KNOTTED TORI IN THE FOUR-SPHERE
WITH DIFFEOMORPHIC EXTERIORS

ANASS NIFA

Abstract. For every positive integer R, there are at least R smooth tori in the standard four-sphere with orientation-preservingly diffeomorphic compact exteriors, no two of which are carried to one another by an ambient homeomorphism. We construct these exteriors by circle surgery on graphs of powers of the trefoil longitude. A circle action identifies the required fillings with the standard four-sphere, for both normal-framing classes. The quotient of the exterior group by its center is a hyperbolic triangle group; rotation numbers distinguish the meridian classes of the fillings. Thus the number of topological embedding types represented by a smooth torus exterior has no uniform finite bound. The same examples admit exterior diffeomorphisms inducing every integral shear in the product-circle direction; the resulting diffeomorphisms of the filled manifolds restrict to the identity on the inserted torus neighborhoods. For the same exteriors, we compute the commutator length of every nonzero power of the corresponding central element. This gives the exact singular spanning genus of the specified surgery direction. The conclusion concerns arbitrarily large finite families, not an infinite family with one fixed exterior.

Contents

1. Introduction

Let $T \subset S^4$ be a smooth embedded torus, and write

$$E(T) = S^4 \setminus \operatorname{int} \nu T.$$

Choose an orientation of T. The induced oriented normal plane bundle has Euler number $[T]\cdot[T] = 0$, since $H_2(S^4;\mathbb{Z}) = 0$. Evaluation on $[T]$ identifies $H^2(T;\mathbb{Z})$ with $\mathbb{Z}$, so its Euler class is zero. The classification of oriented plane bundles by their Euler class therefore gives $\nu T \cong T\times D^2$. Consequently $\partial E(T) \cong T^3$. The exterior has a distinguished conjugacy class of normal meridians, defined up to inversion if orientations are omitted. A homeomorphism of pairs preserves this class. An unmarked diffeomorphism of exteriors is not required to do so.

For classical knots in S^3, Gordon–Luecke proved that homeomorphic complements imply equivalent knots [10]. For tori in S^4, Larson asked whether inequivalent embeddings can have the same complement [17, Question 3.4]. The following theorem gives arbitrarily large finite families with one smooth exterior and distinguishes their members even under ambient homeomorphisms.

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Theorem 1.1. *For every integer $k \geq 1$, set $N = 2^k$. There are a compact connected oriented smooth four-manifold E_N, with $\partial E_N \cong T^3$, and smooth tori*

$$T_{N,0}, T_{N,1}, \dots, T_{N,N-1} \subset S^4$$

with the following properties.

(i) *Every $E(T_{N,r})$ is orientation-preservingly diffeomorphic to E_N.*
(ii) *If a homeomorphism of S^4 takes $T_{N,r}$ to $T_{N,r'}$, then $r = r'$.*
(iii) *Every $T_{N,r}$ is knotted.*

The homeomorphism in (ii) *need not preserve orientations or parametrizations.*

For a compact oriented smooth four-manifold E, let $\mathcal{T}(E)$ be the set of ambient-homeomorphism classes of smooth tori $T \subset S^4$ whose exteriors are orientation-preservingly diffeomorphic to E. The theorem implies

$$\forall R \geq 1 \quad \exists E \quad |\mathcal{T}(E)| \geq R. \tag{1.1}$$

It does not assert that $|\mathcal{T}(E)|$ is infinite for some fixed E. The displayed tori are separate embeddings, not the components of a prescribed disjoint link. The construction uses a framed circle surgery. Either normal-framing class gives the theorem; one is fixed for the entire family. No diffeomorphism between the two exteriors obtained from the two framing classes is asserted.

1.1. **The construction and the meridian obstruction.** Let K be the right-handed trefoil, let $P = E(K)$, and fix its positive meridian μ and preferred longitude λ. Choose an embedded longitude parallel $\ell : S^1 \to \operatorname{int} P$. For $N \geq 2$, define E_N by circle surgery in $P \times S^1$ on

$$\gamma_N(t) = \big(\ell(Nt), e^{it}\big). \tag{1.2}$$

The second coordinate proves injectivity and makes the differential injective; compactness of S^1 then proves that this map is an embedding. The unchanged boundary has coordinates (μ, λ, s), and Proposition 2.3 gives

$$\Gamma_N = \pi_1 E_N \cong \pi_1 P / \langle\!\langle [\lambda^N, \pi_1 P] \rangle\!\rangle, \qquad s \longmapsto \lambda^{-N}. \tag{1.3}$$

Here $[\lambda^N, \pi_1 P]$ denotes the set of commutators $[\lambda^N, g]$ with $g \in \pi_1 P$. This is the circle-surgery description of Kuhrman's point-pushing construction [16, Proposition 4.4 and Remark 4.7]; its centralization presentation is contained in the proof of his Proposition 4.9. We give the framed construction and the entire peripheral homomorphism in Section 2. The abstract group Γ_N is already the group of the $6N$-twist-spun trefoil: [9, equation (1)], with $(m, n) = (6N, 1)$, gives $\pi_1 P / \langle\!\langle [\mu^{6N}, \pi_1 P] \rangle\!\rangle$, which equals (1.3) because $\lambda = h\mu^{-6}$ and h is central. We verify this identification after Proposition 2.3. It is an identification of groups, not of four-manifold exteriors.

Attach $T^2 \times D^2$ to E_N with meridian $\mu + r\lambda$. The core of the attachment defines $T_{N,r}$ whenever the filled manifold is identified with S^4. For $r \geq 0$ and $\gcd(N, 6r - 1) = 1$, Proposition 3.1 proves that the filling is the standard smooth four-sphere. The boundary filling and the interior circle surgery have disjoint attaching regions. Performing the boundary filling first places the surgery circle in

$$S^3_{1/r}(K) \times S^1$$

and is isotopic to a specified orbit of a circle action. The Seifert fibration is the one in Moser's torus-knot filling calculation [22]. The local extension across circle surgery is proved for both framing classes. The homotopy-sphere consequence of the Fintushel–Pao circle-action classification then applies [8, 23, 9]. The group and homology calculations establish the homotopy-sphere hypothesis of that theorem.

The distinction of embeddings is independent of that classification theorem. For this twist-spun knot group, the quotient by the center and the meridian order are known from Hillman [11, Corollary 3]. In Lemma 4.1 we verify, with the present generators,

$$\Gamma_N / \mathrm{Z}(\Gamma_N) \cong \Delta(2, 3, 6N). \tag{1.4}$$

The image u of μ has order $6N$. For a suitable orientation of the circle at infinity, the image of the rth meridian has rotation number

$$\operatorname{rot}\big(u^{1-6r}\big) = \frac{1-6r}{6N} \pmod{\mathbb{Z}}.$$

The boundary-extension theorem for quasi-isometries [14, Propositions 2.20 and 2.23] implies that each automorphism of this triangle group preserves all rotation numbers up to one common sign. Since (1.4) is a characteristic quotient, an exterior-group automorphism carrying one normal meridian to another, up to conjugacy and inversion, forces

$$1-6r \equiv \pm(1-6r') \pmod{6N}.$$

The negative sign would give $1 \equiv -1 \pmod 6$. The positive sign gives $6N \mid 6(r-r')$, hence $r \equiv r' \pmod N$. For $N = 2^k$, the coprimality condition holds for every r; the range $0 \le r < N$ therefore gives Theorem 1.1.

1.2. **Peripheral diffeomorphisms and singular genus.** The coordinate s determines a tangent circle s_r on each filling torus. Write λ_r for its other tangent coordinate and μ_r for its normal meridian.

For an oriented parametrized circle $c \subset \partial E$, let $g_{\mathrm{sing}}(c; E)$ be the least genus of a compact connected oriented surface mapping to E, with c as its only boundary component. The parametrization and a product boundary collar are fixed, and the surface interior maps to $\operatorname{int} E$. If no such surface exists, set $g_{\mathrm{sing}}(c; E) = +\infty$. For the null-homologous curves used here, this is the commutator-length description of singular slope genus in Liu–Ni–Sun–Wang [18, Definition 3.2 and Remark 3.3]; see Lemma 6.6.

Theorem 1.2. *For every torus in Theorem 1.1 and every integer j, the affine torus surgery specified in Corollary 5.3, with meridian $\mu_r + js_r$, gives S^4. The diffeomorphism to the original filling is the identity on the abstract inserted $T^2 \times D^2$, and therefore identifies its core torus with $T_{N,r}$. Moreover,*

$$g_{\mathrm{sing}}(s_r; E(T_{N,r})) = \frac{N}{2} + 1. \tag{1.5}$$

The same minimum holds for proper smooth immersions with the indicated boundary collar. Every properly embedded oriented spanning surface, smooth or locally flat, has genus at least $N/2 + 1$.

The exterior diffeomorphism is induced by

$$(x, e^{is}) \longmapsto (x, v(x)^j e^{is}),$$

where $v : P \to S^1$ has meridional degree one and is identically one near ℓ. This map is the identity near the circle surgery. On the outer boundary it is

$$(\mu, \lambda, s) \longmapsto (\mu, \lambda, s + j\mu).$$

On first homology it sends μ to $\mu + js$ and fixes λ and s. Unlike the longitude fillings in Theorem 1.1, these surgeries identify the parametrized filling tori.

For the genus assertion, put $z = \lambda^N \in \Gamma_N$. For every $N \ge 2$, Theorem 6.5 proves

$$\operatorname{cl}_{\Gamma_N}(z^q) = 1 + \left\lceil \frac{|q|(N-1)}{2} \right\rceil \quad (q \ne 0), \qquad \operatorname{scl}_{\Gamma_N}(z) = \frac{N-1}{2}. \tag{1.6}$$

An explicit representation into $\widetilde{\mathrm{PSL}}_2(\mathbb{R})$ sends z to the translation $x \mapsto x + N - 1$. The Milnor–Wood inequality gives the lower bound [25, 5]. The corresponding commutator length of an integral translation in the full group of lifts of circle homeomorphisms is already determined by Eisenbud–Hirsch–Neumann [7, Theorem 2.3]; see also [20, Theorem 3.1]. That calculation gives a lower bound under a homomorphism, not a factorization in Γ_N. A branched covering of a once-punctured torus, obtained from Bertram's permutation factorization [4], gives the matching upper bound. Coverings with prescribed boundary monodromy are also treated by Rivin [24, Section 5]. The equality concerns surface maps and immersions; it gives only a lower bound for embedded surfaces. The proof of

Theorem 1.1 does not use (1.6). Unbounded singular slope genus itself is already present in the plumbing examples of Liu–Ni–Sun–Wang [18, Remark 5.10 and Proposition 5.11]. For their plumbing torus with both companion knots of genus one, the primitive slopes $(2j+1)\xi+\eta$ have singular genus $j+2$ for $j \geq 0$. In the present construction, the exact genus calculation concerns the specified cosmetic-surgery direction on the tori with one common smooth exterior.

1.3. **Previous results and the conclusions proved here.** Meier–Zupan [19, Theorem 1.3] construct infinitely many pairwise nonisotopic ribbon disks in B^4 with diffeomorphic exteriors and a fixed boundary knot. Their [19, Proposition 2.1] identifies isotopy of these disks with diffeomorphism of the disk pairs. For each generalized square knot in that construction, all doubled disks are isotopic to the same spun two-knot [19, Proposition 4.2]. Thus doubling those disks does not retain their distinction. The objects in Theorem 1.1 are closed tori in S^4, and their distinction holds under arbitrary ambient homeomorphisms.

Examples distinguished only by smooth isotopy do not by themselves give the inequivalence in Theorem 1.1. Darolles constructs smooth tori and Klein bottles in $\mathbb{C}^* \times \mathbb{C}^*$ with the same complement and distinguishes them by a smooth-isotopy invariant [6]. Konno–Mallick–Taniguchi [15, Theorem 1.13] construct surfaces in once-stabilized corks which are related by diffeomorphisms equal to the identity on the boundary, but are not smoothly isotopic relative to that boundary. These results concern different ambient manifolds or a different equivalence relation. Here the ambient manifold is the standard S^4, and the conclusion excludes all ambient homeomorphisms between distinct members of each displayed family.

Torus surgery on spun and turned torus knots was studied by Iwase [13]. His classification concerns those torus-knot constructions and the manifolds obtained from their fillings. More recently, Akhmedov–Akhmedov study torus-exterior replacement using prescribed nonabelian peripheral homomorphisms [1]. Their constructions concern the groups and homeomorphism types of the resulting four-manifolds. The result proved here fixes one smooth exterior and distinguishes meridians of fillings that are diffeomorphic to the standard S^4.

Larson's nontrivial torus surgeries returning S^4 [17, Corollary 3.2] do not by themselves distinguish the surgery-dual tori: an extension of the gluing map over the exterior identifies those duals. His Question 3.4 explicitly separates this issue from the existence of nontrivial surgeries. Kuhrman's point-pushed examples likewise provide the graph construction and centralization groups used in (1.3). The additional assertions proved here are the specified collection of smooth S^4 fillings of one exterior, the obstruction to identifying their meridians, and the exact calculation (1.6) for the same exterior groups. Neither the point-pushing operation nor the interpretation of singular genus as commutator length is introduced here. The twist-spin identification above makes Hillman's comparison particularly relevant. His [11, Corollary 3] gives the quotient by the center and the order of the meridian for twist spins of torus knots. His [11, Theorems 8 and 10] distinguish monodromy or meridian data in branched twist spins; the latter uses a hyperbolic orbifold quotient. We do not claim either the abstract groups or the use of such quotients as new. The assertion here is the construction of one smooth torus exterior with the specified standard S^4 fillings, followed by the calculation that their normal meridians cannot be identified by an automorphism of that exterior group. The genus theorem computes an additional invariant of the same marked examples.

The existence assertion in Larson's question is realized by Theorem 1.1, with the finite quantifiers in (1.1). The necessary congruence $r \equiv r' \pmod N$ is not an equivalence criterion when the indices have the same residue. No infinite family with one fixed exterior, and no smooth distinction between embeddings of a single topological pair, is asserted.

The constructions and peripheral presentations are given in Section 2. Section 3 proves the smooth filling theorem. Section 4 distinguishes the meridians and proves Theorem 1.1. The exterior diffeomorphisms and the genus calculation are established in Sections 5 and 6. The proofs use neither four-move invariance nor a theorem on cork doubles.

Conventions. Manifolds and maps are smooth unless another category is stated. An equivalence of embedded tori is an ambient homeomorphism of pairs; it need not be isotopic to the identity. A diffeomorphism denoted $\cong_+$ preserves orientation. The unknotted torus is the torus obtained by spinning an unknot in a three-ball; a torus is knotted if it is not ambiently homeomorphic to that torus. Boundaries are oriented by the outward-normal-first convention. Tubular neighborhoods are closed disk bundles. Gluings use fixed product collars; corners are rounded in their normal two-dimensional quadrants. Angular coordinates have period 2π. An integral matrix acts on the column of angular coordinates. Its columns are the images in first homology of the ordered source coordinate circles. Additive notation denotes homology classes; multiplicative notation denotes fundamental-group elements. We use $[u,v] = uvu^{-1}v^{-1}$ and write $\langle\!\langle R\rangle\!\rangle$ for the normal closure of a set R of group elements. Homology has integral coefficients.

2. Circle surgery and the exterior

2.1. **Framed circle surgery.** Let V be an oriented four-manifold, and let $\gamma : S^1 \hookrightarrow \operatorname{int} V$ be an oriented embedded circle. Orient the normal bundle so that the tangent orientation followed by the normal orientation is the orientation of V. A frame on a cut interval has endpoint discrepancy in $SO(3)$. A path in $SO(3)$ joining that discrepancy to the identity corrects the frame near one endpoint and produces a frame on S^1. Thus the normal bundle is trivial. Relative to one frame, the homotopy classes of oriented framings are

$$[S^1, SO(3)] = \pi_1 SO(3) \cong \mathbb{Z}/2.$$

Let $\tau : S^1 \times D^3 \to V$ be an orientation-preserving tubular parametrization supplied by a framing. Give $D^2 \times S^2$ its product orientation and use the attachment

$$\partial D^2 \times S^2 \longrightarrow \partial(V \setminus \operatorname{int} \nu\gamma), \qquad (e^{it}, u) \longmapsto \tau(e^{it}, -u). \tag{2.1}$$

The boundary orientation of $S^1 \times D^3$ is the negative of the product orientation on $S^1 \times S^2$. Passing to the exterior reverses it. The boundary of $D^2 \times S^2$ has the product orientation on $S^1 \times S^2$. Since the antipodal map on S^2 has degree -1, (2.1) reverses the two boundary orientations. Inserting the loop $u \mapsto R_t u$, where R_t is one full rotation, changes the normal-framing class. Homotopic framings give isotopic attachments. A boundary isotopy extends over a collar by

$$(y, v) \longmapsto (f_{\chi(v)}(y), v), \qquad (y, v) \longmapsto (f^{-1}_{\chi(v)}(y), v), \tag{2.2}$$

where $f_0 = \mathrm{id}$, $\chi = 1$ near the boundary, and $\chi = 0$ beyond a smaller collar. The two formulas are inverse; both equal the identity at the inner end of the collar. Thus the two framings specify the two circle surgeries up to orientation-preserving diffeomorphism. The tubular-neighborhood theorem [12, Chapter 4] identifies sufficiently small tubular parametrizations with homotopic frames by an isotopy; (2.2) extends the resulting boundary isotopy.

Circle surgery is compatible with diffeomorphisms as follows. Let $H : V \to V'$ be an orientation-preserving diffeomorphism, set $\tau' = H \circ \tau$, and let j_τ denote the map in (2.1). Then

$$H \circ j_\tau = j_{\tau'}. \tag{2.3}$$

The map H on the exterior and id on $D^2 \times S^2$ descend to a diffeomorphism of the surgeries. The maps H^{-1} and id define its inverse. This identity specifies the transported framing.

Lemma 2.1. *Let $c_u : S^1 \to Y$ be a smooth homotopy of maps to a smooth manifold Y, for $0 \le u \le 1$. Then*

$$j_u(t) = (c_u(t), e^{it}) \in Y \times S^1$$

is an isotopy of embeddings. If its trace is contained in a compact subset of the interior, it extends to a compactly supported ambient isotopy. The derivative of that isotopy transports any chosen normal framing.

Proof. For $u \in [0,1]$,

$$j_u(t) = j_u(t') \implies e^{it} = e^{it'} \implies t = t' \text{ in } S^1, \qquad dj_u(\partial_t) = (dc_u(\partial_t), ie^{it}) \neq 0.$$

Thus j_u is an injective immersion of a compact manifold, hence an embedding. For the extension assertion, assume that the trace lies in a compact subset of the interior. The map $J(u,t) = (u, j_u(t))$ is an embedding into $[0,1] \times (Y \times S^1)$. Along its image define the section

$$v\big(u, j_u(t)\big) = \partial_u j_u(t)$$

of the bundle pulled back from $T(Y \times S^1)$. Tubular charts and a partition of unity extend v to a smooth section $V(u,p)$ of that bundle. Choose its support in a fixed compact subset of the interior containing the trace. The time-dependent field V_u has a flow H_u for $0 \le u \le 1$, with $H_0 = \mathrm{id}$. Both $H_u(j_0(t))$ and $j_u(t)$ solve

$$\dot{p}(u) = V_u(p(u)), \qquad p(0) = j_0(t).$$

Uniqueness gives $H_u \circ j_0 = j_u$. The inverse flow makes every H_u a diffeomorphism. The induced normal-bundle map is

$$T_{j_0(t)}(Y \times S^1)/dj_0(T_tS^1) \longrightarrow T_{j_u(t)}(Y \times S^1)/dj_u(T_tS^1), \qquad [v] \longmapsto [dH_u(v)].$$

It carries the chosen frame to a frame of j_u. Formula (2.3) applies at $u = 1$. ☐

2.2. **The trefoil and its peripheral curves.** Fix the right-handed trefoil K and put $P = S^3 \setminus \operatorname{int} \nu K$. Its group has the presentations

$$G = \langle x, y \mid xyx = yxy \rangle = \langle a, b \mid a^2 = b^3 \rangle, \quad a = xyx, \quad b = xy. \tag{2.4}$$

In the first presentation, $(xyx)^2 = (xyx)(yxy) = (xy)^3$. In the second presentation set

$$x = b^{-1}a, \qquad y = a^{-1}b^2.$$

These definitions give $xy = b$, $xyx = a$ and $yxy = a^{-1}b^3 = a$. Conversely, substitution in the first presentation gives $(xy)^{-1}(xyx) = x$ and $(xyx)^{-1}(xy)^2 = y$. Hence the two homomorphisms are inverse. Choose the Wirtinger generator x to be the positive meridian and put

$$\mu = b^{-1}a, \qquad h = a^2 = b^3, \qquad \lambda = h\mu^{-6}. \tag{2.5}$$

The element h commutes with both generators. In the Seifert fibration of the trefoil exterior it is the regular fiber. The fiber on the boundary has longitudinal degree one and linking number six with K; hence its slope is $6\mu + \lambda$. This is the torus framing of the $(2,3)$-curve, and gives the preferred longitude in (2.5); see [22, Section 3]. The curve $h\mu^{-6}$ has longitudinal degree one and zero linking number; it is therefore the preferred peripheral longitude.

Abelianization of (2.4) gives

$$G^{\mathrm{ab}} = \mathbb{Z}\langle \mu \rangle, \qquad [a] = 3[\mu], \quad [b] = 2[\mu], \quad [h] = 6[\mu], \quad [\lambda] = 0. \tag{2.6}$$

Also,

$$G/\langle\!\langle \mu \rangle\!\rangle = \langle a, b \mid a^2 = b^3,\ b^{-1}a = 1 \rangle = 1,$$

since $a = b$ and $a^2 = a^3$ imply $a = b = 1$. Thus μ normally generates G.

Fix the boundary parametrization before making the constructions below. The action

$$\theta \cdot (z_1, z_2) = (e^{3i\theta} z_1, e^{2i\theta} z_2) \qquad ((z_1, z_2) \in S^3 \subset \mathbb{C}^2) \tag{2.7}$$

has a regular orbit equal to K; take its tubular neighborhood invariant. The induced free action on ∂P is a principal circle bundle over S^1. Its Euler class belongs to $H^2(S^1;\mathbb{Z}) = 0$, so the bundle is trivial and admits a section. The section and the action give product coordinates in which the action is translation in one factor. Its orbit has primitive class $6\mu + \lambda$ by (2.5); an integral change of coordinates therefore gives angles (μ, λ) in which the action is translation by $(6\theta, \theta)$. These coordinates, and an invariant inward collar obtained from an invariant metric, are fixed throughout. Choose the based peripheral loops in this collar. A longitude parallel chosen below supplies a base path from this collar to its graph by its parallelism annulus.

2.3. **Surgery on the longitude graph**. Choose an embedded longitude parallel $\ell : S^1 \to \operatorname{int} P$, with an annulus to the preferred longitude on ∂P. For $N \geq 2$, define

$$\gamma_N(t) = (\ell(Nt), e^{it}) \subset \operatorname{int}(P \times S^1_s). \tag{2.8}$$

Fix either framing of γ_N and set

$$E_N = \big((P \times S^1_s) \setminus \operatorname{int} \nu\gamma_N\big) \cup (D^2 \times S^2). \tag{2.9}$$

The notation suppresses the fixed framing. All assertions below apply to either choice. The boundary of E_N is the unchanged $\partial P \times S^1_s$, parametrized by (μ, λ, s).

Lemma 2.2. *Filling this boundary by the original $\nu K \times S^1_s$, meridian to meridian, gives the standard S^4, for either circle-surgery framing.*

Proof. Let $Q = (P \times S^1) \setminus \operatorname{int} \nu\gamma_N$, let $W_0 = \nu K \times S^1$, and let $j_\tau : \partial D^2 \times S^2 \to \partial Q$ be the circle attachment. The two attaching boundary components of Q are disjoint. The manifold in the lemma is the quotient

$$\big(Q \sqcup W_0 \sqcup (D^2 \times S^2)\big)/\sim,$$

where $\sim$ consists of the meridional attachment of W_0 and j_τ. Forming the first attachment identifies $Q \cup W_0$ with the complement of γ_N in $S^3 \times S^1$.

Since $\pi_1(S^3) = 1$, the map $t \mapsto \ell(Nt)$ has a null-homotopy. Relative smooth approximation [12, Chapter 2] makes it smooth with endpoints fixed. Lemma 2.1 gives an ambient isotopy to $\{p\} \times S^1$. By (2.3), this identifies the circle surgeries with their transported framings. The tubular-neighborhood theorem permits using a product tube $B^3 \times S^1$ at the endpoint; its two framings are considered separately below.

The complement of this tube is $D^3 \times S^1$. With the product framing, the replacement is

$$(D^3 \times S^1) \cup (S^2 \times D^2) = \partial(D^3 \times D^2) \cong S^4,$$

with corners rounded. The antipodal factor in (2.1) extends to $x \mapsto -x$ on D^3. The second framing changes the boundary attachment by

$$C(x, e^{it}) = (R_t x, e^{it}).$$

It extends over $D^3 \times S^1$ by the same formula, with inverse $C^{-1}(x, e^{it}) = (R_{-t}x, e^{it})$. Therefore both surgeries are diffeomorphic to S^4. Composition with a reflection of S^4, if necessary, gives an orientation-preserving identification. $\square$

Proposition 2.3. *The fundamental group and peripheral map of E_N are*

$$\Gamma_N \cong (G \times \langle s \rangle)/\langle\!\langle \lambda^N s \rangle\!\rangle \cong G/\langle\!\langle [\lambda^N, G] \rangle\!\rangle, \tag{2.10}$$

$$\mu \longmapsto \mu, \qquad \lambda \longmapsto \lambda, \qquad s \longmapsto \lambda^{-N}. \tag{2.11}$$

Equivalently,

$$\Gamma_N = \langle a, b \mid a^2 = b^3, [\mu^{6N}, a] = [\mu^{6N}, b] = 1 \rangle. \tag{2.12}$$

Moreover $H_1(E_N) = \mathbb{Z}\langle \mu \rangle$.

Proof. Set $Q = (P \times S^1) \setminus \operatorname{int} \nu\,\gamma_N$ and fix a basepoint on the unchanged outer boundary. The inclusion $Q \hookrightarrow P \times S^1$ induces an isomorphism on fundamental groups. Indeed, relative transversality makes a loop disjoint from γ_N because $1 + 1 < 4$, and makes a null-homotopy of a loop disjoint from it because $2 + 1 < 4$. Radial deformation in the punctured normal disks replaces deletion of γ_N by deletion of its open tube. Hence

$$\pi_1 Q \cong G \times \langle s \rangle.$$

The generator of $\pi_1(\partial \nu\,\gamma_N)$ maps to $\lambda^N s$, up to conjugacy. Since $\pi_1(D^2 \times S^2) = 1$, van Kampen gives the first quotient in (2.10).

Let $H = G/\langle\!\langle [\lambda^N, G] \rangle\!\rangle$. The maps

$$(G \times \langle s \rangle)/\langle\!\langle \lambda^N s \rangle\!\rangle \longrightarrow H, \qquad g \longmapsto g, \quad s \longmapsto \lambda^{-N},$$

and $H \to (G \times \langle s \rangle)/\langle\!\langle \lambda^N s \rangle\!\rangle$ induced by $g \mapsto g$ are well-defined: λ^N is central in the first target, and $\lambda^N = s^{-1}$ is central in the second. Their compositions fix all generators. They are inverse isomorphisms. The unchanged inclusion of the outer boundary gives (2.11).

The peripheral elements commute, and h is central. Thus

$$\lambda^N = h^N \mu^{-6N}, \qquad [\lambda^N, g] = 1 \iff [\mu^{6N}, g] = 1 \quad (g \in G).$$

It suffices to impose centrality on the generators a, b, proving (2.12). Finally the additional relators belong to $[G, G]$. The induced abelianization is therefore $G^{\mathrm{ab}} = \mathbb{Z}\langle \mu \rangle$ by (2.6). $\square$

Identification of the abstract group. Let $K^{6N,1}$ denote the $6N$-twist-spun trefoil. The presentation in [9, equation (1)], with the parameter $\beta = 1$, is

$$\pi_1 E(K^{6N,1}) \cong \langle a, b, \eta \mid a^2 = b^3,\ [\eta, a] = [\eta, b] = 1,\ \mu^{6N}\eta = 1 \rangle. \tag{2.13}$$

The assignment $a \mapsto a$, $b \mapsto b$, $\eta \mapsto \mu^{-6N}$ defines a homomorphism from this presentation to (2.12). Conversely, in (2.13), the equality $\mu^{6N} = \eta^{-1}$ makes μ^{6N} central, so the assignments on a, b define a homomorphism in the other direction. The two composites fix a, b, η, with $\eta = \mu^{-6N}$ in the second composite. They are inverse isomorphisms. The two exterior boundaries are different:

$$\partial E_N \cong T^3, \qquad \partial E(K^{6N,1}) \cong S^2 \times S^1.$$

Their fundamental groups are $\mathbb{Z}^3$ and $\mathbb{Z}$, respectively. In particular, the group identification does not give a homeomorphism of exteriors or identify their peripheral homomorphisms.

2.4. **The family of fillings.** For $r \geq 0$, give $W_r = D^2 \times S^1 \times S^1$ its product orientation. Write (μ_r, λ_r, s_r) for the angular coordinates on its boundary and for the corresponding coordinate circles. Orient a knot tube by (ρ, μ, λ), where ρ is its normal radial coordinate. The orientation on P is inherited from S^3, and its outward normal is $-\partial_\rho$. Thus (μ, λ) is negative for ∂P, whereas (μ_r, λ_r) is positive for the boundary of its solid-torus filling. Consequently (μ, λ, s) is negative for ∂E_N and (μ_r, λ_r, s_r) is positive for ∂W_r. Define the boundary map by

$$\varphi_r(\mu_r, \lambda_r, s_r) = (\mu_r, \lambda_r + r\mu_r, s_r), \qquad A_r = \begin{pmatrix} 1 & 0 & 0 \\ r & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.14}$$

The matrix has determinant one, so the map reverses these boundary orientations, and $A_r^{-1} = A_{-r}$. In particular, φ_0 is the original meridional attachment. Put

$$M_{N,r} = E_N \cup_{\varphi_r} W_r, \qquad \widehat{T}_{N,r} = \{0\} \times S^1 \times S^1 \subset W_r. \tag{2.15}$$

In the group of the unchanged exterior, the meridian is

$$m_r = \mu\lambda^r = h^r \mu^{1-6r}. \tag{2.16}$$

The tangent curves λ_r, s_r map to λ, λ^{-N}, respectively. All boundary representatives in (2.14) are affine and constant in the collar parameter.

3. Fillings diffeomorphic to the four-sphere

Proposition 3.1. *If $N \geq 2$, $r \geq 0$, and $\gcd(N, 6r-1) = 1$, then*

$$M_{N,r} \cong_+ S^4.$$

The assertion holds for both choices of the framing used to define E_N.

3.1. The fundamental group and homology.

Lemma 3.2. *Under the hypotheses of Proposition 3.1, $M_{N,r}$ is a homotopy four-sphere.*

Proof. The homomorphism $\pi_1(\partial W_r) \to \pi_1(W_r)$ is surjective, with kernel generated by μ_r. Van Kampen and (2.16) give

$$\pi_1 M_{N,r} = \Gamma_N / \langle\!\langle \mu\lambda^r \rangle\!\rangle. \tag{3.1}$$

In this quotient,

$$\mu = \lambda^{-r}, \qquad h = \mu^6\lambda = \lambda^{1-6r}. \tag{3.2}$$

The elements λ^N and λ^{1-6r} are central. Choose $u, v \in \mathbb{Z}$ with $uN + v(1-6r) = 1$. Then $\lambda = (\lambda^N)^u(\lambda^{1-6r})^v$ and $\mu = \lambda^{-r}$ are central. The image of μ normally generates (3.1), so this group is cyclic. Its abelianization is

$$H_1(E_N)/\langle[\mu\lambda^r]\rangle = \mathbb{Z}\langle[\mu]\rangle/\langle[\mu]\rangle = 0.$$

A cyclic group equals its abelianization; hence $\pi_1 M_{N,r} = 1$.

The disjoint attachments described in the proof of Lemma 2.2 also give

$$Y_r = S^3_{1/r}(K), \qquad M_{N,r} = \big((Y_r \times S^1) \setminus \operatorname{int} \nu\gamma_N\big) \cup (D^2 \times S^2), \tag{3.3}$$

where $Y_0 = S^3$ and $1/r$ denotes the meridian slope $\mu + r\lambda$. The attaching maps and their collars are the same in both orders of construction. Since $\chi(Y_r \times S^1) = 0$,

$$\chi(M_{N,r}) = \chi(Y_r \times S^1) - \chi(S^1 \times D^3) + \chi(D^2 \times S^2) = 2; \tag{3.4}$$

the two boundary pieces $S^1 \times S^2$ have Euler characteristic zero.

Write $M = M_{N,r}$. It is a closed connected oriented four-manifold. Simple connectivity, duality and the universal coefficient theorem give

$$H_1(M) = H_3(M) = 0, \qquad H_2(M) \cong H^2(M) \cong \operatorname{Hom}(H_2(M), \mathbb{Z}).$$

The group $H_2(M)$ is finitely generated because M is compact. The displayed isomorphism therefore makes it free abelian. Its rank is $\chi(M) - 2 = 0$, so $H_2(M) = 0$. Hurewicz now gives successively $\pi_2(M) = 0$, $\pi_3(M) = 0$ and $\pi_4(M) \cong H_4(M) \cong \mathbb{Z}$. A map $S^4 \to M$ representing the positive generator induces an isomorphism on every homology group. The homological Whitehead theorem for simply connected CW complexes makes it a homotopy equivalence. □

We use the following smooth classification theorem.

Theorem 3.3 (Fintushel–Pao). *A smooth homotopy four-sphere admitting an effective smooth circle action is diffeomorphic to the standard S^4.*

This is the homotopy-sphere consequence of Fintushel's classification [8] and Pao's replacement theorem [23], with the three-dimensional Poincaré theorem of Perelman understood; see [21] for the latter theorem and [9, Section 2.1] for the circle-action conclusion used here. Lemma 3.2 verifies the homotopy-sphere hypothesis. Lemma 3.4 and the action below verify the effective smooth circle-action hypothesis for both framing classes.

3.2. **Circle surgery on an orbit.**

Lemma 3.4. *Let V be a connected oriented four-manifold with an effective smooth S^1-action, and let $\gamma \subset \operatorname{int} V$ be a one-dimensional orbit. Circle surgery on γ, with either oriented normal-framing class, admits an effective smooth S^1-action extending the given action outside a tubular neighborhood. The new action has fixed points.*

Proof. Let α be the action and let $p \in \gamma$. An averaged Riemannian metric makes every $\alpha(\theta)$ an isometry. The stabilizer is a finite cyclic subgroup of S^1, say of order $k \geq 1$. Its representation on the normal three-dimensional space is orientation-preserving and faithful. Indeed, an element in its kernel has derivative the identity on both the normal three-dimensional space and the orbit tangent line. For an isometry f, the set

$$A_f = \{q \in \operatorname{int} V \;:\; f(q) = q,\ df_q = \mathrm{id}\}$$

is closed in $\operatorname{int} V$ and is open there by $f(\exp_q v) = \exp_{f(q)}(df_q v)$. The interior of a connected manifold with boundary is connected: a path can be moved inward on a boundary collar. Thus a nonempty A_f equals $\operatorname{int} V$, and continuity gives $f = \mathrm{id}$ on V. Effectiveness excludes a nonidentity element in this kernel.

An orientation-preserving orthogonal transformation of a real three-dimensional space has an invariant axis and acts on its orthogonal plane by a rotation. For the faithful cyclic stabilizer representation, a generator therefore rotates by $2\pi l/k$, where $\gcd(k, l) = 1$ if $k > 1$. For $k = 1$, put $l = 0$. An invariant tubular neighborhood is the associated bundle

$$(\mathbb{R} \times D^3)/\big((t, x) \sim (t + 2\pi/k, R_{-2\pi l/k}x)\big).$$

The maps

$$[t, x] \longmapsto (e^{ikt}, R_{lt}x), \qquad (e^{i\phi}, y) \longmapsto [\phi/k, R_{-l\phi/k}y] \tag{3.5}$$

are well-defined inverses; changing the lift of ϕ by 2π applies exactly the displayed relation. In these coordinates,

$$\alpha_l(\theta)(e^{i\phi}, x) = (e^{i(\phi+k\theta)}, R_{l\theta}x). \tag{3.6}$$

For $\varepsilon \in \{0, 1\}$ set

$$C_\varepsilon(e^{i\phi}, x) = (e^{i\phi}, R_{\varepsilon\phi}x), \qquad l_\varepsilon = l - \varepsilon k.$$

Let $\tau : S^1 \times D^3 \to V$ be the inverse of the coordinates in (3.5), followed by the invariant tubular embedding. Choose the orientation of the normal three-disk so that τ preserves orientation. The parametrizations $\tau \circ C_0$ and $\tau \circ C_1$ represent the two normal-framing classes. On the replacement define

$$\beta_\varepsilon(\theta)(z, u) = (e^{ik\theta}z, R_{l_\varepsilon\theta}u). \tag{3.7}$$

For every θ, ϕ, x,

$$\alpha_l(\theta)C_\varepsilon(e^{i\phi}, x) = C_\varepsilon\alpha_{l_\varepsilon}(\theta)(e^{i\phi}, x) = (e^{i(\phi+k\theta)}, R_{\varepsilon\phi+l\theta}x). \tag{3.8}$$

Let $j_\varepsilon(e^{i\phi}, u) = \tau C_\varepsilon(e^{i\phi}, -u)$. Rotations commute with the antipodal map, so (3.8) gives

$$\alpha(\theta) \circ j_\varepsilon = j_\varepsilon \circ \big(\beta_\varepsilon(\theta)|_{\partial D^2 \times S^2}\big).$$

The two actions preserve the radial collar coordinates. They therefore define a smooth action on the glued manifold. The group law holds on both pieces and hence on their quotient.

Since $\gcd(k, l_\varepsilon) = 1$, an element acting trivially on $D^2 \times S^2$ satisfies $k\theta, l_\varepsilon\theta \in 2\pi\mathbb{Z}$, and thus $\theta \in 2\pi\mathbb{Z}$. The new action is effective. If u_+, u_- are the poles of the rotation axis, then $(0, u_+)$ and $(0, u_-)$ are fixed points of (3.7). When $l_\varepsilon = 0$, its entire core $\{0\} \times S^2$ is fixed. $\square$

3.3. **The Seifert action and its orbit.** Throughout this subsection, assume $N \geq 2$, $r \geq 1$, and $\gcd(N, 6r-1) = 1$, as in Proposition 3.1. Put

$$A = 6r - 1 > 0, \qquad \gcd(A, 6) = 1. \tag{3.9}$$

Use the action and the fixed invariant boundary coordinates of (2.7). Its boundary action is $(\mu, \lambda) \mapsto (\mu + 6\theta, \lambda + \theta)$.

Reverse the action parameter on P. The inverse meridian–longitude block of A_r gives

$$\begin{pmatrix} 1 & 0 \\ -r & 1 \end{pmatrix} \begin{pmatrix} -6 \\ -1 \end{pmatrix} = \begin{pmatrix} -6 \\ 6r-1 \end{pmatrix} = \begin{pmatrix} -6 \\ A \end{pmatrix}. \tag{3.10}$$

The boundary action therefore extends over the filling solid torus, with coordinates $(z, e^{iv}) \in D^2 \times S^1$, by

$$\sigma_\theta(z, e^{iv}) = (e^{-6i\theta} z, e^{i(v + A\theta)}). \tag{3.11}$$

The equality $\gcd(6, A) = 1$ makes this action effective. Its core

$$\kappa_r(t) = (0, e^{it})$$

is an orbit with isotropy of order A, traversed positively A times by the action parameter. The other exceptional multiplicities are two and three. This explicit extension is the Seifert filling calculation of [22, Section 3], in the slope convention of (2.14).

The curve ℓ is joined by an annulus in P to λ. Under the filling, λ is the coordinate v in (3.11). The radial annulus

$$(t, \rho) \longmapsto (\rho e^{i\phi_0}, e^{it}) \qquad (0 \leq \rho \leq 1)$$

joins this longitude to κ_r. Concatenating the two annulus homotopies and smoothing the parameter gives a homotopy of oriented circles in Y_r. By Lemma 2.1, its N-fold first-coordinate traversal gives an ambient isotopy of the surgery circle to

$$t \longmapsto (\kappa_r(Nt), e^{it}) \subset Y_r \times S^1. \tag{3.12}$$

The normal framing is transported by that isotopy.

On $Y_r \times S^1$ use

$$\theta \cdot (y, e^{is}) = \left(\sigma_{N\theta}(y), e^{i(s + A\theta)}\right). \tag{3.13}$$

An element in the kernel satisfies $N\theta, A\theta \in 2\pi\mathbb{Z}$, since σ is effective. Choose integers u, v with $uN + vA = 1$. Then $\theta = u(N\theta) + v(A\theta) \in 2\pi\mathbb{Z}$, proving effectiveness. On $\kappa_r \times S^1$, the orbit through $(\kappa_r(0), 1)$ is

$$\theta \longmapsto (\kappa_r(AN\theta), e^{iA\theta}).$$

With $t = A\theta$, its image is (3.12). The stabilizer has order A.

Choose $0 < \delta < \min\{1, \pi\}$ sufficiently small for the solid-torus chart, and define

$$(s, \eta, z) \longmapsto \left((z, e^{i(Ns + \eta)}), e^{is}\right), \qquad s \in \mathbb{R}/2\pi\mathbb{Z}, \quad |\eta| < \delta, \quad |z| < \delta.$$

Its inverse takes s from the last factor, retains z, and uses the unique representative $\eta = v - Ns \in (-\delta, \delta)$. Uniqueness follows from $2\delta < 2\pi$. Restricting $(\eta, \Re z, \Im z)$ to a ball gives a normal three-disk tube. Formula (3.13) becomes

$$s \longmapsto s + A\theta, \qquad v - Ns \longmapsto v - Ns, \qquad z \longmapsto e^{-6Ni\theta} z. \tag{3.14}$$

Thus the orbit tube has weights $(k, l) = (A, -6N)$. Since $\gcd(A, 6) = 1$ and $\gcd(A, N) = 1$, one has $\gcd(A, 6N) = 1$.

Proof of Proposition 3.1. For $r = 0$, Lemma 2.2 gives the assertion. Suppose $r \geq 1$. The homotopy obtained from the two parallelism annuli is constant near its endpoints after a smooth reparametrization; its concatenation is therefore smooth. Lemma 2.1 and (2.3) identify the defining framed surgery on γ_N with the corresponding framed surgery on (3.12). This circle is an orbit of (3.13), whose action is effective because $\gcd(N, A) = 1$. Lemma 3.4 supplies an effective smooth circle action on $M_{N,r}$ for either transported framing. Lemma 3.2 verifies that $M_{N,r}$ is a homotopy four-sphere. Theorem 3.3 gives a diffeomorphism $M_{N,r} \to S^4$. If its degree is -1, composition with a reflection of S^4 changes that degree to $+1$. □

4. The meridian classes

4.1. **The quotient by the center.** The quotient and meridian-order assertions below are the specialization of Hillman [11, Corollary 3] to the $6N$-twist-spun trefoil. The proof retains the present peripheral generators. Let $q_N = \mu^{6N}$ in Γ_N, and put $C_N = \langle h, q_N\rangle$. Both generators are central. From (2.12),

$$\Gamma_N/C_N \cong \Delta_N := \langle \bar{a}, \bar{b} \mid \bar{a}^2 = \bar{b}^3 = (\bar{b}^{-1}\bar{a})^{6N} = 1\rangle. \tag{4.1}$$

Lemma 4.1. *For $N \geq 2$,*

$$Z(\Gamma_N) = C_N, \qquad \Gamma_N/Z(\Gamma_N) \cong \Delta(2, 3, 6N).$$

The quotient has a faithful cocompact action on $\mathbb{H}^2$, in which the image $u = \bar{\mu}$ has order exactly $6N$ and boundary rotation number $1/(6N)$ after a choice of boundary orientation.

Proof. Set $m = 6N$. Since $N \geq 2$, one has $1/2 + 1/3 + 1/m < 1$. Let $D \subset \mathbb{H}^2$ be the compact triangle of angles $\pi/2, \pi/3, \pi/m$. Label its side reflections r_1, r_2, r_3 so that r_1r_2, r_1r_3, r_3r_2 have orders 2, 3, m, respectively. The hyperbolic polygon theorem [2, Chapter 9] gives a faithful discrete reflection action with fundamental triangle D. The orientation-preserving subgroup has the presentation

$$\langle r_1r_2, r_1r_3 \mid (r_1r_2)^2 = (r_1r_3)^3 = ((r_1r_3)^{-1}(r_1r_2))^m = 1\rangle.$$

It identifies (4.1) with that subgroup by

$$\bar{a} = r_1r_2, \qquad \bar{b} = r_1r_3, \qquad u = \bar{b}^{-1}\bar{a} = r_3r_2.$$

A fundamental domain is the union of two reflected copies of D, hence is compact. The element u is rotation through $\pm 2\pi/m$. It has order exactly m; choose the orientation of $\partial\mathbb{H}^2$ for which its rotation number is $1/m$.

Let $g \in Z(\Delta_N)$. Commutation with $\bar{a}$ implies that g fixes the unique fixed point of $\bar{a}$ in $\mathbb{H}^2$; commutation with $\bar{b}$ gives the same conclusion for its distinct fixed point. An orientation-preserving isometry fixing two distinct interior points is the identity. Thus $Z(\Delta_N) = 1$.

Both h and q_N are central, so their normal closure in Γ_N equals $C_N = \langle h, q_N\rangle$. This is the kernel of the quotient in (4.1). Every central element of Γ_N has central image in Δ_N, hence belongs to C_N. The reverse containment follows from the definitions. Therefore $Z(\Gamma_N) = C_N$. □

For $\varphi \in \mathrm{Aut}(\Gamma_N)$, centrality gives $\varphi(Z(\Gamma_N)) \subseteq Z(\Gamma_N)$. Applying the same inclusion to φ^{-1} gives equality. Thus every such φ induces an automorphism of the quotient Δ_N in Lemma 4.1.

4.2. **Rotation numbers and automorphisms.** For $g \in \mathrm{Homeo}^+(S^1)$, choose a lift $\widetilde{g} : \mathbb{R} \to \mathbb{R}$ with $\widetilde{g}(x + 1) = \widetilde{g}(x) + 1$. Its rotation number is

$$\mathrm{rot}(g) = \lim_{j\to\infty} \frac{\widetilde{g}^{\,j}(0)}{j} \pmod{\mathbb{Z}}. \tag{4.2}$$

Existence of this limit and independence modulo $\mathbb{Z}$ of the lift are the Poincaré rotation-number theorem; we use the conventions in [5, Section 2.3.3].

Lemma 4.2. *Let Δ be a discrete group acting faithfully, properly and cocompactly on $\mathbb{H}^2$ by orientation-preserving isometries. If $\psi \in \mathrm{Aut}(\Delta)$, there is $\varepsilon \in \{1, -1\}$ such that*

$$\mathrm{rot}(\psi(g)) = \varepsilon\, \mathrm{rot}(g) \quad \text{in } \mathbb{R}/\mathbb{Z} \qquad (g \in \Delta),$$

where both rotation numbers are computed using the given boundary action.

Proof. A proper cocompact action on $\mathbb{H}^2$ makes Δ finitely generated and its orbit map a quasi-isometry [14, Proposition 2.23]. Fix a finite symmetric generating set and its word metric d. There is $L \geq 1$ such that

$$L^{-1}d(g,h) \leq d(\psi(g),\psi(h)) \leq Ld(g,h) \qquad (g,h \in \Delta) \ :$$

take L at least the word lengths of the images of the generators under both ψ and ψ^{-1}. The induced quasi-isometry therefore extends to a homeomorphism of the hyperbolic boundary [14, Proposition 2.20]. Under the orbit-map identification with $\partial\mathbb{H}^2$, write this homeomorphism as $H : S^1 \to S^1$. The identity $\psi(gx) = \psi(g)\psi(x)$, applied to sequences converging to a boundary point, gives

$$H \circ g = \psi(g) \circ H \qquad (g \in \Delta). \tag{4.3}$$

Suppose H preserves orientation. Choose a lift $\widetilde{H}$ with $\widetilde{H}(x+1) = \widetilde{H}(x) + 1$ and put $M = \sup_x |\widetilde{H}(x) - x| < \infty$; finiteness follows because $\widetilde{H}(x) - x$ is continuous and one-periodic. For a lift F of g, set $G = \widetilde{H}F\widetilde{H}^{-1}$, a lift of $\psi(g)$. Put $x_0 = \widetilde{H}^{-1}(0)$ and choose $k \in \mathbb{Z}_{\geq 0}$ with $|x_0| \leq k$. Monotonicity and $F^j(x+k) = F^j(x) + k$ give

$$|F^j(x_0) - F^j(0)| \leq k, \qquad |G^j(0) - F^j(0)| \leq M + k.$$

Division by j and (4.2) give $\mathrm{rot}(\psi(g)) = \mathrm{rot}(g)$.

If H reverses orientation, let $r(x) = -x$ on the covering line. Conjugation by r takes the translation limit of F to its negative, since $(rFr)^j(0) = -F^j(0)$. The map $H \circ r$ preserves orientation. Applying its conjugacy invariance to rgr and using $HgH^{-1} = (Hr)(rgr)(Hr)^{-1}$ gives $\mathrm{rot}(\psi(g)) = -\mathrm{rot}(g)$. The sign depends only on H, not on g. □

4.3. **Homeomorphisms and normal meridians.**

Lemma 4.3. *If a homeomorphism of pairs $(S^4, T) \to (S^4, T')$ carries one smooth torus to another, it induces an isomorphism of complement groups taking the conjugacy class of a normal meridian of T to that of a normal meridian of T' or its inverse.*

Proof. Let $h : (S^4, T) \to (S^4, T')$ be the homeomorphism and fix $p \in T$. Let B^2_ρ denote the open Euclidean disk of radius ρ. Choose fixed product charts χ, χ' centered at $p, h(p)$, respectively, which identify the surfaces with $B^2_1 \times \{0\}$ in $B^2_1 \times B^2_1$. Put

$$U_j = \chi(B^2_{2^{-j}} \times B^2_{2^{-j}}), \qquad V_j = \chi'(B^2_{2^{-j}} \times B^2_{2^{-j}}) \qquad (j \geq 1).$$

Each family is cofinal among neighborhoods of its center. Its local complements have fundamental group $\mathbb{Z}$, since $B^2_\rho \times (B^2_\rho \setminus \{0\})$ retracts onto a normal circle. The inclusions within either family preserve that circle generator.

Choose U_j so small that $h(U_j) \subset V_1$, and then choose $V_k \subset h(U_j)$. Up to base paths, the inclusion homomorphisms factor as

$$\pi_1(V_k \setminus T') \longrightarrow \pi_1(h(U_j) \setminus T') \longrightarrow \pi_1(V_1 \setminus T').$$

All three groups are infinite cyclic. If the two homomorphisms are multiplication by integers a, b, their composite is multiplication by ab. The inclusions within the fixed V-chart preserve a normal generator, so $ab = 1$ after those generators are chosen. Consequently $a = b = 1$ or $a = b = -1$. The map induced by h from $\pi_1(U_j \setminus T)$ to the middle group is also an isomorphism. It follows that h takes a local normal generator to a normal generator or its inverse. Inclusion into the global complement and changes of base path introduce only conjugation. Radial deformation of the punctured normal disks identifies the open-complement group with the compact-exterior group. This proves the stated claim without requiring h to preserve a selected smooth tube. □

Proposition 4.4. *Suppose $M_{N,r}$ and $M_{N,r'}$ are standard as in Proposition 3.1. Choose orientation-preserving identifications with S^4, and let $T_{N,r}, T_{N,r'}$ be the images of the core tori. If an ambient homeomorphism identifies these tori, then $r \equiv r' \pmod{N}$.*

Proof. Deleting a smaller normal tube of a filling core leaves E_N with an added collar. Collar reparametrization identifies its fundamental group with Γ_N, taking its meridian to m_r in (2.16). By Lemma 4.3, an ambient homeomorphism supplies $\varphi \in \mathrm{Aut}(\Gamma_N)$ and $\delta \in \{1, -1\}$ such that

$$\varphi(m_r) \sim m_{r'}^{\delta}.$$

Here $\sim$ denotes conjugacy. Since the center is characteristic, φ descends to an automorphism $\bar{\varphi}$ of Δ_N. Equations (2.16) and (4.1) give

$$\bar{m}_r = u^{1-6r}, \qquad \mathrm{rot}(\bar{m}_r) = \frac{1-6r}{6N} \pmod{\mathbb{Z}}. \tag{4.4}$$

The power formula follows by conjugating the elliptic isometry u to a rotation. Lemma 4.2 and conjugacy invariance give, for one $\varepsilon \in \{1, -1\}$,

$$\varepsilon \frac{1-6r}{6N} = \delta \frac{1-6r'}{6N} \quad \text{in } \mathbb{R}/\mathbb{Z}.$$

Hence

$$1 - 6r \equiv \varepsilon\delta(1 - 6r') \pmod{6N}. \tag{4.5}$$

If $\varepsilon\delta = -1$, reduction modulo six gives $1 \equiv -1 \pmod 6$, a contradiction. Thus $\varepsilon\delta = 1$ and $6N \mid 6(r-r')$, equivalently $N \mid r - r'$. □

Proof of Theorem 1.1. Fix $k \geq 1$ and $N = 2^k$. For every $r \geq 0$, the integer $6r - 1$ is odd, so $\gcd(N, 6r - 1) = 1$. For each $0 \leq r < N$, choose the orientation-preserving diffeomorphism $\Phi_r : M_{N,r} \to S^4$ supplied by Proposition 3.1, and define

$$T_{N,r} = \Phi_r(\widehat{T}_{N,r}).$$

The compact exterior of $\widehat{T}_{N,r}$ is E_N with a boundary collar attached. A smooth strictly increasing reparametrization of the extended collar, equal to the identity at its interior end, gives an orientation-preserving diffeomorphism with E_N. This proves (i). If two displayed tori are ambiently homeomorphic, Proposition 4.4 gives $r \equiv r' \pmod N$. Since $0 \leq r, r' < N$, this implies $r = r'$, proving (ii).

For comparison, write $S^4 = (B^3 \times S^1) \cup (S^2 \times D^2)$ and let $U \subset \mathrm{int}\, B^3$ be an unknot. The spun unknot torus $U \times S^1$ is the unknotted torus of our convention. Its exterior has fundamental group

$$\big(\pi_1(B^3 \setminus \mathrm{int}\, \nu U) \times \langle s \rangle\big)/\langle\!\langle s \rangle\!\rangle \cong (\mathbb{Z} \times \mathbb{Z})/\langle s \rangle \cong \mathbb{Z}.$$

Here $\pi_1(B^3 \setminus \mathrm{int}\, \nu U) = \mathbb{Z}$: adjoining a ball along the outer sphere gives the solid-torus exterior of U, without changing the group. On the other hand, Γ_N surjects onto the nonabelian group Δ_N of Lemma 4.1. If its two generating rotations commuted, each would fix the unique interior fixed point of the other. Their fixed points are distinct, so they do not commute. Thus $\Gamma_N \not\cong \mathbb{Z}$, and none of the displayed tori is unknotted. This proves (iii). □

Example 4.5. For $N = 2$, the common characteristic quotient is $\Delta(2, 3, 12)$. The unordered signed rotation classes of the two meridians are

$$r = 0 : \{1/12, 11/12\}, \qquad r = 1 : \{5/12, 7/12\}.$$

These disjoint sets distinguish the two tori under arbitrary ambient homeomorphisms.

Remark 4.6. Proposition 4.4 gives a necessary condition. It does not classify the fillings with $r \equiv r' \pmod N$. For $N = 2^k$, Proposition 3.1 applies to every $r \geq 0$, whereas Proposition 4.4 distinguishes the displayed N residue classes. In particular, no infinite family with one fixed exterior is claimed.

5. Peripheral diffeomorphisms and the filling tori

Lemma 5.1. *There is a smooth map $v : P \to S^1$ such that*

$$v = e^{i\mu} \quad \textit{on a boundary collar,} \qquad v = 1 \quad \textit{on a neighborhood of } \ell.$$

Its cohomology class generates $H^1(P;\mathbb{Z})$ and has meridional degree one.

Proof. Let $F \subset P$ be a properly embedded oriented Seifert surface, obtained by truncating a Seifert surface in S^3. Its normal line bundle is oriented, hence trivial. Choose its normal interval $[-\eta, \eta]$ so that crossing from $-\eta$ to η has positive intersection with F. Let $f : [-\eta, \eta] \to [0, 1]$ be smooth, equal to zero near $-\eta$ and one near η. Define $v_0 = e^{2\pi i f}$ on this product neighborhood and $v_0 = 1$ on its complement. These formulas agree near its two ends, so v_0 is smooth. The degree of v_0 on a loop is its algebraic intersection with F; therefore

$$(v_0)_*([\mu]) = 1, \qquad (v_0)_*([\lambda]) = 0.$$

These values determine the generator of $H^1(P;\mathbb{Z})$ by (2.6).

Choose disjoint neighborhoods U_∂ and U_ℓ, respectively a boundary collar and a tube of ℓ. The map $e^{i\mu}/v_0 : U_\partial \to S^1$ induces zero on both generators of its fundamental group. The map $v_0 : U_\ell \to S^1$ also induces zero on fundamental groups, since $[\ell] = [\lambda] = 0$ in $H_1(P)$. The lifting criterion for $\mathbb{R} \to S^1$ therefore gives smooth functions H_∂, H_ℓ with

$$e^{iH_\partial} = e^{i\mu}/v_0 \text{ on } U_\partial, \qquad e^{iH_\ell} = v_0 \text{ on } U_\ell.$$

Smoothness follows from local smooth inverses of the covering map. Choose smooth cutoffs χ_∂, χ_ℓ supported in these neighborhoods and equal to one on smaller ones. Extending their products with H_∂, H_ℓ by zero, put

$$v = v_0 \exp(i\chi_\partial H_\partial - i\chi_\ell H_\ell). \tag{5.1}$$

On the smaller boundary collar this equals $e^{i\mu}$; near ℓ it equals one. The maps $v_t = v_0 \exp(it\chi_\partial H_\partial - it\chi_\ell H_\ell)$ give a homotopy from v_0 to v, so $[v] = [v_0]$ in $H^1(P;\mathbb{Z})$. □

Proposition 5.2. *For every $j \in \mathbb{Z}$, there is an orientation-preserving diffeomorphism $U_j : E_N \to E_N$ with the exact boundary-collar value*

$$(\mu, \lambda, s) \longmapsto (\mu, \lambda, s + j\mu), \qquad [U_j]_\partial = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ j & 0 & 1 \end{pmatrix}. \tag{5.2}$$

It is the identity on the piece $D^2 \times S^2$ inserted in (2.9).

Proof. Use the function in Lemma 5.1 and define

$$U_j(x, e^{is}) = (x, v(x)^j e^{is}) \qquad (x \in P).$$

For all $j, k \in \mathbb{Z}$, one has $U_j U_k = U_{j+k}$ and $U_0 = \mathrm{id}$; hence $U_j^{-1} = U_{-j}$. If $v = e^{iH}$ in a local chart, then

$$DU_j = \begin{pmatrix} I_3 & 0 \\ j\,dH & 1 \end{pmatrix}, \qquad \det DU_j = 1.$$

Thus U_j is an orientation-preserving diffeomorphism. It equals the identity on $U'_\ell \times S^1$, where U'_ℓ is a neighborhood of ℓ on which $v = 1$. Choose the surgery tube in this set. The identity on $D^2 \times S^2$ then agrees with U_j on the entire attaching collar, so the maps glue, with inverse given by U_{-j} and the identity. On the unchanged outer collar, $v = e^{i\mu}$ gives exactly (5.2). □

Corollary 5.3. *For r satisfying Proposition 3.1 and $j \in \mathbb{Z}$, define*

$$u_j(\mu_r, \lambda_r, s_r) = (\mu_r, \lambda_r, s_r + j\mu_r).$$

Then $E_N \cup_{\varphi_r \circ u_j} W_r$ is diffeomorphic to $M_{N,r} \cong S^4$ by a map equal to the identity on the abstract W_r. The meridian of the abstract W_r is μ_r. Its image under u_j is $\mu_r + js_r$, expressed in the original filling coordinates; its image under $\varphi_r \circ u_j$ is $\mu + r\lambda + js$ in $H_1(\partial E_N;\mathbb{Z})$.

Proof. The first homology column of u_j is $(1, 0, j)^t$, and

$$A_r \begin{pmatrix} 1 \\ 0 \\ j \end{pmatrix} = \begin{pmatrix} 1 \\ r \\ j \end{pmatrix}.$$

The two affine maps $U_j|_{\partial E_N} \circ \varphi_r$ and $\varphi_r \circ u_j$ both send (μ_r, λ_r, s_r) to $(\mu_r, \lambda_r + r\mu_r, s_r + j\mu_r)$. Therefore

$$[U_j]_\partial A_r = A_r[u_j]. \tag{5.3}$$

The new surgery is $E_N \cup_{\varphi_r u_j} W_r$. Define a map to $E_N \cup_{\varphi_r} W_r$ by U_{-j} on E_N and id on W_r. It respects the boundary equivalence relation because

$$U_{-j}\varphi_r u_j = \varphi_r.$$

All maps are product maps on smaller attaching collars, so the quotient map is smooth. Its inverse is U_j on E_N and id on W_r. It preserves orientation and restricts to the identity on all of the abstract W_r, in particular on its core torus. Proposition 3.1 identifies the target with S^4. ☐

The longitude fillings in (2.14) and the fiber surgeries in Corollary 5.3 have different conclusions. The former have inequivalent core tori for different residues of r modulo N, by Proposition 4.4. The maps in Corollary 5.3 instead restrict to the identity on the abstract filling tori.

6. Commutator length and singular genus

6.1. Definitions and the Euler-number bound. For $g \in [H, H]$, let $\mathrm{cl}_H(g)$ be the least integer $l \geq 0$ such that g is a product of l commutators in H. The stable commutator length is

$$\mathrm{scl}_H(g) = \lim_{q\to\infty} \frac{\mathrm{cl}_H(g^q)}{q}.$$

The inequality $\mathrm{cl}_H(g^{p+q}) \leq \mathrm{cl}_H(g^p) + \mathrm{cl}_H(g^q)$ for $p, q \geq 1$ implies, by the subadditive lemma, that the limit exists and equals $\inf_{q\geq 1} \mathrm{cl}_H(g^q)/q$. We compute it for

$$z = \lambda^N \in \mathrm{Z}(\Gamma_N). \tag{6.1}$$

By (2.6) and Proposition 2.3, the image of λ in Γ_N^{ab} is zero. Thus $\lambda, z \in [\Gamma_N, \Gamma_N]$, so their commutator lengths are defined.

The universal cover $\widetilde{\mathrm{PSL}}_2(\mathbb{R})$ acts on the universal cover of $\mathbb{R}P^1$. Parametrize the latter by

$$x \longmapsto [\sin(\pi x) : \cos(\pi x)], \qquad x \in \mathbb{R}. \tag{6.2}$$

Let T be its central generator acting by $x \mapsto x + 1$. Its image in $\mathrm{SL}_2(\mathbb{R})$ is $-I$. For $\widetilde{g} \in \widetilde{\mathrm{PSL}}_2(\mathbb{R})$, write

$$\mathrm{trans}(\widetilde{g}) = \lim_{j\to\infty} \frac{\widetilde{g}^{\,j}(0)}{j}.$$

It is invariant under conjugacy and satisfies $\mathrm{trans}(T^q) = q$.

We use the Milnor–Wood inequality [25] in the form stated in [5, Theorem 2.52]. If $g \geq 1$ and

$$\prod_{i=1}^{g} [U_i, V_i] = T^e \quad \text{in } \widetilde{\mathrm{PSL}}_2(\mathbb{R}), \qquad e \in \mathbb{Z}, \tag{6.3}$$

then $|e| \leq 2g - 2$. Indeed, the projective images define a representation of the closed genus-g surface group, and e is its Euler number, up to the common orientation convention. Its absolute value is independent of that convention.

For comparison, put

$$\mathcal{H} = \{F \in \mathrm{Homeo}^+(\mathbb{R}) : F(x + 1) = F(x) + 1\}.$$

Eisenbud–Hirsch–Neumann [7, Theorem 2.3], in the formulation of [20, Theorem 3.1], prove that $F \in \mathcal{H}$ is a product of $g \geq 1$ commutators if and only if

$$\min_{x \in \mathbb{R}} |F(x) - x| < 2g - 1.$$

For the integral translation $T^e(x) = x + e$, with $e \neq 0$, this gives

$$\mathrm{cl}_{\mathcal{H}}(T^e) = 1 + \left\lceil \frac{|e|}{2} \right\rceil. \tag{6.4}$$

Indeed, for integers e, g, the displayed strict inequality is equivalent to $|e| \leq 2g - 2$. For every homomorphism $\rho : H_1 \to H_2$ and $a \in [H_1, H_1]$,

$$\mathrm{cl}_{H_2}(\rho(a)) \leq \mathrm{cl}_{H_1}(a),$$

because ρ takes a product of g commutators to a product of g commutators. Thus (6.4) supplies only the lower-bound direction for our discrete group. Lemma 6.4 constructs the required products inside that group. The proof of the lower bound below uses (6.3).

Proposition 6.1. *For every $N \geq 2$ there is a homomorphism*

$$\rho_N : \Gamma_N \longrightarrow \widetilde{\mathrm{PSL}}_2(\mathbb{R}) \quad \textit{such that} \quad \rho_N(z) = T^{N-1}.$$

In particular, z has infinite order.

Proof. Put $\sigma = \sqrt{3}/2$ and $\kappa = 1/2$. For $u \in \mathbb{R}$, define

$$R_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \quad D_u = \begin{pmatrix} e^u & 0 \\ 0 & e^{-u} \end{pmatrix}, \quad A = R_{\pi/2}, \quad B_u = D_u R_{\pi/3} D_u^{-1}.$$

Multiplication gives

$$A^2 = B_u^3 = -I, \qquad M_u := B_u^{-1} A = \begin{pmatrix} e^{2u}\sigma & \kappa \\ -\kappa & e^{-2u}\sigma \end{pmatrix}. \tag{6.5}$$

In particular,

$$\det M_u = 1, \qquad \mathrm{tr}\, M_u = \sqrt{3}\cosh(2u). \tag{6.6}$$

For $N \geq 2$, the inequalities $0 < \pi/(6N) < \pi/6$ give $1 < 2\cos(\pi/(6N))/\sqrt{3} < 2/\sqrt{3}$. There is therefore a unique $u_N > 0$ with

$$\cosh(2u_N) = \frac{2\cos(\pi/(6N))}{\sqrt{3}}. \tag{6.7}$$

Indeed, the right-hand side is greater than one, and $\cosh(2u)$ is continuous and strictly increasing from one to infinity on $u \geq 0$.

For $0 \leq u \leq u_N$, set

$$\theta_u = \arccos\big(\sqrt{3}\cosh(2u)/2\big), \qquad s_u = \sin\theta_u > 0, \quad c_u = \cos\theta_u.$$

Then $\theta_0 = \pi/6$, $\theta_{u_N} = \pi/(6N)$ and $\pi/(6N) \leq \theta_u \leq \pi/6$. Define

$$Q_u = \sqrt{\frac{s_u}{\kappa}} \begin{pmatrix} 1 & (c_u - e^{2u}\sigma)/s_u \\ 0 & \kappa/s_u \end{pmatrix}. \tag{6.8}$$

The inequalities $s_u > 0$ and $\kappa > 0$ make Q_u smooth, and

$$\det Q_u = \frac{s_u}{\kappa}\frac{\kappa}{s_u} = 1.$$

At $u = 0$, one has $\sin\theta_0 = \kappa$ and $\cos\theta_0 = \sigma$; substitution gives $Q_0 = I$. Moreover,

$$M_u Q_u = Q_u R_{\theta_u}. \tag{6.9}$$

To prove this identity, put $e = (1,0)^t$ and $f_u = (c_u e - M_u e)/s_u$. The columns before multiplication by $\sqrt{s_u/\kappa}$ in (6.8) are e, f_u and have determinant κ/s_u. The first-column identity is $M_u e = c_u e - s_u f_u$. The Cayley–Hamilton identity $M_u^2 - 2c_u M_u + I = 0$ gives the second one, $M_u f_u = s_u e + c_u f_u$. Scaling both columns does not change these identities and makes the determinant equal to one. This proves (6.9).

Let $\widetilde{R}_\theta$ be the lift of the rotation path, acting in the coordinate (6.2) by $x \mapsto x + \theta/\pi$. Put $\widetilde{A} = \widetilde{R}_{\pi/2}$, choose the lift $\widetilde{D}_u$ with $\widetilde{D}_0 = 1$, and define

$$\widetilde{B}_u = \widetilde{D}_u \widetilde{R}_{\pi/3} \widetilde{D}_u^{-1}.$$

Then $\widetilde{A}^2 = \widetilde{R}_\pi = T$ and $\widetilde{B}_u^3 = \widetilde{D}_u T \widetilde{D}_u^{-1} = T$, since T is central. Lift Q_u continuously with $\widetilde{Q}_0 = 1$. The two paths

$$\widetilde{B}_u^{-1}\widetilde{A}, \qquad \widetilde{Q}_u \widetilde{R}_{\theta_u} \widetilde{Q}_u^{-1}$$

have the same image in $\mathrm{SL}_2(\mathbb{R})$ by (6.9). Their quotient lies in the discrete kernel $\langle T^2 \rangle$. At $u = 0$ both equal $\widetilde{R}_{\pi/6}$, so path-lifting uniqueness makes them equal for every $u \in [0, u_N]$.

The assignments $a \mapsto \widetilde{A}$, $b \mapsto \widetilde{B}_{u_N}$ define a homomorphism $\widetilde{\rho}_N : G \to \widetilde{\mathrm{PSL}_2}(\mathbb{R})$. At the endpoint, the preceding equality gives

$$\widetilde{\rho}_N(\mu)^{6N} = \widetilde{Q}_{u_N} \widetilde{R}_{\pi/(6N)}^{6N} \widetilde{Q}_{u_N}^{-1} = \widetilde{Q}_{u_N} T \widetilde{Q}_{u_N}^{-1} = T. \tag{6.10}$$

Since $\widetilde{\rho}_N(h) = T$, equation (2.5) implies

$$\widetilde{\rho}_N(\lambda^N) = T^N T^{-1} = T^{N-1}. \tag{6.11}$$

This value is central. Thus $\widetilde{\rho}_N$ annihilates $[\lambda^N, G]$ and factors through Γ_N, giving ρ_N. The translation T^{N-1} has infinite order, so $z = \lambda^N$ has infinite order as well. □

Corollary 6.2. *For $q \neq 0$,*

$$\mathrm{cl}_{\Gamma_N}(z^q) \geq 1 + \left\lceil \frac{|q|(N-1)}{2} \right\rceil.$$

Proof. Suppose $z^q = \prod_{i=1}^g [u_i, v_i]$ in Γ_N. Since $q \neq 0$ and z has infinite order by Proposition 6.1, one has $g \geq 1$. Applying ρ_N gives

$$\prod_{i=1}^g [\rho_N(u_i), \rho_N(v_i)] = T^{q(N-1)}.$$

The Milnor–Wood inequality (6.3) therefore yields $|q|(N-1) \leq 2g - 2$. Since g is integral,

$$g \geq \left\lceil 1 + \frac{|q|(N-1)}{2} \right\rceil = 1 + \left\lceil \frac{|q|(N-1)}{2} \right\rceil.$$

This holds for every such product, hence for commutator length. □

6.2. **Surface coverings and the upper bound.** Bertram's permutation theorem states that every even permutation of D letters is a product of two cycles of length D [4]. The full-cycle formulation is also stated in [3, p. 275]. Permutations are composed from right to left. For the degrees used below, the following identities also cover $D < 5$:

$$\begin{array}{r|l} D = 2 & \mathrm{id} = (12)(12), \\ D = 3 & \mathrm{id} = (123)(132), \quad (123) = (132)^2, \quad (132) = (123)^2. \end{array}$$

For $D = 4$, every even permutation is conjugate in S_4 to the identity, a three-cycle, or a product of two disjoint transpositions. The identities

$$\mathrm{id} = (1234)(1432), \qquad (132) = (1234)(1243), \qquad (13)(24) = (1234)^2 \tag{6.12}$$

give the required representatives. Conjugating both factors gives the factorization of every permutation of the same cycle type.

Lemma 6.3. *Let $N \geq 2$ and $q \geq 1$ be integers, and let $\varepsilon \in \{0, 1\}$ be congruent to $q(N-1)$ modulo two. There is a connected oriented surface S with q boundary components and a degree-Nq branched covering*

$$S \longrightarrow S_{1,1}$$

of a once-punctured torus, such that every boundary component has degree N, there are exactly ε simple interior branch points, and

$$g(S) = 1 + \frac{q(N-1) + \varepsilon}{2}. \tag{6.13}$$

Proof. Set $D = Nq$ and let $C \in S_D$ consist of q disjoint N-cycles. Its sign is $(-1)^{q(N-1)}$. Set $P_0 = 1$ if $\varepsilon = 0$ and let P_0 be a transposition if $\varepsilon = 1$. Then $\operatorname{sgn}(CP_0) = 1$. By Bertram's theorem, including the small-degree identities above, write $(CP_0)^{-1} = UV$, with U, V both D-cycles. Define B by $B(U^{-i}(1)) = V^i(1)$ for $0 \leq i < D$. Both lists enumerate all D letters, so this defines a permutation, and $BU^{-1}B^{-1} = V$. Put $A = U$. Hence

$$[A, B]CP_0 = UVCP_0 = 1. \tag{6.14}$$

Let $B_0 = S_{1,2}$ be an oriented genus-one surface with two boundary components and presentation

$$\pi_1 B_0 = \langle a, b, c, d \mid [a, b]cd = 1 \rangle,$$

where c, d are its oriented boundary curves with base paths. Equation (6.14) defines a permutation representation by $(a, b, c, d) \mapsto (A, B, C, P_0)$. Let $\rho : \pi_1 B_0 \to S_D$ denote this representation, and set $H = \rho^{-1}(\operatorname{Stab}(1))$. The permutation A is a D-cycle, so the action is transitive and $[\pi_1 B_0 : H] = D$. The connected covering corresponding to H has degree D and the prescribed permutation monodromy. A cycle of length j in the monodromy of a boundary curve corresponds to one boundary component covering that curve with degree j. Thus the boundary over c has exactly q components, all of degree N.

Fill the d-boundary of B_0 by a disk. If $P_0 = 1$, its preimage has D degree-one boundary components; fill each by a disk mapping with degree one. If P_0 is a transposition, there are $D - 2$ degree-one components and one degree-two component; use $z \mapsto z$ on the former disks and $z \mapsto z^2$ on the latter. To specify smooth matching, parametrize each boundary covering as $e^{it} \mapsto e^{ijt}$ and use logarithmic radial collars in which the disk map is $(t, u) \mapsto (jt, ju)$. Rescaling the source collar by j identifies this with the collar of the covering. These choices give a smooth oriented branched covering of $S_{1,1}$, with exactly ε simple branch points and with the asserted remaining boundary degrees.

The initial covering has Euler characteristic $D\chi(B_0) = -2D$. The number of disks added is $D - \varepsilon$. Consequently

$$2 - 2g(S) - q = -2D + D - \varepsilon = -Nq - \varepsilon,$$

which gives (6.13). □

Lemma 6.4. *Suppose H is a group, $c = [u, v] \in H$, and $z_0 = c^N$ is central, where $N \geq 2$. Then for every $q \neq 0$,*

$$\operatorname{cl}_H(z_0^q) \leq 1 + \left\lceil \frac{|q|(N-1)}{2} \right\rceil.$$

Proof. Suppose first $q > 0$, and let $S \to S_{1,1}$ be the covering in Lemma 6.3. A retraction of $S_{1,1}$ to two generating circles defines a map to a connected CW complex with fundamental group H, taking its generators to u, v. The boundary class is c^δ for one $\delta \in \{1, -1\}$, according to the boundary-orientation convention. Since each boundary degree is N, all q boundary components of S map to conjugates of z_0^δ with the same sign. The oriented surface relation therefore gives

$$\prod_{i=1}^{g(S)} [a_i, b_i] = \left(\prod_{j=1}^{q} h_j z_0^\delta h_j^{-1} \right)^{-1} = z_0^{-\delta q}.$$

The second equality uses centrality. Reversing a product of commutators and replacing $[a,b]^{-1}$ by $[b,a]$ does not change its length. Hence

$$\mathrm{cl}_H(z_0^q) \le g(S) = 1 + \frac{q(N-1)+\varepsilon}{2} = 1 + \left\lceil \frac{q(N-1)}{2} \right\rceil.$$

For $q < 0$, apply this inequality to $|q|$ and use $\mathrm{cl}_H(g^{-1}) = \mathrm{cl}_H(g)$. □

Theorem 6.5. *For every $N \ge 2$ and $q \ne 0$,*

$$\mathrm{cl}_{\Gamma_N}(z^q) = 1 + \left\lceil \frac{|q|(N-1)}{2} \right\rceil, \qquad \mathrm{scl}_{\Gamma_N}(z) = \frac{N-1}{2}, \quad z = \lambda^N. \tag{6.15}$$

Proof. Apply the Seifert algorithm to the closed positive braid σ_1^3. Its surface has two disks and three bands attached compatibly with the orientation; the bands connect the disks. Its boundary is the trefoil, with one component. Its Euler characteristic is $2-3=-1$, so it has genus one. Truncate this surface in P. Its boundary is a degree-one parallel of K with linking number zero, hence is the preferred longitude λ. The genus-one surface relation expresses λ as a single commutator in G, after interchanging the two generators or conjugating both if required by the boundary path. Its image is therefore a single commutator in Γ_N.

Since $\lambda^N = z$ is central, Lemma 6.4 gives the upper bound in (6.15); Corollary 6.2 gives the reverse bound. For $q > 0$, the resulting formula also implies

$$\frac{N-1}{2} + \frac{1}{q} \le \frac{\mathrm{cl}_{\Gamma_N}(z^q)}{q} < \frac{N-1}{2} + \frac{2}{q}.$$

Taking $q \to \infty$ proves the stable formula. □

6.3. Surfaces with prescribed boundary.

Lemma 6.6. *Let E be a connected smooth four-manifold with boundary, and let $c : S^1 \hookrightarrow \partial E$ be parametrized and oriented. Fix its product collar. If $[c] \in [\pi_1 E, \pi_1 E]$, then*

$$g_{\mathrm{sing}}(c; E) = \mathrm{cl}_{\pi_1 E}([c]).$$

Every minimum-genus surface map can be replaced by a proper smooth immersion of the same genus, retaining that collar. No relative normal Euler number is prescribed.

Proof. Choose a basepoint and a base path to c. The boundary word of a genus-g oriented surface with one boundary component is, up to inversion, a product of g commutators. Every surface map with boundary c therefore gives $\mathrm{cl}_{\pi_1 E}([c]) \le g$; base-path changes only conjugate this word.

Conversely, suppose $[c] = \prod_{i=1}^{g} [u_i, v_i]$. Map a spine of a genus-g surface $S_{g,1}$ to based loops representing the u_i, v_i, and compose with a retraction onto that spine. Its boundary loop is freely homotopic to c, with the compatible orientation. Attaching the corresponding annulus to $S_{g,1}$ changes neither its genus nor its number of boundary components, and makes the boundary map equal to the parametrized c.

To impose the boundary condition, write an inward collar as $\partial E \times [0, 3\eta)$ and choose smooth functions $\beta : [0,3\eta) \to [0,1]$ and $a : S_{g,1} \to [0, \eta/2)$ such that $\beta = 1$ on $[0,\eta]$, $\beta = 0$ on $[2\eta, 3\eta)$, and a vanishes exactly on $\partial S_{g,1}$. On points whose current image is (y,t) in this collar, replace it by $(y, t + a(x)\beta(t))$; leave all other images unchanged. The replacement is continuous, fixes the boundary, and sends every interior point into $\operatorname{int} E$. It is a homotopy obtained by multiplying a by a parameter in $[0,1]$.

On a sufficiently small domain collar $S^1 \times [0,\delta)$, write the map as $(y(t,u), b(t,u))$ in the target collar. Continuity and compactness permit choosing this collar so that $y(t,u) = \exp_{c(t)} v(t,u)$ in a common normal-coordinate neighborhood. Here $v(t,0)=0$, $b(t,0)=0$, and $b(t,u)>0$ for $u>0$. Choose $0<\delta<\eta$ and a smooth function $\zeta : [0,\delta) \to [0,1]$ which equals one near zero and zero near δ. Replace the map on this collar by

$$\left(\exp_{c(t)}\big((1-\zeta(u))v(t,u)\big), (1-\zeta(u))b(t,u) + \zeta(u)u\right).$$

The second coordinate is positive for $u > 0$. The formula equals $(c(t), u)$ near $u = 0$ and the preceding map near $u = \delta$; multiplying ζ by a parameter gives a homotopy relative to the boundary. It imposes the prescribed product collar without changing the surface.

Relative smooth approximation [12, Chapter 2] makes this map smooth, fixed on that smaller collar. The compact part outside the collar has image at positive distance from ∂E; choose subsequent perturbations smaller than that distance. For linear maps $\mathbb{R}^2 \to \mathbb{R}^4$, the rank-one stratum has codimension $(2-1)(4-1) = 3$ and the rank-zero stratum has codimension eight. Relative jet transversality [12, Chapter 3] makes the one-jet transverse to both strata. Since $\dim S_{g,1} = 2$, their inverse images are empty. The resulting map is a proper immersion, fixed on the prescribed smaller collar. It need not be injective, and no normal Euler number was specified. Together with the first inequality this proves the genus equality. □

Proof of Theorem 1.2. Corollary 5.3 proves the surgery assertion, including the restriction to the inserted neighborhood. By (2.11) and (2.14), the prescribed curve s_r represents z^{-1}. Equations (6.15) and Lemma 6.6 therefore give

$$g_{\mathrm{sing}}(s_r; E_N) = \mathrm{cl}_{\Gamma_N}(z^{-1}) = 1 + \left\lceil \frac{N-1}{2} \right\rceil = \frac{N}{2} + 1,$$

since $N = 2^k$ is even. The immersion assertion is part of Lemma 6.6. Every smooth or locally flat embedded oriented surface is a surface map, so its genus satisfies the same lower bound.

A normal push-off of s_r is a section of the normal circle bundle over this curve. Relative to the chosen trivialization, it is a map $S^1 \to S^1$ of some degree a, and its boundary class is $s_r + a\mu_r$. Its homology image in E_N is $a[\mu]$, by (2.6) and (2.16). The oriented boundary of a surface map has zero homology class; hence $a = 0$. In that case the normal section is homotopic to the chosen section, so the boundary loop represents the same conjugacy class z^{-1}. Thus changing the normal push-off cannot lower the genus bound. □

Corollary 6.7. *For every pair of integers $R_0, G_0 \geq 1$, there are at least R_0 pairwise inequivalent smooth tori in S^4 with one common smooth exterior, each admitting all the fiber surgeries of Theorem 1.2, for which the specified fiber direction cannot bound a connected oriented surface of genus at most G_0 in the exterior, even as a continuous map.*

Proof. Choose $k \geq 1$ with $N = 2^k \geq \max(R_0, 2G_0)$. Select R_0 distinct indices in $\{0, \dots, N-1\}$. Theorem 1.1 gives the common exterior and inequivalence for these indices. Theorem 1.2 gives all the indicated surgeries and

$$g_{\mathrm{sing}}(s_r; E_N) = N/2 + 1 \geq G_0 + 1 > G_0.$$

Therefore none of the specified directions has an oriented spanning surface map of genus at most G_0. □

Remark 6.8. For a primitive tangent slope $b\lambda_r + qs_r$, where $b, q \in \mathbb{Z}$ and $\gcd(b, q) = 1$, fix its constant-normal-angle affine representative. Its image in Γ_N is λ^{b-Nq}. If $b - Nq = Na \neq 0$, then Theorem 6.5 and Lemma 6.6 give

$$g_{\mathrm{sing}}(b\lambda_r + qs_r; E_N) = 1 + \left\lceil \frac{|a|(N-1)}{2} \right\rceil.$$

If $b - Nq = 0$, the boundary element is the identity, and Lemma 6.6 gives singular genus zero. No formula for the other primitive tangent slopes, and no equality between singular and embedded genus, is asserted. The common-exterior theorem distinguishes topological embedding classes; it does not distinguish smooth embeddings of a fixed topological pair.

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Toulouse, France
Email address: nifa.anass@gmail.com