishop–Gromov imply

$$\operatorname{Vol}_{g_j} B_1(p_j) \ge \frac{\operatorname{Vol}_{g_j} M_j}{(D+1)^4} \ge \frac{v_0}{(D+1)^4} > 0.$$

Thus a marked subsequence converges to a compact noncollapsed four-dimensional limit (Y, d, p).

Lemma 14.1. *For this sequence, $Y \setminus \{p\}$ is locally isometric to a smooth Riemannian four-manifold.*

Proof. Let $s_m = 3t_m/4$ for $m \ge 1$. The inequalities $4R_i < t_i/100$ and $t_{i+1} = t_i/2$ imply

$$K_i \cap \overline{B}^{d_S}_{s_m}(p) = \emptyset \quad (i \le m), \qquad K_i \subset B^{d_S}_{s_m}(p) \quad (i > m).$$

Indeed, the distance from p of a point of K_i lies in $[t_i - 2R_i, t_i + 2R_i]$; its lower endpoint is greater than $3t_m/4$ for $i \le m$, and its upper endpoint is less than $3t_m/4$ for $i > m$. The same separation holds for the closed balls of radius $4R_i$.

The round coordinate ball $B^{d_S}_{s_m}(p)$ is disjoint from all replacement regions in M_m. Define the compact smooth domain

$$N_m = M_m \setminus B^{d_S}_{s_m}(p).$$

It consists of $S^4 \setminus B^{d_S}_{s_m}(p)$ with $K_1, \dots, K_m$ replaced by $W_1, \dots, W_m$. No replacement of index $i > m$ intersects this domain. Thus, for $j \ge m$, the common coordinates and fixed boundary identifications define a smooth embedding $e_{m,j} : N_m \to M_j$. Their differentials on $\operatorname{Int} N_m$ are injective maps between four-dimensional tangent spaces and hence are isomorphisms. The inverse function theorem shows that $e_{m,j}(\operatorname{Int} N_m)$ is open in M_j and that the common coordinates are ambient charts there. These embeddings agree on every common sphere chart, replacement domain, and collar. Their identifications give

$$N_m \subset \operatorname{Int} N_{m+1}, \qquad \mathcal{N} = \bigcup_{m \ge 1} \operatorname{Int} N_m.$$

Equip $\mathcal{N}$ with the atlas induced by these inclusions. Any two points belong to some $\operatorname{Int} N_m$ and are separated by disjoint chart neighborhoods there, which remain open in $\mathcal{N}$. A countable union of countable chart bases for the $\operatorname{Int} N_m$ is a base for $\mathcal{N}$. Thus $\mathcal{N}$ is Hausdorff and second countable and is a smooth four-manifold without boundary.

The tensors g_0 and h_i^W agree on collars of ∂W_i: there the parameters are $A = 0$ and $\delta = \delta_i$ for every $j \ge i$. They therefore define a smooth metric g_∞ on $\mathcal{N}$. For the embeddings just defined, on N_m one has

$$e_{m,j}^* g_j \longrightarrow g_\infty \quad \text{in } C^\infty(N_m) \quad (j \to \infty). \tag{14.3}$$

This follows from Lemma 12.1 on the sphere part and equality of the tensors on the finitely many replacement domains and collars. The exterior of the $m+1$ disjoint round balls $B_{s_m}(p), \operatorname{Int} K_1, \dots, \operatorname{Int} K_m$ is connected. A shorter round geodesic has at most one open interval in each ball; replace that interval by an arc on the boundary three-sphere. The finitely many boundary arcs avoid the other balls because the closed balls are disjoint. This gives an exterior path between any two exterior points. Each attached W_i is connected and meets that exterior along its boundary; hence N_m is connected. Its intrinsic g_∞-distance is finite and induces the compact manifold topology. Equation (14.3) gives uniform Lipschitz bounds for $e_{m,j}$ on each such compact metric space.

Realize $M_j \to Y$ in a common compact metric space as in Proposition 2.3. Arzelà–Ascoli on each N_m and a diagonal subsequence give uniform limits $e_m : N_m \to Y$. Compatibility gives $e_{m+1}|_{N_m} = e_m$, so these maps define a continuous map $e : \mathcal{N} \to Y$.

The functions in Lemma 13.2 restrict, under $e_{m,j}$, to one function $\lambda_\infty : N_m \to (0, \infty)$ independent of $j \ge m$. These functions agree on overlaps and define a positive function on $\mathcal{N}$. On the sphere part, $\varrho/2 \le \lambda_\infty \le 2\varrho$; on W_i, $\lambda_\infty = t_i$. The separation by s_m therefore gives

$$\sup_{\mathcal{N} \setminus N_m} \lambda_\infty \le 2s_m, \qquad \sup_{M_j \setminus e_{m,j}(N_m)} \lambda_j \le 2s_m \quad (j \ge m). \tag{14.4}$$

For the second inequality, points in a replacement domain of index $i > m$ have value $t_i \le t_m/2$, and all other points of the complement have round distance less than s_m; the exterior comparison in Lemma 13.2 applies. In particular, for $a > 0$, choose m with $2s_m < a$. Then $\{\lambda_\infty \ge a\}$ is a closed subset of N_m, and is compact.

Since $\lambda_j(p_j) = 0$, $\operatorname{Lip} \lambda_j \le 32$, and $\operatorname{diam} M_j \le D$,

$$0 \le \lambda_j \le 32D.$$

Their graphs in the common compact realization times $[0, 32D]$ have a Hausdorff-convergent subsequence. Their limit projects onto Y. For two limit graph points $(y, a), (z, b)$, passage to the Lipschitz inequality gives $|a-b| \le 32d_Y(y,z)$; hence the graph defines a 32-Lipschitz function $\lambda : Y \to [0, 32D]$. Choose this subsequence inside the one defining e. The identities on each N_m and Lemma 13.2 give

$$\lambda(y) \le 32d(p,y), \qquad d(p,y) \le C_*\lambda(y), \qquad \lambda \circ e = \lambda_\infty. \tag{14.5}$$

Thus $\lambda^{-1}(0) = \{p\}$ and $e(\mathcal{N}) \subset Y \setminus \{p\}$.

Fix $u \in \mathcal{N}$. Choose a coordinate map $x : O_3 \to B_{3a}(0) \subset \mathbb{R}^4$ with $x(u) = 0$ and $\overline{O}_3 \subset \operatorname{Int} N_m$ for some m. Put $O_2 = x^{-1}(B_{2a}(0))$. By (14.3), there are $0 < c < C$ such that, on $\overline{O}_2$, both g_∞ and all sufficiently late pulled-back metrics satisfy

$$c\sum_{b=1}^{4} dx_b^2 \le g \le C\sum_{b=1}^{4} dx_b^2.$$

Choose $0 < b < a\sqrt{c}/(4\sqrt{C})$ and put $O_1 = x^{-1}(B_b(0))$. A curve starting in O_1 and leaving O_2 has length at least $\sqrt{c}(2a-b) > \sqrt{c}\,a$ before its first exit. Any two points of O_1 are joined by their coordinate segment with length at most $2\sqrt{C}\,b < \sqrt{c}\,a/2$. Consequently their distances, both in M_j and in $(\mathcal{N}, g_\infty)$, are infima over curves contained in O_2. On $\overline{O}_2$ the tensors satisfy $(1-\eta_j)g_\infty \le e_{m,j}^* g_j \le (1+\eta_j)g_\infty$, with $\eta_j \to 0$. Comparison of the lengths of every such curve and passage to the infimum yield

$$d_Y(e(v), e(w)) = d_{g_\infty}(v,w) \qquad (v, w \in O_1).$$

Thus e is an isometry on O_1 onto its image.

If $v \ne u$ in $\mathcal{N}$, choose the preceding chart with $v \notin \overline{O}_2$. For all sufficiently large j, the images of u and v under their compatible embeddings into M_j are separated by the first-exit length at u. Passing to the limit gives $d_Y(e(u), e(v)) > 0$. Hence e is injective. Choose $0 < \rho < \sqrt{c}\,b/4$ and let $y \in B_\rho^Y(e(u))$. Take $y_j \in M_j$ with $y_j \to y$. For large j, $d_{M_j}(e_{m,j}(u), y_j) < 2\rho < \sqrt{c}\,b/2$. The first-exit estimate for $x^{-1}(B_{b/2}(0))$ forces $y_j \in e_{m,j}(x^{-1}(B_{b/2}(0)))$. A subsequence of their coordinates converges to a point $v \in x^{-1}(\overline{B}_{b/2}(0)) \subset O_1$. Uniform convergence of $e_{m,j}$ gives $e(v) = y$. Therefore $B_\rho^Y(e(u)) \subset e(O_1)$. The first-exit estimate also gives $B_\rho^{g_\infty}(u) \subset O_1$. The distance identity on O_1 consequently makes $e : B_\rho^{g_\infty}(u) \to B_\rho^Y(e(u))$ an isometry. In particular $e(\mathcal{N})$ is open in Y.

Finally, let $y \ne p$ and put $a = \lambda(y)/2 > 0$. Let $y_j \in M_j$ converge to y. Every subsequential limit b of $\lambda_j(y_j)$ gives a limit graph point (y, b) and therefore equals $\lambda(y)$. Boundedness of these values implies $\lambda_j(y_j) \to \lambda(y)$. Thus $\lambda_j(y_j) > a$ for large j. Choose m with $2s_m < a$. Equation (14.4) implies $y_j \in e_{m,j}(N_m)$ for all sufficiently large j. Compactness of N_m and uniform convergence of $e_{m,j}$ then give $y = e(v)$ for some $v \in N_m$. Thus $e(\mathcal{N}) = Y \setminus \{p\}$, proving the assertion. □

Proof of Theorem 1.3. Equation (9.2), the rebasing estimate above, and Lemma 9.3 give the curvature, noncollapse, and convergence assertions. Proposition 10.4 gives the scalar and Ricci integral bound. For every fixed $A > 0$, Proposition 13.3 supplies a correspondence $C_{A,r}$ between $\overline{B}_{Ar}^Y(p)$ and $\overline{B}_{Ar}^{Y_0}(p)$ such that

$$(p,p) \in C_{A,r}, \qquad \operatorname{dis}_{r^{-1}d, r^{-1}d_0} C_{A,r} = r^{-1}\operatorname{dis}_{d,d_0} C_{A,r} \le 2C_A r^2 \longrightarrow 0. \tag{14.6}$$

Compose it with the pointed correspondences of (12.3). The composition is surjective onto both factors: choose an intermediate point using the surjectivity of each correspondence. For two composed pairs, the triangle inequality bounds the distance discrepancy by the sum of the two distortions. The distinguished pair is retained. Thus the rescaled closed A-balls in Y converge to the closed A-ball in the stated product cone, for every $A > 0$ and for the full limit $r \downarrow 0$. This proves the unique-tangent assertion of (1.12).

Apply Proposition 2.3 to the fixed domains W_i. By Lemma 2.1, choose generators $c_i \in H_2(W_i;\mathbb{Z})$ whose images under the quotient maps generate $\widetilde{H}_2(W_i/\partial W_i;\mathbb{Z})$. The classes $a_i = (f_i)_* c_i$ in $H_2(Y;\mathbb{Z})$ are independent: applying $(\pi_i)_*$ to a finite integral relation isolates its i-th coefficient in an infinite cyclic group. Tensoring the same homomorphisms with $\mathbb{Q}$ gives independence over $\mathbb{Q}$ by the calculation in (2.5). The localization estimate gives

$$\sup_{w \in W_i} d_{M_j}(p_j, w) \le 2t_i + C_P R_i \longrightarrow 0 \qquad (j \ge i).$$

After passage to the limit, every neighborhood U of p contains $f_i(W_i)$ for all sufficiently large i. Therefore the image of $H_2(U;\mathbb{Z})$ in $H_2(Y;\mathbb{Z})$ contains all the corresponding independent a_i. A Euclidean-ball neighborhood has zero second homology. Hence $p \in S_{\text{top}}(Y)$. If $p \in V \subset U$ are open and $V \hookrightarrow U$ is null-homotopic, its composition with $U \hookrightarrow Y$ induces the zero map in degree two, contrary to the infinite-rank assertion for V. This proves failure of local contractibility. Every open topological cone over a compact space is contractible by (1.14); it therefore cannot be homeomorphic to an open neighborhood of p. Lemma 14.1 proves $S_{\text{top}}(Y) = \{p\}$.

At any $y \neq p$, normal coordinates for the smooth local metric give $g_{ab}(rz) \to \delta_{ab}$ uniformly on every bounded rescaled coordinate set. Curves leaving a fixed larger coordinate neighborhood have a fixed positive length, so the local distance comparisons imply $\operatorname{Tan}(Y, y) = \{\mathbb{R}^4\}$. At p, the unique tangent splits $\mathbb{R}^2$. Thus no point lies in $\mathcal{S}_1(Y)$. Since $\mathcal{S}_0(Y) \subset \mathcal{S}_1(Y)$,

$$\mathcal{S}_0(Y) = \mathcal{S}_0^{\mathrm{CC}}(Y) = \varnothing, \qquad \operatorname{Int}_Y(Y \setminus \mathcal{S}_0^{\mathrm{CC}}(Y)) = Y.$$

The space Y is not a manifold, so this is a counterexample to [4, Conjecture 0.7] in dimension four. □

The join $S^1 * S^1_{2\pi q}$ has the same topological quotient as $S^1 * S^1$. The map $[u, z, w] \mapsto (\cos u\, z, \sin u\, w)$ is a continuous bijection from that compact quotient to $S^3 \subset \mathbb{C}^2$, and hence a homeomorphism. The infinite-rank image in degree two excludes a topologically conical neighborhood of p, as proved above. No formula for $H_*(Y, Y \setminus \{p\}; \mathbb{Z})$ in terms of a fixed link is asserted or used.

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