OPERATIONAL AND FREE-ENERGY CONVERGENCE DO NOT DETERMINE KINETIC GEOMETRY IN LOGARITHMIC QUANTUM TRANSPORT

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ABSTRACT. For every primitive GNS-symmetric quantum Markov generator on $M_d(\mathbb{C})$, $d \geq 2$, we construct reversible extensions on a countable direct sum with a common all-time diamond-norm semigroup limit and a common trace-norm Γ-limit of relative entropies. Their entropy-constrained logarithmic current actions nevertheless have a continuum of distinct Γ-limits. The additional cotangent form is $\chi(E - F_0(B))\|Q_0 C e_h\|^2$, where $\chi = \limsup_n b_n/\varphi_n^2$ is determined by the auxiliary rates. The joint action–Fisher functional has instead the original core limit. For canonically embedded faithful initial data, the corresponding energy–dissipation functionals Γ-converge, and their almost minimizers converge to the core semigroup. Every recovery family attaining a strict kinetic saving along a faithful C^2 curve below the entropy cap has integrated Fisher cost bounded below by a positive multiple of $\log^2(1/\delta)$. At fixed coupling, the current action need not be lower semicontinuous and can strictly exceed the metric energy of its induced distance. A second construction has a faithful stationary limit, vanishing generator and all-time semigroup differences in diamond norm, and entropy Γ-convergence, but collapsing distances between fixed faithful states. A bound in terms of the diamond norm of the entire auxiliary generator gives a complementary obstruction.

Contents

2020 *Mathematics Subject Classification.* Primary 49Q22; Secondary 46L57, 47D07, 49J45, 81P17.

Key words and phrases. Quantum Markov semigroups; noncommutative optimal transport; logarithmic transport metric; Γ-convergence; energy–dissipation principle; relative entropy.

1. Introduction

1.1. **Operational, kinetic, and energy–dissipation limits.** Convergence of a quantum Markov semigroup and of its relative entropy does not determine the limit of its logarithmic current action. The examples below distinguish the kinetic action, the induced distance, and the joint action–Fisher functional.

Fix a primitive GNS-symmetric generator $\mathcal{K}_*$ on $M_d(\mathbb{C})$, $d \geq 2$, and an eigenvector e_h of its stationary density. The family in Theorem B, indexed by $\varepsilon > 0$ and $k \geq 0$, satisfies

$$\sup_{k\geq 0} \sup_{t\geq 0} \|S_t^{\varepsilon,k} - S_t^0\|_\diamond \leq C\varepsilon, \qquad 0 < \varepsilon \leq \varepsilon_0, \tag{1.1}$$

where $C, \varepsilon_0 > 0$ depend only on the core and e_h. The comparison semigroup retains every return channel and is independent of k. Its restriction to the core is $e^{t\mathcal{K}_*}$. For each fixed k, the relative entropies have the same trace-norm Γ-limit, whereas, for every $E, T > 0$,

$$\mathcal{A}_{\varepsilon,k,E} \xrightarrow{\Gamma} \mathcal{A}^{\mathrm{eff}}_{k,E} \qquad (\varepsilon \downarrow 0) \tag{1.2}$$

in the uniform trace-norm topology on $[0, T]$. The limiting cotangent form is

$$g_B^0(C) + k\big(E - F_0(B)\big)\|Q_0 C e_h\|^2. \tag{1.3}$$

Distinct k give distinct local metrics at every faithful core state with $F_0(B) < E$. The estimate (1.1) is uniform in k; the variational limit (1.2) is asserted for fixed k.

For the same family, Theorem C gives

$$\mathcal{A}_{\varepsilon,k,E} + \int_0^T \mathcal{I}_{\varepsilon,k}(\rho_t)\, dt \xrightarrow{\Gamma} \mathcal{A}_{0,E} + \int_0^T \mathcal{I}_0(B_t)\, dt. \tag{1.4}$$

Here k is fixed and $\mathcal{A}_{0,E}$ is the original core action. For canonically embedded faithful initial data, the associated current energy–dissipation functionals have the corresponding Γ-limit; their almost minimizers converge to $e^{t\mathcal{K}_*}x$ (Corollary 6.6 and Proposition 6.8).

The additional mobility in (1.3) is obtained by placing mass m/φ_n at an auxiliary level with $\log(1/\delta) = o(\varphi_n)$ and $b_n/\varphi_n^2 \to \chi$. The mass tends to zero, its entropy cost tends to m, and its cotangent contribution tends to $\chi m\|Q_0 C e_h\|^2$ (Lemma 4.6). Such kinetic recovery need not recover the entropy. Indeed, recovery of the time-integrated entropy implies the original core-action lower bound (Corollary 4.8).

There is also a quantitative Fisher obstruction. Let B be a faithful C^2 core curve on $[0, T]$ with $\sup_t F_0(B_t) < E$ and $G = \mathcal{A}_{0,E}(B) - \mathcal{A}^{\mathrm{eff}}_{\chi,E}(B) > 0$. Every uniformly trace-convergent family $\rho^\delta \to B \oplus 0$ attaining the effective action satisfies

$$\liminf_{\delta\downarrow 0} \frac{\int_0^T \mathcal{I}_\delta(\rho_t^\delta)\, dt}{\log^2(1/\delta)} \geq \frac{G}{4M_C} > 0, \tag{1.5}$$

with M_C as in Proposition 6.10. No matching upper rate is claimed.

These statements concern the current action defined in Section 2. At positive coupling it is not, in general, the metric energy of the induced distance: Corollary 8.9 proves a strict gap and failure of lower semicontinuity with fixed endpoints. The energy–dissipation limit uses a fixed time interval, canonically prepared faithful initial data, and no entropy tilt. The proximal limit fixes a positive step. The faithful-reference example and the complete-generator bound are stated in Sections 9 and 10, respectively.

1.2. **The weak-coupling extension.** Let $\mathcal{H}_0 = \mathbb{C}^d$, $d \geq 2$, and let $\mathcal{K}_*$ be a primitive GNS-symmetric quantum Markov generator with faithful stationary state

$$\sigma_0 = \mathrm{diag}(\pi_1, \dots, \pi_d), \qquad \pi_i > 0, \qquad \sum_i \pi_i = 1.$$

Fix a coordinate vector e_h in this eigenbasis, and put $P = E_{hh}$ and $Q_0 = I_{\mathcal{H}_0} - P$. On

$$\mathcal{H} = \mathcal{H}_0 \oplus \ell^2(\{v_1, v_2, \dots\}), \qquad P_0 = \mathrm{proj}_{\mathcal{H}_0}, \tag{1.6}$$

extend the core channels by zero and adjoin the reversible pairs

$$a_n^\delta = q(h, v_n) = \delta b_n e^{-\varphi_n}, \qquad q(v_n, h) = b_n. \tag{1.7}$$

The core channels and the sequences (b_n) and (φ_n) are fixed; only $\delta > 0$ varies. Assume that, for constants $c_*, C_* > 0$,

$$\varphi_n \geq c_* n > 0, \qquad 0 < b_n \leq C_* \varphi_n^2, \qquad \chi = \limsup_{n\to\infty} \frac{b_n}{\varphi_n^2}. \tag{1.8}$$

Thus $0 \leq \chi < \infty$. Set

$$A_\varphi = \sum_n b_n e^{-\varphi_n}, \qquad Z_\delta = 1 + \delta \pi_h \sum_n e^{-\varphi_n}, \qquad \beta_\delta = Z_\delta^{-1}. \tag{1.9}$$

Throughout, $D(A\|B)$ denotes Umegaki relative entropy in its extended-value sense. For positive trace-class operators A, B with $\operatorname{supp} A \leq \operatorname{supp} B$, we write

$$D(A\|B) = \operatorname{Tr}\big(A(\log A - \log B)\big),$$

allowing the value $+\infty$; if the support inclusion fails, then $D(A\|B) = +\infty$. This notation refers to the extended relative entropy, not to a subtraction of two possibly infinite traces. More explicitly, if $A = \sum_i a_i |u_i\rangle\langle u_i|$ and $B = \sum_j b_j |v_j\rangle\langle v_j|$, with positive eigenvalues listed on their supports, then, under the support inclusion, it is the spectral sum

$$D(A\|B) = \sum_{i,j} a_i |\langle u_i, v_j\rangle|^2 (\log a_i - \log b_j).$$

Its negative part is at most $\operatorname{Tr} B/e$, since $a \log(b/a) \leq b/e$ for $0 < a < b$ and $\sum_i |\langle u_i, v_j\rangle|^2 \leq 1$. Thus the sum is well defined in $\mathbb{R} \cup \{+\infty\}$. We use $0 \log 0 = 0$, $D(0\|B) = 0$, and the same convention for positive subnormalized blocks. On states this is the usual Umegaki relative entropy; in particular, it is jointly lower semicontinuous for trace-norm convergence [8, Lemma 4]. The extended semigroup has stationary density

$$\sigma_\delta = \beta_\delta \left(\sigma_0 + \delta \pi_h \sum_n e^{-\varphi_n} E_{v_n v_n} \right), \quad F_\delta(\rho) = D(\rho\|\sigma_\delta), \quad F_0(B) = D(B\|\sigma_0). \tag{1.10}$$

Core operators are identified with their zero extensions to $\mathcal{H}$.

Write g_B^0 for the logarithmic cotangent form of the finite core, and g_ρ^δ for the form obtained by adjoining the pairs (1.7). For each δ, let $\mathscr{T}_\delta$ be a real linear space of bounded Hermitian operators such that

$$\mathscr{T}_{\mathrm{fin}} + \mathbb{R}I \subseteq \mathscr{T}_\delta \subseteq \mathscr{T}_\delta^{\mathrm{eq}} := \{C = C^* \in \mathcal{B}(\mathcal{H}) : g_{\sigma_\delta}^\delta(C) < \infty\}. \tag{1.11}$$

Here $\mathscr{T}_{\mathrm{fin}}$ consists of the Hermitian matrices with finite coordinate support. The reduced action $\mathcal{A}_{\delta,E}$ is the infimum of the integrated squared current norm over currents satisfying the weak continuity equation on a prescribed trace-continuous state curve, with $F_\delta(\rho_t) \leq E$ at every time. It is $+\infty$ if these conditions cannot be satisfied. The state domain and the test-integrability condition are specified in Definition 2.6.

1.3. **The effective kinetic limit.**

For $E > 0$, let

$$K_E^0 = \{B \in M_d : B \geq 0,\ \operatorname{Tr} B = 1,\ F_0(B) \leq E\}.$$

For $B \in K_E^0$ and a Hermitian core potential C, define

$$\widehat{g}_{B,E,\chi}(C) = g_B^0(C) + \chi(E - F_0(B))\|Q_0 C e_h\|^2, \tag{1.12}$$

$$a_{\chi,E}(B, V) = \sup_{\substack{C=C^* \\ C_{hh}=0}} \{2\operatorname{Tr}(CV) - \widehat{g}_{B,E,\chi}(C)\}, \qquad V = V^*, \quad \operatorname{Tr} V = 0. \tag{1.13}$$

The condition $C_{hh} = 0$ fixes the additive scalar in the potential. For an absolutely continuous curve $B : [0, T] \to K_E^0$, put

$$\mathcal{A}^{\mathrm{eff}}_{\chi,E}(B) = \int_0^T a_{\chi,E}(B_t, \dot{B}_t)\, dt. \tag{1.14}$$

Regard this as a functional on $C([0, T]; \mathfrak{S}(\mathcal{H}))$, with value $+\infty$ on all other curves, in particular on curves not supported in $\mathcal{H}_0$.

Theorem A. *Assume* (1.8). *For every $E, T > 0$ and every family of test spaces satisfying* (1.11),

$$\mathcal{A}_{\delta,E} \xrightarrow{\Gamma} \mathcal{A}^{\mathrm{eff}}_{\chi,E} \qquad (\delta \downarrow 0) \tag{1.15}$$

on $C([0, T]; \mathfrak{S}(\mathcal{H}))$ with the uniform trace-norm topology. If $\delta_j \to 0$ and $\sup_j \mathcal{A}_{\delta_j,E}(\rho^j) < \infty$, then (ρ^j) has a uniformly trace-convergent subsequence. Every limit is a core-supported curve with values in K_E^0.

The lower inequality holds for arbitrary quantum approximating curves. For every B of finite effective action with faithful endpoints, there is a recovery family with endpoints $\iota_\delta B_0$ and $\iota_\delta B_T$, where

$$\iota_\delta B = \beta_\delta \left(B + \delta \pi_h \sum_n e^{-\varphi_n} E_{v_n v_n} \right). \tag{1.16}$$

Recovery without prescribed endpoints holds for every curve of finite effective action, including curves through singular states.

For $B > 0$, let $\mathbb{K}_B$ denote the Onsager operator representing g_B^0 on trace-zero Hermitian matrices, and put

$$\mathbb{R}_h C = \tfrac{1}{2}(PCQ_0 + Q_0 CP). \tag{1.17}$$

Then

$$a_{\chi,E}(B, V) = \mathrm{Tr}\big(V[\mathbb{K}_B + \chi(E - F_0(B))\mathbb{R}_h]^{-1} V\big)\,. \tag{1.18}$$

The inverse is taken on the trace-zero Hermitian space. At singular states, (1.13) defines the extended quadratic form. The auxiliary profile enters the limit only through χ; the ratio b_n/φ_n^2 need not converge. For $\chi = 0$, the effective action is the core action restricted to K_E^0.

The unit-time distance $d_{\chi,E}$ of this action is finite on K_E^0. It induces the matrix norm topology and makes K_E^0 a compact geodesic metric space. If $B_0, B_1 \in K_E^0$ are faithful, then

$$\mathcal{W}^{\mathscr{T}_\delta}_{\delta,E}(\iota_\delta B_0, \iota_\delta B_1) \longrightarrow d_{\chi,E}(B_0, B_1). \tag{1.19}$$

For a fixed faithful base state and a fixed positive step, the entropy proximal functionals also Γ-converge. Every sequence of almost minimizers converges to the unique effective minimizer (Corollary 5.2). Attainment of the moving infima is not required.

1.4. **One operational limit and a continuum of kinetic limits.** Let $\mathcal{L}_{0,*}$ retain the core and every return channel, with all outward rates equal to zero. The difference $\mathcal{L}_{\delta,*} - \mathcal{L}_{0,*}$ is bounded and has diamond norm $2\delta A_\varphi$. If $b_n \geq 1$ and the total outward rate is ε, Theorem 7.1 gives

$$\sup_{t \geq 0} \|S_t^\varepsilon - S_t^0\|_\diamond \leq C\varepsilon$$

for sufficiently small ε, with C depending only on the core and e_h.

Keep $b_n = n^2$. For each $k \geq 0$, define

$$\varphi_n^{(k)} = \begin{cases} n/\sqrt{k}, & k > 0, \\ n^2, & k = 0, \end{cases} \qquad A_k = \sum_n n^2 e^{-\varphi_n^{(k)}}, \qquad a_n^{\varepsilon,k} = \frac{\varepsilon}{A_k} n^2 e^{-\varphi_n^{(k)}}. \tag{1.20}$$

These extensions have the same core, enlarged Hilbert space, return channels, and total outward rate ε. Their comparison generator $\mathcal{L}_{0,*}$ is independent of k.

Theorem B. *For the family* (1.20)*, there are constants $C, \varepsilon_0 > 0$, depending only on the core and e_h, such that*

$$\|\mathcal{L}_{\varepsilon,k,*} - \mathcal{L}_{0,*}\|_\diamond = 2\varepsilon, \qquad \sup_{k\ge 0}\sup_{t\ge 0}\|S_t^{\varepsilon,k} - S_t^0\|_\diamond \le C\varepsilon \quad (0 < \varepsilon \le \varepsilon_0). \tag{1.21}$$

For every fixed $k \ge 0$ and $E, T > 0$, the constrained current actions Γ-converge, as $\varepsilon \downarrow 0$, to the core action with cotangent form

$$g_B^0(C) + k(E - F_0(B))\|Q_0 C e_h\|^2. \tag{1.22}$$

The relative entropies Γ-converge in trace norm to F_0 on the core and to $+\infty$ elsewhere; the embeddings (1.16)*, with $\delta = \varepsilon/A_k$, recover both the states and their entropies. Distinct values of k give distinct local metrics at faithful core states with $F_0(B) < E$.*

For the symmetric two-level core with equilibrium $I_2/2$, the local coefficient in the direction $E_{01} + E_{10}$ at equilibrium is $4/(1 + kE)$. On faithful core states below the cap, the entropy-gradient equation of the limiting metric is

$$\dot B = \mathcal{K}_* B - \frac{k(E - F_0(B))}{2}[P, [P, \log B]]. \tag{1.23}$$

The physical core limit is $\dot B = \mathcal{K}_* B$. At each positive coupling, the quadratic-exponential profile also has physical trajectories for which entropy dissipation strictly exceeds squared speed in the constrained metric (Theorem 8.1).

In Theorem B, the variational limit is taken for fixed k, while the diamond estimate is uniform in k. The parameter tending to zero multiplies the outward generator; the unbounded return generator remains fixed. The common stationary limit $\sigma_0 \oplus 0$ has kernel $\mathcal{H}_0^\perp$. Corollary 5.2 fixes its proximal step independently of the coupling.

1.5. **Fisher coercivity, action–Fisher convergence, and prepared EDP limits.** Let $\mathcal{I}_\delta$ be the channel Fisher information defined in (2.21), and let $\mathcal{I}_0$ be its finite-core counterpart. Write $\mathcal{A}_{0,E}$ for the original finite-core action on K_E^0; thus $\mathcal{A}_{0,E} = \mathcal{A}_{0,E}^{\mathrm{eff}}$ with $\chi = 0$. On the same curve space as in Theorem A, set

$$\mathscr{B}_{\delta,E}(\rho) = \mathcal{A}_{\delta,E}(\rho) + \int_0^T \mathcal{I}_\delta(\rho_t)\,dt, \qquad \mathscr{B}_{0,E}(B) = \mathcal{A}_{0,E}(B) + \int_0^T \mathcal{I}_0(B_t)\,dt. \tag{1.24}$$

The second functional is $+\infty$ outside core-supported curves.

Theorem C. *Assume* (1.8)*. For every $E, T > 0$ and every family of test spaces satisfying* (1.11)*,*

$$\mathscr{B}_{\delta,E} \xrightarrow{\Gamma} \mathscr{B}_{0,E} \qquad (\delta \downarrow 0) \tag{1.25}$$

in the uniform trace-norm topology. If $\delta_j \to 0$ and $\sup_j \mathscr{B}_{\delta_j,E}(\rho^j) < \infty$, then (ρ^j) has a uniformly trace-convergent subsequence. Every limit is core-supported and takes values in K_E^0. Recovery includes singular limiting states. For a curve of finite $\mathscr{B}_{0,E}$ with faithful endpoints, recovery may be chosen with the prescribed endpoints $\iota_\delta B_0$ and $\iota_\delta B_T$.

The limit in Theorem C is independent of χ. Its principal estimate is

$$\sum_n b_n \Lambda(\rho_{\nu_n\nu_n}, \delta e^{-\varphi_n}) \le \frac{4}{\log^2(1/\delta)}\mathcal{I}_\delta^{\mathrm{aux}}(\rho) + \sqrt{\delta}\sum_n b_n e^{-\varphi_n/2}, \qquad 0 < \delta < 1.$$

Thus bounded integrated Fisher information makes the auxiliary current pairing vanish in the limiting continuity equation. More precisely, Proposition 6.10 applies to faithful C^2 core curves lying strictly below the cap for which $\mathcal{A}_{0,E}(B) > \mathcal{A}_{\chi,E}^{\mathrm{eff}}(B)$: every uniformly trace-convergent family attaining the effective action satisfies the positive $\log^2(1/\delta)$ lower bound in (6.14). For a fixed canonically embedded faithful initial state, the full current energy–dissipation functionals also Γ-converge; their almost minimizers converge to $e^{t\mathcal{K}_*}x$ (Corollary 6.6 and Proposition 6.8). The initial state is prepared canonically; the entropy is untilted, and the time interval is fixed.

1.6. **Robustness and the boundary of the instability.** Theorem 9.1 concerns a different family. A fixed faithful infinite-dimensional reversible reference admits perturbations of both rates of one pair for which the bounded generator differences and the all-time semigroup differences tend to zero in diamond norm. The stationary states converge in trace norm and in both directions of relative entropy, but distances between fixed faithful states collapse. The reference generator remains unbounded, and the reference-state ratios are not uniformly bounded. The associated energies nevertheless Γ-converge to the faithful reference energy (Corollary 9.6).

For an auxiliary Lindblad generator $\mathcal{Q}_*$ with positive rates and diamond norm η, Proposition 10.1 gives

$$g_{\rho,\mathrm{aux}}(C) \leq \tfrac{\eta}{4}\|Ce_h\|^2$$

uniformly over all states and core tests. Hence a nonzero limiting auxiliary cotangent term cannot persist when the entire auxiliary generator of this edge form tends to zero in diamond norm. Moreover, fixed positive returns and vanishing total outward rate force every trace limit of the stationary states to be core-supported (Proposition 10.3).

1.7. **Related work.** Carlen–Maas [3] constructed the logarithmic transport metric for finite-dimensional quantum Markov semigroups with detailed balance. Mittnenzweig–Mielke [10] obtained entropic gradient structures for Lindblad equations and their couplings to macroscopic systems. Brooks–Maas [2] proved that a finite-dimensional ergodic Lindblad equation admits a Riemannian relative-entropy gradient representation if and only if it satisfies BKM detailed balance. These existence results concern a fixed evolution. Theorems A and B concern limits of the specified logarithmic current construction. For the Carlen–Maas noncommutative transport distance, Wirth [16] proved a Hamilton–Jacobi dual formula. The time-dual representation in Lemma 4.2 is derived directly for the present capped effective action. In the tracial Dirichlet-form setting, Wirth [17] obtained a metric entropy flow under a gradient estimate; the test domains and hypotheses there differ from (1.11).

Variational convergence of gradient structures is studied through energy–dissipation functionals and their tilts in [11, 12]. Mielke–Stephan [12] treated linear fast–slow reaction systems with cosh dissipation. Gladbach–Kopfer–Maas–Portinale [7] obtained effective dynamical transport actions on periodic graphs through a cell formula. Peletier–Schlottke [14] gave an energy–dissipation limit which is not the functional of a gradient system. Deng–Gao–Wang [5] obtained different homogenized dynamics from different time-discrete metric structures. Theorem A instead determines a kinetic-action limit on moving quantum entropy sublevels. Its coefficient is the limsup in (1.8); Theorem B realizes all finite values of that coefficient with one common all-time semigroup limit. Theorem C and Corollary 6.6 give, for the same families, the joint action–Fisher limit and its prepared-initial energy–dissipation form. The auxiliary Fisher estimate separates this limit from the kinetic limit; no tilted EDP convergence is asserted.

Notation. $\mathfrak{S}_p(\mathcal{H})$ denotes the Schatten class, and $\mathfrak{S}(\mathcal{H}) = \{\rho \in \mathfrak{S}_1(\mathcal{H}) : \rho \geq 0,\ \mathrm{Tr}\,\rho = 1\}$. The trace is unnormalized. Hilbert inner products are conjugate-linear in the first variable; on Hilbert–Schmidt operators the real inner product is $\langle U, V\rangle_{\mathbb{R}} = \mathrm{Re}\,\mathrm{Tr}(U^*V)$. The symbols I and I_0 denote the identities on $\mathcal{H}$ and $\mathcal{H}_0$, respectively. Matrix coordinates refer to the chosen eigenbasis of the core and the auxiliary basis (e_{v_n}). A matrix has finite coordinate support if it equals $P_K C P_K$ for some finite coordinate projection P_K. For a bounded map T on trace class,

$$\|T\|_\diamond = \sup_{m\geq 1}\|T \otimes \mathrm{id}_{M_m}\|_{\mathfrak{S}_1\to\mathfrak{S}_1}.$$

Faithfulness means trivial kernel. It does not imply a positive lower bound by the identity on an infinite-dimensional space. Metric derivatives and curves of maximal slope have the conventions of [1]; for extended metrics they are considered within a finite-distance component.

2. The logarithmic calculus and the level extension

2.1. **The finite core.** The generator $\mathcal{K}_*$ is *GNS symmetric* at σ_0 if its Heisenberg adjoint $\mathcal{K}$ is self-adjoint for $\langle A, B\rangle_{\sigma_0} = \mathrm{Tr}(\sigma_0 A^* B)$. It is *primitive* if $e^{t\mathcal{K}_*}B \to \sigma_0$ as $t \to \infty$ for every core state B. Fix a modular representation [3, Theorem 3.1 and Remark 3.3], indexed by a finite set $\mathcal{J}_0$ with an involution $j \mapsto j'$, such that

$$L_{j'} = L_j^*, \qquad \omega_{j'} = -\omega_j, \qquad [L_j, \log \sigma_0] = \omega_j L_j, \tag{2.1}$$

$$\mathcal{K}_* B = -\sum_{j\in\mathcal{J}_0} [L_j^*, e^{-\omega_j/2} L_j B - e^{\omega_j/2} B L_j]. \tag{2.2}$$

Nonnegative coefficients are absorbed in the L_j. Their common commutant is $\mathbb{C}I_0$: a matrix commuting with all the L_j is fixed by the Heisenberg semigroup, and primitivity makes every fixed matrix scalar.

For $U \in \mathcal{B}(\mathcal{H})$ put $\mathcal{D}_U(\rho) = U\rho U^* - \{U^*U, \rho\}/2$. Reindexing by j' gives

$$\mathcal{K}_* = 2\sum_{j\in\mathcal{J}_0} e^{-\omega_j/2}\mathcal{D}_{L_j}. \tag{2.3}$$

The identity includes self-adjoint channels. Extend each L_j by zero to $\mathcal{H}$, and denote the resulting bounded generator by $\widetilde{\mathcal{K}}_*$. Its loss operator on the core is

$$G_0 = 2\sum_{j\in\mathcal{J}_0} e^{-\omega_j/2} L_j^* L_j. \tag{2.4}$$

It commutes with σ_0 and is positive definite. Indeed, if $G_0 v = 0$, then $L_j v = L_j^* v = 0$ for all j. The projection onto $\mathbb{C}v$ commutes with every channel. Since $d \geq 2$, primitivity implies $v = 0$.

2.2. **The extended semigroup.** On edge hv_n set

$$L^\delta_{n,+} = \ell^\delta_n E_{v_n h}, \quad L^\delta_{n,-} = (L^\delta_{n,+})^*, \quad \ell^\delta_n = \frac{(a^\delta_n b_n)^{1/4}}{\sqrt{2}}, \quad \omega^\delta_{n,\pm} = \pm(\varphi_n - \log\delta). \tag{2.5}$$

Together with the extended core channels, these satisfy (2.1) with reference σ_δ. Their generator is

$$\mathcal{L}_{\delta,*} = \widetilde{\mathcal{K}}_* + \sum_{n\geq 1}\left(a^\delta_n \mathcal{D}_{E_{v_n h}} + b_n \mathcal{D}_{E_{h v_n}}\right), \tag{2.6}$$

since $2(\ell^\delta_n)^2 e^{-\omega^\delta_{n,+}/2} = a^\delta_n$ and $2(\ell^\delta_n)^2 e^{\omega^\delta_{n,+}/2} = b_n$. The return part of (2.6) is the generator of (2.7); the core and outward parts are bounded perturbations. In particular, (2.6) does not denote an operator-norm convergent series.

Lemma 2.1. *Let D_t be the identity on $\mathcal{H}_0$ and let $D_t e_{v_n} = e^{-b_n t/2} e_{v_n}$. Then*

$$T_t\rho = D_t \rho D_t + P\sum_n (1 - e^{-b_n t})\rho_{v_n v_n} \tag{2.7}$$

defines a strongly continuous semigroup of completely positive trace-preserving maps on $\mathfrak{S}_1(\mathcal{H})$. If $\mathcal{G}_*$ is its generator, then

$$\mathrm{Dom}\,\mathcal{G}_* = \left\{X \in \mathfrak{S}_1(\mathcal{H}) : \lim_{t\downarrow 0}\frac{T_t X - X}{t} \text{ exists in } \mathfrak{S}_1(\mathcal{H})\right\}.$$

Finite-coordinate matrices form a generator core. On that core,

$$\mathcal{G}_* X = \sum_{n\geq 1} b_n\left(X_{v_n v_n} P - \frac{1}{2}\{E_{v_n v_n}, X\}\right), \tag{2.8}$$

where the sum is finite for each finite-coordinate X. More precisely, if P_R projects onto the core and the first R auxiliary coordinates and

$$\mathcal{E}_R X = P_R X P_R + P\,\mathrm{Tr}((I - P_R)X),$$

then $\mathcal{E}_R \operatorname{Dom} \mathcal{G}_* \subset \operatorname{Dom} \mathcal{G}_*$ *and, for every* $X \in \operatorname{Dom} \mathcal{G}_*$,

$$\mathcal{E}_R X \longrightarrow X, \qquad \mathcal{G}_* \mathcal{E}_R X = \mathcal{E}_R \mathcal{G}_* X \longrightarrow \mathcal{G}_* X$$

in trace norm.

Proof. For $X \in \mathfrak{S}_1(\mathcal{H})$, $\sum_n |X_{\nu_n \nu_n}| \le \|X\|_1$. The series in (2.7) therefore converges in trace norm. After finite amplification, diagonal compression gives $\sum_n \|X_{\nu_n \nu_n}\|_1 \le \|X\|_1$, so the same conclusion holds for the amplified series. Each summand is completely positive, and, for $X \ge 0$,

$$\operatorname{Tr}(D_t X D_t) + \sum_n (1 - e^{-b_n t}) X_{\nu_n \nu_n} = \operatorname{Tr} X.$$

Thus T_t is a channel. The identities $D_t P = P$ and $D_t D_s = D_{t+s}$ give $T_t T_s = T_{t+s}$. On each finite coordinate subspace containing the core, $T_t X \to X$ as $t \downarrow 0$. Density and contractivity imply strong continuity on trace class. Differentiation on that subspace gives (2.8).

Direct substitution gives

$$\mathcal{E}_R T_t X = T_t \mathcal{E}_R X = D_t P_R X P_R D_t + P \left(\sum_{n \le R} (1 - e^{-b_n t}) X_{\nu_n \nu_n} + \sum_{n > R} X_{\nu_n \nu_n} \right).$$

Moreover,

$$\|\mathcal{E}_R X - X\|_1 \le \|P_R X P_R - X\|_1 + |\operatorname{Tr}((I - P_R)X)| \longrightarrow 0.$$

The contractions $\mathcal{E}_R$ therefore converge strongly to the identity and commute with T_t. For $X \in \operatorname{Dom} \mathcal{G}_*$,

$$\mathcal{E}_R X \in \operatorname{Dom} \mathcal{G}_*, \qquad \mathcal{G}_* \mathcal{E}_R X = \mathcal{E}_R \mathcal{G}_* X,$$

and

$$\|\mathcal{E}_R X - X\|_1 + \|\mathcal{G}_* \mathcal{E}_R X - \mathcal{G}_* X\|_1 \longrightarrow 0.$$

Since $\mathcal{E}_R X$ has finite coordinate support, these matrices form a generator core. □

Proposition 2.2. *For every* $\delta > 0$*, the operator* (2.6) *generates a strongly continuous semigroup of completely positive trace-preserving maps on* $\mathfrak{S}_1(\mathcal{H})$*. Its unique stationary density is* σ_δ*, and its Heisenberg semigroup is GNS symmetric with respect to this density. Moreover,*

$$\operatorname{Dom} \mathcal{L}_{\delta,*} = \operatorname{Dom} \mathcal{L}_{0,*} = \operatorname{Dom} \mathcal{G}_*,$$

their difference extends to a bounded map on trace class, and, for every $T > 0$,

$$\|\mathcal{L}_{\delta,*} - \mathcal{L}_{0,*}\|_\diamond = 2\delta A_\varphi, \qquad \sup_{0 \le t \le T} \|S_t^\delta - S_t^0\|_\diamond \le 2\delta A_\varphi T. \tag{2.9}$$

The restriction of S_t^0 *to core-supported operators is* $e^{t\mathcal{K}_*}$.

Proof. Since $\varphi_n \ge c_* n$ and $b_n \le C_* \varphi_n^2$, $A_\varphi < \infty$. The outward generator is bounded and equals

$$B_\delta(\rho) = \rho_{hh} \sum_n a_n^\delta E_{\nu_n \nu_n} - \frac{\delta A_\varphi}{2} \{P, \rho\}. \tag{2.10}$$

The bounded perturbation theorem [6, Theorem III.1.3] gives

$$\mathcal{L}_{\delta,*} = \mathcal{G}_* + \widetilde{\mathcal{K}}_* + B_\delta, \qquad \operatorname{Dom} \mathcal{L}_{\delta,*} = \operatorname{Dom} \mathcal{G}_*.$$

The same holds with $B_\delta = 0$. Since the two perturbations are bounded,

$$V_t := e^{t(\widetilde{\mathcal{K}}_* + B_\delta)} = \lim_{q \to \infty} (e^{t\widetilde{\mathcal{K}}_*/q} e^{tB_\delta/q})^q$$

in operator norm. The products $(T_{t/q}V_{t/q})^q$ are contractions. The sum generator has dense domain $\operatorname{Dom}\mathcal{G}_*$, and $\lambda - \mathcal{G}_* - \widetilde{\mathcal{K}}_* - B_\delta$ is surjective for sufficiently large λ by the bounded perturbation theorem. The product formula [6, Corollary III.5.8] therefore gives

$$S_t^\delta X = \lim_{q\to\infty}(T_{t/q}V_{t/q})^q X \qquad (X \in \mathfrak{S}_1(\mathcal{H})).$$

Each factor is a channel. The same limits hold after every finite amplification. The positive cone is trace-norm closed and the trace is continuous; hence each amplified limit is positive and trace preserving.

The identity

$$\sum_n b_n(\sigma_\delta)_{v_nv_n} = \beta_\delta\delta\pi_h A_\varphi < \infty$$

and dominated convergence in (2.7) imply $\sigma_\delta \in \operatorname{Dom}\mathcal{G}_*$. The core term vanishes, and $a_n^\delta(\sigma_\delta)_{hh} = b_n(\sigma_\delta)_{v_nv_n}$ for each n. Thus $\mathcal{L}_{\delta,*}\sigma_\delta = 0$.

Let $S_t^{\delta,R}$ retain the core and the first R reversible pairs, with the other auxiliary coordinates frozen. The modular identities imply GNS symmetry at σ_δ on every finite coordinate compression containing these pairs, hence on the full space by trace-class approximation of the GNS pairings. Let $\widehat{S}_t^{\delta,R}$ retain all returns but only the first R outward channels. Duhamel's formula gives

$$\|\widehat{S}_t^{\delta,R} - S_t^\delta\|_\diamond \le 2t\sum_{n>R} a_n^\delta \longrightarrow 0$$

locally uniformly in t. For X supported on the core and the first R levels, $S_t^{\delta,R}X = \widehat{S}_t^{\delta,R}X$. Density and contractivity give $S_t^{\delta,R}X \to S_t^\delta X$ in trace norm for every X, also at each finite amplification. For bounded A, B, duality and preservation of adjoints give

$$\operatorname{Tr}(\sigma_\delta A^*(S_t^{\delta,R})^*B) = \operatorname{Tr}\big(S_t^{\delta,R}(\sigma_\delta A^*)B\big),$$
$$\operatorname{Tr}(\sigma_\delta((S_t^{\delta,R})^*A)^*B) = \operatorname{Tr}\big(A^*S_t^{\delta,R}(B\sigma_\delta)\big).$$

Both inputs on the right are trace class. Passing to the limit in

$$\operatorname{Tr}(\sigma_\delta A^*(S_t^{\delta,R})^*B) = \operatorname{Tr}(\sigma_\delta((S_t^{\delta,R})^*A)^*B)$$

proves GNS symmetry of $(S_t^\delta)^*$.

Let ρ be stationary. Then $\rho \in \operatorname{Dom}\mathcal{L}_{\delta,*}$. For $B = P_0\rho P_0$ and $p_n = \rho_{v_nv_n}$, its coordinate equations give

$$(G_0 + \delta A_\varphi P + b_nI_0)P_0\rho e_{v_n} = 0, \qquad (b_n + b_m)\rho_{v_nv_m} = 0 \quad (n \ne m),$$

$$b_np_n = a_n^\delta B_{hh}.$$

The off-diagonal blocks vanish, $\sum_n b_np_n < \infty$, and the core equation becomes $\mathcal{K}_*B + \delta A_\varphi\mathcal{D}_P(B) = 0$. Both Heisenberg summands are nonpositive and self-adjoint for $\langle\cdot,\cdot\rangle_{\sigma_0}$. Therefore

$$\ker(\mathcal{K} + \delta A_\varphi\mathcal{D}_P) = \ker\mathcal{K} \cap \ker\mathcal{D}_P = \mathbb{C}I_0.$$

The predual kernel is one-dimensional and contains σ_0. It follows that $B = r\sigma_0$ and $p_n = r\delta\pi_h e^{-\varphi_n}$. Normalization gives $r = \beta_\delta$.

Finally, $\|B_\delta\|_\diamond \le 2\delta A_\varphi$ by the Lindblad representation. Equality follows from $\|B_\delta(P)\|_1 = 2\delta A_\varphi$. The identity

$$(S_t^\delta - S_t^0)X = \int_0^t S_{t-s}^\delta B_\delta S_s^0 X\,ds \qquad (X \in \mathfrak{S}_1(\mathcal{H}))$$

holds as a trace-norm Bochner integral. At each finite amplification, contractivity bounds it by $2\delta A_\varphi t\|X\|_1$. Taking the operator norm and then the supremum over the ancilla dimension proves (2.9). When the outward rates vanish, the core is invariant and its generator is $\mathcal{K}_*$. □

The auxiliary commutators are Hilbert–Schmidt for every bounded test, and

$$\sum_n \left(\|L^\delta_{n,+}\|_2^2 + \|L^\delta_{n,-}\|_2^2\right) = \sqrt{\delta}\sum_n b_n e^{-\varphi_n/2} < \infty. \tag{2.11}$$

The finiteness of (2.11) does not give a uniform bound for g^δ_ρ over all states.

2.3. **Mobility and raw currents.** For $\rho \in \mathfrak{S}_1(\mathcal{H})$, $\rho \geq 0$, define the logarithmic mobility by

$$\mathsf{M}_{\rho,j}U = \int_0^1 e^{(u-1/2)\omega_j}\rho^u U \rho^{1-u}\, du, \qquad U \in \mathfrak{S}_2(\mathcal{H}). \tag{2.12}$$

The integrand is norm-continuous for $0 < u < 1$ and bounded on $(0,1)$. Thus (2.12) is a Bochner integral in $\mathcal{B}(\mathfrak{S}_2)$. Each integrand is a positive operator there. For a state ρ, $\|\mathsf{M}_{\rho,j}\| \leq e^{|\omega_j|/2}$. The values assigned to the powers at $u = 0, 1$ do not affect the integral. The scalar logarithmic mean is

$$\Lambda(a,b) = \int_0^1 a^u b^{1-u}\, du = \frac{a-b}{\log a - \log b} \quad (a, b > 0), \tag{2.13}$$

with $\Lambda(a,a) = a$ and $\Lambda(a,0) = \Lambda(0,a) = 0$. For $r > 0$ and a positive finite matrix B, $\Lambda(r,B)$ is defined by spectral calculus, with its continuous extension when $r = 0$ or B is singular. In a spectral basis $\rho e_a = r_a e_a$, substitution in (2.12) gives

$$\mathsf{M}_{\rho,j}E_{ab} = \Lambda(e^{\omega_j/2}r_a, e^{-\omega_j/2}r_b)E_{ab}.$$

This coefficient is positive when $r_a, r_b > 0$ and zero when $r_a r_b = 0$. In particular, $\mathsf{M}_{\rho,j}$ is injective at every faithful state; in infinite dimension its inverse need not be bounded.

Lemma 2.3 (Uniform remote logarithmic-mean asymptotics). *Let $s_j \to \infty$, let $c_j > 0$, and let $L_j \in \mathbb{R}$ satisfy $L_j/s_j \to 0$. Let Θ be a parameter set and let $u_j : \Theta \to (0,\infty)$ satisfy*

$$\sup_{\theta\in\Theta} \frac{|\log u_j(\theta)|}{s_j} \longrightarrow 0. \tag{2.14}$$

For every compact interval $K \Subset (0,\infty)$,

$$\sup_{\theta\in\Theta,\,\lambda\in K} \left| \frac{s_j}{c_j u_j(\theta)} \Lambda\big(c_j u_j(\theta), c_j e^{-(s_j+L_j)}\lambda\big) - 1 \right| \longrightarrow 0. \tag{2.15}$$

Consequently, if $B_j(\theta)$ are positive matrices in a fixed finite dimension independent of j, whose spectra all lie in K, then

$$\sup_{\theta\in\Theta} \left\| \frac{s_j}{c_j u_j(\theta)} \Lambda\big(c_j u_j(\theta), c_j e^{-(s_j+L_j)}B_j(\theta)\big) - I \right\| \longrightarrow 0. \tag{2.16}$$

In particular, if $c_j u_j(\theta)/s_j \to q(\theta)$ uniformly, then the operator logarithmic means converge uniformly to $q(\theta)I$.

Proof. For all sufficiently large j, put

$$\varepsilon_j = \frac{|L_j| + \sup_{\theta\in\Theta}|\log u_j(\theta)| + \max_{\lambda\in K}|\log\lambda|}{s_j}.$$

Then $s_j > 0$ and $\varepsilon_j \to 0$. Homogeneity gives

$$\frac{s_j}{c_j u_j}\Lambda(c_j u_j, c_j e^{-(s_j+L_j)}\lambda) = \frac{1 - e^{-(s_j+L_j)}\lambda/u_j}{1 + (L_j + \log u_j - \log\lambda)/s_j}.$$

For $\varepsilon_j < 1$ the denominator is at least $1 - \varepsilon_j$, and

$$0 \leq e^{-(s_j+L_j)}\lambda/u_j \leq e^{-(1-\varepsilon_j)s_j}.$$

Consequently the absolute difference of the preceding quotient from one is at most

$$\frac{\varepsilon_j + e^{-(1-\varepsilon_j)s_j}}{1-\varepsilon_j},$$

uniformly in θ and $\lambda \in K$. This tends to zero. Spectral calculus proves (2.16).

For large fixed j, condition (2.14) bounds u_j on Θ. If $c_j u_j/s_j \to q$ uniformly, one such index shows that q is bounded. Uniform convergence then bounds $c_j u_j/s_j$ uniformly for all large j. Multiplication in (2.16) proves the last assertion. □

Let $\mathcal{J} = \mathcal{J}_0 \sqcup \{(n,+),(n,-) : n \geq 1\}$ and let $\mathscr{H}_\nabla = \bigoplus_{j\in\mathcal{J}} \mathfrak{S}_2(\mathcal{H})$ be a real Hilbert space. For $C = C^* \in \mathcal{B}(\mathcal{H})$ put

$$D_\rho^\delta C = \big((\mathsf{M}_{\rho,j}^\delta)^{1/2}[L_j^\delta, C]\big)_j, \qquad g_\rho^\delta(C) = \sum_j \|(\mathsf{M}_{\rho,j}^\delta)^{1/2}[L_j^\delta, C]\|_2^2. \tag{2.17}$$

The vector $D_\rho^\delta C$ is defined in $\mathscr{H}_\nabla$ precisely when the sum is finite. Superscripts are omitted when the coupling is fixed.

Lemma 2.4. *Fix a channel j. On the positive trace class, the maps $\rho \mapsto \mathsf{M}_{\rho,j}$ and $\rho \mapsto \mathsf{M}_{\rho,j}^{1/2}$ are continuous from trace norm to operator norm on $\mathfrak{S}_2$. For every $U \in \mathfrak{S}_2$, the function $\rho \mapsto \langle U, \mathsf{M}_{\rho,j} U\rangle_{\mathbb{R}}$ is concave. For positive trace-class ρ, η and $b \geq 0$,*

$$0 \leq \rho \leq b\eta \implies \mathsf{M}_{\rho,j} \leq b\mathsf{M}_{\eta,j}, \qquad \mathsf{M}_{b\rho,j} = b\mathsf{M}_{\rho,j}. \tag{2.18}$$

Proof. Let $\rho_n \to \rho$ in trace norm. For $0 < u < 1$, continuous functional calculus gives $\rho_n^u \to \rho^u$ and $\rho_n^{1-u} \to \rho^{1-u}$ in operator norm. Their norms are uniformly bounded on a bounded trace set. Dominated convergence in (2.12) gives $\mathsf{M}_{\rho_n,j} \to \mathsf{M}_{\rho,j}$ in operator norm on $\mathfrak{S}_2$. Continuity of the positive square root gives the same conclusion for $\mathsf{M}^{1/2}$.

For $0 < u < 1$, Lieb's theorem [9] makes

$$\rho \longmapsto \operatorname{Tr}(U^*\rho^u U \rho^{1-u})$$

concave in finite dimension. Apply it to the compressions of two positive trace-class operators and of U by one increasing sequence of finite-rank projections. The operators converge in trace norm and U in Hilbert–Schmidt norm. Convergence of the powers in operator norm permits passage to the limit. Integration against the positive weight $e^{(u-1/2)\omega_j}$ proves concavity on the positive trace class.

Let L_A and R_A denote left and right multiplication. If $0 \leq \rho \leq b\eta$, operator monotonicity of powers and commutation of left and right multiplication give

$$\mathsf{L}_{\rho^u}\mathsf{R}_{\rho^{1-u}} \leq b^u \mathsf{L}_{\eta^u}\mathsf{R}_{\rho^{1-u}} \leq b\mathsf{L}_{\eta^u}\mathsf{R}_{\eta^{1-u}}.$$

Integrating proves the first assertion in (2.18). Homogeneity follows from $(b\rho)^u = b^u\rho^u$ for $b > 0$; the case $b = 0$ is immediate. □

For a positive bounded operator M on a real Hilbert space, define its inverse quadratic form by

$$a_M(J) = \sup_U \{2\langle J, U\rangle - \langle U, MU\rangle\}. \tag{2.19}$$

Lemma 2.5. *The action $a_M(J)$ is finite precisely when $J \in \operatorname{Ran} M^{1/2}$. In that case,*

$$a_M(J) = \min\{\|\xi\|^2 : M^{1/2}\xi = J\},$$

and the minimizer is the unique solution orthogonal to $\ker M$. *For positive bounded M, M_0,*

$$M \geq aM_0, \quad a > 0 \implies a_M(J) \leq a^{-1} a_{M_0}(J). \tag{2.20}$$

For each channel, $(\rho, J) \mapsto a_{\mathsf{M}_{\rho,j}}(J)$ is jointly convex. Measurable state and raw-current fields of finite action have strongly measurable least-norm square-root representatives.

Proof. Suppose $a_M(J) = A < \infty$. Optimization over scalar multiples of U gives

$$|\langle J, U\rangle| \le \sqrt{A}\,\|M^{1/2}U\|.$$

Thus $M^{1/2}U \mapsto \langle J, U\rangle$ is a well-defined bounded linear functional on $\operatorname{Ran} M^{1/2}$. Its continuous extension to $\overline{\operatorname{Ran} M^{1/2}} = (\ker M)^\perp$ has a unique Riesz vector ξ there. For every U, $\langle J, U\rangle = \langle \xi, M^{1/2}U\rangle$; hence $J = M^{1/2}\xi$. Conversely, for such a vector,

$$2\langle J, U\rangle - \|M^{1/2}U\|^2 = \|\xi\|^2 - \|M^{1/2}U - \xi\|^2.$$

Density of $\operatorname{Ran} M^{1/2}$ in $(\ker M)^\perp$ gives $a_M(J) = \|\xi\|^2$. Every other solution differs from ξ by a vector in $\ker M$, proving the least-norm assertion.

If $M \ge aM_0$, then

$$a_M(J) \le \sup_U \{2\langle J, U\rangle - a\langle U, M_0U\rangle\} = a^{-1}a_{M_0}(J).$$

For fixed U, the expression defining $a_{\mathsf{M}_{\rho,j}}(J)$ is affine in J and convex in ρ, by Lemma 2.4. Its supremum is jointly convex.

For the least representative $J = M^{1/2}\xi$, spectral calculus gives

$$(M + rI)^{-1/2}J = \big(M(M + rI)^{-1}\big)^{1/2}\xi \longrightarrow \xi.$$

Indeed, the scalar multiplier $\sqrt{\lambda/(\lambda + r)}$ is at most one, tends to one for $\lambda > 0$, and $\xi \perp \ker M$. For measurable state and current fields, Lemma 2.4 and continuous functional calculus make each regularized expression strongly measurable. Their pointwise norm limit is strongly measurable. The countable channel representatives have finite summed square norm; their finite partial sums converge in the separable space $\mathscr{H}_\nabla$. □

For Hermitian C, adjoint pairing gives

$$\mathsf{M}_{\rho,j'}(U^*) = (\mathsf{M}_{\rho,j}U)^*, \qquad [L_{j'}, C] = -[L_j, C]^*.$$

The involution $J_j \mapsto -J_{j'}^*$ preserves the action and the real continuity pairings. Averaging a raw current with its image does not increase the action. Raw currents with Hermitian prescribed velocity may therefore be chosen so that $J_{j'} = -J_j^*$.

2.4. **Channel Fisher information.** For each state and channel define

$$J_j^{\mathrm{ent}}(\rho) = e^{-\omega_j/2}L_j\rho - e^{\omega_j/2}\rho L_j, \qquad \mathcal{I}_\delta(\rho) = \sum_{j\in\mathcal{J}} a_{\mathsf{M}_{\rho,j}}(J_j^{\mathrm{ent}}(\rho)). \tag{2.21}$$

The parameters in the channels and mobilities are those of the given coupling. The sum takes values in $[0, +\infty]$ and uses the inverse quadratic form (2.19), including at singular states. When it is finite, let $\zeta_{\rho,j}$ be the least-norm solution of $\mathsf{M}_{\rho,j}^{1/2}\zeta_{\rho,j} = J_j^{\mathrm{ent}}(\rho)$. The finite-core counterpart, with only the original core channels and states on $\mathcal{H}_0$, is denoted by $\mathcal{I}_0$. The auxiliary-channel sum is $\mathcal{I}_\delta^{\mathrm{aux}}$.

2.5. **Admissible curves.** For a test space (1.11), set

$$\mathfrak{S}_{\mathscr{T}_\delta} = \{\rho \in \mathfrak{S}_1(\mathcal{H}) : \rho \ge 0,\ \operatorname{Tr}\rho = 1,\ g_\rho^\delta(C) < \infty\ (C \in \mathscr{T}_\delta)\}, \qquad K_{\delta,E}^{\mathscr{T}_\delta} = \mathfrak{S}_{\mathscr{T}_\delta} \cap \{F_\delta \le E\}.$$

Definition 2.6. Let $T > 0$. A state-current pair (ρ, ξ) on $[0, T]$ is *admissible* if $\rho : [0, T] \to \mathfrak{S}_{\mathscr{T}_\delta}$ is trace-norm continuous, $\xi \in L^2(0, T; \mathscr{H}_\nabla)$, and, for every $C \in \mathscr{T}_\delta$,

$$\int_0^T g_{\rho_t}^\delta(C)\,dt < \infty, \tag{2.22}$$

$$\operatorname{Tr} C(\rho_t - \rho_s) = -\int_s^t \langle D_{\rho_u}^\delta C, \xi_u\rangle_{\mathbb{R}}\,du \qquad (0 \le s \le t \le T). \tag{2.23}$$

Its action is $\int_0^T \|\xi_t\|^2\,dt$. For $\rho \in C([0,T];\mathfrak{S}(\mathcal{H}))$, the reduced action $\mathcal{A}_{\delta,E}(\rho)$ is the infimum of these actions over all admissible currents on ρ, subject to $F_\delta(\rho_t) \le E$ for every $t \in [0,T]$. An empty infimum is $+\infty$. Dependence of $\mathcal{A}_{\delta,E}$ on the chosen test space and the time interval is suppressed in the notation. The time interval is fixed; a family $(\mathscr{T}_\delta)_\delta$ is specified whenever a variational limit is taken, and this family may vary with δ as in (1.11). For $x, y \in K^{\mathscr{T}_\delta}_{\delta,E}$, define

$$(\mathcal{W}^{\mathscr{T}_\delta}_{\delta,E}(x,y))^2 = \inf_{\rho_0=x,\,\rho_1=y} \mathcal{A}_{\delta,E}(\rho), \qquad T = 1. \tag{2.24}$$

The same infimum without the entropy constraint defines the uncapped extended pseudometric $\mathcal{W}^{\mathscr{T}_\delta}_\delta$ on $\mathfrak{S}_{\mathscr{T}_\delta}$. It has the symmetry and triangle properties supplied by the action, but it need not separate distinct states; Proposition 8.7 gives an explicit nonseparation example. By contrast, the entropy-capped functional $\mathcal{W}^{\mathscr{T}_\delta}_{\delta,E}$ is separating by Lemma 3.2. Thus the constraint in $\mathcal{W}^{\mathscr{T}_\delta}_{\delta,E}$ applies to the whole connecting curve.

For an admissible pair put $J_j = \mathsf{M}^{1/2}_{\rho,j}\xi_j$. If $-\sum_j [L_j^*, J_j]$ converges in trace norm to a Hermitian V, then, for every bounded Hermitian C with $g_\rho(C) < \infty$,

$$-\langle D_\rho C, \xi\rangle_{\mathbb{R}} = \mathrm{Tr}(CV). \tag{2.25}$$

For a finite set $F \subset \mathcal{J}$, put $V_F = -\sum_{j\in F}[L_j^*, J_j]$. Each L_j has finite rank, so $[L_j, C] \in \mathfrak{S}_2$ and $[L_j^*, J_j] \in \mathfrak{S}_1$. Self-adjointness of $\mathsf{M}^{1/2}_{\rho,j}$ and cyclicity of the trace give

$$\begin{aligned} -\sum_{j\in F} \langle \mathsf{M}^{1/2}_{\rho,j}[L_j, C], \xi_j\rangle_{\mathbb{R}} &= -\sum_{j\in F} \operatorname{Re}\mathrm{Tr}\big([L_j, C]^* J_j\big) \\ &= \operatorname{Re}\mathrm{Tr}(CV_F). \end{aligned}$$

Moreover,

$$\sum_j \left|\langle \mathsf{M}^{1/2}_{\rho,j}[L_j, C], \xi_j\rangle_{\mathbb{R}}\right| \le g_\rho(C)^{1/2}\|\xi\|_{\mathscr{H}_\mathcal{J}} < \infty.$$

Passing along the trace-norm convergent partial sums gives $-\langle D_\rho C, \xi\rangle_{\mathbb{R}} = \operatorname{Re}\mathrm{Tr}(CV)$, since $|\mathrm{Tr}(C(V_F - V))| \le \|C\|\,\|V_F - V\|_1$. As $C = C^*$ and $V = V^*$, $\overline{\mathrm{Tr}(CV)} = \mathrm{Tr}(VC) = \mathrm{Tr}(CV)$, which proves (2.25). If $J_{j'} = -J_j^*$ and F is invariant under $j \mapsto j'$, then

$$\big(-[L_j^*, J_j]\big)^* = -[L_{j'}^*, J_{j'}], \qquad V_F = V_F^*.$$

In this case the finite-sum trace is already real. Finite raw action supplies the measurable least representatives of Lemma 2.5.

Lemma 2.7. *Let ρ be a trace-continuous state curve on a compact interval and suppose that $\rho_t \le b\,\sigma_\delta$ for one constant $b < \infty$. Then*

$$g^\delta_{\rho_t}(C) \le b\; g^\delta_{\sigma_\delta}(C) \qquad (C \in \mathscr{T}^{\mathrm{eq}}_\delta).$$

Every such test satisfies (2.22).

Proof. For each channel, (2.18) gives

$$\|\mathsf{M}^{1/2}_{\rho_t,j}[L_j, C]\|_2^2 \le b\|\mathsf{M}^{1/2}_{\sigma_\delta,j}[L_j, C]\|_2^2.$$

Summation proves the claimed bound. Its right side is finite and independent of t, so integration over the compact interval proves (2.22). ☐

Lemma 2.8. *Let (ρ, ξ) be admissible on $[0,T]$ and let $C(t) = \sum_{a=1}^m c_a(t)C_a$, where $C_a \in \mathscr{T}_\delta$ and $c_a \in C^1([0,T];\mathbb{R})$. Put*

$$\mathcal{L}_\rho(C) = \mathrm{Tr}(C(T)\rho_T) - \mathrm{Tr}(C(0)\rho_0) - \int_0^T \mathrm{Tr}(C'(t)\rho_t)\,dt.$$

Then $D^\delta_\rho C \in L^2(0,T;\mathscr{H}_\nabla)$ *and*

$$\mathcal{L}_\rho(C) = -\int_0^T \langle D^\delta_{\rho_t} C(t), \xi_t\rangle_{\mathbb{R}}\, dt. \tag{2.26}$$

If $F_\delta(\rho_t) \le E$ *for all* t*, then*

$$2\mathcal{L}_\rho(C) - \int_0^T g^\delta_{\rho_t}(C(t))\, dt \le \mathcal{A}_{\delta,E}(\rho). \tag{2.27}$$

Proof. For each a, the scalar function $t \mapsto \operatorname{Tr}(C_a\rho_t)$ is absolutely continuous, with derivative $-\langle D^\delta_{\rho_t} C_a, \xi_t\rangle_{\mathbb{R}}$ in L^1 by (2.22) and Cauchy–Schwarz. The finite-span estimate

$$g^\delta_{\rho_t}(C(t)) \le m \sum_{a=1}^m |c_a(t)|^2 g^\delta_{\rho_t}(C_a)$$

gives square integrability. Scalar integration by parts proves (2.26). Finally,

$$2\mathcal{L}_\rho(C) - \int_0^T g^\delta_{\rho_t}(C(t))\, dt = \int_0^T \left(\|\xi_t\|^2 - \|\xi_t + D^\delta_{\rho_t} C(t)\|^2\right) dt \le \int_0^T \|\xi_t\|^2\, dt.$$

Taking the infimum over admissible currents proves (2.27). □

2.6. **The core Onsager operator and auxiliary currents.** For $B \ge 0$ on $\mathcal{H}_0$, define $\mathbb{K}_B$ on $\mathfrak{h}_0 = \{C = C^* \in M_d : \operatorname{Tr} C = 0\}$ by

$$\operatorname{Tr}(C\mathbb{K}_B D) = \langle D^0_B C, D^0_B D\rangle_{\mathbb{R}}. \tag{2.28}$$

Equivalently, $\mathbb{K}_B D = \sum_{j\in\mathcal{J}_0}[L_j^*, \mathsf{M}_{B,j}[L_j, D]]$. This sum is Hermitian and trace zero by adjoint pairing. It is nonnegative and continuous in B, including at singular matrices, by Lemma 2.4. If $B > 0$, it is positive definite on $\mathfrak{h}_0$: a zero of its quadratic form commutes with all the channels. On every compact set of positive definite core matrices its least eigenvalue has a positive lower bound. The maps $B \mapsto \mathsf{M}_{B,j}$, $B \mapsto \mathsf{M}_{B,j}^{1/2}$, and $B \mapsto \mathbb{K}_B^{-1}$ are smooth on $B > 0$: finite-dimensional functional calculus and differentiation under the integral apply on every compact spectral interval in $(0,\infty)$. Inverses of $\mathbb{K}_B$ are used only when $B > 0$.

For $B > 0$, differentiation with respect to u in $e^{(u-1/2)\omega_j} B^u L_j B^{1-u}$ gives

$$\mathsf{M}_{B,j}[L_j, \log B - \log\sigma_0] = e^{-\omega_j/2} L_j B - e^{\omega_j/2} B L_j. \tag{2.29}$$

Consequently $\mathcal{K}_* B = -\mathbb{K}_B(\log B - \log\sigma_0)$, where scalar parts of the potential are immaterial.

Lemma 2.9. *Let* $\mathcal{H}_K$ *be a finite coordinate subspace containing* e_h*, with* $e_v \perp \mathcal{H}_K$*. Fix* $B > 0$ *on* $\mathcal{H}_K$ *and* $p, a, b > 0$*, and set*

$$\rho = B \oplus pE_{vv}, \qquad \ell = (ab)^{1/4}/\sqrt{2}, \qquad \omega = \log(b/a), \qquad L_+ = \ell E_{vh}, \quad L_- = L_+^*.$$

For $R = |e_v\rangle\langle v|$, $v \in \mathcal{H}_K$,

$$\mathsf{M}_{\rho,+}(R) = (ab)^{-1/2} R\Lambda(bp, aB). \tag{2.30}$$

If $w \in \mathcal{H}_K \cap e_h^\perp$*, the current pair*

$$J_+ = -\ell^{-1}|e_v\rangle\langle w|, \qquad J_- = -J_+^* \tag{2.31}$$

has negative divergence $V_w = |e_h\rangle\langle w| + |w\rangle\langle e_h|$ *and action*

$$a_{\mathsf{M}_{\rho,+}}(J_+) + a_{\mathsf{M}_{\rho,-}}(J_-) = 4\langle w, \Lambda(bp, aB)^{-1} w\rangle. \tag{2.32}$$

For a real scalar $\dot r$*, the pair* $J_+ = \dot r E_{vh}/(2\ell)$, $J_- = -J_+^*$ *has negative divergence* $\dot r(E_{vv} - P)$ *and action*

$$\dot r^2 \langle e_h, \Lambda(bp, aB)^{-1} e_h\rangle. \tag{2.33}$$

The formulas remain valid after adjoining further diagonal blocks to ρ.

Proof. Left multiplication by ρ^u on R is multiplication by p^u; right multiplication by ρ^{1-u} is multiplication by B^{1-u}. Thus

$$\mathsf{M}_{\rho,+}(R) = R\int_0^1 e^{(u-1/2)\omega} p^u B^{1-u}\,du = (ab)^{-1/2} R\Lambda(bp, aB).$$

The row space is invariant for this self-adjoint positive operator and therefore reducing. Its inverse is right multiplication by $\sqrt{ab}\Lambda(bp, aB)^{-1}$. Since $\ell^2 = \sqrt{ab}/2$, each of the currents in (2.31) has cost $2\langle w, \Lambda(bp, aB)^{-1} w\rangle$. Moreover,

$$-[\ell E_{hv}, J_+] - [\ell E_{vh}, J_-] = |e_h\rangle\langle w| + |w\rangle\langle e_h|,$$

because $w \perp e_h$. For $J_+ = \dot r E_{vh}/(2\ell)$ and $J_- = -\dot r E_{hv}/(2\ell)$, direct multiplication gives

$$-[\ell E_{hv}, J_+] - [\ell E_{vh}, J_-] = \dot r(E_{vv} - P).$$

Each channel has cost $\frac{1}{2}\dot r^2\langle e_h, \Lambda(bp, aB)^{-1} e_h\rangle$. Their sum is (2.33). The orthogonal complement of the displayed row and column spaces is invariant under left and right multiplication by every power of ρ. Adjoining diagonal blocks therefore changes neither the currents nor their inverse-mobility costs. □

3. The entropy of the moving sublevels

Let N vanish on $\mathcal{H}_0$ and satisfy $Ne_{v_n} = ne_{v_n}$. For a state ρ, write

$$s = \mathrm{Tr}((I - P_0)\rho), \qquad p_n = \rho_{v_n v_n}, \qquad M = \sum_n \varphi_n p_n, \qquad L = \log(1/\delta).$$

For $s < 1$ put $B_\rho = P_0\rho P_0/(1 - s)$.

Lemma 3.1. *Fix $E > 0$. As $\delta \downarrow 0$, uniformly over states with $F_\delta(\rho) \le E$,*

$$F_\delta(\rho) = F_0(B_\rho) + M + Ls + o(1). \tag{3.1}$$

The compression B_ρ is defined for all sufficiently small δ; $s \to 0$ uniformly, and

$$M + Ls \le E - F_0(B_\rho) + o(1), \qquad \|\rho - (B_\rho \oplus 0)\|_1 \longrightarrow 0 \tag{3.2}$$

uniformly on the same sublevels. For all sufficiently large L, the absolute remainder in (3.1) *is at most $C_E \log(e + L)/L$, where $C_E < \infty$ is independent of ρ and δ.*

Proof. Diagonal measurement decreases relative entropy [13]. For probability vectors p, π and bounded real V, relative entropy against $\pi_i e^{V_i}/\sum_j \pi_j e^{V_j}$ gives

$$\sum_i V_i p_i \le D(p\|\pi) + \log\sum_i \pi_i e^{V_i}.$$

Take $V = 0$ on the core and $V_{v_n} = \frac{1}{2}\min\{\varphi_n + L, R\}$ on the tail. For $L > 0$, monotone convergence as $R \to \infty$ yields

$$M + Ls \le 2F_\delta(\rho) + 2\log\left[\beta_\delta\left(1 + \sqrt{\delta}\,\pi_h\sum_n e^{-\varphi_n/2}\right)\right] \le 2E + C. \tag{3.3}$$

Hence $s \le C_E/L$ and $\mathrm{Tr}(N\rho) \le M/c_* \le C_E$. For $\gamma > 0$,

$$\mathrm{Tr}\, e^{-\gamma N} = d + \frac{1}{e^\gamma - 1} < \infty.$$

The Gibbs entropy bound gives $S(\rho) \le S(\mathrm{diag}\,\rho) \le \gamma\,\mathrm{Tr}(N\rho) + \log\mathrm{Tr}\, e^{-\gamma N} < \infty$.

Let $\rho^{\mathrm{bl}} = P_0\rho P_0 + (I - P_0)\rho(I - P_0)$. Concavity of entropy and the entropy triangle inequality for the isometry $v \mapsto P_0 v \otimes e_0 + (I - P_0)v \otimes e_1$ give

$$0 \le S(\rho^{\mathrm{bl}}) - S(\rho) \le h_2(s), \qquad h_2(s) = -s\log s - (1 - s)\log(1 - s).$$

For $s > 0$, the normalized tail block has number mean at most $M/(c_* s)$. Comparison with the geometric law of mean μ bounds the entropy of every probability distribution on $\mathbb{N}_0$ with that mean by

$$(1+\mu)\log(1+\mu) - \mu\log\mu \le 1 + \log(1+\mu).$$

Applying it to the shifted tail index and using the direct-sum entropy identity gives

$$|S(\rho) - S(B_\rho)| \le h_2(s) + s\log d + s\left[1 + \log\left(1 + \frac{M}{c_* s}\right)\right]. \tag{3.4}$$

At $s = 0$ the right-hand side is defined by continuity. For $0 < s \le 1/2$, $h_2(s) \le s\log(e/s)$. The functions $s \mapsto s\log(e/s)$ and $s \mapsto s\log(1 + C/s)$ are increasing on this interval for $C \ge 0$. Using $s \le C_E/L$ and $M \le C_E$ in (3.4) therefore bounds it by $C'_E \log(e+L)/L$ for all sufficiently large L.

For $H_0 = -\log\sigma_0$, the reference energy is finite and

$$\begin{aligned} F_\delta(\rho) - F_0(B_\rho) - M - Ls = &-s\operatorname{Tr}(H_0 B_\rho) - s\log\pi_h + \log Z_\delta \\ &+ S(B_\rho) - S(\rho). \end{aligned}$$

The first two terms are $O(s)$, and $\log Z_\delta = O(\delta)$. This proves (3.1) with the stated remainder. Finally, for $Q = I - P_0$, Schatten Hölder gives

$$\begin{aligned} \|\rho - P_0\rho P_0\|_1 &\le \|Q\rho^{1/2}\|_2\|\rho^{1/2}\|_2 + \|P_0\rho^{1/2}\|_2\|\rho^{1/2}Q\|_2 \\ &\le \sqrt{s} + \sqrt{s(1-s)} \le 2\sqrt{s}. \end{aligned}$$

Since $P_0\rho P_0 = (1-s)B_\rho$, $\|\rho - (B_\rho \oplus 0)\|_1 \le 2\sqrt{s} + s$. Together with (3.1), this proves (3.2). □

The estimate (3.4) is a number-energy entropy bound; related continuity estimates are given in [15].

Lemma 3.2. *For every $\delta > 0$ and $C \in \mathscr{T}_{\text{fin}}$, there is a finite constant $c_{\delta,C}$ such that, for every state ρ,*

$$g^\delta_\rho(C) \le c_{\delta,C}(1 + F_\delta(\rho)). \tag{3.5}$$

Consequently, every finite-entropy state belongs to the finite-test domain, and $\mathscr{T}_{\text{fin}} + \mathbb{R}I \subseteq \mathscr{T}^{\text{eq}}_\delta$. For each $E > 0$ and every test space (1.11)*, $\mathcal{W}^{\mathscr{T}_\delta}_{\delta,E}$ is a separating extended metric on $K^{\mathscr{T}_\delta}_{\delta,E}$. For a fixed core test C and fixed E, the values of $g^\delta_\rho(C)$ on $\{F_\delta \le E\}$ are bounded uniformly for sufficiently small δ.*

Proof. Choose a finite coordinate set K containing $\mathcal{H}_0$ and the support of C. For a diagonal unitary U which is the identity on K, every channel is multiplied by a scalar phase under conjugation, or is unchanged. For a finite channel set J, this implies $g^\delta_{U\rho U^*,J}(C) = g^\delta_{\rho,J}(C)$. Concavity in Lemma 2.4 implies that averaging such conjugations can only increase this finite sum.

Successively average the phases of coordinates outside K. The averaged states converge in trace norm to $B_K \oplus \operatorname{diag}(p_n)_{v_n \notin K}$, with $B_K = P_K\rho P_K$. Indeed the assertion holds exactly on each finite compression, and contractivity of the averages and density of finite-coordinate matrices give the trace-norm limit. Continuity of each channel, followed by monotone convergence of the nonnegative channel sums, gives the same inequality for g^δ.

For a level outside K, the row calculation in Lemma 2.9, followed by homogeneity, gives

$$b_n\langle Ce_h, \Lambda(p_n, \delta e^{-\varphi_n}B_K)Ce_h\rangle \le b_n\Lambda(p_n, \delta e^{-\varphi_n})\|Ce_h\|^2. \tag{3.6}$$

This also holds for singular B_K and $p_n = 0$, by continuity; here $0 \le B_K \le I_K$. The finitely many remaining channels have a bound independent of the state.

For fixed δ, choose n_0 such that, for $n \ge n_0$, $\varphi_n + L - \varphi_n/2 \ge \varphi_n/4$ and $\delta e^{-\varphi_n} \le e^{-\varphi_n/2}$. On $p_n \ge e^{-\varphi_n/2}$,

$$b_n\Lambda(p_n, \delta e^{-\varphi_n}) \le 4C_*\varphi_n p_n.$$

On the complementary set it is at most $C_*\varphi_n^2 e^{-\varphi_n/2}$. The latter sequence is summable: for every $a>0$, $\varphi_n^2 e^{-a\varphi_n} \le C_a e^{-ac_* n/2}$. Applying the entropy variational bound with potential $\varphi_n/2$ on the tail gives

$$M \le 2F_\delta(\rho) + 2\log\left[\beta_\delta\left(1+\delta\pi_h \sum_n e^{-\varphi_n/2}\right)\right].$$

This proves (3.5).

For $0<\delta<1$, split instead at $p_n = e^{-(L+\varphi_n)/2}$. Then

$$\sum_n b_n \Lambda(p_n, \delta e^{-\varphi_n}) \le 2C_*(M+Ls) + C_* \sum_n \varphi_n^2 e^{-(L+\varphi_n)/2}. \tag{3.7}$$

For core tests, the other channels are fixed core channels; their bounds are independent of δ. Lemma 3.1 proves the asserted uniform bound, also when $C_{hh}\neq 0$.

For every admissible capped pair, Cauchy–Schwarz gives

$$|\operatorname{Tr} C(\rho_1-\rho_0)|^2 \le c_{\delta,C}(1+E)\int_0^1 \|\xi_t\|^2\,dt.$$

Finite Hermitian tests separate states. Symmetry follows by time reversal. If two connectors have positive actions A_1, A_2, give them durations $\sqrt{A_i}/(\sqrt{A_1}+\sqrt{A_2})$ and concatenate. The resulting action is $(\sqrt{A_1}+\sqrt{A_2})^2$. The same conclusion for a zero action follows by a limiting choice of positive durations. Taking infima proves the triangle inequality. Constant curves have zero action and satisfy the test conditions at each state of $K_{\delta,E}^{\mathcal{T}_\delta}$. ☐

Corollary 3.3. *As $\delta \downarrow 0$, the energies F_δ Γ-converge in trace norm to F_0 on core-supported states and to $+\infty$ elsewhere. If $\delta_j\to 0$ and $\sup_j F_{\delta_j}(\rho^j)<\infty$, then (ρ^j) is relatively compact in trace norm. For every core state B,*

$$F_\delta(\iota_\delta B) = \beta_\delta F_0(B), \qquad \iota_\delta B \longrightarrow B\oplus 0. \tag{3.8}$$

No faithfulness assumption on B is required.

Proof. For every bounded-energy sequence, Lemma 3.1 gives concentration on the finite core and relative compactness in trace norm. For a convergent such sequence, (3.1) and continuity of F_0 give the lower inequality. Any limit outside the core is therefore incompatible with a finite energy lower limit.

The auxiliary block of $\iota_\delta B$ equals that of σ_δ; their core blocks are $\beta_\delta B$ and $\beta_\delta\sigma_0$. The direct-sum formula for relative entropy gives

$$D(\iota_\delta B\|\sigma_\delta) = \beta_\delta D(B\|\sigma_0).$$

This identity holds also for singular B, since $\sigma_0>0$. Moreover $\beta_\delta\to 1$. It gives recovery at every core state. ☐

4. The action limit

4.1. **The effective action.** For the remainder of this section fix $E, T>0$ and set $s_E(B)=\chi(E-F_0(B))$ on K_E^0.

Lemma 4.1. *For each Hermitian potential C, the function $B\mapsto \widehat{g}_{B,E,\chi}(C)$ is continuous and concave on K_E^0. For $B\in K_E^0$ and $V=V^*$ with $\operatorname{Tr} V=0$, the dual action $a_{\chi,E}(B,V)$ is the infimum of*

$$\sum_{j\in\mathcal{J}_0} a_{\mathsf{M}_{B,j}}(J_j) + \frac{4\|w\|^2}{\chi(E-F_0(B))} \tag{4.1}$$

over currents satisfying

$$V = -\sum_{j\in\mathcal{J}_0}[L_j^*, J_j] + V_w, \qquad V_w = |e_h\rangle\langle w| + |w\rangle\langle e_h|, \qquad w\perp e_h. \tag{4.2}$$

When the denominator vanishes, the last term is 0 *for* $w = 0$ *and* $+\infty$ *otherwise. A finite infimum is attained in normalized currents. For* $B > 0$ *it is given by* (1.18)*. The cost* (4.1) *is jointly convex in* B, J*, and* w*.*

Proof. Lemma 2.4 and continuity of entropy in finite dimension give continuity. Concavity follows from mobility concavity and concavity of s_E. Adjoint-pair averaging allows the restriction $J_{j'} = -J_j^*$ without increasing the cost. Normalize $v = -2w/\sqrt{s_E(B)}$ when $s_E(B) > 0$; when it is zero, take $v = 0$. Regard $Q_0\mathcal{H}_0$ as a real Hilbert space with scalar product $\mathrm{Re}\langle\cdot,\cdot\rangle$, and put $\mathscr{H}_{\nabla,0} = \bigoplus_{j\in\mathcal{J}_0} \mathfrak{S}_2(\mathcal{H}_0)$. The real linear map

$$\mathcal{G}_B : \mathfrak{h}_0 \longrightarrow \mathscr{H}_{\nabla,0} \oplus (Q_0\mathcal{H}_0)_{\mathbb{R}}, \qquad \mathcal{G}_B C = (D_B^0 C, \sqrt{s_E(B)}\, Q_0 C e_h)$$

has squared norm $\widehat{g}_{B,E,\chi}(C)$. Since $\mathrm{Tr}(CV_w) = 2\,\mathrm{Re}\langle Q_0 C e_h, w\rangle$, equation (4.2) is equivalent to

$$\mathrm{Tr}(CV) = -\langle \mathcal{G}_B C, (\xi, v)\rangle_{\mathbb{R}} \quad (C \in \mathfrak{h}_0), \qquad J_j = \mathsf{M}_{B,j}^{1/2}\xi_j.$$

If the supremum is finite, scalar optimization gives $|\mathrm{Tr}(CV)| \le \sqrt{a_{\chi,E}(B,V)}\,\|\mathcal{G}_B C\|$. Riesz representation on the finite-dimensional range of $\mathcal{G}_B$ gives a unique least vector with the displayed pairings. Lemma 2.5 identifies its squared norm with the infimum in (4.1). Adjoint-pair averaging makes the raw divergence Hermitian without changing its pairings.

On trace-zero Hermitian matrices, $\mathcal{G}_B^*\mathcal{G}_B = \mathbb{K}_B + s_E(B)\mathbb{R}_h$. For $B > 0$, $\mathbb{K}_B$ is positive definite by (2.28); inversion proves (1.18). At every finite-action point, the least normalized current is

$$-\lim_{r\downarrow 0} \mathcal{G}_B(\mathcal{G}_B^*\mathcal{G}_B + rI)^{-1}V.$$

This gives a measurable choice when B, V are measurable, also when the rank changes. The raw core costs are jointly convex by Lemma 2.5. The map $(w, s) \mapsto 4\|w\|^2/s$ is convex and nonincreasing in $s > 0$; its extension at zero is lower semicontinuous. Composition with the nonnegative concave function $s_E(B)$ proves joint convexity of the remaining term. □

Lemma 4.2. *For* $B \in C([0,T]; K_E^0)$ *and* $C \in C^1([0,T]; M_d^{\mathrm{sa}})$*, let*

$$\mathcal{L}_B(C) = \mathrm{Tr}(C(T)B_T) - \mathrm{Tr}(C(0)B_0) - \int_0^T \mathrm{Tr}(C'(t)B_t)\,dt.$$

Then, with values in $[0, +\infty]$*,*

$$\mathcal{A}_{\chi,E}^{\mathrm{eff}}(B) = \sup_{\substack{C\in C^1([0,T];M_d^{\mathrm{sa}})\\ C_{hh}=0}} \left\{ 2\mathcal{L}_B(C) - \int_0^T \widehat{g}_{B_t,E,\chi}(C(t))\,dt \right\}. \tag{4.3}$$

The effective action is lower semicontinuous for uniform matrix convergence. Every curve of finite action belongs to $W^{1,2}([0,T]; M_d^{\mathrm{sa}})$*.*

Proof. On the compact set K_E^0 there is $M_E > 0$ such that $\widehat{g}_{B,E,\chi}(C) \le M_E\|C\|_2^2$ for $C \in \mathfrak{h}_0$. Scalar changes of a potential leave its quadratic form and its pairing with a trace-zero velocity unchanged. Hence

$$a_{\chi,E}(B,V) \ge M_E^{-1}\|V\|_2^2.$$

Finite action therefore implies $B \in W^{1,2}$.

Let $\mathfrak{D}$ be the right-hand side of (4.3). If $\mathfrak{D} < \infty$, optimization over scalar multiples of C gives

$$|\mathcal{L}_B(C)| \le \sqrt{\mathfrak{D}} \left(\int_0^T \widehat{g}_{B_t,E,\chi}(C(t))\,dt \right)^{1/2} \le c_E\sqrt{\mathfrak{D}}\,\|C\|_{L^2}.$$

Apply this bound to $C(t) = f(t)A$, with $f \in C_c^\infty(0,T)$ and A ranging over the diagonal units E_{aa}, $a \ne h$, and the Hermitian real and imaginary off-diagonal matrix units. Each resulting distributional derivative is represented by an L^2 function. Since $B_{hh} = 1 - \sum_{a\neq h} B_{aa}$, its derivative is also in L^2. Thus $B \in W^{1,2}$, and integration by parts gives $\mathcal{L}_B(C) = \int_0^T \operatorname{Tr}(C\dot{B})\,dt$.

Suppose $B \in W^{1,2}$. Choose a countable dense set (C_j) in the gauge space, with $C_1 = 0$, and put

$$f_m(t) = \max_{1\le j\le m}\{2\operatorname{Tr}(C_j\dot{B}_t) - \widehat{g}_{B_t,E,\chi}(C_j)\}.$$

Then $0 \le f_m \uparrow a_{\chi,E}(B_t, \dot{B}_t)$ almost everywhere. Taking the least maximizing index gives a bounded measurable field $C^{(m)}$. Approximate it in L^2 by smooth gauge-valued fields. The linear integral converges since $\dot{B} \in L^2$; the quadratic integral converges since the coefficients of $\widehat{g}$ are uniformly bounded. Hence $\mathfrak{D} \ge \int_0^T f_m\,dt$. Monotone convergence gives $\mathfrak{D} \ge \mathcal{A}^{\mathrm{eff}}_{\chi,E}(B)$. The reverse inequality follows from the pointwise definition of $a_{\chi,E}$. If $B \notin W^{1,2}$, both sides are infinite.

Each expression in the supremum is continuous for uniform convergence of B, by continuity of $\widehat{g}$ on K^0_E. The supremum is therefore lower semicontinuous. ☐

4.2. **The cotangent bound and the lower inequality.**

Lemma 4.3. *Let $\rho_\delta \to B \oplus 0$ in trace norm, with $F_\delta(\rho_\delta) \le E$. For every Hermitian core potential C satisfying $C_{hh} = 0$, extended by zero to $\mathcal{H}$,*

$$\limsup_{\delta\downarrow 0} g^\delta_{\rho_\delta}(C) \le g^0_B(C) + \chi(E - F_0(B))\|Q_0Ce_h\|^2. \tag{4.4}$$

For every fixed Hermitian core test, with or without the condition $C_{hh} = 0$, its cotangent norm is uniformly bounded on $\{F_\delta \le E\}$ for sufficiently small δ.

Proof. For each core channel, $\rho_\delta \to B \oplus 0$ implies $\mathsf{M}_{\rho_\delta,j} \to \mathsf{M}_{B\oplus 0,j}$ in operator norm by Lemma 2.4. Its commutator with C is fixed and core-supported. Hence the sum of the core cotangent terms tends to $g^0_B(C)$. Average the phases of the auxiliary coordinates as in Lemma 3.2, retaining only the auxiliary channel sums. Concavity and (3.6) give

$$\sum_n g^\delta_{\rho_\delta,h\nu_n}(C) \le \|Ce_h\|^2 \sum_n b_n\Lambda(p_n, e^{-(L+\varphi_n)}). \tag{4.5}$$

Fix $0 < \theta < 1$ and $\eta > 0$. For some n_0, $b_n \le (\chi + \eta)\varphi_n^2$ whenever $n \ge n_0$. If $p_n \ge e^{-\theta(L+\varphi_n)}$, then

$$b_n\Lambda(p_n, e^{-(L+\varphi_n)}) \le \frac{\chi+\eta}{1-\theta}\frac{\varphi_n^2 p_n}{L+\varphi_n} \le \frac{\chi+\eta}{1-\theta}(L+\varphi_n)p_n.$$

For the other indices $n \ge n_0$, monotonicity of the mean gives the bound $C_*\varphi_n^2 e^{-\theta(L+\varphi_n)}$. Their sum is at most $C_*e^{-\theta L}\sum_n \varphi_n^2 e^{-\theta\varphi_n}$ and tends to zero. For each $n < n_0$, continuity of Λ and $p_n \to 0$ make its contribution tend to zero. Hence Lemma 3.1 implies

$$\limsup_{\delta\downarrow 0} \sum_n b_n\Lambda(p_n, e^{-(L+\varphi_n)}) \le \frac{\chi+\eta}{1-\theta}(E - F_0(B)).$$

Let $\eta, \theta \downarrow 0$. Since $Ce_h = Q_0Ce_h$ in this gauge, (4.4) follows. Uniform boundedness without the gauge restriction is (3.7). ☐

Lemma 4.4. *Let $\delta_j \to 0$. If $\sup_j \mathcal{A}_{\delta_j,E}(\rho^j) < \infty$, then (ρ^j) has a uniformly trace-convergent subsequence, and every limit is a core-supported curve with values in K^0_E. For every sequence $\rho^j \to B$ uniformly in trace norm,*

$$\mathcal{A}^{\mathrm{eff}}_{\chi,E}(B) \le \liminf_j \mathcal{A}_{\delta_j,E}(\rho^j). \tag{4.6}$$

The left side is $+\infty$ when B is not core-supported or does not belong to the domain of the effective action.

Proof. Choose admissible currents with squared norms at most $\mathcal{A}_{\delta_k,E}(\rho^k) + 1/k$. Lemma 3.2 bounds the cotangent norm of every core matrix test uniformly in k. Thus

$$|\operatorname{Tr} C(\rho^k_t - \rho^k_s)| \le C_{E,C}\sqrt{|t-s|}\big(\mathcal{A}_{\delta_k,E}(\rho^k) + 1/k\big)^{1/2}.$$

The core compressions form an equicontinuous bounded family in a finite-dimensional matrix space. Their traces tend uniformly to one by Lemma 3.1. The normalized compressions consequently have the same compactness. Arzelà–Ascoli and (3.2) give uniform trace convergence of a subsequence of the full curves to a core curve B. The entropy budget and continuity of F_0 imply $F_0(B_t) \leq E$ for every t.

Suppose the liminf is finite. Pass to a subsequence along which the actions are bounded and converge to that liminf. For $C \in C^1([0,T]; M_d^{\mathrm{sa}})$ with $C_{hh} = 0$, extend C by zero to $\mathcal{H}$. Lemma 2.8 gives

$$\mathcal{A}_{\delta_k,E}(\rho^k) \geq 2\left[\operatorname{Tr} C(T)\rho_T^k - \operatorname{Tr} C(0)\rho_0^k - \int_0^T \operatorname{Tr}(C'(t)\rho_t^k)dt\right] - \int_0^T g_{\rho_t^k}^{\delta_k}(C(t))dt.$$

A fixed basis of core Hermitian tests and the bounded coefficients of $C(t)$ give a common integrable bound for the last integrand. Lemma 4.3 applies at every time. Reverse Fatou therefore gives

$$\limsup_k \int_0^T g_{\rho_t^k}^{\delta_k}(C(t))\,dt \leq \int_0^T \widehat{g}_{B_t,E,\chi}(C(t))\,dt.$$

Uniform convergence passes the linear terms to $2\mathcal{L}_B(C)$. Taking the supremum and using Lemma 4.2 proves (4.6). $\square$

4.3. **Recovery.**

Lemma 4.5. *Let B have finite effective action. There are smooth core state curves B^j and smooth raw currents J^j, w^j satisfying* (4.2) *with $V = \dot{B}^j$, such that $B^j \to B$ uniformly,*

$$B_t^j > 0, \qquad \inf_{0\leq t\leq T}(E - F_0(B_t^j)) > 0,$$

and the upper limit of their integrated costs (4.1) *is at most $\mathcal{A}_{\chi,E}^{\mathrm{eff}}(B)$.*

Proof. Choose the measurable least currents of Lemma 4.1. They lie in L^2: the core mobilities are uniformly bounded and $s_E(B) \leq \chi E$. For $0 < \eta < 1$, set

$$B^\eta = (1-\eta)B + \eta\sigma_0, \qquad J^\eta = (1-\eta)J, \qquad w^\eta = (1-\eta)w.$$

Then $B^\eta \geq \eta\sigma_0$ and $E - F_0(B^\eta) \geq \eta E$. Joint convexity bounds the integrated cost by $(1-\eta)\mathcal{A}_{\chi,E}^{\mathrm{eff}}(B)$. Extend B^η constantly outside $[0,T]$ and extend both currents by zero. The state extension is continuous at 0, T, so its distributional derivative has no endpoint masses. For a nonnegative smooth kernel κ_r of integral one, let

$$(B^{\eta,r}, J^{\eta,r}, w^{\eta,r}) = \kappa_r * (B^\eta, J^\eta, w^\eta).$$

Differentiation and the finite sums in (4.2) commute with convolution; hence that equation holds for the convolved triple. Convexity gives $B^{\eta,r} \geq \eta\sigma_0$ and $F_0(B^{\eta,r}) \leq (1-\eta)E$. If C denotes the cost in (4.1), Jensen's inequality and Tonelli's theorem give

$$\int_0^T C(B^{\eta,r}, J^{\eta,r}, w^{\eta,r})\,dt \leq \int_{\mathbb{R}} C(B^\eta, J^\eta, w^\eta)\,dt \leq (1-\eta)\mathcal{A}_{\chi,E}^{\mathrm{eff}}(B).$$

For each fixed η, convolution converges uniformly to the continuous curve B^η and in L^2 to the raw currents. Choose its width so that the state error is at most η. On $[0,T]$ the convolved currents are smooth and bounded, and their integrated cost is no greater than $(1-\eta)\mathcal{A}_{\chi,E}^{\mathrm{eff}}(B)$. Taking $\eta \downarrow 0$ gives the asserted sequence. $\square$

Lemma 4.6. *Let B, J, w be a smooth state-current triple as in Lemma 4.5. There are admissible curves ρ^δ with $F_\delta(\rho_t^\delta) \leq E$ for all t, converging uniformly in trace norm to $B \oplus 0$, such that*

$$\limsup_{\delta\downarrow 0} \mathcal{A}_{\delta,E}(\rho^\delta) \leq \int_0^T \left[\sum_{j\in\mathcal{J}_0} a_{\mathsf{M}_{B_t,j}}(J_j(t)) + \frac{4\|w_t\|^2}{\chi(E - F_0(B_t))}\right] dt.$$

One family of curves is admissible for every choice of the test spaces in (1.11).

Proof. If $\chi = 0$, finite cost forces $w = 0$. Take $\rho^\delta = \iota_\delta B$ and use the core currents $\beta_\delta J_j$, with all auxiliary currents zero. Their negative divergence is $\beta_\delta \dot{B}$ and their cost is exactly $\beta_\delta \sum_j a_{\mathsf{M}_{B,j}}(J_j)$ by homogeneity. Each curve has positive finite order bounds against σ_δ; Lemma 2.7 gives all test conditions.

Suppose $\chi > 0$. Choose increasing indices n_j such that $|b_{n_j}/\varphi_{n_j}^2 - \chi| \le 1/j$ and $\varphi_{n_j} \ge j$. For $L = \log(1/\delta)$ sufficiently large, let $j = j(\delta)$ be the least integer with $j \ge \lceil L \rceil$ and $\varphi_{n_j} \ge L^2$, and put $n = n_j$. Such an integer exists since $\varphi_{n_j} \to \infty$. Then

$$n \longrightarrow \infty, \qquad \frac{L}{\varphi_n} \longrightarrow 0, \qquad \frac{b_n}{\varphi_n^2} \longrightarrow \chi. \tag{4.7}$$

For a smooth positive function $m(t) < E - F_0(B_t)$, with strict uniform margin, define

$$\rho_t^\delta = \iota_\delta B_t - r(t)P + r(t)E_{v_n v_n}, \qquad r(t) = \frac{m(t)}{\varphi_n}. \tag{4.8}$$

The core has a positive uniform least eigenvalue. Consequently these are faithful states for small δ, and $\rho_t^\delta \ge (1 - c/\varphi_n)\iota_\delta B_t$ uniformly in time. They converge uniformly to $B_t \oplus 0$.

Write $p_n^\delta = \beta_\delta \delta \pi_h e^{-\varphi_n}$. Direct-sum relative entropy gives

$$\begin{aligned} F_\delta(\rho_t^\delta) - F_\delta(\iota_\delta B_t) = {} & \operatorname{Tr}\big[(\beta_\delta B_t - rP)\log(\beta_\delta B_t - rP) - \beta_\delta B_t \log(\beta_\delta B_t)\big] \\ & + (p_n^\delta + r)\log(p_n^\delta + r) - p_n^\delta \log p_n^\delta + r(\varphi_n + L). \end{aligned} \tag{4.9}$$

The core difference is $O(1/\varphi_n)$. Both scalar entropy terms tend uniformly to zero, since $x\log x \to 0$ as $x \downarrow 0$. The last term tends uniformly to $m(t)$ by (4.7). Hence

$$F_\delta(\rho_t^\delta) = \beta_\delta F_0(B_t) + m(t) + o(1) < E. \tag{4.10}$$

Use the core raw currents $\beta_\delta J_j$ at the loaded state. Equation (4.2) gives their negative divergence exactly:

$$-\sum_{j\in\mathcal{J}_0} [L_j^*, \beta_\delta J_j] = \beta_\delta(\dot{B} - V_w).$$

By (2.20), their cost is at most $\beta_\delta(1 - c/\varphi_n)^{-1}$ times the original core cost. On the single auxiliary pair add

$$J_{n,+}^r = -(\ell_n^\delta)^{-1}|e_{v_n}\rangle\langle\beta_\delta w_t|, \qquad J_{n,-}^r = -(J_{n,+}^r)^*.$$

Lemma 2.9 gives its velocity $\beta_\delta V_{w_t}$ and its action

$$4\beta_\delta^2\langle w_t, T_\delta(t)^{-1}w_t\rangle, \quad T_\delta(t) = \Lambda\big(b_n(p_n^\delta + r),\, \delta b_n e^{-\varphi_n}(\beta_\delta B_t - rP)\big). \tag{4.11}$$

The eigenvalues of $\beta_\delta B_t - rP$ remain in one compact interval $K \Subset (0,\infty)$. Put

$$s_\delta = \varphi_n, \qquad c_\delta = b_n, \qquad u_\delta(t) = p_n^\delta + \frac{m(t)}{\varphi_n}, \qquad L_\delta = L.$$

Since m has a positive minimum and finite maximum, $\sup_t |\log u_\delta(t)|/\varphi_n \to 0$; moreover $L/\varphi_n \to 0$ by (4.7). Hence Lemma 2.3, applied uniformly for $t \in [0,T]$ and $\lambda \in K$, gives

$$T_\delta(t) = \frac{b_n u_\delta(t)}{\varphi_n}\big(I_0 + o_{\mathrm{op}}(1)\big).$$

The scalar coefficient satisfies, uniformly in t,

$$\frac{b_n u_\delta(t)}{\varphi_n} = \frac{b_n}{\varphi_n^2}m(t) + \frac{b_n}{\varphi_n}p_n^\delta \longrightarrow \chi m(t).$$

Indeed $b_n \le C_*\varphi_n^2$ and $p_n^\delta = \beta_\delta \delta \pi_h e^{-\varphi_n}$ make the second term at most $C\varphi_n \delta e^{-\varphi_n} = o(1)$. Thus $T_\delta(t) \to \chi m(t) I_0$ uniformly in operator norm. Let $m_* = \min_{[0,T]} m > 0$. For small δ, $T_\delta(t) \ge \chi m_* I_0/2$ uniformly in time. Hence

$$\sup_t \|T_\delta(t)^{-1} - (\chi m(t))^{-1} I_0\| \longrightarrow 0.$$

Since w is bounded, the integrated reservoir cost tends to $\int_0^T 4\|w_t\|^2/(\chi m(t))\,dt$.

On the same auxiliary pair add

$$J^{\ell}_{n,+} = \frac{m'(t)}{2\varphi_n \ell^{\delta}_n} E_{v_n h}, \qquad J^{\ell}_{n,-} = -(J^{\ell}_{n,+})^*.$$

Its velocity is $r'(E_{v_n v_n} - P)$ and its integrated action is at most

$$\frac{2}{\chi m_* \varphi_n^2} \int_0^T |m'(t)|^2\,dt.$$

The mixed term in the integrated inverse-mobility quadratic form has absolute value at most twice the geometric mean of the two integrated actions. It is therefore $O(\varphi_n^{-1})$. Core and auxiliary currents have disjoint channel supports. Their total negative divergence is exactly $\dot\rho^{\delta}_t$.

For each fixed δ, the whole curve has finite positive upper and lower order bounds against σ_δ. All active currents and divergences have finite channel and matrix support. Lemmas 2.5 and 2.7, together with (2.25), give admissibility for every test space (1.11). Therefore

$$\limsup_{\delta\downarrow 0} \mathcal{A}_{\delta,E}(\rho^{\delta}) \le \int_0^T \left[\sum_j a_{\mathsf{M}_{B_t,j}}(J_j(t)) + \frac{4\|w_t\|^2}{\chi m(t)}\right] dt.$$

For $q \ge 2$, take $m_q(t) = (1-1/q)(E - F_0(B_t))$ and denote the resulting family by $\rho^{\delta,q}$. Let

$$A_c = \int_0^T \sum_j a_{\mathsf{M}_{B_t,j}}(J_j(t))\,dt, \qquad A_r = \int_0^T \frac{4\|w_t\|^2}{\chi(E - F_0(B_t))}\,dt.$$

Choose $d_q \downarrow 0$ with $d_{q+1} < d_q/2$ such that, for $0 < \delta \le d_q$, the family $\rho^{\delta,q}$ lies in the cap and

$$\|\rho^{\delta,q} - (B \oplus 0)\|_{C([0,T];\mathfrak{S}_1)} \le 1/q, \qquad \mathcal{A}_{\delta,E}(\rho^{\delta,q}) \le A_c + \frac{A_r}{1-1/q} + 1/q.$$

Set $\rho^{\delta} = \rho^{\delta,q}$ when $d_{q+1} < \delta \le d_q$. Then $q \to \infty$ as $\delta \downarrow 0$, which gives the assertion. □

4.4. **Prescribed endpoints.**

Lemma 4.7. *Let $B : [0,T] \to K^0_E$ have finite effective action, with B_0 and B_T faithful. The recovery curves may be chosen with*

$$\rho^{\delta}_0 = \iota_\delta B_0, \qquad \rho^{\delta}_T = \iota_\delta B_T.$$

They converge uniformly in trace norm to $B \oplus 0$ and satisfy $\limsup_{\delta\downarrow 0} \mathcal{A}_{\delta,E}(\rho^{\delta}) \le \mathcal{A}^{\mathrm{eff}}_{\chi,E}(B)$.

Proof. First fix a smooth faithful prepared curve and $\chi > 0$. At an endpoint, write $r_* = m/\varphi_n$ and choose $b_* > 0$ smaller than the least eigenvalue of each loaded core matrix and satisfying $\beta_\delta \pi_h \ge 2b_*$ for small δ. Since $p^{\delta}_n = \beta_\delta \delta \pi_h e^{-\varphi_n}$, for $0 < r \le r_*$ one has

$$\Lambda(p^{\delta}_n + r, \delta e^{-\varphi_n} b_*)^{-1} \le \frac{L_n}{r}, \qquad L_n = \log \frac{p^{\delta}_n + r_*}{\delta e^{-\varphi_n} b_*} = L + \varphi_n + O(\log \varphi_n).$$

The loading action from Lemma 2.9 is at most $L_n \dot r^2/(b_n r)$. For $\tau_n = \varphi_n^{-1/2}$, use $r(t) = r_*(t/\tau_n)^2$ on $[0,\tau_n]$. Its action is at most

$$\frac{4 r_* L_n}{b_n \tau_n} = O(\varphi_n^{-3/2}). \tag{4.12}$$

Use the reversed curve at T. Both endpoints of each loading segment lie in the cap, so its affine image lies there by entropy convexity. For large n, $2\tau_n < T$. Place the middle curve on $[\tau_n, T-\tau_n]$ by linear rescaling. For the rescaling $\vartheta_n(u) = T(u-\tau_n)/(T-2\tau_n)$, multiply every raw current by ϑ'_n and compose it with ϑ_n. The continuity equation is preserved and the action is multiplied by

$T/(T-2\tau_n)\to 1$. If ω_B is the trace-norm modulus of continuity of B, then $|\vartheta_n(u)-u|\le\tau_n$ on the middle interval. The full connector differs from $B\oplus 0$ by at most

$$2(1-\beta_\delta)+\frac{2\max m}{\varphi_n}+\omega_B(\tau_n),$$

which tends to zero. If $\chi=0$, the embedded smooth curve already has the required endpoints.

For a general finite-action curve, the approximations in Lemma 4.5 have endpoints B_0^j, B_T^j tending to B_0, B_T. Near each faithful endpoint, the inverse of the original core Onsager operator has a uniform bound. The straight core segment from B_0 to B_0^j has unit-time action at most $c\|B_0-B_0^j\|_2^2$, and the analogous estimate holds at T. Its normalized embedding has at most the same action. For endpoint displacements Δ_j, choose durations $\tau_j=\|\Delta_j\|_2+1/j$. Then $\tau_j\to 0$ and $\|\Delta_j\|_2^2/\tau_j\to 0$. Rescale the intervening curve. All segments remain in the entropy cap by convexity. Their actions tend to zero. For each fixed j, apply the first part to B^j and insert these endpoint segments. Their total duration tends to zero with j, so the middle rescaling factor tends to one. The upper limit of the resulting actions, first in δ and then in j, is at most $\mathcal{A}^{\mathrm{eff}}_{\chi,E}(B)$. Their state errors have the same iterated limit zero: on the middle interval use $B^j\to B$ and the modulus of continuity of B; on the endpoint intervals use that modulus and $B_0^j\to B_0$, $B_T^j\to B_T$. After passing to a subsequence in j, choose thresholds $d_{j+1}<d_j/2$ such that all these errors and the excess action are at most $1/j$ for $0<\delta\le d_j$. Use the j-th construction on $d_{j+1}<\delta\le d_j$. The endpoints are exactly $\iota_\delta B_0, \iota_\delta B_T$ for every chosen coupling. ☐

Proof of Theorem A. Lemma 4.4 gives the lower inequality and asymptotic compactness. Let $\mathcal{A}^{\mathrm{eff}}_{\chi,E}(B)<\infty$. Choose the sequence in Lemma 4.5, and pass to a subsequence whose state error and excess cost are at most $1/j$. For its j-th member, Lemma 4.6 gives a threshold $\delta_j>0$ below which the recovery state error and excess action are at most $1/j$. Choose $\delta_{j+1}<\delta_j/2$ and use that recovery for $\delta_{j+1}<\delta\le\delta_j$. The resulting family tends uniformly to $B\oplus 0$ and has action upper limit at most $\mathcal{A}^{\mathrm{eff}}_{\chi,E}(B)$. Lemma 4.7 gives the same construction with the stated faithful endpoints. At an infinite limiting value the upper inequality is automatic. This proves (1.15). ☐

4.5. **Entropy recovery and kinetic saving.**

Corollary 4.8 (Entropy-recovering families). *Assume* (1.8)*, and fix $E, T>0$. Let $\rho^\delta\in C([0,T];\mathfrak{S}(\mathcal{H}))$ satisfy $\rho^\delta\to B\oplus 0$ uniformly in trace norm, $F_\delta(\rho_t^\delta)\le E$ for every t, and*

$$\int_0^T F_\delta(\rho_t^\delta)\,dt\longrightarrow\int_0^T F_0(B_t)\,dt. \tag{4.13}$$

Then $B_t\in K_E^0$ for every t, and, for every family of test spaces satisfying (1.11)*,*

$$\liminf_{\delta\downarrow 0}\mathcal{A}_{\delta,E}(\rho^\delta)\ge\mathcal{A}_{0,E}(B),\qquad \mathcal{A}_{0,E}\;:=\mathcal{A}^{\mathrm{eff}}_{0,E}. \tag{4.14}$$

Here the right side is the original core action, including its extended value at curves outside its domain. In particular, (4.13) *holds if $F_\delta(\rho_t^\delta)\to F_0(B_t)$ uniformly in time.*

Proof. Put $s_\delta(t)=\mathrm{Tr}((I-P_0)\rho_t^\delta)$, $M_\delta(t)=\sum_n\varphi_n(\rho_t^\delta)_{v_nv_n}$, and $L_\delta=\log(1/\delta)$. For small δ, the normalized core compressions $B_{\rho_t^\delta}$ are defined and converge uniformly to B_t. Lemma 3.1 gives $F_0(B_t)\le E$ and a remainder $e_\delta(t)$ tending uniformly to zero such that

$$F_\delta(\rho_t^\delta)=F_0(B_{\rho_t^\delta})+M_\delta(t)+L_\delta s_\delta(t)+e_\delta(t).$$

Continuity of F_0 on the finite-dimensional state space and (4.13) imply

$$\int_0^T\big(M_\delta(t)+L_\delta s_\delta(t)\big)\,dt\longrightarrow 0.$$

The integrand is nonnegative for $\delta<1$. By (3.7),

$$\int_0^T\sum_n b_n\Lambda\big((\rho_t^\delta)_{v_nv_n},\delta e^{-\varphi_n}\big)\,dt\longrightarrow 0.$$

Consequently, for every $C \in C^1([0,T]; M_d^{\mathrm{sa}})$ extended by zero, (3.6) bounds the integrated auxiliary cotangent form by $\max_t \|C(t)e_h\|^2$ times this vanishing quantity. The core cotangent integrals converge to $\int_0^T g^0_{B_t}(C(t))\,dt$ by fixed-channel continuity and the uniform finite-core bound.

If the action lower limit is finite, pass to a subsequence attaining it and apply (2.27) on that subsequence. Uniform state convergence passes its linear terms to $2\mathcal{L}_B(C)$. Thus

$$\liminf_{\delta\downarrow 0} \mathcal{A}_{\delta,E}(\rho^\delta) \geq 2\mathcal{L}_B(C) - \int_0^T g^0_{B_t}(C(t))\,dt.$$

Take the supremum over the gauge $C_{hh} = 0$ and use Lemma 4.2 with $\chi = 0$. The case of an infinite action lower limit is immediate. This proves (4.14). □

Thus a family attaining a strict saving $\mathcal{A}^{\mathrm{eff}}_{\chi,E}(B) < \mathcal{A}_{0,E}(B)$ cannot also recover the time-integrated entropy of B. This does not conflict with Corollary 3.3: energy Γ-convergence provides energy recovery sequences, not simultaneous recovery by every kinetic recovery family.

5. Distances and proximal functionals

Let $d_{\chi,E}$ be the distance defined by the unit-time effective action. All states in this section are finite core states unless otherwise specified.

Lemma 5.1. *The distance $d_{\chi,E}$ is finite on K^0_E, induces the matrix norm topology, and makes K^0_E a compact geodesic metric space. Its square is jointly convex in the two endpoints. For every $h > 0$ and $x \in K^0_E$, the function*

$$B \longmapsto F_0(B) + \frac{d_{\chi,E}(B,x)^2}{2h}$$

has a unique minimizer in K^0_E.

Proof. There is $M_E < \infty$ such that $\widehat{g}_{B,E,\chi}(C) \leq M_E\|C\|_2^2$ on K^0_E. Testing an admissible effective curve by constant matrices gives $d_{\chi,E}(B,C) \geq c_E\|B-C\|_2$ for a positive c_E. Time reversal and concatenation give symmetry and the triangle inequality. Joint convexity of (4.1) shows that mixing two state-current pairs mixes their endpoints with no greater weighted action. Taking infima proves joint convexity of the squared distance.

For every Hermitian trace-zero V, the stationary core current $-D^0_{\sigma_0}\mathbb{K}^{-1}_{\sigma_0}V$ has velocity V and norm at most $c\|V\|_2$. For $B \in K^0_E$ and $0 < \eta \leq 1$, set $B_t = (1-\eta t^2)B + \eta t^2\sigma_0$, $0 \leq t \leq 1$. Let J be the stationary raw current for $\sigma_0 - B$. At time $t > 0$, $\mathsf{M}_{B_t,j} \geq \eta t^2\mathsf{M}_{\sigma_0,j}$, so the raw current $2\eta t J$ has velocity $\dot{B}_t$ and cost at most

$$(\eta t^2)^{-1}(2\eta t)^2 \sum_j a_{\mathsf{M}_{\sigma_0,j}}(J_j) \leq C\eta.$$

The additional current is zero. Entropy convexity keeps the curve in K^0_E. Taking $\eta = 1$ connects every state to σ_0 with finite action, and arbitrary η gives

$$d_{\chi,E}(B, (1-\eta)B + \eta\sigma_0) \leq c\sqrt{\eta} \qquad (0 < \eta \leq 1).$$

Two η-mixed states dominate $\eta\sigma_0$. The straight segment between them has unit-time action at most $c^2\|B-C\|_2^2/\eta$, by the same stationary-current bound and (2.20). For $B \neq C$, the triangle inequality with $\eta = \min\{\|B-C\|_1, 1\}$ yields

$$d_{\chi,E}(B,C) \leq c_E\sqrt{\|B-C\|_1} \qquad (\|B-C\|_1 \leq 1). \tag{5.1}$$

The two topologies coincide, so K^0_E is compact in the metric.

The bounded cotangent form makes a bounded-action sequence bounded in $W^{1,2}$. Uniform convergence follows after passage to a subsequence. Lemma 4.2 gives the action lower bound, hence a minimizer between every fixed pair. If its unit-time action is $A = d_{\chi,E}(B_0,B_1)^2$ and $A_{s,t}$ is the action on $[s,t]$, then

$$d_{\chi,E}(B_0,B_1) \leq d_{\chi,E}(B_0,B_t) + d_{\chi,E}(B_t,B_1) \leq \sqrt{tA_{0,t}} + \sqrt{(1-t)A_{t,1}} \leq \sqrt{A}.$$

If $A = 0$, the matrix lower bound makes the curve constant. If $A > 0$, equality in Cauchy–Schwarz gives $A_{0,t} = tA$ and $A_{t,1} = (1-t)A$. Apply the same equalities to a partition containing s, t. The middle action is $(t-s)A$ and its distance is $(t-s)\sqrt{A}$. Thus $d_{\chi,E}(B_s, B_t) = (t-s)d_{\chi,E}(B_0, B_1)$ for all $s \le t$.

The proximal objective is lower semicontinuous on the compact cap. For $B_\theta = (1-\theta)B + \theta C$,

$$\begin{aligned}&(1-\theta)F_0(B) + \theta F_0(C) - F_0(B_\theta)\\ &\quad = (1-\theta)D(B\|B_\theta) + \theta D(C\|B_\theta) \ge \tfrac{1}{2}\theta(1-\theta)\|B-C\|_1^2.\end{aligned}$$

The last inequality is Pinsker's inequality applied to both terms. The squared-distance term is convex in the output state. Hence two distinct minimizers are impossible. □

Corollary 5.2. *For faithful $B_0, B_1 \in K_E^0$,*

$$\mathcal{W}_{\delta,E}^{\mathcal{T}_\delta}(\iota_\delta B_0, \iota_\delta B_1) \longrightarrow d_{\chi,E}(B_0, B_1). \tag{5.2}$$

More generally, let $B_0, B_1 \in K_E^0$ be arbitrary, and suppose that $x_\delta, y_\delta \in K_{\delta,E}^{\mathcal{T}_\delta}$ converge in trace norm to $B_0 \oplus 0, B_1 \oplus 0$, respectively. Then

$$d_{\chi,E}(B_0, B_1) \le \liminf_{\delta\downarrow 0} \mathcal{W}_{\delta,E}^{\mathcal{T}_\delta}(x_\delta, y_\delta).$$

Fix a faithful $x \in K_E^0$ and a step $h > 0$. Define

$$\Phi_\delta(y) = F_\delta(y) + \frac{\mathcal{W}_{\delta,E}^{\mathcal{T}_\delta}(y, \iota_\delta x)^2}{2h} \quad (y \in K_{\delta,E}^{\mathcal{T}_\delta}),$$

with value $+\infty$ elsewhere. These functionals Γ-converge in trace norm to

$$\Phi_0(B) = F_0(B) + \frac{d_{\chi,E}(B,x)^2}{2h} \quad (B \in K_E^0),$$

also extended by $+\infty$ elsewhere. If $\Phi_\delta(y_\delta) - \inf \Phi_\delta \to 0$, then y_δ converges to the unique minimizer of Φ_0. The entropy and squared-distance terms converge separately to their values at that minimizer.

Proof. Put

$$W_\delta = \mathcal{W}_{\delta,E}^{\mathcal{T}_\delta}(x_\delta, y_\delta), \qquad \ell = \liminf_{\delta\downarrow 0} W_\delta^2.$$

If $\ell = +\infty$, the distance lower inequality holds. Suppose $\ell < \infty$, and choose $\delta_j \downarrow 0$ such that $W_{\delta_j}^2 \to \ell$. After omitting finitely many indices, these values are finite. By (2.24), there are connecting curves ρ^j on $[0,1]$ with endpoints $x_{\delta_j}, y_{\delta_j}$ and

$$W_{\delta_j}^2 \le \mathcal{A}_{\delta_j,E}(\rho^j) \le W_{\delta_j}^2 + 1/j.$$

Their actions converge to ℓ. Lemma 4.4 gives a subsequence converging uniformly in trace norm to $B \oplus 0$, where $B(0) = B_0$, $B(1) = B_1$, and

$$d_{\chi,E}(B_0, B_1)^2 \le \mathcal{A}_{\chi,E}^{\mathrm{eff}}(B) \le \liminf_j \mathcal{A}_{\delta_j,E}(\rho^j) = \ell.$$

Since $W_\delta \ge 0$, one has $\sqrt{\ell} = \liminf_{\delta\downarrow 0} W_\delta$. Taking square roots proves the lower inequality. For faithful fixed endpoints, apply Lemma 4.7 to an effective minimizing geodesic. This gives the upper inequality and proves (5.2).

The energy lower bound in Corollary 3.3 and the distance lower inequality give the lower bound for Φ_δ. For a faithful target B, the canonical embedding has exactly $F_\delta(\iota_\delta B) = \beta_\delta F_0(B)$, and (5.2) gives recovery of its distance term. For a singular target use $B^\eta = (1-\eta)B + \eta\sigma_0$. Continuity of F_0 and (5.1) imply $\Phi_0(B^\eta) \to \Phi_0(B)$. Choose $\eta_j \downarrow 0$ with $|\Phi_0(B^{\eta_j}) - \Phi_0(B)| \le 1/j$, and $d_{j+1} < d_j/2$ such that, for $0 < \delta \le d_j$,

$$|\Phi_\delta(\iota_\delta B^{\eta_j}) - \Phi_0(B^{\eta_j})| \le 1/j, \qquad \|\iota_\delta B^{\eta_j} - (B^{\eta_j} \oplus 0)\|_1 \le 1/j.$$

The choice $y_\delta = \iota_\delta B^{\eta_j}$ on $d_{j+1} < \delta \le d_j$ proves the upper bound.

The competitor $\iota_\delta x$ bounds $\inf \Phi_\delta$ uniformly. Hence every sequence with objective error tending to zero has bounded energy. The entropy budget gives relative compactness. The lower inequality and recovery of the unique effective minimizer show that every cluster point minimizes Φ_0 and that the infima converge. Lemma 5.1 makes this cluster point unique. Write $u_\delta = F_\delta(y_\delta)$ and $v_\delta = \mathcal{W}_{\delta,E}^{\mathcal{T}_\delta}(y_\delta, \iota_\delta x)^2/(2h)$. Their respective lower limits are at least u_0, v_0, and $u_\delta + v_\delta \to u_0 + v_0$. Consequently $\limsup u_\delta \le u_0 + v_0 - \liminf v_\delta \le u_0$, and similarly for v_δ. Both terms converge. □

Corollary 5.2 fixes $h > 0$ before $\delta \downarrow 0$. Its conclusion concerns almost minimizers; the moving infima need not be attained.

Lemma 5.3. *Let $B \in K_E^0$ be faithful, with $F_0(B) < E$, and let $V \in M_d^{\mathrm{sa}}$ satisfy* $\operatorname{Tr} V = 0$. *Then*

$$\lim_{s\to 0} \frac{d_{\chi,E}(B + sV, B)^2}{s^2} = \operatorname{Tr}\big(V[\mathbb{K}_B + \chi(E - F_0(B))\mathbb{R}_h]^{-1} V\big). \tag{5.3}$$

Proof. The inverse tensor is continuous on a faithful neighbourhood of B contained in $F_0 < E$. The straight segment gives

$$d_{\chi,E}(B + sV, B)^2 \le s^2 \int_0^1 \operatorname{Tr}(V H_{B+tsV} V)\, dt = s^2 \operatorname{Tr}(V H_B V) + o(s^2),$$

where $H_D = [\mathbb{K}_D + \chi(E - F_0(D))\mathbb{R}_h]^{-1}$. For the lower bound, choose curves between B and $B + sV$ whose actions exceed the squared distances by $o(s^2)$. The upper bound makes their actions $O(s^2)$. If $D^{(s)}$ is one such curve, the coercive bound in Lemma 4.2 gives

$$\sup_t \|D_t^{(s)} - B\|_2 \le \|\dot D^{(s)}\|_{L^2} \le \sqrt{M_E \mathcal{A}_{\chi,E}^{\mathrm{eff}}(D^{(s)})} = O(|s|).$$

Thus these curves lie in the same faithful neighbourhood for small s. Put $\epsilon_s = \sup_t \|H_{D_t^{(s)}} - H_B\|/\lambda_{\min}(H_B)$. Then $\epsilon_s \to 0$ and $H_{D_t^{(s)}} \ge (1 - \epsilon_s) H_B$. Set $B_t = D_t^{(s)}$. Therefore

$$\int_0^1 \operatorname{Tr}(\dot B_t H_{B_t} \dot B_t)\, dt \ge (1 - o(1)) \operatorname{Tr}\Big(\Big[\int_0^1 \dot B_t\, dt\Big] H_B \Big[\int_0^1 \dot B_t\, dt\Big]\Big) = (1 - o(1)) s^2 \operatorname{Tr}(V H_B V).$$

This is the lower inequality in (5.3). □

6. Fisher information and energy–dissipation

Throughout this section, (1.8) holds and $E, T > 0$ are fixed. Write $\mathcal{I}_\delta^{\mathrm{aux}}$ for the auxiliary-channel sum in (2.21), and put

$$L = \log(1/\delta), \qquad p_n = \rho_{v_n v_n}, \qquad p_h = \rho_{hh}, \qquad C_\varphi = \sum_n b_n e^{-\varphi_n/2} < \infty.$$

For $x, y \ge 0$, define

$$\psi(x, y) = \begin{cases} (x - y)(\log x - \log y), & x, y > 0, \\ 0, & x = y = 0, \\ +\infty, & xy = 0,\ x + y > 0. \end{cases}$$

6.1. The auxiliary Fisher estimate.

Lemma 6.1. *For every state ρ,*

$$\mathcal{I}_\delta^{\mathrm{aux}}(\rho) \ge \sum_n \psi(b_n p_n, a_n^\delta p_h). \tag{6.1}$$

For each fixed $\delta > 0$, the functionals $\mathcal{I}_\delta$ and $\mathcal{I}_0$ are convex and lower semicontinuous in trace norm.

Proof. Fix an auxiliary pair, and let C be a diagonal Hermitian matrix. With $t = C_{\nu_n\nu_n} - C_{hh}$, its entropy current satisfies

$$\sum_{j\in\{(n,+),(n,-)\}} \langle J_j^{\mathrm{ent}}(\rho), [L_j, C]\rangle_{\mathbb{R}} = (b_n p_n - a_n^{\delta} p_h)t.$$

Coordinate pinching, Lemma 2.4, and the row formula give

$$\sum_{j\in\{(n,+),(n,-)\}} \|\mathsf{M}_{\rho,j}^{1/2}[L_j, C]\|_2^2 \le \Lambda(b_n p_n, a_n^{\delta} p_h)t^2.$$

The dual formula (2.19), optimized over $t \in \mathbb{R}$, therefore bounds the pair Fisher action below by

$$\sup_t\{2(b_n p_n - a_n^{\delta} p_h)t - \Lambda(b_n p_n, a_n^{\delta} p_h)t^2\} = \psi(b_n p_n, a_n^{\delta} p_h).$$

If both scalar fluxes vanish, the supremum is zero. If exactly one vanishes, the quadratic coefficient vanishes while the linear coefficient does not, so the supremum is $+\infty$. If both are positive, it equals their squared difference divided by their logarithmic mean, namely ψ. Summing over finitely many pairs and then increasing the pair set proves (6.1).

For fixed j, the map $\rho \mapsto J_j^{\mathrm{ent}}(\rho)$ is linear and continuous from trace class to Hilbert–Schmidt class. For fixed $U \in \mathfrak{S}_2$, the expression

$$2\langle J_j^{\mathrm{ent}}(\rho), U\rangle_{\mathbb{R}} - \langle U, \mathsf{M}_{\rho,j} U\rangle_{\mathbb{R}}$$

is continuous and convex in ρ, by Lemma 2.4. Its supremum over U is convex and lower semicontinuous. The same properties pass to the increasing finite channel sums. □

Lemma 6.2. *For $0 < \delta < 1$ and every state ρ,*

$$R_\delta(\rho) := \sum_n b_n \Lambda(p_n, \delta e^{-\varphi_n}) \le \frac{4}{L^2}\mathcal{I}_\delta^{\mathrm{aux}}(\rho) + C_\varphi\sqrt{\delta}. \tag{6.2}$$

For every Hermitian core test C, extended by zero to $\mathcal{H}$,

$$g_{\rho,\mathrm{aux}}^{\delta}(C) \le \|Ce_h\|^2\left(\frac{4}{L^2}\mathcal{I}_\delta^{\mathrm{aux}}(\rho) + C_\varphi\sqrt{\delta}\right). \tag{6.3}$$

Proof. Set $v_n = \delta e^{-\varphi_n} < 1$. If $p_n \ge \sqrt{v_n}$, then $\log(p_n/v_n) \ge (L + \varphi_n)/2 \ge L/2$. For $0 < y < x$, $\partial_y \psi(x, y) = 1 - x/y - \log(x/y) < 0$. Thus, when $p_h > 0$ and $p_n \ge \sqrt{v_n} > v_n$, the bound $p_h \le 1$ gives

$$\begin{aligned}\psi(b_n p_n, b_n v_n p_h) &\ge b_n \psi(p_n, v_n)\\ &= b_n \Lambda(p_n, v_n)\log^2(p_n/v_n) \ge \frac{L^2}{4} b_n \Lambda(p_n, v_n).\end{aligned}$$

If $p_h = 0$ in this first case, its Fisher term is infinite and the same inequality holds in the extended sense. For $p_n < \sqrt{v_n}$, monotonicity of the logarithmic mean gives $\Lambda(p_n, v_n) \le \sqrt{v_n}$. Summation and (6.1) prove (6.2).

Average the phases of all auxiliary coordinates as in Lemma 3.2. The resulting state is $P_0\rho P_0 \oplus \mathrm{diag}(p_n)$, and its auxiliary cotangent sum is no smaller. Equation (3.6) gives

$$g_{\rho,\mathrm{aux}}^{\delta}(C) \le \sum_n b_n \langle Ce_h, \Lambda(p_n, v_n P_0\rho P_0)Ce_h\rangle \le \|Ce_h\|^2 R_\delta(\rho),$$

since $0 \le P_0\rho P_0 \le I_0$. Continuity extends the calculation to zero eigenvalues. Combining the two bounds proves (6.3). □

Corollary 6.3. *Let $\delta_k \to 0$ and let ρ^k be measurable state curves with*

$$\sup_k \int_0^T \mathcal{I}_{\delta_k}(\rho_t^k)\, dt < \infty.$$

For every bounded measurable Hermitian core test $C(t)$,

$$\int_0^T g^{\delta_k}_{\rho_t^k,\mathrm{aux}}(C(t))\,dt \longrightarrow 0.$$

Thus the auxiliary components of $D^{\delta_k}_{\rho^k}C$ tend strongly to zero in $L^2(0,T;\mathscr{H}_\nabla)$.

Proof. Let $M = \operatorname{ess\,sup}_{t\in(0,T)} \|C(t)e_h\|^2$. For all sufficiently large k, (6.3) gives

$$\int_0^T g^{\delta_k}_{\rho_t^k,\mathrm{aux}}(C(t))\,dt \le M\left(\frac{4}{\log^2(1/\delta_k)}\int_0^T \mathcal{I}_{\delta_k}(\rho_t^k)\,dt + TC_\varphi\sqrt{\delta_k}\right) \longrightarrow 0.$$

Each finite channel projection of the auxiliary gradient is strongly measurable by Lemma 2.4. The displayed finite integral makes its channel sum finite almost everywhere; the projections then converge in the separable space $\mathscr{H}_\nabla$. The left side is its squared L^2 norm. □

6.2. **The joint variational limit.** The functionals $\mathscr{B}_{\delta,E}$ and $\mathscr{B}_{0,E}$ are those of (1.24). The notation $\mathcal{A}_{0,E} = \mathcal{A}^{\mathrm{eff}}_{0,E}$ means the original finite-core action; the first subscript on the right is $\chi = 0$.

Lemma 6.4. *Suppose $\delta_k \to 0$, $\rho^k \to B$ uniformly in trace norm, and*

$$\sup_k \mathscr{B}_{\delta_k,E}(\rho^k) < \infty.$$

Then B is core-supported, takes values in K_E^0, and

$$\mathcal{A}_{0,E}(B) \le \liminf_k \mathcal{A}_{\delta_k,E}(\rho^k), \tag{6.4}$$

$$\int_0^T \mathcal{I}_0(B_t)\,dt \le \liminf_k \int_0^T \mathcal{I}_{\delta_k}(\rho_t^k)\,dt. \tag{6.5}$$

Every sequence with bounded $\mathscr{B}_{\delta_k,E}$ has a uniformly trace-convergent subsequence.

Proof. Lemma 4.4 gives compactness, core support, and the entropy constraint. For (6.4), take a subsequence attaining the action lower limit, and admissible currents ξ^k such that

$$\int_0^T \|\xi_t^k\|^2\,dt \le \mathcal{A}_{\delta_k,E}(\rho^k) + 1/k.$$

Their core-channel components converge weakly, along a subsequence, in $\bigoplus_{j\in\mathcal{J}_0} L^2(0,T;\mathfrak{S}_2(\mathcal{H}))$. For every Hermitian core test C, Lemma 2.4 and the uniform state convergence give

$$D^{\delta_k,\mathrm{core}}_{\rho^k}C \longrightarrow D^0_B C \quad \text{strongly in } L^2.$$

The limit gradients are core-supported. Compress the weak limit currents to this block; their pairings are unchanged and their norms do not increase. Corollary 6.3 gives, for $0 \le s \le t \le T$,

$$\left|\int_s^t \langle D^{\delta_k,\mathrm{aux}}_{\rho_u^k}C, \xi_u^{k,\mathrm{aux}}\rangle_{\mathbb{R}}\,du\right| \le \|D^{\delta_k,\mathrm{aux}}_{\rho^k}C\|_{L^2}\|\xi^k\|_{L^2} \longrightarrow 0.$$

Passing to the limit in the continuity equation for a fixed real basis of M_d^{sa} proves the core continuity equation for every core test. Weak lower semicontinuity of the compressed current norm yields (6.4).

For each core channel and every t,

$$J_j^{\mathrm{ent}}(\rho_t^k) \longrightarrow J_j^{\mathrm{ent}}(B_t) \quad \text{in } \mathfrak{S}_2, \qquad \mathsf{M}_{\rho_t^k,j} \longrightarrow \mathsf{M}_{B_t,j} \quad \text{in operator norm},$$

where the limiting operators act on the full Hilbert–Schmidt space. For $0 < u < 1$, multiplication on both sides by $(B_t \oplus 0)^u$ and $(B_t \oplus 0)^{1-u}$ annihilates the orthogonal complement of $\mathfrak{S}_2(\mathcal{H}_0)$. Thus $\mathsf{M}_{B_t\oplus 0,j}$ equals $\mathsf{M}_{B_t,j}$ on that block and is zero on its complement. The limiting entropy current is core-supported, so its inverse action is exactly the finite-core inverse action. Formula (2.19) gives its lower semicontinuity. Discarding the auxiliary terms and applying Fatou's lemma proves (6.5). □

Lemma 6.5. *For every faithful core state B,*

$$\mathcal{I}_\delta(\iota_\delta B) = \beta_\delta \mathcal{I}_0(B) + \beta_\delta \delta A_\varphi \Psi_h(B), \tag{6.6}$$

where

$$\Psi_h(B) = \langle e_h, (B - \pi_h I_0)(\log B - \log(\pi_h) I_0) e_h \rangle \geq 0. \tag{6.7}$$

Proof. Homogeneity gives the core contribution $\beta_\delta \mathcal{I}_0(B)$. The entropy potential of $\iota_\delta B$ has core block $H_B = \log B - \log \sigma_0$ and zero auxiliary diagonal. The auxiliary pair $h v_n$ has core velocity

$$V_n = a_n^\delta \beta_\delta \left(\pi_h P - \tfrac{1}{2}\{P, B\} \right).$$

The entropy potential is bounded and supported on the core. On the finite block $\mathcal{H}_0 \oplus \mathbb{C} e_{v_n}$, (2.29) gives $J_j^{\mathrm{ent}} = \mathsf{M}_{\iota_\delta B, j}[L_j, H_B \oplus 0]$. Its Fisher action is therefore

$$-\operatorname{Tr}(H_B V_n) = a_n^\delta \beta_\delta \left(\operatorname{Re}\langle e_h, B H_B e_h \rangle - \pi_h \langle e_h, H_B e_h \rangle \right) = a_n^\delta \beta_\delta \Psi_h(B).$$

Here $\log \sigma_0 e_h = \log(\pi_h) e_h$. The scalar function $(r - \pi_h)(\log r - \log \pi_h)$ is nonnegative for $r > 0$; spectral calculus proves (6.7). Summing the pair contributions gives (6.6). ☐

Proof of Theorem C. Lemma 6.4 proves compactness and the lower inequality. Let B be a smooth core curve such that $B_t \geq a I_0$ and $F_0(B_t) \leq E - \gamma$ for $a, \gamma > 0$. Its least core current is $-D^0_{B_t} \mathbb{K}^{-1}_{B_t} \dot{B}_t$ and is smooth. Multiply its raw representative by β_δ, and set every auxiliary current to zero. At $\iota_\delta B$ the negative divergence is $\beta_\delta \dot{B}$, and homogeneity gives integrated action $\beta_\delta \mathcal{A}_{0,E}(B)$. The core matrices lie between fixed positive multiples of σ_0, while the auxiliary diagonal equals that of σ_δ. Lemma 2.7 and the finite raw divergence prove admissibility for every space (1.11). By (6.6),

$$\mathscr{B}_{\delta,E}(\iota_\delta B) \leq \beta_\delta \mathscr{B}_{0,E}(B) + \beta_\delta \delta A_\varphi \int_0^T \Psi_h(B_t)\, dt.$$

The last integral is finite. Hence the upper limit is at most $\mathscr{B}_{0,E}(B)$.

Let $\mathscr{B}_{0,E}(B) < \infty$ and let J be a least core raw current. For $0 < \eta < 1$, set

$$B^\eta = (1 - \eta) B + \eta \sigma_0, \qquad J^\eta = (1 - \eta) J.$$

Convexity of the raw action and Fisher information gives

$$\mathcal{A}_{0,E}(B^\eta) + \int_0^T \mathcal{I}_0(B^\eta_t)\, dt \leq (1 - \eta) \mathscr{B}_{0,E}(B).$$

Moreover $B^\eta \geq \eta \sigma_0$ and $E - F_0(B^\eta) \geq \eta E$. Extend B^η constantly outside $[0, T]$ and extend J^η by zero. Convolve both with a nonnegative smooth kernel supported in $[-r, r]$. The linear raw continuity equation is preserved. Jensen's inequality does not increase the integrated action. For the convolved state $B^{\eta,r}$, convexity gives

$$\int_0^T \mathcal{I}_0(B^{\eta,r}_t)\, dt \leq \int_0^T \mathcal{I}_0(B^\eta_t)\, dt + r\big(\mathcal{I}_0(B^\eta_0) + \mathcal{I}_0(B^\eta_T) \big).$$

For fixed η, the two endpoint values are finite. Choose $r \downarrow 0$ so that the last term and the uniform state error tend to zero, and then let $\eta \downarrow 0$. This gives smooth faithful curves B^j with positive entropy margin, $B^j \to B$ uniformly, and joint costs with upper limit at most $\mathscr{B}_{0,E}(B)$.

For each fixed j, choose $d_j > 0$, with $d_{j+1} < d_j/2$, such that, for $0 < \delta \leq d_j$,

$$\delta A_\varphi \int_0^T \Psi_h(B^j_t)\, dt \leq 1/j, \qquad \|\iota_\delta B^j - (B^j \oplus 0)\|_{C([0,T];\mathfrak{S}_1)} \leq 1/j.$$

Set $\rho^\delta = \iota_\delta B^j$ when $d_{j+1} < \delta \leq d_j$. Then $j \to \infty$, the state error tends to zero, and the preceding joint-action estimate gives the required upper limit.

At a faithful prescribed endpoint, choose a positive compact convex neighbourhood on which $\mathbb{K}_B^{-1}$, $\mathcal{I}_0$, and Ψ_h are bounded. A straight core connector to its approximating endpoint, with displacement

ΔB and duration τ, has action at most $C\|\Delta B\|_2^2/\tau$ and Fisher integral at most $C\tau$. For the j-th approximation take $\tau_j = \|\Delta B_j\|_2 + 1/j$; then $\tau_j \to 0$ and $\|\Delta B_j\|_2^2/\tau_j \to 0$. Its canonical embedding has the same bounds up to β_δ, with additional Fisher integral at most $C\delta\tau$. Entropy convexity preserves the constraint. If the two connectors have total duration $\tau_0 + \tau_T$, compress the middle curve linearly: its action is multiplied by $T/(T - \tau_0 - \tau_T)$ and its Fisher integral by $(T - \tau_0 - \tau_T)/T$. Both factors tend to one. The same diagonal choice gives exact prescribed faithful endpoints. □

6.3. **Prepared initial data.**

Corollary 6.6. *Let $x \in K_E^0$ be faithful and put $x_\delta = \iota_\delta x$. On curves with $\rho_0 = x_\delta$, define*

$$\mathscr{D}^x_{\delta,E}(\rho) = F_\delta(\rho_T) - F_\delta(x_\delta) + \tfrac{1}{2}\mathcal{A}_{\delta,E}(\rho) + \tfrac{1}{2}\int_0^T \mathcal{I}_\delta(\rho_t)\,dt, \tag{6.8}$$

and give this functional the value $+\infty$ on all other curves. Then, in the uniform trace-norm topology,

$$\mathscr{D}^x_{\delta,E} \xrightarrow{\Gamma} \mathscr{D}^x_{0,E}, \tag{6.9}$$

where

$$\mathscr{D}^x_{0,E}(B) = F_0(B_T) - F_0(x) + \tfrac{1}{2}\mathcal{A}_{0,E}(B) + \tfrac{1}{2}\int_0^T \mathcal{I}_0(B_t)\,dt$$

on core curves with $B_0 = x$, and $+\infty$ elsewhere. The sublevels are asymptotically relatively compact.

Proof. Equation (3.8) gives $F_\delta(x_\delta) = \beta_\delta F_0(x) \to F_0(x)$. For curves with finite functional value,

$$\mathscr{B}_{\delta,E}(\rho) = 2\big(\mathscr{D}^x_{\delta,E}(\rho) - F_\delta(\rho_T) + F_\delta(x_\delta)\big) \le 2\big(\mathscr{D}^x_{\delta,E}(\rho) + E\big).$$

Theorem C therefore gives compactness. For a uniformly convergent sequence with bounded functional values, its initial value tends to $x \oplus 0$ and $\liminf F_\delta(\rho_T) \ge F_0(B_T)$ by Corollary 3.3. The initial entropy converges exactly. Adding these bounds to the joint lower inequality proves the lower inequality in (6.9). For recovery, use the proof of Theorem C with its faithful initial endpoint fixed. Its terminal states have the form $\iota_\delta B_T^j$, with $B_T^j \to B_T$. Continuity of F_0 on all finite-dimensional states gives, for the same $j = j(\delta)$,

$$F_\delta(\iota_\delta B_T^{j(\delta)}) = \beta_\delta F_0(B_T^{j(\delta)}) \longrightarrow F_0(B_T).$$

This proves the upper inequality even when B_T is singular. □

Lemma 6.7. *For a core state B, one has $\mathcal{I}_0(B) < \infty$ if and only if $B > 0$, and $\mathcal{I}_0(B) = 0$ if and only if $B = \sigma_0$. Suppose $B : [0,T] \to K_E^0$ has finite $\mathscr{B}_{0,E}$ and ξ is an admissible core current of finite action. Then $F_0 \circ B$ is absolutely continuous and, for $0 \le s \le t \le T$,*

$$F_0(B_t) - F_0(B_s) = -\int_s^t \langle \zeta_{B_u}, \xi_u\rangle_{\mathbb{R}}\,du, \qquad \|\zeta_{B_u}\|^2 = \mathcal{I}_0(B_u) \quad \text{a.e.} \tag{6.10}$$

Proof. Let S be the support projection of a singular state B. In a spectral basis, the mobility of every matrix unit joining $S\mathcal{H}_0$ to $\ker B$ is zero. On an entry with $r_a = 0 < r_b$,

$$(J_j^{\mathrm{ent}}(B))_{ab} = e^{-\omega_j/2}(L_j)_{ab}\, r_b.$$

Finite inverse action forces $(L_j)_{ab} = 0$. On an entry with $r_b = 0 < r_a$, the same argument gives $(L_j)_{ab} = 0$ from the other term of J_j^{ent}. Thus S commutes with every L_j. Primitivity implies $S = I_0$, a contradiction. Conversely, all channel mobilities are positive definite at $B > 0$, so $\mathcal{I}_0(B) < \infty$. At such a state, (2.29) gives $\zeta_B = D_B^0(\log B - \log\sigma_0)$. Its vanishing is equivalent to $\log B - \log\sigma_0$ commuting with all channels, hence to its being scalar. Trace one then gives $B = \sigma_0$.

Finite core action implies $B \in W^{1,2}$ by Lemma 4.2 with $\chi = 0$. For $\eta > 0$, define

$$F_\eta(B) = \mathrm{Tr}((B + \eta I_0)\log(B + \eta I_0)) - \mathrm{Tr}(B\log\sigma_0), \qquad H_\eta(B) = \log(B + \eta I_0) - \log\sigma_0.$$

On trace-zero velocities, H_η is the differential of F_η. For almost every u, finite Fisher information gives $B_u > 0$. In a spectral basis of such a state, set

$$d_{ab} = \log r_b - \log r_a, \qquad d^\eta_{ab} = \log(r_b + \eta) - \log(r_a + \eta).$$

If $r_a = r_b$, both differences vanish. Otherwise,

$$d_{ab} = \int_{r_a}^{r_b} \frac{dx}{x}, \qquad d^\eta_{ab} = \int_{r_a}^{r_b} \frac{dx}{x+\eta},$$

so $d^\eta_{ab} = \theta d_{ab}$ for some $0 \le \theta \le 1$. Convexity of the square gives

$$|d^\eta_{ab} - \omega_j|^2 \le \theta |d_{ab} - \omega_j|^2 + (1-\theta)\omega_j^2 \le |d_{ab} - \omega_j|^2 + \omega_j^2. \tag{6.11}$$

Multiplying by the channel mobility coefficients and $|(L_j)_{ab}|^2$, then summing, yields

$$\|D^0_B H_\eta(B)\|^2 \le \mathcal{I}_0(B) + C_\omega, \qquad C_\omega = \sum_{j \in \mathcal{J}_0} \omega_j^2 e^{|\omega_j|/2} \|L_j\|_2^2. \tag{6.12}$$

Here the channel mobility coefficient is at most $e^{|\omega_j|/2}$, since $\|B\| \le 1$. Pointwise the gradients tend to ζ_B; their squared differences are bounded by $4\mathcal{I}_0(B) + 2C_\omega$. Dominated convergence therefore gives strong convergence in $L^2(0,T;\mathscr{H}_{\nabla,0})$, where $\mathscr{H}_{\nabla,0}$ denotes the finite-core current space.

Differentiate the continuity identities for a fixed real basis of M_d^{sa}. Outside the union of their finitely many exceptional null sets, linearity gives

$$\mathrm{Tr}(C\dot{B}_u) = -\langle D^0_{B_u} C, \xi_u \rangle_{\mathbb{R}} \qquad (C \in M_d^{\mathrm{sa}}).$$

At each such u take $C = H_\eta(B_u)$. The finite-dimensional Sobolev chain rule then gives

$$F_\eta(B_t) - F_\eta(B_s) = -\int_s^t \langle D^0_{B_u} H_\eta(B_u), \xi_u \rangle_{\mathbb{R}} \, du.$$

The scalar functions $(r+\eta)\log(r+\eta)$ converge uniformly to $r \log r$ on $[0,1]$. Thus $F_\eta \to F_0$ uniformly on the core state space. Strong gradient convergence passes the right side to (6.10). Its integrand is in L^1, proving absolute continuity of $F_0 \circ B$. □

Proposition 6.8. *The functional $\mathscr{D}^x_{0,E}$ is nonnegative, and its unique zero is $B_t = e^{t\mathcal{K}_*}x$. Furthermore,*

$$\inf \mathscr{D}^x_{\delta,E} \longrightarrow 0.$$

If $\mathscr{D}^x_{\delta,E}(\rho^\delta) - \inf \mathscr{D}^x_{\delta,E} \to 0$, then $\rho^\delta \to (e^{t\mathcal{K}_*}x)_{0 \le t \le T}$ uniformly in trace norm.

Proof. Let $\mathscr{D}^x_{0,E}(B) < \infty$ and choose the least core current ξ. Lemma 6.7 gives (6.10); its force has squared norm $\mathcal{I}_0(B_t)$. Consequently,

$$\mathscr{D}^x_{0,E}(B) = \tfrac{1}{2} \int_0^T \|\xi_t - \zeta_{B_t}\|^2 \, dt \ge 0. \tag{6.13}$$

Equality implies $\xi = \zeta_B$ almost everywhere, hence $\dot{B} = \mathcal{K}_* B$ and $B_0 = x$. The solution is $e^{t\mathcal{K}_*}x$. Conversely, stationarity and positivity of the core semigroup imply $e^{t\mathcal{K}_*}x \ge a\sigma_0$ whenever $x \ge a\sigma_0$. The entropy identity (2.29) keeps this faithful trajectory in K^0_E and gives zero functional.

The recovery of this zero in Corollary 6.6 gives $\limsup_\delta \inf \mathscr{D}^x_{\delta,E} \le 0$. Each functional is bounded below by $-E$. For any sequence $\delta_j \downarrow 0$ attaining the lower limit of the infima, choose curves ρ^j with

$$\mathscr{D}^x_{\delta_j,E}(\rho^j) \le \inf \mathscr{D}^x_{\delta_j,E} + 1/j.$$

Their values are bounded above by the preceding recovery bound. Corollary 6.6 gives a uniformly convergent subsequence, with limit B, and

$$\liminf_j \inf \mathscr{D}^x_{\delta_j,E} \ge \mathscr{D}^x_{0,E}(B) \ge 0.$$

Thus the infima converge to zero, and every cluster point of almost minimizers is the unique zero just identified. The entire family converges uniformly in trace norm. □

Remark 6.9. Equation (6.9) concerns the current action and the channel Fisher information (2.21). It does not identify current action with metric energy at positive coupling. The initial states are the fixed canonical embeddings of a faithful core state. No tilt of the entropy is included, and no simultaneous coupling and time-step limit is asserted.

6.4. **The Fisher cost of kinetic recovery.** For a fixed profile, Theorems A and C give

$$\Gamma\text{-}\lim_{\delta\downarrow 0} \mathcal{A}_{\delta,E} = \mathcal{A}^{\mathrm{eff}}_{\chi,E},$$

$$\Gamma\text{-}\lim_{\delta\downarrow 0} \left(\mathcal{A}_{\delta,E} + \int_0^T \mathcal{I}_\delta \, dt \right) = \mathcal{A}_{0,E} + \int_0^T \mathcal{I}_0 \, dt.$$

The second limit is independent of χ.

Proposition 6.10. *Let $B : [0,T] \to K^0_E$ be a C^2 faithful core curve with $\sup_t F_0(B_t) < E$, and suppose*

$$G = \mathcal{A}_{0,E}(B) - \mathcal{A}^{\mathrm{eff}}_{\chi,E}(B) > 0.$$

Choose $C(t) = \mathbb{K}^{-1}_{B_t} \dot{B}_t$ modulo the scalar making $C_{hh}(t) = 0$, and put $M_C = \max_t \|C(t)e_h\|^2$. Then $M_C > 0$. Every family converging uniformly in trace norm to $B \oplus 0$ and satisfying $\mathcal{A}_{\delta,E}(\rho^\delta) \to \mathcal{A}^{\mathrm{eff}}_{\chi,E}(B)$ obeys

$$\liminf_{\delta\downarrow 0} \frac{\int_0^T \mathcal{I}_\delta(\rho^\delta_t) \, dt}{\log^2(1/\delta)} \geq \frac{G}{4M_C} > 0. \tag{6.14}$$

Proof. The field C is C^1, and

$$\mathbb{K}_{B_t} C(t) = \dot{B}_t, \qquad 2\,\mathrm{Tr}(C(t)\dot{B}_t) - g^0_{B_t}(C(t)) = \mathrm{Tr}(\dot{B}_t \mathbb{K}^{-1}_{B_t} \dot{B}_t).$$

If $M_C = 0$, the additional term of $\widehat{g}_{B_t,E,\chi}(C(t))$ vanishes for every t. Testing the effective dual action with $C(t)$ gives $a_{\chi,E}(B_t, \dot{B}_t) \geq a_{0,E}(B_t, \dot{B}_t)$. The reverse inequality follows from $\widehat{g} \geq g^0$. Integration would give $G = 0$. Thus $M_C > 0$.

Lemma 2.8 and fixed-channel continuity give

$$\mathcal{A}_{\delta,E}(\rho^\delta) \geq \mathcal{A}_{0,E}(B) - o(1) - \int_0^T g^\delta_{\rho^\delta_t,\mathrm{aux}}(C(t)) \, dt.$$

For $L = \log(1/\delta)$, (6.3) yields

$$\mathcal{A}_{\delta,E}(\rho^\delta) \geq \mathcal{A}_{0,E}(B) - o(1) - \frac{4M_C}{L^2} \int_0^T \mathcal{I}_\delta(\rho^\delta_t) \, dt - TM_C C_\varphi \sqrt{\delta}.$$

Along couplings for which the Fisher integral is finite, rearrangement and convergence of the actions give (6.14). At infinite Fisher values the inequality holds in the extended sense. □

7. Uniform convergence in diamond norm

In this section let $b_n \geq 1$ and let $a_n > 0$ satisfy $\sum_n a_n = \varepsilon$. No relation between the two sequences is required. The generator is constructed from (2.7) and the bounded core and outward terms. The comparison generator $\mathcal{L}_{0,*}$ retains the same core and all the same returns.

Theorem 7.1. *Fix the primitive GNS-symmetric core and e_h. There are constants $\varepsilon_0, C > 0$ such that, for every choice of return rates $b_n \geq 1$ and outward rates $a_n > 0$ with $\sum_n a_n = \varepsilon$,*

$$\|\mathcal{L}_{\varepsilon,*} - \mathcal{L}_{0,*}\|_\diamond = 2\varepsilon, \qquad \sup_{t\geq 0} \|S^\varepsilon_t - S^0_t\|_\diamond \leq C\varepsilon \quad (0 < \varepsilon \leq \varepsilon_0). \tag{7.1}$$

The constants are independent of both rate sequences subject to these conditions.

For $\eta \geq 0$ and $t \geq 0$, write

$$\mathcal{K}^{\mathrm{kill}}_{\eta,*} = \mathcal{K}_* - \frac{\eta}{2}\{P, \cdot\}, \qquad N_{\eta,t} = e^{-t(G_0+\eta P)/2},$$

$$U_{\eta,t} = e^{t\mathcal{K}^{\mathrm{kill}}_{\eta,*}}, \qquad C_{\eta,t} = U_{\eta,t} - \mathrm{Ad}_{N_{\eta,t}}.$$

Lemma 7.2 (Finite Dyson subchannels). *Fix $R \geq 1$ and put $\mathcal{H}_R = \mathcal{H}_0 \oplus \mathrm{span}\{e_{\nu_1}, \dots, e_{\nu_R}\}$ and $\varepsilon_R = \sum_{n\leq R} a_n$. Let $S^{\varepsilon,R}_t$ be the finite-dimensional semigroup obtained by retaining the core and precisely the first R outward/return pairs. Fix $T_* > 1$ and define on $\mathfrak{S}_1(\mathcal{H}_R)$*

$$\mathcal{U}_R(X) = C_{\varepsilon_R,T_*}(P_0XP_0),$$
$$\mathcal{V}_R(X) = \sum_{n\leq R} X_{\nu_n\nu_n} \int_0^1 b_n e^{-b_n u} U_{\varepsilon_R,T_*-u}(P)\, du.$$

Then $\mathcal{U}_R$ and $\mathcal{V}_R$ are completely positive and

$$S^{\varepsilon,R}_{T_*} - \mathcal{U}_R - \mathcal{V}_R \geq_{\mathrm{CP}} 0. \tag{7.2}$$

Proof. Put

$$H_R = G_0 + \varepsilon_R P + \sum_{n\leq R} b_n E_{\nu_n\nu_n}, \qquad \mathcal{N}_t = \mathrm{Ad}_{e^{-tH_R/2}}.$$

Let $\mathcal{J}_\alpha$ run over the jump maps $2e^{-\omega_j/2}\,\mathrm{Ad}_{L_j}$, $a_n\,\mathrm{Ad}_{E_{\nu_n h}}$, and $b_n\,\mathrm{Ad}_{E_{h\nu_n}}$, with $j \in \mathcal{J}_0$ and $n \leq R$. The Dyson expansion is

$$S^{\varepsilon,R}_t = \sum_{q=0}^{\infty} \sum_{\alpha_1,\dots,\alpha_q} \int_{0<t_1<\dots<t_q<t} \mathcal{N}_{t-t_q}\mathcal{J}_{\alpha_q}\mathcal{N}_{t_q-t_{q-1}} \cdots \mathcal{J}_{\alpha_1}\mathcal{N}_{t_1}\, dt_1 \cdots dt_q,$$

where the term $q = 0$ is $\mathcal{N}_t$ and the integrand for $q = 1$ is $\mathcal{N}_{t-t_1}\mathcal{J}_{\alpha_1}\mathcal{N}_{t_1}$. Since $\|\mathcal{N}_t\| \leq 1$, the sum of the norms of the terms of order q is at most $t^q(\sum_\alpha \|\mathcal{J}_\alpha\|)^q/q!$. Thus the series is absolutely norm convergent, and every integrand is completely positive.

The sum over nonempty sequences consisting only of core jumps is

$$\big(U_{\varepsilon_R,T_*} - \mathrm{Ad}_{N_{\varepsilon_R,T_*}}\big)(P_0XP_0) = \mathcal{U}_R(X).$$

For a first jump $\nu_n \to h$ at $u \in [0,1]$, followed only by core jumps, the sum is

$$X_{\nu_n\nu_n} \int_0^1 b_n e^{-b_n u} U_{\varepsilon_R,T_*-u}(P)\, du.$$

Summing over $n \leq R$ gives $\mathcal{V}_R$. The two index sets are disjoint. Their complement in the Dyson expansion is a norm-convergent sum of completely positive terms. This proves (7.2), as well as complete positivity of $\mathcal{U}_R$ and $\mathcal{V}_R$. □

Lemma 7.3 (Completely positive order under truncation). *Let $P_R \uparrow I$ be finite-coordinate projections on a separable Hilbert space. Suppose that Φ_R, Ψ_R are uniformly bounded maps on $\mathfrak{S}_1(P_R\mathcal{H})$, with $\Phi_R - \Psi_R$ completely positive, and identify their outputs with operators on $\mathcal{H}$ by the natural inclusion. Assume that there are bounded maps Φ, Ψ on $\mathfrak{S}_1(\mathcal{H})$ such that, for every finite ancilla dimension m and every trace-class X supported in one finite coordinate compression,*

$$(\Phi_R \otimes \mathrm{id}_m)X \longrightarrow (\Phi \otimes \mathrm{id}_m)X, \qquad (\Psi_R \otimes \mathrm{id}_m)X \longrightarrow (\Psi \otimes \mathrm{id}_m)X$$

in trace norm once R contains that compression. Then $\Phi - \Psi$ is completely positive.

Proof. Fix m. For positive finite-coordinate X,

$$((\Phi - \Psi) \otimes \mathrm{id}_m)X = \lim_R((\Phi_R - \Psi_R) \otimes \mathrm{id}_m)X \geq 0,$$

since the positive cone is trace-norm closed. For arbitrary $X \geq 0$, its compressions $(P_R \otimes I_m)X(P_R \otimes I_m)$ converge in trace norm to X. Writing $X = (X_{ab})_{a,b=1}^m$ and $A = \Phi - \Psi$, one has

$$\|(A \otimes \mathrm{id}_m)X\|_1 \leq \sum_{a,b} \|A(X_{ab})\|_1 \leq m^2 \|A\| \, \|X\|_1.$$

Thus this fixed amplification is bounded and its value at X is the trace-norm limit of positive operators. This holds for every m. □

Proof of Theorem 7.1. On the core, the jump map is

$$\mathcal{J}_*(X) = 2 \sum_{j \in \mathcal{J}_0} e^{-\omega_j/2} L_j X L_j^*.$$

The Dyson expansion of $U_{\varepsilon,t}$ about $\mathrm{Ad}_{N_{\varepsilon,t}}$ has completely positive terms. Consequently $C_{\varepsilon,t} \geq_{\mathrm{CP}} 0$. Moreover $U_{\varepsilon,t}$ is trace nonincreasing, since $\mathrm{Tr}(\mathcal{K}_{\varepsilon,*}^{\mathrm{kill}} X) = -\varepsilon \, \mathrm{Tr}(PX)$ for $X \geq 0$.

Primitivity gives $U_{0,t} \to R_{\sigma_0}$ in norm, where $R_{\sigma_0}(X) = \sigma_0 \, \mathrm{Tr}\, X$. Since $G_0 > 0$, $N_{0,t} \to 0$. The Choi matrix of R_{σ_0} is $I_d \otimes \sigma_0 > 0$. Thus [4] gives $t_0 > 0$ such that $C_{0,t} \geq_{\mathrm{CP}} \frac{1}{2} R_{\sigma_0}$ for $t \geq t_0$. Uniform continuity in ε on $[t_0, t_0 + 1]$ gives $\varepsilon_0 > 0$ with

$$C_{\varepsilon,t} \geq_{\mathrm{CP}} a R_{\sigma_0}, \qquad a = \tfrac{1}{4}, \qquad t_0 \leq t \leq t_0 + 1, \quad 0 \leq \varepsilon \leq \varepsilon_0. \tag{7.3}$$

Put $T_* = t_0 + 1$ and

$$A_n = \int_0^1 b_n e^{-b_n u} U_{\varepsilon, T_* - u}(P) \, du.$$

Then $A_n \geq 0$ and $\mathrm{Tr}\, A_n \leq 1$. Define

$$\mathcal{U}(X) = C_{\varepsilon, T_*}(P_0 X P_0), \qquad \mathcal{V}(X) = \sum_n X_{\nu_n \nu_n} A_n.$$

The second series converges absolutely in trace norm. After amplification, diagonal compression gives $\sum_n \|X_{\nu_n \nu_n}\|_1 \leq \|X\|_1$; hence the amplified series also converges absolutely. Its summands are completely positive, so $\mathcal{V}$ is a completely positive contraction.

For inputs supported in a fixed finite coordinate compression, the truncation argument of Proposition 2.2 gives $S_{T_*}^{\varepsilon,R} X \to S_{T_*}^{\varepsilon} X$. That argument uses only $\sum_n a_n < \infty$. Since $\varepsilon_R \to \varepsilon$, $\mathcal{U}_R X \to \mathcal{U} X$. Only finitely many diagonal entries contribute to $\mathcal{V}_R X$, and $U_{\varepsilon_R, T_* - u} \to U_{\varepsilon, T_* - u}$ uniformly for $u \in [0, 1]$. Thus $\mathcal{V}_R X \to \mathcal{V} X$. These convergences hold at every finite amplification. Lemmas 7.2 and 7.3 imply

$$S_{T_*}^{\varepsilon} - \mathcal{U} - \mathcal{V} \geq_{\mathrm{CP}} 0. \tag{7.4}$$

By (7.3), $\mathcal{U} \geq_{\mathrm{CP}} a\sigma_0 \, \mathrm{Tr}(P_0 \, \cdot)$. For $0 \leq u \leq 1$, $U_{\varepsilon, T_* - u}(P) \geq a\sigma_0$, and $\int_0^1 b_n e^{-b_n u} \, du \geq 1 - e^{-1}$ because $b_n \geq 1$. Hence $A_n \geq a(1 - e^{-1})\sigma_0$ for every n. It follows that

$$S_{T_*}^{\varepsilon} \geq_{\mathrm{CP}} d_* R_{\sigma_0}, \qquad d_* = a(1 - e^{-1}) \in (0, 1), \tag{7.5}$$

with R_{σ_0} now defined on the full trace class and taking values in the core. The map $(S_{T_*}^{\varepsilon} - d_* R_{\sigma_0})/(1 - d_*)$ is a channel. For every finite ancilla and every X with zero system partial trace,

$$\|(S_t^{\varepsilon} \otimes \mathrm{id})X\|_1 \leq (1 - d_*)^{\lfloor t/T_* \rfloor} \|X\|_1.$$

Indeed,

$$(R_{\sigma_0} \otimes \mathrm{id}_m)X = \sigma_0 \otimes \mathrm{Tr}_{\mathcal{H}} X = 0, \qquad \mathrm{Tr}_{\mathcal{H}}((S_t^{\varepsilon} \otimes \mathrm{id}_m)X) = \mathrm{Tr}_{\mathcal{H}} X.$$

The channel decomposition in (7.5) gives the contraction factor $1 - d_*$ at time T_*. Iteration and contractivity on the remaining interval give the estimate for every $t \geq 0$.

The outward perturbation has diamond norm 2ε, with equality on P. Its amplified image has zero system partial trace. Duhamel's formula therefore gives

$$\|S_t^{\varepsilon} - S_t^0\|_{\diamond} \leq 2\varepsilon \int_0^t (1 - d_*)^{\lfloor (t-s)/T_* \rfloor} \, ds \leq \frac{2T_*}{d_*} \varepsilon.$$

The constants T_*, d_*, ε_0 depend only on the core and e_h. □

7.1. **A common semigroup limit.**

Proof of Theorem B. The returns in (1.20) satisfy $b_n = n^2 \geq 1$, and the outward rates sum to ε. Theorem 7.1 gives (1.21) with constants independent of k. For each fixed k, set $\delta = \varepsilon/A_k$. If $k > 0$, then $b_n/(\varphi_n^{(k)})^2 = k$; if $k = 0$, this ratio is $1/n^2$. The hypotheses of Theorem A hold for each profile, with $\chi = k$. Its conclusion gives (1.22). Corollary 3.3 gives the common limiting energy and the state-and-entropy recovery.

Since $b_n \geq 1$, the equilibrium mass outside the core is bounded by $\pi_h \sum_n a_n^{\varepsilon,k}/b_n \leq \pi_h\varepsilon$. For every core state B, its canonical embedding therefore satisfies

$$\|\iota_{\varepsilon,k}B - (B \oplus 0)\|_1 \leq 2\pi_h\varepsilon,$$

and

$$\sup_{k\geq 0}\sup_{t\geq 0} \|S_t^{\varepsilon,k}\iota_{\varepsilon,k}B - (e^{t\mathcal{K}_*}B \oplus 0)\|_1 \leq (C + 2\pi_h)\varepsilon.$$

The return-only generator in this comparison is the same for all k.

Fix $B > 0$ with $F_0(B) < E$. For $k_2 > k_1$, the two effective Onsager operators differ by the nonzero positive operator $(k_2 - k_1)(E - F_0(B))\mathbb{R}_h$. The inverses of two distinct positive definite operators are distinct. Their difference is self-adjoint and nonzero, so its quadratic form is nonzero on some trace-zero Hermitian velocity. Lemma 5.3 identifies this quadratic form with the difference of the two local squared-distance coefficients. Thus the limiting metrics are distinct. Such states exist in every positive cap, since $F_0(\sigma_0) = 0$. □

The variational limit holds for each fixed k. The estimate (1.21) is uniform for $k \geq 0$. The common stationary limit $\sigma_0 \oplus 0$ is faithful on the core and has kernel $\mathcal{H}_0^\perp$ on the enlarged space.

7.2. **A two-level formula.** For

$$\mathcal{K}_* = \mathcal{D}_{E_{10}} + \mathcal{D}_{E_{01}}, \qquad \sigma_0 = I_2/2, \qquad X = E_{01} + E_{10},$$

put $B_z = \sigma_0 + zX$ for $z \in (-1/2, 1/2)$, and

$$b(z) = \tfrac{1}{2}\log\frac{1/2 + z}{1/2 - z}, \qquad \mu_0(z) = \frac{z}{2b(z)} \quad (z \neq 0), \qquad \mu_0(0) = \tfrac{1}{4}.$$

Proposition 7.4. *For every $k \geq 0$, $E > 0$, and $z \in (-1/2, 1/2)$ with $F_0(B_z) < E$,*

$$\lim_{s\to 0}\frac{d_{k,E}(B_{z+s}, B_z)^2}{s^2} = \frac{1}{\mu_0(z) + k(E - F_0(B_z))/4}. \tag{7.6}$$

In particular,

$$\lim_{z\to 0}\frac{d_{k,E}(B_z, \sigma_0)^2}{z^2} = \frac{4}{1 + kE}. \tag{7.7}$$

Proof. For $z \neq 0$, $\log B_z - \log\sigma_0$ equals $b(z)X$ modulo a scalar, and $\mathcal{K}_*B_z = -zX$. Equation (2.29) therefore gives $\mathbb{K}_{B_z}X = zX/b(z)$. Also $\mathbb{R}_hX = X/2$ and $\operatorname{Tr}X^2 = 2$. Substitution in Lemma 5.3 gives (7.6). Continuity of $\mathbb{K}_B$ and $z/b(z) \to 1/2$ give $\mathbb{K}_{\sigma_0}X = X/2$. Applying Lemma 5.3 directly at $B = \sigma_0$, with velocity X, proves (7.7). □

The distances in (7.6)–(7.7) are the full effective distances, not distances restricted to the coherent line. The metric differentiation is taken after the weak-coupling limit.

7.3. **The limiting finite-dimensional entropy equation.**

Proposition 7.5. *Let $k \geq 0$, $E > 0$, and let B_0 be a faithful core state with $F_0(B_0) < E$. The relative-entropy gradient equation for $d_{k,E}$ is*

$$\dot{B} = \mathcal{K}_*B - \frac{k(E - F_0(B))}{2}[P, [P, \log B]], \qquad B(0) = B_0. \tag{7.8}$$

It has a unique solution for all $t \geq 0$, with $B_t > 0$ and $F_0(B_t) < E$. It satisfies

$$\frac{d}{dt}F_0(B_t) = -\mathcal{I}_0(B_t) - k(E - F_0(B_t))\|Q_0 \log B_t e_h\|^2, \quad \mathcal{I}_0(B) = g_B^0(\log B - \log \sigma_0). \tag{7.9}$$

Proof. The entropy differential on trace-zero velocities is represented by $\log B - \log \sigma_0$. Since P commutes with σ_0,

$$\mathbb{R}_h(\log B - \log \sigma_0) = \tfrac{1}{2}[P, [P, \log B]].$$

Equation (2.29) and the inverse tensor in (1.18) give (7.8). Pairing its velocity with the entropy differential gives (7.9). The vector field is smooth on faithful states below the cap, hence has a unique local solution.

Let v be a unit eigenvector for the least eigenvalue λ of B. Then

$$\left\langle v, \frac{P \log BQ_0 + Q_0 \log BP}{2} v \right\rangle = \log \lambda \, \|Pv\|^2 - \langle Pv, \log B \, Pv\rangle \leq 0,$$

because $\log B \geq \log \lambda \, I_0$. The additional term in (7.8) cannot decrease the least eigenvalue. In (2.3), the positive jump term has nonnegative expectation at v, while the loss has expectation $-\lambda\langle v, G_0 v\rangle$. The variational formula for the least eigenvalue gives its lower right Dini derivative as the minimum of $\langle v, \dot{B}_t v\rangle$ over unit vectors in the least eigenspace. The preceding bounds hold for every such vector. Hence this derivative is at least $-\|G_0\|\lambda_{\min}(B_t)$, and

$$\lambda_{\min}(B_t) \geq e^{-t\|G_0\|}\lambda_{\min}(B_0).$$

Trace is preserved, and (7.9) keeps the entropy strictly below E. On every bounded time interval the solution remains in a compact subset of the domain of the vector field. The local solution consequently extends to all $t \geq 0$. □

For the coherent two-level family, (7.8) reduces to

$$\dot{z} = -z - \frac{k}{2}(E - F_0(B_z))b(z),$$

whereas the limiting physical semigroup satisfies $\dot{z} = -z$. Equation (7.8) is the entropy flow of the limiting finite-dimensional metric. Corollary 5.2 gives convergence at a fixed proximal step, rather than a joint coupling–time-step limit.

8. Entropy dissipation and metric speed

Assume in this section that

$$a_n = \varepsilon n^2 e^{-n}, \qquad b_n = n^2, \qquad A = \sum_n n^2 e^{-n}, \qquad Z_\varepsilon = 1 + \varepsilon \pi_h \sum_n e^{-n}. \tag{8.1}$$

Use the notation of Section 2 with $\delta = \varepsilon$, and write $\mathscr{T} = \mathscr{T}_\varepsilon$. Here ε is the profile prefactor; the total outward rate is εA, not the parameter denoted by ε in Theorem 7.1. Superscripts on the gradient and mobility are omitted when ε is fixed. The auxiliary channels are $L_{n,+} = \ell_n E_{v_n h}$ and $L_{n,-} = L_{n,+}^*$, where $\ell_n = (\varepsilon n^4 e^{-n})^{1/4}/\sqrt{2}$.

The Fisher information $\mathcal{I}_\varepsilon$ and its least-norm entropy-current representative ζ_ρ are those of (2.21), with $\delta = \varepsilon$.

Theorem 8.1. *For every finite primitive GNS-symmetric core, every $E > 0$, and every $\varepsilon > 0$, there are a faithful state x_ε with $F_\varepsilon(x_\varepsilon) < E$ and constants $T_0, \Delta > 0$ such that the trajectory $\rho_t = S_t^\varepsilon x_\varepsilon$ satisfies*

$$|\dot{\rho}_t|^2_{\mathcal{W}^{\mathscr{T}_\varepsilon}_{\varepsilon,E}} \leq \mathcal{I}_\varepsilon(\rho_t) - \Delta < \mathcal{I}_\varepsilon(\rho_t) = -\frac{d}{dt}F_\varepsilon(\rho_t) \tag{8.2}$$

for almost every $t \in (0, T_0)$ and every test space (1.11). *This trajectory is not a 2-curve of maximal slope for F_ε with respect to this metric and any strong upper gradient.*

For each $\varepsilon_0 > 0$, the states x_ε may be obtained from one fixed core perturbation, with T_0 and Δ independent of $0 < \varepsilon \leq \varepsilon_0$ and of $\mathscr{T}_\varepsilon$.

8.1. **Entropy identities and metric speed.** Let d be an extended metric, and work in a finite-distance component. A 2-curve of maximal slope for F with respect to a strong upper gradient G is locally absolutely continuous and satisfies, on every compact subinterval [1],

$$F(\rho_t) + \frac{1}{2}\int_s^t |\dot\rho_u|_d^2 du + \frac{1}{2}\int_s^t G(\rho_u)^2 du \le F(\rho_s). \tag{8.3}$$

The strong upper-gradient inequality is

$$|F(\rho_t) - F(\rho_s)| \le \int_s^t G(\rho_u)|\dot\rho_u|_d\, du. \tag{8.4}$$

Lemma 8.2. *Let ρ be a locally absolutely continuous metric curve. Suppose that $F \circ \rho$ is continuously differentiable and nonincreasing, and that* (8.3) *and* (8.4) *hold on every compact subinterval. Then*

$$|\dot\rho_t|_d^2 = -\frac{d}{dt}F(\rho_t) \qquad \text{for almost every } t.$$

Proof. For every compact subinterval,

$$\frac{1}{2}\int_s^t \left(|\dot\rho_u|_d^2 + G(\rho_u)^2\right) du \le F(\rho_s) - F(\rho_t) \le \int_s^t G(\rho_u)|\dot\rho_u|_d\, du \le \frac{1}{2}\int_s^t \left(|\dot\rho_u|_d^2 + G(\rho_u)^2\right) du.$$

The first inequality makes both squared integrals finite. Subtracting the middle integral therefore gives $\int_s^t (G(\rho_u) - |\dot\rho_u|_d)^2\, du = 0$ and $F(\rho_s) - F(\rho_t) = \int_s^t |\dot\rho_u|_d^2\, du$. Lebesgue differentiation proves the assertion. □

8.2. **A semigroup trajectory.** Choose $i \neq h$ and a nonzero real z such that

$$B_0 = \sigma_0 + z(E_{hi} + E_{ih}) > 0, \qquad D(B_0\|\sigma_0) < E. \tag{8.5}$$

Put

$$x_\varepsilon = Z_\varepsilon^{-1}\left(B_0 + \varepsilon\pi_h \sum_n e^{-n} E_{\nu_n\nu_n}\right), \qquad \rho_t = S_t^\varepsilon x_\varepsilon. \tag{8.6}$$

Then

$$F_\varepsilon(x_\varepsilon) = Z_\varepsilon^{-1} D(B_0\|\sigma_0) < E. \tag{8.7}$$

Lemma 8.3. *The trajectory* (8.6) *is continuously differentiable in trace norm on bounded time intervals. It has the block form*

$$\rho_t = B(t) \oplus \operatorname{diag}(p_n(t)), \tag{8.8}$$

where

$$\dot B = \mathcal{K}_* B - \frac{\varepsilon A}{2}\{P, B\} + P\sum_n b_n p_n, \tag{8.9}$$

$$\dot p_n = a_n B_{hh} - b_n p_n. \tag{8.10}$$

There are constants $0 < \underline{c} < 1 < \overline{c}$, independent of ε, such that

$$\underline{c}\sigma_\varepsilon \le \rho_t \le \overline{c}\sigma_\varepsilon \qquad (t \ge 0). \tag{8.11}$$

On every bounded time interval,

$$-\frac{d}{dt}F_\varepsilon(\rho_t) = \|\zeta_t\|^2 = \mathcal{I}_\varepsilon(\rho_t), \qquad \zeta_t = D_{\rho_t}H_t, \quad H_t = \log\rho_t - \log\sigma_\varepsilon. \tag{8.12}$$

The state-current pair (ρ, ζ) is admissible for every test space (1.11)*. Moreover, for $0 \le s < t < \infty$ and every allowed test space, its reduced current action is*

$$\mathcal{A}_{\varepsilon,E}(\rho|_{[s,t]}) = \int_s^t \mathcal{I}_\varepsilon(\rho_u)\, du. \tag{8.13}$$

Proof. The core block of x_ε is finite-dimensional and $\sum_n b_n(x_\varepsilon)_{v_nv_n} < \infty$. Differentiation of (2.7) in trace norm therefore gives $x_\varepsilon \in \operatorname{Dom}\mathcal{G}_* = \operatorname{Dom}\mathcal{L}_{\varepsilon,*}$. The block space in (8.8) is invariant under the return semigroup and both bounded perturbations. Multiplication gives (8.9)–(8.10). Choose $\underline{c}, \overline{c}$ so that $\underline{c}\sigma_0 \le B_0 \le \overline{c}\sigma_0$. The initial state satisfies (8.11). Positivity of S_t^ε and stationarity of σ_ε preserve these inequalities.

Write $\pi_n^\varepsilon = \varepsilon\pi_h e^{-n}/Z_\varepsilon$. Then

$$\underline{c}\pi_n^\varepsilon \le p_n(t) \le \overline{c}\pi_n^\varepsilon, \qquad \sum_n b_n p_n(t) \le \overline{c}\frac{\varepsilon\pi_h A}{Z_\varepsilon}.$$

Equations (8.9)–(8.10) imply $|\dot p_n(t)| \le Ca_n$ on bounded intervals. The derivative series converges uniformly in ℓ^1. The core is positive, and

$$H_t = \big(\log B(t) - \log(Z_\varepsilon^{-1}\sigma_0)\big) \oplus \operatorname{diag}\big(\log(p_n(t)/\pi_n^\varepsilon)\big)$$

defines a bounded Hermitian operator, uniformly on compact time intervals. It is the bounded extension of the logarithmic difference on the algebraic sum of the core and the auxiliary eigenspaces. Its core block and each auxiliary coefficient are continuous.

The direct-sum formula for relative entropy is the finite-core term plus $\sum_n p_n \log(p_n/\pi_n^\varepsilon)$. The latter summands are bounded by $C\pi_n^\varepsilon$, and their derivatives by Ca_n because $|\dot p_n| \le Ca_n$ and $|\log(p_n/\pi_n^\varepsilon)| \le C$. Both series converge uniformly. The finite-core term is continuously differentiable on the positive compact core family. Since $\operatorname{Tr}\dot B + \sum_n \dot p_n = 0$,

$$\frac{d}{dt}F_\varepsilon(\rho_t) = \operatorname{Tr}(H_t\dot\rho_t). \tag{8.14}$$

For every channel, differentiation of $e^{(u-1/2)\omega_j}\rho^u L_j\rho^{1-u}$ in u gives

$$\mathsf{M}_{\rho,j}[L_j, \log\rho - \log\sigma_\varepsilon] = J_j^{\mathrm{ent}}(\rho). \tag{8.15}$$

At the block states in (8.8), this is a finite-dimensional identity on the core and the individual active leaf.

Let $H_t^0 = P_0H_tP_0$, $d_n(t) = (H_t)_{v_nv_n}$ and $c_h = \pi_h/Z_\varepsilon$. At equilibrium the paired contribution of edge hv_n is

$$a_n\left\langle (H_t^0 - d_n(t)I)e_h, \Lambda(c_h, Z_\varepsilon^{-1}\sigma_0)(H_t^0 - d_n(t)I)e_h\right\rangle \le Ca_n.$$

This follows from (3.6) and homogeneity of the mean. Order comparison gives the same bound at ρ_t. Hence

$$g_{\rho_t}(H_t) < \infty, \qquad \sum_{n>R, s=\pm} \|\mathsf{M}_{\rho_t,n,s}^{1/2}[L_{n,s}, H_t]\|_2^2 \le C\sum_{n>R} a_n \longrightarrow 0$$

uniformly in time. Formula (8.15) identifies its squared norm with (2.21).

The negative divergence of the entropy current on one auxiliary pair is

$$a_n\left(B_{hh}E_{v_nv_n} - \tfrac12\{P, B\}\right) + b_np_n(P - E_{v_nv_n}).$$

Its trace norm is at most $C(a_n + b_np_n) \le C'a_n$. The sum of the negative divergences therefore converges in trace norm to $\dot\rho_t$, as given by (8.9)–(8.10). Each finite channel component of ζ_t is continuous. The uniform channel-tail bound above gives $\zeta \in C([0,T]; \mathscr{H}_\nabla)$ on every bounded interval. In particular, $\zeta \in L^2$ there. Equations (2.25) and (8.14) prove (8.12). Lemma 2.7 gives the remaining test conditions for every allowed $\mathscr{T}_\varepsilon$.

Let H_t^R retain H_t^0 and the first R tail diagonal entries. The difference $H_t - H_t^R$ is diagonal on the remaining tail and bounded by a constant independent of R, t. Its gradient norm is at most $C\sum_{n>R} a_n$ in square, by the same equilibrium calculation and order comparison. $D_{\rho_t}H_t^R \to \zeta_t$ in $\mathscr{H}_\nabla$. Thus $\zeta_t \in \overline{D_{\rho_t}\mathscr{T}_{\mathrm{fin}}}$. Fix a real matrix-unit basis on each finite coordinate compression. The union of these bases is countable and independent of t. For any admissible current ξ, differentiating its continuity identities for these tests and using linearity gives a single null set outside which

$$\langle D_{\rho_t}C, \xi_t - \zeta_t\rangle_{\mathbb{R}} = 0 \qquad (C \in \mathscr{T}_{\mathrm{fin}}).$$

Every finite Hermitian test is a finite real linear combination of these basis elements. Closure of the gradient range then gives $\langle \zeta_t, \xi_t - \zeta_t\rangle_{\mathbb{R}} = 0$ almost everywhere. Hence

$$\|\xi_t\|^2 = \|\zeta_t\|^2 + \|\xi_t - \zeta_t\|^2.$$

This proves the least-norm assertion. Integrating the identity for any admissible current, and using the admissible entropy current for the opposite inequality, proves (8.13). □

For the same initial states, Theorem 7.1 gives constants $C, \varepsilon_* > 0$ such that

$$\sup_{t\ge 0} \|S_t^\varepsilon x_\varepsilon - e^{t\mathcal{K}_*} B_0\|_1 \le C\varepsilon A + 2(1 - Z_\varepsilon^{-1}) \qquad (0 < \varepsilon \le \varepsilon_*). \tag{8.16}$$

Indeed, the total outward rate is εA, $\|x_\varepsilon - (B_0 \oplus 0)\|_1 = 2(1 - Z_\varepsilon^{-1})$, and $S_t^0(B_0 \oplus 0) = e^{t\mathcal{K}_*} B_0 \oplus 0$.

8.3. **A finite-core correction current.** Let $Q_0 = P_0 - P$, and set

$$\ell(t) = Q_0 \log B(t) e_h. \tag{8.17}$$

It is nonzero at $t = 0$: its i-th entry is z times the strictly positive divided difference of the logarithm on the two eigenvalues of the hi block of B_0. Multiplication by Z_ε^{-1} adds only a scalar to the core logarithm and does not change this entry.

Let $\mathbb{K}_B$ be the finite-core Onsager operator in (2.28).

Lemma 8.4. *Let $B > 0$ on $\mathcal{H}_0$ and $w \in Q_0\mathcal{H}_0$. Set*

$$V_w = |e_h\rangle\langle w| + |w\rangle\langle e_h|, \qquad \xi_w = -D_B^0(\mathbb{K}_B^{-1} V_w),$$

and extend ξ_w by zero to the auxiliary channels. This current has velocity V_w. On every compact set of positive core matrices, there is $K < \infty$ such that

$$\|\xi_w\|^2 \le K\|w\|^2. \tag{8.18}$$

At $B = B(t)$ on the trajectory (8.6),

$$\langle \zeta_t, \xi_w\rangle_{\mathbb{R}} = -\operatorname{Tr}(H_t V_w) = -2\operatorname{Re}\langle \ell(t), w\rangle. \tag{8.19}$$

Proof. The positivity and compact-set inverse bound follow from (2.28). Since $\|V_w\|_2^2 = 2\|w\|^2$, $\|\xi_w\|^2 = \operatorname{Tr}(V_w \mathbb{K}_B^{-1} V_w)$ gives

$$\|\xi_w\|^2 \le 2\|\mathbb{K}_B^{-1}\|\,\|w\|^2.$$

Taking $K = 2\sup_B \|\mathbb{K}_B^{-1}\|$ on the stated compact set proves (8.18).

By (2.28), for each Hermitian core test C,

$$-\langle D_B^0 C, \xi_w\rangle_{\mathbb{R}} = \operatorname{Tr}(C V_w).$$

The raw divergence has finite channel and matrix support. Equation (2.25) therefore gives the same identity for every bounded Hermitian test of finite cotangent norm. Substituting the core restriction of H_t yields (8.19), since $Q_0 \log \sigma_0 e_h = 0$. □

Fix $T_0, m > 0$ such that

$$m < \inf_{0\le t\le T_0} (E - F_\varepsilon(\rho_t)), \qquad \inf_{0\le t\le T_0} \|\ell(t)\|^2 \ge r_* > 0. \tag{8.20}$$

Choose K as in Lemma 8.4, and put

$$\theta = \frac{2}{K + 4/m}, \qquad w_t = -\theta\ell(t), \qquad \gamma = \frac{4}{K + 4/m}. \tag{8.21}$$

Then (8.19) gives

$$\begin{aligned} \|\zeta_t - \xi_{w_t}\|^2 + \frac{4\|w_t\|^2}{m} &\le \mathcal{I}_\varepsilon(\rho_t) + \left[-4\theta + (K + 4/m)\theta^2\right]\|\ell(t)\|^2 \\ &= \mathcal{I}_\varepsilon(\rho_t) - \gamma\|\ell(t)\|^2. \end{aligned} \tag{8.22}$$

8.4. **Realization by original currents.**

Lemma 8.5. *Assume* (8.20)*, and let* $[s,t] \subset [0,T_0]$*,* $s<t$*. There is a sequence of admissible state-current pairs joining* ρ_s *to* ρ_t *on* $[s,t]$*, with states of relative entropy at most* E*, converging uniformly in trace norm to* $\rho|_{[s,t]}$*, whose actions have upper limit at most*

$$\int_s^t \left[\mathcal{I}_\varepsilon(\rho_u) - \gamma\|\ell(u)\|^2\right] du. \tag{8.23}$$

Every state on these curves is faithful. Each curve has positive finite two-sided order bounds with respect to σ_ε*, and the same pairs are admissible for all test spaces* (1.11).

Proof. Let n tend to infinity and set, on the interior part of the connector,

$$\rho_u^n = \rho_u - \frac{m}{n}P + \frac{m}{n}E_{v_n v_n}. \tag{8.24}$$

The core eigenvalues have a common positive lower bound. Hence

$$\rho_u^n \geq (1 - C/n)\rho_u > 0, \qquad \sup_u \|\rho_u^n - \rho_u\|_1 = 2m/n. \tag{8.25}$$

The changed state is bounded above by a finite multiple of σ_ε for each fixed n, uniformly in u.

Write $r = m/n$. Direct-sum relative entropy gives the exact increment

$$\begin{aligned} F_\varepsilon(\rho_u^n) - F_\varepsilon(\rho_u) = {} & \operatorname{Tr}\left[(B - rP)\log(B - rP) - B\log B\right] \\ & + (p_n + r)\log(p_n + r) - p_n \log p_n + r\log\frac{(\sigma_\varepsilon)_{hh}}{\pi_n^\varepsilon}. \end{aligned} \tag{8.26}$$

The first term is $O(r)$ on the positive core compact set. The second tends uniformly to zero, since p_n is uniformly comparable to π_n^ε. The last is $(m/n)(n - \log\varepsilon)$. Therefore

$$F_\varepsilon(\rho_u^n) = F_\varepsilon(\rho_u) + m + o(1) \tag{8.27}$$

uniformly. By (8.20), the loaded curve lies in the cap for sufficiently large n.

Let J be the raw current corresponding to $\zeta - \xi_w$. It has velocity $\dot\rho - V_w$. Transfer J unchanged to ρ^n. Equations (8.25) and (2.20) bound its least square-root cost by $(1 - C/n)^{-1}\|\zeta - \xi_w\|^2$.

Add the pair (2.31) on edge hv_n. By Lemma 2.9, it has velocity V_{w_u} and cost $4\langle w_u, T_n(u)^{-1}w_u\rangle$, where

$$T_n(u) = \Lambda\left(n^2(p_n(u) + m/n),\ \varepsilon n^2 e^{-n}(B(u) - (m/n)P)\right). \tag{8.28}$$

The spectra of $B(u) - (m/n)P$ remain in a fixed compact interval $K \Subset (0,\infty)$. Apply Lemma 2.3 with

$$s_n = n, \qquad c_n = n^2, \qquad L_n = \log(1/\varepsilon), \qquad u_n(u) = p_n(u) + m/n.$$

The order bounds (8.11) imply $p_n(u) \asymp \varepsilon e^{-n}$ uniformly, so $\sup_u |\log u_n(u)|/n \to 0$ and $np_n(u) \to 0$. Consequently

$$\frac{c_n u_n(u)}{s_n} = np_n(u) + m \longrightarrow m$$

uniformly. The operator conclusion of Lemma 2.3 therefore proves

$$T_n(u) \longrightarrow mI \tag{8.29}$$

uniformly in operator norm. The added action tends to $4\|w_u\|^2/m$.

The core correction has no auxiliary-edge component. By Lemma 8.3, the original entropy-current energy on edge hv_n is at most Ca_n. Its transferred energy has the same bound up to $1 + o(1)$. If A_n^{old} and A_n^{add} denote the two integrated costs on this pair, then $A_n^{\mathrm{old}} = O(a_n)$ and $A_n^{\mathrm{add}} = O(1)$. Their mixed term has absolute value at most $2\sqrt{A_n^{\mathrm{old}}A_n^{\mathrm{add}}} = O(\sqrt{a_n}) \to 0$. Since m is constant, the total raw current satisfies

$$-\sum_j [L_j^*, J_j^{\mathrm{tot,n}}] = \dot\rho - V_w + V_w = \dot\rho^n.$$

Equation (8.22) gives (8.23) for the interior curve.

At either fixed endpoint, the affine loading curve is

$$B - rP, \qquad p_n + r, \qquad 0 \le r \le m/n.$$

Choose $b_* > 0$ so small that the core is at least $b_* I$ and $p_n \ge 2\varepsilon e^{-n} b_*$. This is possible by (8.11). Equation (2.33) bounds its length by

$$\frac{1}{n}\int_0^{m/n} \frac{dr}{\sqrt{\Lambda(p_n + r, \varepsilon e^{-n} b_*)}} \le \frac{2\sqrt{(m/n)L_n}}{n} = O(n^{-1}), \tag{8.30}$$

where

$$L_n = \log \frac{p_n + m/n}{\varepsilon e^{-n} b_*} = O(n).$$

Indeed, for $r > 0$,

$$\Lambda(p_n + r, \varepsilon e^{-n} b_*)^{-1} = \frac{\log((p_n + r)/(\varepsilon e^{-n} b_*))}{p_n + r - \varepsilon e^{-n} b_*} \le \frac{L_n}{r}.$$

Set $\tau_n = n^{-1/2}$ and parametrize a loading leg by

$$r(\tau) = \frac{m}{n}\left(\frac{\tau}{\tau_n}\right)^2, \qquad 0 \le \tau \le \tau_n.$$

Its action is at most $4(m/n)L_n/(n^2\tau_n) = O(n^{-3/2})$. The reversed parameter gives the unloading leg. For large n, $2\tau_n < t - s$. On $[s + \tau_n, t - \tau_n]$, set

$$\vartheta_n(u) = s + \frac{t - s}{t - s - 2\tau_n}(u - s - \tau_n).$$

Compose the middle state with ϑ_n and multiply every composed raw current by ϑ_n'. Its action is multiplied by $(t - s)/(t - s - 2\tau_n) \to 1$. The first loading leg joins ρ_s to the loaded state at s, and the last leg joins the loaded state at t to ρ_t. All endpoints agree exactly. If $\omega_\rho(a) = \sup\{\|\rho_u - \rho_v\|_1 : u, v \in [s,t],\ |u - v| \le a\}$, the full connector, denoted by $\widehat{\rho}^n$, satisfies

$$\sup_{u \in [s,t]} \|\widehat{\rho}^n_u - \rho_u\|_1 \le 2m/n + \omega_\rho(\tau_n) \longrightarrow 0.$$

Indeed $|\vartheta_n(u) - u| \le \tau_n$ on the middle interval, and each loading leg is within $2m/n$ of the corresponding fixed endpoint. Entropy convexity keeps each affine loading image in the cap.

For each fixed n, all parts of the connector satisfy fixed two-sided bounds by σ_ε. Lemma 2.7 proves (2.22) for every $C \in \mathscr{T}_\varepsilon^{\mathrm{eq}}$. The transferred raw divergence is the trace-norm convergent entropy divergence minus a finite-core divergence. The added divergences have finite matrix support. Their sums are the prescribed derivatives, so (2.25) proves all the continuity equations. Lemma 2.5 gives measurable least representatives. □

8.5. **Metric speed and uniformity.**

Proof of Theorem 8.1. For the state (8.6), Lemma 8.3 gives continuous bounded Fisher information on compact intervals and original admissible entropy currents. Time rescaling therefore gives

$$\mathcal{W}^{\mathscr{T}}_{\varepsilon,E}(\rho_s, \rho_t)^2 \le (t - s)\int_s^t \mathcal{I}_\varepsilon(\rho_u)\,du.$$

Thus the orbit is locally Lipschitz in this metric. Choose a short interval on which (8.20) holds. Lemma 8.5 gives

$$\mathcal{W}^{\mathscr{T}}_{\varepsilon,E}(\rho_s, \rho_t)^2 \le (t - s)\int_s^t \left[\mathcal{I}_\varepsilon(\rho_u) - \gamma\|\ell(u)\|^2\right] du. \tag{8.31}$$

Its metric derivative exists almost everywhere by the metric differentiation theorem [1, Theorem 1.1.2]. Divide (8.31) by $(t-s)^2$ and apply Lebesgue differentiation to its continuous integrand. This gives

$$|\dot\rho_t|^2_{\mathcal{W}^{\mathscr{T}}_{\varepsilon,E}} \le \mathcal{I}_\varepsilon(\rho_t) - \gamma\|\ell(t)\|^2.$$

Take $\Delta = \gamma r_* > 0$. If this orbit were a 2-curve of maximal slope for a strong upper gradient, Lemma 8.2 would give $|\dot\rho_t|^2 = \mathcal{I}_\varepsilon(\rho_t)$ almost everywhere, contrary to the positive deficit.

Fix $\varepsilon_0 > 0$. The same B_0 in (8.5) works for every $0 < \varepsilon \le \varepsilon_0$. By (8.7), the initial slack is at least $E - D(B_0\|\sigma_0) > 0$. Choose one smaller $m > 0$. The core matrices satisfy

$$\frac{\underline{c}}{Z_{\varepsilon_0}}\sigma_0 \le B(t) \le \overline{c}\sigma_0.$$

Equations (8.9)–(8.10) give a bound for $\|\dot B(t)\|$ independent of this range of ε. The logarithm is Lipschitz on the displayed positive compact set. Since $\ell(0)$ is independent of ε and nonzero, one common $T_0 > 0$ gives a common lower bound $r_* > 0$. The least eigenvalue of $\mathbb{K}_B$ has a common positive lower bound, so K in (8.18), and then γ and Δ, can be chosen uniformly. Every constructed connector is admissible for the maximal test class, so none of these constants depends on $\mathscr{T}$. ☐

8.6. **The initial entropy as constraint.**

Corollary 8.6. *For the initial states* (8.6)*, set*

$$E_\varepsilon = F_\varepsilon(x_\varepsilon).$$

The strict speed inequality of Theorem 8.1 holds for $\mathcal{W}^{\mathscr{T}_\varepsilon}_{\varepsilon,E_\varepsilon}$ on a positive-time interval. For every $\varepsilon_0 > 0$, the interval and a positive deficit can be chosen uniformly for $0 < \varepsilon \le \varepsilon_0$ and all test spaces (1.11)*.*

Proof. By Lemma 6.7, $\mathcal{I}_0(B_0) > 0$ because $B_0 \ne \sigma_0$. Homogeneity of the finite-core mobility gives

$$\mathcal{I}_{\mathrm{core}}(B_0/Z_\varepsilon) = Z_\varepsilon^{-1}\mathcal{I}_{\mathrm{core}}(B_0).$$

Continuity on the common positive core compact set yields a short interval and $c_1 > 0$, independent of $0 < \varepsilon \le \varepsilon_0$, such that $\mathcal{I}_\varepsilon(\rho_t) \ge c_1$. Shrink the interval so that $\|\ell(t)\|^2 \ge r_* > 0$ there.

Choose $t_* > 0$ with $[t_*, 2t_*]$ contained in that interval. By (8.12),

$$E_\varepsilon - F_\varepsilon(\rho_t) = \int_0^t \mathcal{I}_\varepsilon(\rho_u)\,du \ge c_1 t_* \qquad (t_* \le t \le 2t_*).$$

Choose one $0 < m < c_1 t_*$. Lemma 8.5 and (8.31) apply on this interval, with all constants uniform as above. ☐

8.7. **Nonseparation of the uncapped pseudometric.**

Proposition 8.7. *Let $\varepsilon > 0$, let $i \ne h$ be a core coordinate, and let $z \in \mathbb{C}$ satisfy $0 < |z| < \sqrt{(\sigma_\varepsilon)_{hh}(\sigma_\varepsilon)_{ii}}$. For every test space* (1.11)*, the uncapped extended pseudometric from Definition 2.6 satisfies*

$$\mathcal{W}^{\mathscr{T}}_\varepsilon\left(\sigma_\varepsilon + zE_{hi} + \overline{z}E_{ih}, \sigma_\varepsilon\right) = 0. \tag{8.32}$$

The two states are distinct, faithful, and of finite relative entropy. They admit connecting curves of action tending to zero, each faithful and bounded above and below by positive multiples of σ_ε, with only one active auxiliary channel pair.

Proof. Put $r_n = n^{-1/2}$. On $[0,1]$ define

$$r(t) = \begin{cases} r_n(3t)^2, & 0 \le t \le 1/3,\\ r_n, & 1/3 \le t \le 2/3,\\ r_n(3-3t)^2, & 2/3 \le t \le 1,\end{cases} \qquad z(t) = \begin{cases} z, & 0 \le t \le 1/3,\\ (2-3t)z, & 1/3 \le t \le 2/3,\\ 0, & 2/3 \le t \le 1.\end{cases}$$

Set

$$\rho^n_t = \sigma_\varepsilon - r(t)P + r(t)E_{v_n v_n} + z(t)E_{hi} + \overline{z(t)}E_{ih}.$$

For large n, its core matrix $B_n(t)$ has eigenvalues in a fixed compact subset of $(0, \infty)$. Thus ρ^n is faithful and has the prescribed endpoints. On the middle interval, Lemma 2.9 with $w = \overline{\dot z(t)} e_i$ gives the exact velocity and action $4\langle w, T_n(t)^{-1} w\rangle$, where

$$T_n(t) = \Lambda\big(n^2(\pi_n^\varepsilon + r_n), \varepsilon n^2 e^{-n} B_n(t)\big).$$

Apply Lemma 2.3 with $s_n = n$, $c_n = n^2$, $L_n = \log(1/\varepsilon)$, and $u_n = \pi_n^\varepsilon + n^{-1/2}$. The spectra of $B_n(t)$ lie in one compact subset of $(0, \infty)$, $|\log u_n|/n \to 0$, and

$$\frac{c_n u_n}{s_n} = n\pi_n^\varepsilon + n^{1/2} = n^{1/2}(1 + o(1)).$$

Hence $n^{-1/2} T_n(t) \to I_0$ uniformly in operator norm. In particular $\|T_n(t)^{-1}\| \le C n^{-1/2}$ on the middle interval, and its integrated action is $O(n^{-1/2})$.

On each outer interval use the loading current in Lemma 2.9. As in (8.30), its action density is at most $L_n \dot r^2/(n^2 r)$, with $L_n = O(n)$. The parametrization gives integrated action at most $12 r_n L_n/n^2 = O(n^{-3/2})$ on each interval. Consequently $\mathcal{A}_{\varepsilon,\infty}(\rho^n) \to 0$, where $\mathcal{A}_{\varepsilon,\infty}$ denotes the reduced action without a cap. Each curve differs from σ_ε on finitely many coordinates and has finite positive upper and lower equilibrium-order bounds. Equation (8.26) gives finite relative entropy, with maximum $\sqrt{n} + O(1)$. The currents and divergences have finite support. Lemma 2.7 and (2.25) prove admissibility for every space (1.11). Taking the uncapped infimum proves (8.32). □

Remark 8.8. On the loaded middle portion (8.24) of the connectors in Lemma 8.5, one has

$$\frac{(\rho_t^n)_{v_n v_n}}{(\sigma_\varepsilon)_{v_n v_n}} \asymp \frac{m e^n}{n\varepsilon}.$$

Thus each curve is bounded above by a multiple of σ_ε, but no single multiple bounds the family as $n \to \infty$. A path constraint $\rho_t \le C\sigma_\varepsilon$ with fixed C excludes this family. Define the auxiliary jump activity by

$$\mathrm{Act}_\varepsilon(\eta) = \sum_j \big(a_j \eta_{hh} + b_j \eta_{v_j v_j}\big).$$

On the loaded middle curve (8.24),

$$\mathrm{Act}_\varepsilon(\rho_u^n) = \mathrm{Act}_\varepsilon(\rho_u) + mn - \frac{m\varepsilon A}{n} = mn + O(1).$$

For its selected pair, put $x_n = n^2(p_n(u) + m/n)$ and $y_n = \varepsilon n^2 e^{-n}(B_{hh}(u) - m/n)$. For fixed $\varepsilon > 0$, the order bounds (8.11) give, uniformly on the middle interval,

$$x_n = mn + o(1), \qquad y_n \asymp \varepsilon n^2 e^{-n} = o(1), \qquad \log(x_n/y_n) = n - \log n + O(1).$$

Lemma 6.1 gives

$$\mathcal{I}_\varepsilon(\rho_u^n) \ge \psi(x_n, y_n) = mn^2(1 + o(1)).$$

For each fixed n, both quantities are finite on the whole connector: the selected block is faithful and finite-dimensional, while the remaining pair contributions are bounded by a constant depending on n times the summable sequence (a_j).

8.8. **Failure of lower semicontinuity at fixed coupling.**

Corollary 8.9. *For the profile* (8.1)*, every* $\varepsilon, E > 0$ *admits a faithful semigroup trajectory* $\rho : [0, T_0] \to K_{\varepsilon,E}^{\mathscr{T}_\varepsilon}$ *and faithful capped curves* $\widehat{\rho}^n$ *with the same endpoints, admissible for every test space* (1.11)*, such that*

$$\widehat{\rho}^n \longrightarrow \rho \quad \textit{uniformly in trace norm}, \qquad \limsup_{n\to\infty} \mathcal{A}_{\varepsilon,E}(\widehat{\rho}^n) < \mathcal{A}_{\varepsilon,E}(\rho) < \infty.$$

In particular, the fixed-coupling reduced current action is not lower semicontinuous for uniform trace convergence, even with these endpoints fixed. For this trajectory one also has

$$\int_0^{T_0} |\dot\rho_t|^2_{\mathcal{W}_{\varepsilon,E}^{\mathscr{T}_\varepsilon}} \, dt < \mathcal{A}_{\varepsilon,E}(\rho) = \int_0^{T_0} \mathcal{I}_\varepsilon(\rho_t)\, dt. \tag{8.33}$$

Proof. Choose the trajectory and interval in Theorem 8.1, and the constants in (8.20)–(8.21). Equation (8.13) identifies its reduced action with the integrated Fisher information. Lemma 8.5, applied to $[0, T_0]$, gives exact-endpoint uniformly trace-convergent connectors and

$$\limsup_n \mathcal{A}_{\varepsilon,E}(\widehat{\rho}^n) \leq \mathcal{A}_{\varepsilon,E}(\rho) - \gamma \int_0^{T_0} \|\ell(t)\|^2\,dt \leq \mathcal{A}_{\varepsilon,E}(\rho) - \gamma r_* T_0.$$

All constants on the last line are positive. Integrating (8.2) proves (8.33). □

The distance construction can therefore lower the kinetic cost of a prescribed trajectory through nearby connecting curves. This corollary does not identify the full lower-semicontinuous relaxation of the current action with metric energy. The kinetic Γ-limit in Theorem A and the joint current action–Fisher limit in Theorem C do not require such an identification.

9. Faithful stationary limits

Fix the finite core $\mathcal{K}_*$ and the eigenvector e_h as in Section 2. For a reversible level extension with stationary state σ, write $\mathcal{W}_E^{\mathscr{T}}$ for the current distance of Definition 2.6, with entropy $D(\cdot\|\sigma)$. The test space satisfies

$$\mathscr{T}_{\mathrm{fin}} + \mathbb{R}I \subseteq \mathscr{T} \subseteq \{C = C^* \in \mathcal{B}(\mathcal{H}) \,:\, g_\sigma(C) < \infty\}.$$

Theorem 9.1. *Every primitive GNS-symmetric finite core admits a conservative GNS-symmetric extension $\mathcal{L}_{\mathrm{ref},*}$ and a sequence of extensions $(\mathcal{L}_{n,*})_{n\geq n_0}$, for some $n_0 \in \mathbb{N}$, on the same $\mathcal{H}$, with common generator domain, such that*

$$\|\mathcal{L}_{n,*} - \mathcal{L}_{\mathrm{ref},*}\|_\diamond \leq 6e^{-n}, \qquad \sup_{t\geq 0} \|S_t^n - S_t^{\mathrm{ref}}\|_\diamond \leq Ce^{-n}, \tag{9.1}$$

$$\|\sigma_n - \sigma_{\mathrm{ref}}\|_1 \longrightarrow 0, \qquad D(\sigma_n\|\sigma_{\mathrm{ref}}) + D(\sigma_{\mathrm{ref}}\|\sigma_n) \longrightarrow 0. \tag{9.2}$$

The stationary states in (9.2)*, including σ_{ref}, are faithful on $\mathcal{H}$. The energies $F_n = D(\,\cdot\,\|\sigma_n)$ Γ-converge in trace norm to $F_{\mathrm{ref}} = D(\,\cdot\,\|\sigma_{\mathrm{ref}})$; moreover, $F_n(\rho) \to F_{\mathrm{ref}}(\rho)$ at every fixed state of finite reference entropy. Both rates of one auxiliary pair change; the core channels are unchanged. For every $E > 0$, there are fixed distinct faithful states x, y, with $D(x\|\sigma_{\mathrm{ref}}), D(y\|\sigma_{\mathrm{ref}}) < E$, such that, for all sufficiently large n,*

$$0 < \mathcal{W}_{n,E}^{\mathscr{T}_n}(x,y)^2 \leq \frac{C_x}{n}, \qquad 0 < \mathcal{W}_{\mathrm{ref},E}^{\mathscr{T}_{\mathrm{ref}}}(x,y) < \infty. \tag{9.3}$$

These conclusions hold for every choice of the stated bounded test spaces. The constants in (9.1) *are independent of n, and those in* (9.3) *are independent of the test spaces and of sufficiently large n.*

9.1. **The rates and their invariant states.** Let $\varepsilon_0, T_*, d_* > 0$ be the constants in the proof of Theorem 7.1. Set

$$A_* := \sum_{j\geq 1} j^4 e^{-j^6}, \qquad 0 < \alpha < \min\{A_*, \varepsilon_0/2\}, \qquad w_j = \frac{\alpha}{A_*} e^{-j^6}, \qquad b_j = j^4, \quad \bar{a}_j = b_j w_j. \tag{9.4}$$

Thus $\sum_j \bar{a}_j = \alpha$. Define

$$Z_* = 1 + \pi_h \sum_j w_j, \qquad \sigma_{\mathrm{ref}} = Z_*^{-1}\left(\sigma_0 + \pi_h \sum_j w_j E_{v_j v_j}\right), \qquad c_h = (\sigma_{\mathrm{ref}})_{hh}. \tag{9.5}$$

Choose $n_0 \in \mathbb{N}$ such that, for every $n \geq n_0$,

$$w_n + \frac{e^{-n}}{b_n} \leq 1, \qquad \alpha + 2e^{-n} \leq \varepsilon_0.$$

Such an n_0 exists since $w_n \to 0$, $b_n = n^4$, and $\alpha < \varepsilon_0/2$. The perturbed family is indexed by $n \geq n_0$ throughout this section; the auxiliary coordinate index j still ranges over all positive integers. For $n \geq n_0$, put

$$\varepsilon_n = e^{-n}, \qquad \lambda_n = 1 + \frac{\varepsilon_n}{b_n},$$

and let

$$a_j^{(n)} = \bar{a}_j, \quad b_j^{(n)} = b_j \quad (j \neq n), \qquad a_n^{(n)} = \lambda_n(\bar{a}_n + \varepsilon_n), \quad b_n^{(n)} = \lambda_n b_n = b_n + \varepsilon_n. \tag{9.6}$$

These rates define $\mathcal{L}_{\mathrm{ref},*}$ and $\mathcal{L}_{n,*}$ through (2.6) with the fixed core.

Lemma 9.2. *The extensions in* (9.4)–(9.6) *are conservative and GNS symmetric. Their stationary states are* σ_{ref} *and*

$$\sigma_n = \frac{\sigma_{\mathrm{ref}} + r_n E_{v_n v_n}}{1 + r_n}, \qquad r_n = \frac{c_h \varepsilon_n}{b_n}. \tag{9.7}$$

They satisfy (9.2). *If* $F_{\mathrm{ref}}(\rho) = D(\rho \| \sigma_{\mathrm{ref}}) < \infty$, *then*

$$F_n(\rho) = F_{\mathrm{ref}}(\rho) + \log(1 + r_n) - \rho_{v_n v_n} \log\left(1 + \frac{r_n}{(\sigma_{\mathrm{ref}})_{v_n v_n}}\right). \tag{9.8}$$

Proof. The return semigroups have the form (2.7). Their core and outward perturbations are bounded. The construction and GNS-symmetry argument of Proposition 2.2 apply. The selected ratio is

$$\frac{a_n^{(n)}}{b_n^{(n)}} = w_n + \frac{\varepsilon_n}{b_n}.$$

Multiplying the reference weights by the common normalization $(1 + r_n)^{-1}$ and adding $r_n/(1 + r_n)$ at v_n gives

$$\frac{(\sigma_n)_{v_n v_n}}{(\sigma_n)_{hh}} = w_n + \frac{\varepsilon_n}{b_n}.$$

Every other auxiliary ratio and the core ratios are unchanged. This proves (9.7); all its diagonal entries are positive. Also $\sum_j b_j^{(n)} (\sigma_n)_{v_j v_j} < \infty$, so the density belongs to the generator domain and is stationary.

Write $\theta_n = r_n/(1 + r_n)$ and $H_* = -\log \sigma_{\mathrm{ref}}$. Then $\sigma_n = (1 - \theta_n)\sigma_{\mathrm{ref}} + \theta_n E_{v_n v_n}$. Convexity of relative entropy and $\sigma_n \geq \sigma_{\mathrm{ref}}/(1 + r_n)$ imply

$$\begin{aligned}
&\|\sigma_n - \sigma_{\mathrm{ref}}\|_1 \leq 2\theta_n, \\
&D(\sigma_{\mathrm{ref}} \| \sigma_n) \leq \log(1 + r_n), \\
&D(\sigma_n \| \sigma_{\mathrm{ref}}) \leq \theta_n (H_*)_{v_n v_n} = O(e^{-n} n^2),
\end{aligned}$$

since $(H_*)_{v_n v_n} = n^6 + \log(Z_* A_*/(\alpha \pi_h))$. This proves (9.2). The logarithmic reference difference is the bounded diagonal operator

$$\log \sigma_{\mathrm{ref}} - \log \sigma_n = \log(1 + r_n) I - \log\left(1 + \frac{r_n}{(\sigma_{\mathrm{ref}})_{v_n v_n}}\right) E_{v_n v_n}.$$

For each fixed n this operator is bounded. The spectral definition of relative entropy therefore gives

$$F_n(\rho) - F_{\mathrm{ref}}(\rho) = \mathrm{Tr}\big(\rho(\log \sigma_{\mathrm{ref}} - \log \sigma_n)\big)$$

whenever $F_{\mathrm{ref}}(\rho) < \infty$. This is (9.8); no difference of two infinite traces is taken. □

9.2. **The complete norm estimate.**

Lemma 9.3. *The generators in* (9.6) *have the same domain as* $\mathcal{L}_{\mathrm{ref},*}$ *and satisfy* (9.1). *Each semigroup converges in trace norm to its displayed stationary state from every initial state.*

Proof. Set

$$u_n = a_n^{(n)} - \bar{a}_n = \varepsilon_n \left(1 + \frac{\bar{a}_n + \varepsilon_n}{b_n}\right).$$

For every $n \geq n_0$, the choice of n_0 gives $u_n \leq 2\varepsilon_n$, and

$$\mathcal{L}_{n,*} - \mathcal{L}_{\mathrm{ref},*} = u_n \mathcal{D}_{E_{v_n h}} + \varepsilon_n \mathcal{D}_{E_{h v_n}}. \tag{9.9}$$

The jump map has amplified norm $\|U\|^2$, and each of left and right multiplication by $U^*U/2$ has amplified norm at most $\|U\|^2/2$. Hence $\|\mathcal{D}_U\|_\diamond \le 2\|U\|^2$, so (9.9) has norm at most $6\varepsilon_n$. The two return generators differ on their finite-coordinate cores by $\varepsilon_n \mathcal{D}_{E_{hv_n}}$. This is bounded; the core property in Lemma 2.1 and the bounded perturbation theorem identify their closed realizations and give equal domains. Adding the bounded core and outward terms preserves that domain.

All returns are at least one. For every $n \ge n_0$, the total outward rates are α and $\alpha + u_n \le \alpha + 2e^{-n} \le \varepsilon_0$. Equation (7.5), whose constants are independent of the outward distribution and the returns above one, gives

$$S^n_{T_*} \ge_{\mathrm{CP}} d_* R_{\sigma_0}, \qquad S^{\mathrm{ref}}_{T_*} \ge_{\mathrm{CP}} d_* R_{\sigma_0}.$$

At every finite amplification, an operator X of zero system partial trace therefore satisfies

$$\|(S^n_t \otimes \mathrm{id})X\|_1 \le (1 - d_*)^{\lfloor t/T_* \rfloor} \|X\|_1.$$

The amplified image of (9.9) has zero system partial trace. Duhamel's formula yields

$$\sup_{t \ge 0} \|S^n_t - S^{\mathrm{ref}}_t\|_\diamond \le \frac{T_*}{d_*} \|\mathcal{L}_{n,*} - \mathcal{L}_{\mathrm{ref},*}\|_\diamond.$$

This is (9.1). Applying the same contraction to the difference of an arbitrary state and the stationary state proves convergence and uniqueness of the stationary density. □

9.3. **Fixed endpoints and exact currents.**

Lemma 9.4. *For each fixed model in* (9.4)–(9.6) *and each* $C \in \mathscr{T}_{\mathrm{fin}}$,

$$\sup_{\rho \in \mathfrak{S}(\mathcal{H})} g_\rho(C) < \infty.$$

Its capped distance is separating on its state domain for every allowed test space.

Proof. Retain a finite coordinate block containing the core, the support of C, and the selected perturbed level. Phase averaging as in Lemma 3.2 bounds the remaining cotangent sum by a constant times

$$\sum_j j^4 \Lambda(p_j, (\alpha/A_*) e^{-j^6}).$$

For all sufficiently large j, split at $p_j = e^{-j^6/2}$. On the upper part, the logarithmic denominator is at least $j^6/4$, so the sum is bounded by $4\sum_j j^{-2} p_j$. On the lower part, the mean is at most a fixed multiple of $e^{-j^6/2}$; the bound is summable after multiplication by j^4. The remaining finite channel set has uniformly bounded mobilities. This proves the supremum bound.

For any admissible unit-time connector, its continuity equation gives

$$|\operatorname{Tr} C(\rho_1 - \rho_0)|^2 \le \big(\sup_\rho g_\rho(C)\big) \int_0^1 \|\xi_t\|^2 \, dt.$$

Finite tests separate states. Enlarging the test space can only increase the distance. Constant curves, reversal, and concatenation supply the other extended-metric properties on the state domain. □

Proof of Theorem 9.1. Lemmas 9.2 and 9.3 give the semigroup and stationary-state assertions. The energy assertions follow from Corollary 9.6, whose proof uses only Lemma 9.2. Fix $E > 0$, choose $i \ne h$, and take $z \in \mathbb{C} \setminus \{0\}$ sufficiently small that

$$x = \sigma_{\mathrm{ref}} + zE_{hi} + \bar{z}E_{ih} > 0, \qquad y = \sigma_{\mathrm{ref}}, \qquad F_z = F_{\mathrm{ref}}(x) < E.$$

Since $x_{v_n v_n} = y_{v_n v_n} = (\sigma_{\mathrm{ref}})_{v_n v_n}$, formula (9.8) yields

$$F_n(x),\ F_n(y) \le F_z + \log(1 + r_n) < E$$

for all sufficiently large n. Thus the fixed endpoints belong to the moving entropy caps used below. The core coherence segment from x to y stays in the cap by entropy convexity. Its core block has a positive uniform least eigenvalue. The inverse core Onsager operator therefore supplies a finite-action core current. The complementary diagonal is fixed. The states are individually order-comparable with equilibrium and their divergences have finite matrix support, so the segment is admissible for all allowed tests. Lemma 9.4 proves $0 < \mathcal{W}_{\mathrm{ref},E}(x,y) < \infty$.

For the pair without its common factor λ_n, write

$$a = \bar{a}_n + \varepsilon_n, \qquad b = n^4, \qquad p^0 = (\sigma_{\mathrm{ref}})_{v_n v_n}, \qquad m = n^{-2}.$$

On $[0,1]$ define

$$s(t) = \begin{cases} 3mt, & 0 \le t \le 1/3, \\ m, & 1/3 \le t \le 2/3, \\ 3m(1-t), & 2/3 \le t \le 1, \end{cases} \qquad z(t) = \begin{cases} z, & 0 \le t \le 1/3, \\ (2-3t)z, & 1/3 \le t \le 2/3, \\ 0, & 2/3 \le t \le 1. \end{cases}$$

Set

$$\rho_t^n = \sigma_{\mathrm{ref}} - s(t)P + s(t)E_{v_n v_n} + z(t)E_{hi} + \overline{z(t)}E_{ih}.$$

For large n, its core block is at least $b_* I_0$ for one fixed $b_* > 0$. The states are faithful and have endpoints x, y. With $\ell = (ab)^{1/4}/\sqrt{2}$, use

$$J_+^\ell = \frac{\dot{s}}{2\ell} E_{v_n h}, \quad J_-^\ell = -(J_+^\ell)^*, \qquad J_+^r = -\ell^{-1}|e_{v_n}\rangle\langle w|, \quad J_-^r = -(J_+^r)^*, \quad w = \overline{\dot{z}(t)}e_i.$$

The first pair is active on the outer thirds, the second on the middle third. Lemma 2.9 gives their exact negative divergences and the action bounds

$$\frac{\dot{s}^2}{\Lambda(bs, ab_*)}, \qquad \frac{4|\dot{z}|^2}{\Lambda(bm, ab_*)}.$$

Each outer interval has $|\dot{s}| = 3m$. Since $\Lambda(bs, ab_*)^{-1} = \log(bs/(ab_*))/(bs - ab_*)$, the two outer actions have sum at most

$$\frac{6m}{b}\int_0^R \frac{\log u}{u-1}\, du, \qquad R = \frac{bm}{ab_*}.$$

The quotient at $u = 1$ is defined by continuity, and the integral at zero is improper. By monotone convergence,

$$\int_0^1 \frac{-\log u}{1-u}\, du = \sum_{q=0}^\infty \int_0^1 u^q(-\log u)\, du = \sum_{q=1}^\infty q^{-2} \le 1 + \int_1^\infty x^{-2}\, dx = 2.$$

On $[1,2]$, $\log u \le u - 1$. On $[2,R]$, $\log u/(u-1) \le 2\log u/u$. Thus, for $R \ge 2$, the integral is at most $3 + \log^2 R$. On the middle interval $|\dot{z}| = 3|z|$ and its duration is $1/3$. The total action is at most

$$\frac{6m}{b}(3 + \log^2 R) + \frac{12|z|^2}{\Lambda(bm, ab_*)}. \tag{9.10}$$

Here $a = \bar{a}_n + e^{-n} = e^{-n}(1 + o(1))$. Set

$$L_n = -\log a + 4\log n - n.$$

Then $L_n/n \to 0$ and $a = n^4 e^{-(n+L_n)}$. Lemma 2.3, with $s_n = n$, $c_n = n^4$, $u_n = n^{-2}$, and the fixed scalar $\lambda = b_*$, yields

$$\frac{1}{n}\Lambda(bm, ab_*) \longrightarrow 1.$$

Also $R = bm/(ab_*)$ satisfies $\log R = n + 2\log n + O(1)$. Therefore the two terms in (9.10) are $O(n^{-4})$ and $O(n^{-1})$.

Let B be the core block before the subtraction of sP. The exact entropy increment of that subtraction and the auxiliary loading is

$$\begin{aligned}&\mathrm{Tr}[(B - sP)\log(B - sP) - B\log B]\\&\quad + (p^0 + s)\log(p^0 + s) - p^0\log p^0 - s\log(w_n + \varepsilon_n/b_n).\end{aligned}$$

The core term is $O(s)$ on the fixed positive compact family. For $0 \le s \le n^{-2}$, the scalar entropy difference is $O(n^{-2}\log n)$, whereas $-s\log(w_n + \varepsilon_n/b_n) = O(n^{-1})$. Equation (9.8) bounds the entropy of the unsubtracted core segment by $F_z + \log(1 + r_n)$. Hence

$$\sup_t F_n(\rho_t^n) \le F_z + O(n^{-1}) < E$$

for all sufficiently large n.

Restoring λ_n leaves the modular weight and the stationary state unchanged and multiplies the two channel amplitudes by $\sqrt{\lambda_n}$. Divide the displayed raw currents by $\sqrt{\lambda_n}$. Their divergence is unchanged and their action is divided by λ_n. Each curve is order-comparable with σ_n with finite positive constants depending on n; its currents and divergences have finite support. Thus it is admissible for every allowed bounded test space. Formula (9.10) proves the upper bound in (9.3); Lemma 9.4 proves positivity. □

Remark 9.5. In Theorem 9.1,

$$\frac{(\sigma_n)_{v_n v_n}}{(\sigma_{\mathrm{ref}})_{v_n v_n}} \asymp \frac{e^{n^6 - n}}{n^4}.$$

The maximal reference entropy of the displayed connectors is of order n^4, whereas their maximal moving entropy is at most $F_z + O(n^{-1})$. Their stationary limit is faithful, but no uniform relative-order or reference-entropy bound holds. This family is not the finite-core concentration family of Theorems A and B.

Nor does the smallness of the additive generator difference (9.9) contradict Proposition 10.1. At a block state, the row mobility of the full pair is $\Lambda(bp, aB)$. In general,

$$\Lambda((b + \Delta b)p, (a + \Delta a)B) - \Lambda(bp, aB) \ne \Lambda(\Delta b\, p, \Delta a\, B).$$

For example, in the scalar case $p = B = 1/2$, $(a, b) = (1, 4)$ and $(\Delta a, \Delta b) = (4, 1)$ give

$$\Lambda(5/2, 5/2) = 5/2 > 2\Lambda(2, 1/2) = 3/\log 4,$$

by the strict arithmetic-mean bound for unequal arguments. In this construction, adding the two Lindblad rate pairs changes the modular ratio of the total pair; its logarithmic cotangent form is not the sum of the two separately constructed cotangent forms. Proposition 10.1 bounds the form attached to the entire small auxiliary generator, not the change of the form of a fixed unbounded reference under this perturbation.

9.4. **The faithful-reference energy limit.**

Corollary 9.6. *For the stationary states in Lemma 9.2,*

$$D(\,\cdot\,\|\sigma_n) \xrightarrow{\Gamma} D(\,\cdot\,\|\sigma_{\mathrm{ref}}) \tag{9.11}$$

in the trace-norm topology on $\mathfrak{S}(\mathcal{H})$. Every constant sequence $\rho_n = \rho$ with $F_{\mathrm{ref}}(\rho) < \infty$ recovers the limiting entropy, in fact $F_n(\rho) \to F_{\mathrm{ref}}(\rho)$.

Proof. Let $\rho_n \to \rho$ in trace norm. By Lemma 9.2, $\sigma_n \to \sigma_{\mathrm{ref}}$ in the same norm. Joint lower semicontinuity of relative entropy [8, Lemma 4] gives

$$F_{\mathrm{ref}}(\rho) \le \liminf_n F_n(\rho_n).$$

For $F_{\mathrm{ref}}(\rho) < \infty$, formula (9.8) and $r_n \geq 0$ give

$$F_n(\rho) \leq F_{\mathrm{ref}}(\rho) + \log(1 + r_n) \longrightarrow F_{\mathrm{ref}}(\rho).$$

Together with the preceding lower bound applied to the constant sequence, this proves both recovery and pointwise convergence at every finite-entropy state. If $F_{\mathrm{ref}}(\rho) = +\infty$, the upper recovery inequality is automatic. This proves (9.11). □

10. Bounds for complete auxiliary perturbations

Let $\widetilde{\mathcal{K}}_*$ denote the core generator extended by zero, as in Section 2. It has no auxiliary channels, and its semigroup fixes every $E_{\nu_n\nu_n}$.

Proposition 10.1. *Let $a_n, b_n > 0$, with $\sum_n a_n < \infty$, and set*

$$\mathcal{Q}_* = \sum_n a_n \mathcal{D}_{E_{\nu_n h}} + \sum_n b_n \mathcal{D}_{E_{h\nu_n}}.$$

Suppose that $\mathcal{Q}_*$ extends to a bounded map on trace class with finite diamond norm η. Then

$$\sum_n a_n \leq \eta/2, \qquad \sup_n b_n \leq \eta/2. \tag{10.1}$$

For every state ρ and every Hermitian core test C extended by zero,

$$g_{\rho,\mathrm{aux}}(C) \leq \frac{\eta}{4}\|Ce_h\|^2. \tag{10.2}$$

Proof. Evaluation on diagonal matrix units gives

$$\mathcal{Q}_*(P) = \sum_n a_n(E_{\nu_n\nu_n} - P), \qquad \mathcal{Q}_*(E_{\nu_n\nu_n}) = b_n(P - E_{\nu_n\nu_n}).$$

Their trace norms are $2\sum_n a_n$ and $2b_n$, proving (10.1). Put $B = P_0\rho P_0$, $p_n = \rho_{\nu_n\nu_n}$, $s = \sum_n p_n$, and $v = Ce_h$. Phase averaging and the row-mobility identity give

$$\begin{aligned} g_{\rho,\mathrm{aux}}(C) &\leq \sum_n \langle v, \Lambda(b_n p_n, a_n B)v\rangle \\ &\leq \frac{1}{2}\left[\left(\sum_n b_n p_n\right)\|v\|^2 + \left(\sum_n a_n\right)\langle v, Bv\rangle\right] \\ &\leq \frac{\eta}{4}\left[s\|v\|^2 + \langle v, Bv\rangle\right] \leq \frac{\eta}{4}\|v\|^2. \end{aligned}$$

The second inequality follows from $\Lambda(x, aB) \leq (xI_0 + aB)/2$ by spectral calculus, including at zero eigenvalues. The last inequality uses $B \leq (\operatorname{Tr} B)I_0 = (1 - s)I_0$. □

Under (1.8), $\chi > 0$ implies that (b_n) is unbounded. Thus $\mathcal{L}_{\delta,*} - \widetilde{\mathcal{K}}_*$ has no bounded extension on trace class. More generally, (10.2) makes the auxiliary cotangent form tend uniformly to zero on core tests whenever the complete auxiliary generator tends to zero in diamond norm.

Proposition 10.2. *Let (S_t) and (U_t) be completely positive trace-preserving semigroups on trace class. Suppose that, for one state τ,*

$$S_t E_{\nu_n\nu_n} \longrightarrow \tau \quad (t \to \infty) \quad \textit{for every } n, \qquad U_t E_{\nu_n\nu_n} = E_{\nu_n\nu_n} \quad (t \geq 0).$$

Then

$$\sup_{t\geq 0}\|S_t - U_t\|_\diamond = 2.$$

Proof. Fix n. Contractivity of the two-outcome measurement $\{E_{\nu_n\nu_n}, I - E_{\nu_n\nu_n}\}$ gives

$$\sup_{t\geq 0}\|S_t - U_t\|_\diamond \geq \lim_{t\to\infty}\|(S_t - U_t)E_{\nu_n\nu_n}\|_1 = \|\tau - E_{\nu_n\nu_n}\|_1 \geq 2(1 - \tau_{\nu_n\nu_n}).$$

Since $\sum_n \tau_{\nu_n\nu_n} \leq 1$, these diagonal entries tend to zero. Taking the supremum over n gives the lower bound two. Complete trace-norm contractivity of each channel gives the upper bound two. □

Proposition 10.3. *Fix $b_n > 0$. Let reversible extensions have outward rates a_n^δ with $\sum_n a_n^\delta \to 0$ and stationary states satisfying*

$$(\sigma_\delta)_{v_n v_n} = (\sigma_\delta)_{hh} \frac{a_n^\delta}{b_n}.$$

Every trace-norm limit of these stationary states is core-supported. If $b_n \geq b_* > 0$ for all n, then

$$\operatorname{Tr}((I - P_0)\sigma_\delta) \leq b_*^{-1} \sum_n a_n^\delta.$$

For extensions of the fixed primitive core (2.2)*, every trace-norm limit is $\sigma_0 \oplus 0$. Under the additional bound $b_n \geq b_* > 0$, the whole family converges to this state.*

Proof. For each fixed n, $0 < a_n^\delta \leq \sum_j a_j^\delta \to 0$ and $0 \leq (\sigma_\delta)_{hh} \leq 1$. Detailed balance gives $(\sigma_\delta)_{v_n v_n} \to 0$. If $\sigma_{\delta_j} \to \tau$ in trace norm, then $\tau_{v_n v_n} = 0$ for every n. For a positive operator, $|\langle v, \tau e_{v_n} \rangle|^2 \leq \langle v, \tau v \rangle \tau_{v_n v_n} = 0$. Hence τ and its cross blocks vanish off the core.

If $b_n \geq b_* > 0$, summing detailed balance yields

$$\operatorname{Tr}((I - P_0)\sigma_\delta) = (\sigma_\delta)_{hh} \sum_n \frac{a_n^\delta}{b_n} \leq b_*^{-1} \sum_n a_n^\delta \longrightarrow 0.$$

For extensions of the fixed primitive core, the stationary calculation in Proposition 2.2 gives a core block $c_\delta \sigma_0$ and no cross blocks. Every trace-norm cluster point therefore equals $\sigma_0 \oplus 0$. Under the lower bound on the returns, the tail mass s_δ tends to zero and $c_\delta = 1 - s_\delta$. The block decomposition gives

$$\|\sigma_\delta - (\sigma_0 \oplus 0)\|_1 = 2 s_\delta \leq 2 b_*^{-1} \sum_n a_n^\delta \longrightarrow 0.$$

□

Statements and Declarations

Funding. The author received no funding for this work.

Competing interests. The author declares no competing interests.

Data availability. No datasets were generated or analysed during the current study.

Code availability. The results are analytical and do not require computational code.

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