# HOMOCYCLIC JACOBIANS OF PLANAR BICONNECTED GRAPHS

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Abstract. For every $r \geq 1$, we construct a simple planar biconnected graph with $2r+2$ vertices, $4r+1$ edges, maximum degree at most four, and Jacobian $(\mathbb{Z}/8\mathbb{Z})^r$. This disproves the bounded-multiplicity conjecture of Gaudet, Jensen, Ranganathan, Wawrykow and Weisman. We give explicit generators and compute the monodromy pairing: its matrix is $B_r/8$ modulo $\mathbb{Z}$, where $B_r$ is tridiagonal with diagonal $4, \dots, 4, 5$ and adjacent entries $-1$. For every finite tree $T$ with at least two vertices and every integer $m \geq 2$, we also prove

$$\exp \operatorname{Jac}(T[\overline{K_m}]) = m^2 \operatorname{lcm}_{v \in V(T)} \deg_T(v).$$

For paths with at least three vertices, these products have vertex connectivity $m$ and exponent $2m^2$. They disprove the bounded-exponent finiteness conjecture attributed to Baker and Shokrieh, including its analogue for connected regular matroids.

## 1. Introduction

The Jacobian of a finite connected graph is a finite abelian group with a canonical nondegenerate symmetric pairing into $\mathbb{Q}/\mathbb{Z}$. Its order is the number of spanning trees. Its exponent is the least positive integer annihilating the group. We use the divisor and Laplacian descriptions of this group; see [1, 3, 2, 9], and [13, Theorem 3.4 and Proposition 3.7] for the pairing. The terms *critical group* and *sandpile group* denote the same finite abelian group.

If parallel edges are allowed, every finite abelian group occurs as a graph Jacobian. Indeed, a cycle of length $a \geq 2$ has Jacobian $\mathbb{Z}/a\mathbb{Z}$, and

$$\operatorname{Jac}(G_1 \vee \cdots \vee G_k) \cong \bigoplus_{i=1}^{k} \operatorname{Jac}(G_i);$$

see [5, Section 2.3]. Here the cycle of length two has two parallel edges, and $\vee$ identifies one chosen vertex in each graph. The trivial group is the Jacobian of a tree. If $k \geq 2$ and each $G_i$ has at least two vertices, deleting the identified vertex disconnects the wedge. In this paper, a graph is *biconnected* if it has at least three vertices and deletion of any one vertex leaves a connected graph.

Gaudet, Jensen, Ranganathan, Wawrykow and Weisman conjectured that, for each $a \geq 2$, there is an integer $b_a$ such that no biconnected graph has Jacobian $(\mathbb{Z}/a\mathbb{Z})^b$ with $b > b_a$ [5, arXiv version 2, Conjecture 36]. A group $(\mathbb{Z}/a\mathbb{Z})^b$ is called *homocyclic*. We construct such graphs for $a = 8$ and every $b \geq 1$, subject also to simplicity, planarity and a degree bound of four.

Lheem, Li, Quines and Zhang attribute to Baker and Shokrieh the conjecture

$$\#\{[G] \,:\, G \text{ is biconnected}, \ \exp \operatorname{Jac}(G) \leq b\} < \infty \qquad (b \geq 1), \tag{1.1}$$

where $[G]$ denotes the isomorphism class. They prove the cases $b = 2, 3$ and state the corresponding conjecture for connected regular matroids [8, Conjectures 1.1–1.2 and Section 3.1]. Conjecture numbers refer throughout to the specified revised arXiv versions. The examples below contradict both finiteness assertions at exponent eight.

All graphs considered from now on are finite, undirected and simple. Let $T = (V, E)$ be a tree, $n = |V| \geq 2$, and $\delta_v = \deg_T(v)$. For an integer $m \geq 2$, define

$$V(G_m(T)) = V \times \{1, \dots, m\}, \qquad (v, a) \sim (w, b) \iff \{v, w\} \in E. \tag{1.2}$$

2020 *Mathematics Subject Classification.* 05C25, 05C50, 05C05, 05C40, 05B35.

*Key words and phrases.* Graph Jacobian, critical group, homocyclic group, monodromy pairing, exponent, vertex connectivity.

Thus $G_m(T) = T[\overline{K_m}]$, where $\overline{K_m}$ is the edgeless graph on $m$ vertices. Write $V_v = \{v\} \times \{1, \dots, m\}$ for the *layer* over $v$. Then

$$|V(G_m(T))| = mn, \qquad |E(G_m(T))| = m^2(n-1), \qquad \deg_{G_m(T)}(v,a) = m\delta_v. \tag{1.3}$$

A path in $T$ gives a path through any prescribed vertices in its successive layers. Distinct vertices in one layer are joined by a path of length two through a neighbouring layer. Such a layer exists since $n \geq 2$. Hence $G_m(T)$ is connected. We write $\kappa(G)$ for the least cardinality of a set of vertices whose deletion disconnects $G$ or leaves at most one vertex.

Let $P_t$ be the path on $\{1, \dots, t\}$ and put $G_{m,t} = G_m(P_t)$. For $t \geq 2$, write $a_i = (i, 1)$, $b_i = (i, 2)$ and set

$$H_t = G_{2,t} + \{a_1, b_1\}, \qquad D_i = \mathbf{e}_{a_i} - \mathbf{e}_{a_{i+1}} \quad (1 \leq i < t). \tag{1.4}$$

Here $\mathbf{e}_v$ denotes the divisor with coefficient one at $v$ and zero elsewhere. For $r \geq 1$, define the symmetric matrix $B_r$ by

$$(B_r)_{ij} = \begin{cases} 4, & i = j < r, \\ 5, & i = j = r, \\ -1, & |i-j| = 1, \\ 0, & \text{otherwise.} \end{cases} \tag{1.5}$$

In particular, $B_1 = (5)$.

**Theorem 1.1.** *For every $t \geq 2$, the graph $H_t$ is simple, planar and biconnected. It has $2t$ vertices, $4t-3$ edges and maximum degree at most four. The homomorphism*

$$(\mathbb{Z}/8\mathbb{Z})^{t-1} \longrightarrow \operatorname{Jac}(H_t), \qquad (c_i) \longmapsto \sum_{i=1}^{t-1} c_i[D_i], \tag{1.6}$$

*is an isomorphism. In these generators, the monodromy pairing is*

$$\langle [D_i], [D_j] \rangle = \frac{(B_{t-1})_{ij}}{8} \pmod{\mathbb{Z}}. \tag{1.7}$$

**Theorem 1.2.** *For every tree $T$ with $n \geq 2$ vertices and every integer $m \geq 2$,*

$$\exp \operatorname{Jac}(G_m(T)) = m^2 \operatorname{lcm}_{v \in V(T)} \delta_v, \tag{1.8}$$

$$|\operatorname{Jac}(G_m(T))| = m^{mn-2} \prod_{v \in V(T)} \delta_v^{m-1}, \tag{1.9}$$

$$\kappa(G_m(T)) = m. \tag{1.10}$$

**Corollary 1.3.** *For each fixed $m \geq 2$, the graphs $G_{m,t}$, $t \geq 3$, are pairwise nonisomorphic, $m$-connected and bipartite, and*

$$\exp \operatorname{Jac}(G_{m,t}) = 2m^2, \qquad |\operatorname{Jac}(G_{m,t})| = m^{mt-2} 2^{(m-1)(t-2)}. \tag{1.11}$$

*For $m = 2$, they are planar and have maximum degree four.*

1.1. **Earlier results.** The construction $G[\overline{K_2}]$ is the double graph of Munarini, Perelli Cippo, Scagliola and Zagaglia Salvi [11]. For general $m$, the Laplacian spectral formula is stated in [12, Theorem 4.1], where it is attributed to Marino–Zagaglia Salvi [10]. In particular, (1.9) is the tree case of [10, Theorem 20]; we include its spectral proof for completeness. When $T = P_2$, the product is $K_{m,m}$, included among the complete multipartite graphs treated in [7].

Dong, Guo and Jiang [4] study graphs with independent parts of sizes $n_1, \dots, n_t$ and all edges between consecutive parts. Their graph for $n_1 = \cdots = n_t = m$ is $G_{m,t}$. Their Proposition 2.1 gives an integral reduction of the Laplacian; they determine the critical groups for two through six parts. The present results concern the exponent for every underlying tree and the group with pairing of $H_t$ for every $t \geq 2$. The latter graph does not belong to that bipartite family, since $a_1, b_1, a_2$ form a triangle.

For odd $u$, let $\langle u/8\rangle$ denote the pairing $(x,y)\mapsto uxy/8 \pmod{\mathbb{Z}}$ on $\mathbb{Z}/8\mathbb{Z}$. Corollary 6.5 identifies the pairing of $H_{r+1}$ with

$$\langle 5/8\rangle^{\perp\lceil r/2\rceil} \perp \langle 7/8\rangle^{\perp\lfloor r/2\rfloor}.$$

The two cyclic forms are $\mathcal{C}_8$ and $\mathcal{B}_8$ in [5, Section 2.4]. This is a realization of the specified orthogonal sums, not a classification of pairings on homocyclic groups. For $r=1$, the graph is $H_2 = K_{1,1,2}$.

1.2. **Organization.** Section 2 gives the potential criterion for orders of divisor classes. The proofs of (1.8), (1.10) and (1.9) are in Sections 3, 4 and 5, respectively. Sections 4 and 5 prove Corollary 1.3. Section 6 proves Theorem 1.1 directly from the inverse Laplacian and the pairing on the divisors $D_i$; Section 7 gives the matroid consequences. Appendix A contains the Smith normal form and matrix-tree arguments. Appendix B gives a second determination of $\mathrm{Jac}(H_t)$ by reduction modulo two. Appendix B is not used in the proof of Theorem 1.1.

## 2. Divisors and rational potentials

Let $G=(V,E)$ be connected, with $N=|V|\geq 2$. Set

$$\mathrm{Div}^0(G)=\{D\in\mathbb{Z}^V : \textstyle\sum_v D_v=0\}, \qquad \mathrm{Jac}(G)=\mathrm{Div}^0(G)/L_G\mathbb{Z}^V, \tag{2.1}$$

where

$$(L_Gx)_v=\deg_G(v)x_v-\sum_{w\sim v}x_w. \tag{2.2}$$

Vectors in $L_G\mathbb{Z}^V$ are called principal divisors. A rational potential of $D$ is a vector $x\in\mathbb{Q}^V$ such that $L_Gx=D$. The symmetry and zero row sums of $L_G$, together with

$$x^{\mathsf{T}}L_Gx=\sum_{\{v,w\}\in E}(x_v-x_w)^2 \quad (x\in\mathbb{R}^V), \tag{2.3}$$

give

$$\ker_{\mathbb{Q}} L_G=\mathbb{Q}\mathbf{1}, \qquad L_G\mathbb{Q}^V=\{D\in\mathbb{Q}^V : \textstyle\sum_v D_v=0\}. \tag{2.4}$$

Indeed, equality to zero in (2.3) implies equality of the coordinates on each edge, hence on $V$. The image has dimension $N-1$ and is contained in the displayed hyperplane, which has the same dimension.

**Lemma 2.1.** *Delete row and column $o\in V$ from $L_G$, obtaining $L_o$. Then $L_o$ is positive definite and*

$$\mathrm{Jac}(G)\cong\mathbb{Z}^{V\setminus\{o\}}/L_o\mathbb{Z}^{V\setminus\{o\}}, \qquad |\mathrm{Jac}(G)|=\det L_o. \tag{2.5}$$

*Proof.* For $y\in\mathbb{R}^{V\setminus\{o\}}$, extend $y$ to $x$ by $x_o=0$. Then $y^{\mathsf{T}}L_oy=x^{\mathsf{T}}L_Gx\geq 0$; equality implies $x\in\mathbb{R}\mathbf{1}$ and $x_o=0$, hence $y=0$. Restriction $\rho : \mathrm{Div}^0(G)\to\mathbb{Z}^{V\setminus\{o\}}$ is an isomorphism, with

$$(\rho^{-1}y)_v=y_v\ (v\neq o), \qquad (\rho^{-1}y)_o=-\sum_{v\neq o}y_v.$$

For $v\neq o$, $\rho(L_G\mathbf{e}_v)=L_o\mathbf{e}_v$, and $\rho(L_G\mathbf{e}_o)=-\sum_{v\neq o}L_o\mathbf{e}_v$. Thus $\rho(L_G\mathbb{Z}^V)=L_o\mathbb{Z}^{V\setminus\{o\}}$. The order formula follows from Lemma A.1, since $\det L_o>0$. ☐

**Lemma 2.2.** *The matrices $L_G$ and* $\mathrm{diag}(L_o,0)$ *are integrally congruent. In particular, $L_G$ and $L_o$ have the same rank over every field.*

*Proof.* Place $o$ last and let $\mathbf{1}\in\mathbb{Z}^{N-1}$. The zero row sums give

$$L_G=\begin{pmatrix} L_o & -L_o\mathbf{1}\\ -\mathbf{1}^{\mathsf{T}}L_o & \mathbf{1}^{\mathsf{T}}L_o\mathbf{1}\end{pmatrix}.$$

For $S=\begin{pmatrix} I_{N-1} & \mathbf{1}\\ 0 & 1\end{pmatrix}$, one has $\det S=1$ and $S^{\mathsf{T}}L_GS=\mathrm{diag}(L_o,0)$. The matrix $S$ is invertible over every field. ☐

**Lemma 2.3.** *Let $D \in \mathrm{Div}^0(G)$ and $L_G x = D$, with $x \in \mathbb{Q}^V$. For every positive integer $q$,*

$$q[D] = 0 \iff q(x_v - x_w) \in \mathbb{Z} \quad \text{for all } v, w \in V. \tag{2.6}$$

*In particular, $qx \in \mathbb{Z}^V$ implies $q[D] = 0$.*

*Proof.* If $qD = L_G y$ with $y \in \mathbb{Z}^V$, then $qx - y \in \ker_{\mathbb{Q}} L_G = \mathbb{Q}\mathbf{1}$. Hence $q(x_v - x_w) = y_v - y_w \in \mathbb{Z}$. Conversely, fix $o \in V$ and put $y = q(x - x_o \mathbf{1})$. The assumed differences imply $y \in \mathbb{Z}^V$, and $L_G y = qD$. □

**Corollary 2.4.** *The exponent of* $\mathrm{Jac}(G)$ *is the least positive integer $q$ such that $qL_o^{-1}$ is integral. Equivalently, it is the least common multiple of the reduced denominators of the entries of $L_o^{-1}$.*

*Proof.* The divisors $\mathbf{e}_v - \mathbf{e}_o$, $v \neq o$, form a basis of $\mathrm{Div}^0(G)$. Their potentials with coordinate zero at $o$ are the columns of $L_o^{-1}$, extended by zero. Apply Lemma 2.3 to these basis divisors. We use denominator one for a zero entry. □

**Lemma 2.5.** *For $D, E \in \mathrm{Div}^0(G)$ and $L_G x_E = E$, the formula*

$$\langle [D], [E] \rangle = D^{\mathsf{T}} x_E \pmod{\mathbb{Z}} \tag{2.7}$$

*defines a nondegenerate symmetric bilinear pairing on* $\mathrm{Jac}(G)$.

*Proof.* Changing $x_E$ by a constant does not change $D^{\mathsf{T}} x_E$. For $y, z \in \mathbb{Z}^V$, replacing $D$ by $D + L_G y$ changes its value by $y^{\mathsf{T}} E \in \mathbb{Z}$, and replacing $E$ by $E + L_G z$ changes it by $D^{\mathsf{T}} z \in \mathbb{Z}$. The formula is therefore well defined. The sum of two potentials is a potential of the sum, which proves bilinearity. If $L_G x_D = D$, symmetry follows from

$$D^{\mathsf{T}} x_E = x_D^{\mathsf{T}} L_G x_E = E^{\mathsf{T}} x_D.$$

Suppose $[D]$ pairs to zero with every class. Normalize $x_{D,o} = 0$. Pairing with $\mathbf{e}_v - \mathbf{e}_o$ gives $x_{D,v} \in \mathbb{Z}$ for every $v \neq o$. Thus $D = L_G x_D$ is principal. The induced map to $\mathrm{Hom}(\mathrm{Jac}(G), \mathbb{Q}/\mathbb{Z})$ is injective. It is an isomorphism: by Lemma A.1 the two groups have the same order, since $\mathrm{Hom}(\mathbb{Z}/s\mathbb{Z}, \mathbb{Q}/\mathbb{Z})$ has exactly $s$ elements. □

The sign in (2.7) is the positive-Laplacian convention of [13].

**Lemma 2.6.** *Let $q \geq 1$, let $C \in \mathrm{Mat}_q(\mathbb{Z})$ be nonsingular, and put $A = \mathbb{Z}^q / C\mathbb{Z}^q$. If $8A = 0$, then*

$$A \cong (\mathbb{Z}/2\mathbb{Z})^a \oplus (\mathbb{Z}/4\mathbb{Z})^b \oplus (\mathbb{Z}/8\mathbb{Z})^c \tag{2.8}$$

*for nonnegative integers $a, b, c$, with*

$$a + b + c = q - \mathrm{rank}_{\mathbb{F}_2}(C \bmod 2), \qquad |\det C| = 2^{a+2b+3c}. \tag{2.9}$$

*Proof.* The invariant factors in Lemma A.1 divide eight, hence belong to $\{1, 2, 4, 8\}$. Let $a, b, c$ be the multiplicities of $2, 4, 8$, respectively. Then $a + b + c$ is the dimension of

$$A/2A = \mathbb{Z}^q / (C\mathbb{Z}^q + 2\mathbb{Z}^q) \cong \mathbb{F}_2^q / (C \bmod 2)\mathbb{F}_2^q.$$

The order of $A$ is $2^{a+2b+3c} = |\det C|$. □

## 3. Exponents of tree products

**Lemma 3.1.** *Let $T$ be a nonempty finite tree and $s \in \mathbb{Z}^{V(T)}$ satisfy $\sum_v s_v = 0$. There exists $z \in \mathbb{Z}^{V(T)}$ with $L_T z = s$.*

*Proof.* Root $T$ at $o$. For $v \neq o$, let $p(v)$ be its parent, let $T_v$ be the component of $T - \{vp(v)\}$ containing $v$, and put $S_v = \sum_{w \in V(T_v)} s_w$. If $o = v_0, v_1, \dots, v_k = v$ is the unique path from $o$ to $v$, set

$$z_o = 0, \qquad z_v = \sum_{j=1}^{k} S_{v_j}. \tag{3.1}$$

Then $z_v - z_{p(v)} = S_v$. Let $\mathrm{Ch}(v)$ denote the children of $v$. The disjoint union $V(T_v) = \{v\} \sqcup \bigsqcup_{w \in \mathrm{Ch}(v)} V(T_w)$ gives

$$(L_T z)_v = S_v - \sum_{w \in \mathrm{Ch}(v)} S_w = s_v \quad (v \neq o). \tag{3.2}$$

At the root, $(L_T z)_o = -\sum_{w \in \mathrm{Ch}(o)} S_w = -\sum_{v \neq o} s_v = s_o$. This also covers the one-vertex tree. □

For $G_m(T)$, the Laplacian is

$$(L_{G_m(T)} x)_{v,a} = m\delta_v x_{v,a} - \sum_{w \sim v} \sum_{b=1}^{m} x_{w,b}. \tag{3.3}$$

All $\delta_v$ are positive, since $T$ is connected and $n \geq 2$.

**Lemma 3.2.** *Let $D \in \mathrm{Div}^0(G_m(T))$, set $s_v = \sum_{a=1}^m D_{v,a}$, and choose an integral solution of $L_T z = s$. Then*

$$x_{v,a} = \frac{z_v}{m^2} + \frac{mD_{v,a} - s_v}{m^2 \delta_v} \tag{3.4}$$

*is a rational potential of $D$.*

*Proof.* The vector $s$ has total sum zero, so Lemma 3.1 applies. Since $\sum_a (mD_{v,a} - s_v) = 0$, one has $\sum_a x_{v,a} = z_v/m$. Therefore

$$\begin{aligned}(L_{G_m(T)} x)_{v,a} &= \frac{\delta_v z_v}{m} + D_{v,a} - \frac{s_v}{m} - \frac{1}{m} \sum_{w \sim v} z_w \\ &= D_{v,a} + \frac{(L_T z)_v - s_v}{m} = D_{v,a}.\end{aligned}$$

□

*Proof of* (1.8). Put $d_T = \mathrm{lcm}_v\, \delta_v$ and $E = m^2 d_T$. For every $D$ and its potential (3.4),

$$Ex_{v,a} = d_T z_v + \frac{d_T}{\delta_v}(mD_{v,a} - s_v) \in \mathbb{Z}. \tag{3.5}$$

Hence $ED = L_{G_m(T)}(Ex)$ is principal, and

$$\exp \mathrm{Jac}(G_m(T)) \mid m^2 d_T. \tag{3.6}$$

Assume $n \geq 3$. Let $w$ satisfy $\delta_w \geq 2$, choose a neighbour $u$ of $w$, and let $U$ be the component of $T - \{uw\}$ containing $u$. For $z = \mathbf{1}_U$, the only edge with unequal endpoint values is $uw$; thus

$$L_T z = \mathbf{e}_u - \mathbf{e}_w.$$

Take $D = \mathbf{e}_{(u,1)} - \mathbf{e}_{(w,1)}$ in Lemma 3.2. Since $z_w = 0$, $s_w = -1$ and $D_{w,2} = 0$, one obtains

$$x_{w,2} = \frac{1}{m^2 \delta_w}. \tag{3.7}$$

Choose $w' \sim w$ with $w' \neq u$. The vertices $w, w'$ belong to the same component of $T - \{uw\}$. Hence $z_{w'} = 0$; also $D_{w',a} = s_{w'} = 0$, so $x_{w',1} = 0$. By Lemma 2.3,

$$q[D] = 0 \quad \Longrightarrow \quad \frac{q}{m^2 \delta_w} \in \mathbb{Z}.$$

Thus $m^2 \delta_w$ divides $\exp \operatorname{Jac}(G_m(T))$ for every nonleaf $w$. There is at least one nonleaf, since $2(n-1) > n$. Leaf degrees are one, and consequently $m^2 d_T \mid \exp \operatorname{Jac}(G_m(T))$.

For $n = 2$, write $V(T) = \{u, w\}$. Take $z_u = 1$, $z_w = 0$ and $D = \mathbf{e}_{(u,1)} - \mathbf{e}_{(w,1)}$. Formula (3.4) gives $x_{u,2} = 0$ and $x_{w,2} = 1/m^2$. Lemma 2.3 implies $m^2 \mid \exp \operatorname{Jac}(G_m(T))$. Since $d_T = 1$, this and (3.6) finish the proof. □

**Corollary 3.3.** *If every vertex of $T$ has degree at most an integer $\Delta \geq 1$, then*

$$\exp \operatorname{Jac}(G_m(T)) \mid m^2 \operatorname{lcm}(1, \dots, \Delta).$$

*Proof.* Each $\delta_v$ divides $\operatorname{lcm}(1, \dots, \Delta)$. Apply (1.8). □

## 4. Vertex connectivity and planarity

**Proposition 4.1.** *For every $T$ and $m$ in Theorem 1.2, $\kappa(G_m(T)) = m$.*

*Proof.* Let $S \subseteq V(G_m(T))$ satisfy $|S| < m$. Then $V_v \setminus S \neq \emptyset$ for every $v$. For vertices in distinct layers, choose the unique path between the corresponding vertices of $T$ and a vertex outside $S$ in every intermediate layer. Formula (1.2) gives a path in $G_m(T) - S$. Distinct vertices in one layer are joined through a vertex outside $S$ in a neighbouring layer. Thus $G_m(T) - S$ is connected and has at least $mn - (m - 1) = m(n - 1) + 1 > 1$ vertices. Hence $\kappa(G_m(T)) \geq m$. If $v$ is a leaf with neighbour $w$, deletion of $V_w$ leaves the $m \geq 2$ vertices of $V_v$ isolated. Thus $\kappa(G_m(T)) \leq |V_w| = m$. □

**Lemma 4.2.** *Let a finite graph be embedded in $S^2$, and let $F$ be a face with a homeomorphism $h : \overline{\mathbb{D}} \to \overline{F}$ satisfying $h(\mathbb{D}) = F$, where $\mathbb{D}$ is the open unit disk. Let $a, b$ be distinct vertices in $h(\partial \mathbb{D})$. The embedding extends after adjoining two vertices $c, d$ and the edges $ac, bc, ad, bd$. The extended embedding has a face $F'$ bounded by $a, c, b, d, a$, with*

$$(\overline{F'}, F') \cong (\overline{\mathbb{D}}, \mathbb{D}).$$

*Proof.* Choose a circle homeomorphism $\psi$ taking $(-1, 0), (1, 0)$ to $h^{-1}(a), h^{-1}(b)$, respectively. Such a map is obtained by choosing a homeomorphism on each of the two complementary closed arcs with these endpoint values. Its radial extension $r\zeta \mapsto r\psi(\zeta)$ is a homeomorphism of $\overline{\mathbb{D}}$, with inverse $r\zeta \mapsto r\psi^{-1}(\zeta)$. After composition, assume $h(-1, 0) = a$ and $h(1, 0) = b$. Define

$$\gamma_\pm(s) = (2s - 1,\ \pm s(1 - s)), \qquad 0 \leq s \leq 1. \tag{4.1}$$

Each arc is injective, and their interiors are disjoint. For $0 < s < 1$, put $u = s(1 - s) \in (0, 1/4]$. Then $|\gamma_\pm(s)|^2 = 1 - 4u + u^2 < 1$. Adjoin $c = h(\gamma_+(1/2))$, $d = h(\gamma_-(1/2))$ and the four half-arcs as the new edges. Their interiors lie in $F$ and hence are disjoint from the original graph.

Put

$$K = \{(x, y) : -1 \leq x \leq 1,\ |y| \leq (1 - x^2)/4\}.$$

The map

$$f : \overline{\mathbb{D}} \longrightarrow K, \qquad f(u, v) = \left(u, \frac{1}{4}\sqrt{1 - u^2}\, v\right) \tag{4.2}$$

is continuous and bijective. For $|u| < 1$, its inverse has second coordinate $4y/\sqrt{1 - u^2}$; at $u = \pm 1$, both domain and image have only second coordinate zero. Compactness makes $f$ a homeomorphism. The strict disk inequality gives $f(\mathbb{D}) = K^\circ$. The set $F' = h(K^\circ)$ is connected and open in the complement of the extended graph, while $\overline{F'} \setminus F' = h(\partial K)$ belongs to that graph. Thus $F'$ is also relatively closed in the complement and is a connected component, hence a face. Its boundary is the indicated four-cycle, and $h \circ f$ is the required homeomorphism of pairs. □

**Proposition 4.3.** *For every $t \geq 2$, the graphs $G_{2,t}$ and $H_t$ are planar. Each admits an embedding with a face as in Lemma 4.2 whose boundary contains $a_t, b_t$.*

*Proof.* Embed $G_{2,2}$ as the quadrilateral $a_1, a_2, b_1, b_2, a_1$. Either complementary face satisfies the assertion. For $H_2$, embed the diagonal $a_1 b_1$ in one of those faces; the other still satisfies the assertion for $a_2, b_2$. Assume an embedding with the stated face has been constructed for $t$. Apply Lemma 4.2 with $a = a_t$, $b = b_t$, $c = a_{t+1}$ and $d = b_{t+1}$. The added edges are exactly the four edges between layers $t$ and $t+1$. The new face satisfies the same hypothesis with boundary vertices $a_{t+1}, b_{t+1}$. Induction proves both assertions. □

*Graph properties and exponent in Corollary 1.3.* For $t \geq 3$, the degrees of $P_t$ are $1, 1$ and $t-2$ occurrences of 2. Formula (1.8) gives exponent $2m^2$. Proposition 4.1 gives connectivity $m$. The unions of the odd layers and of the even layers form a bipartition. The vertex count $mt$ is strictly increasing in $t$. There are $2m$ vertices of degree $m$ and $m(t-2)$ of degree $2m$. For $m = 2$, the maximum degree is four, and Proposition 4.3 gives planarity. The order formula is proved in Section 5. □

**Corollary 4.4.** *There are infinitely many pairwise nonisomorphic simple planar bipartite biconnected graphs of maximum degree four and Jacobian exponent eight. Thus* (1.1) *is false for $b = 8$. For every fixed $m \geq 2$, there are infinitely many pairwise nonisomorphic simple $m$-connected graphs of exponent $2m^2$.*

*Proof.* Take $G_{2,t}$, $t \geq 3$, in the first assertion, and $G_{m,t}$ in the second. The preceding proof establishes all the properties used. □

For the first family, a divisor of order eight is $D = \mathbf{e}_{(1,1)} - \mathbf{e}_{(2,1)}$. In fact, on $G_{2,t}$, $t \geq 3$, set

$$(x_{1,1}, x_{1,2}) = (1/2, 0), \qquad (x_{2,1}, x_{2,2}) = (-1/8, 1/8), \qquad x_{i,a} = 0 \quad (i \geq 3).$$

Formula (3.3) gives $(Lx)_{1,1} = 1$, $(Lx)_{1,2} = 0$, $(Lx)_{2,1} = -1$ and $(Lx)_{2,2} = 0$. All remaining coordinates vanish, since the sum on layer two is zero. Moreover, $8x$ is integral and $x_{2,2} - x_{3,1} = 1/8$. Lemma 2.3 proves the assertion.

## 5. Orders and invariant factors

*Proof of* (1.9). Let $U$ be the space of real functions constant on each layer, and let $W_v$ be the space of functions supported on $V_v$ with sum zero. Then

$$\mathbb{R}^{V(G_m(T))} = U \perp \bigoplus_{v \in V(T)} W_v, \qquad \dim U = n, \quad \dim W_v = m-1. \tag{5.1}$$

For $x_{v,a} = y_v$, formula (3.3) gives $(Lx)_{v,a} = m(L_T y)_v$. For $x \in W_v$, it gives $Lx = m\delta_v x$. Thus the summands are invariant; the nonzero spectrum is the nonzero spectrum of $mL_T$, together with $m\delta_v$ repeated $m-1$ times for each $v$. Since $\tau(T) = 1$, Lemma A.2 gives $\prod_{i=1}^{n-1} \lambda_i(T) = n$ and therefore

$$|\operatorname{Jac}(G_m(T))| = \frac{m^{n-1} n}{mn} \prod_v (m\delta_v)^{m-1} = m^{mn-2} \prod_v \delta_v^{m-1}. \tag{5.2}$$

For $T = P_t$, this is $m^{mt-2} 2^{(m-1)(t-2)}$, completing the proof of Corollary 1.3. □

*Remark* 5.1. The splitting (5.1) is over $\mathbb{R}$, not over $\mathbb{Z}$. In one layer,

$$\mathbb{Z}\mathbf{1} + \{z \in \mathbb{Z}^m : \textstyle\sum_a z_a = 0\} = \{z \in \mathbb{Z}^m : \textstyle\sum_a z_a \equiv 0 \pmod m\}.$$

The inclusion from left to right follows by summing coordinates. For the reverse inclusion subtract $(\sum_a z_a/m)\mathbf{1}$. The lattice has index $m$, since the sum map $\mathbb{Z}^m \to \mathbb{Z}/m\mathbb{Z}$ is surjective. Thus the real splitting does not induce an integral direct-sum decomposition of the Jacobian.

**Proposition 5.2.** *For even $t \geq 4$,*

$$\operatorname{Jac}(G_{2,t}) \cong \mathbb{Z}/4\mathbb{Z} \oplus (\mathbb{Z}/8\mathbb{Z})^{t-2}. \tag{5.3}$$

*Proof.* Let $L = L_{G_{2,t}}$, let $A_{P_t}$ be the adjacency matrix of the path and let $J_2$ be the all-one matrix of size two. All vertex degrees are even, so

$$L \bmod 2 = A_{P_t} \otimes J_2. \tag{5.4}$$

Over $\mathbb{Z}$, put $a_t = \det A_{P_t}$, with $a_0 = 1$ and $a_1 = 0$. Expansion along the last row and then the last column of the minor gives $a_t = -a_{t-2}$ for $t \geq 2$. For even $t$, $a_t = (-1)^{t/2}$, so $A_{P_t}$ is invertible modulo two. Over $\mathbb{F}_2$,

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} J_2 \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \operatorname{diag}(1, 0). \tag{5.5}$$

Taking Kronecker products with $I_t$ and permuting coordinates transforms (5.4) into $\operatorname{diag}(A_{P_t}, 0_t)$. Lemma 2.2 therefore gives

$$\operatorname{rank}_{\mathbb{F}_2}(L_o \bmod 2) = t, \qquad \dim_{\mathbb{F}_2}(\operatorname{Jac}(G_{2,t})/2\operatorname{Jac}(G_{2,t})) = t - 1.$$

By (1.11), the exponent is eight and the order is $2^{3t-4}$. In the notation of Lemma 2.6,

$$a + b + c = t - 1, \qquad a + 2b + 3c = 3t - 4.$$

Subtracting the second equality from three times the first gives $2a + b = 1$. Nonnegativity implies $a = 0$, $b = 1$ and $c = t - 2$. $\square$

## 6. Homocyclic Jacobians and their pairings

The next identity is a specialization of the Sherman–Morrison formula; see [6].

**Lemma 6.1.** *Let $u, v$ be distinct nonadjacent vertices of a connected graph $G$ with the same neighbour set, of cardinality $d$. Put $G^+ = G + \{u, v\}$. Choose $o \notin \{u, v\}$, and let $A, A^+$ be the reduced Laplacians at $o$. With $b = \mathbf{e}_u - \mathbf{e}_v$ in the reduced coordinates,*

$$(A^+)^{-1} = A^{-1} - \frac{b\,b^\top}{d(d+2)}, \tag{6.1}$$

$$\det A^+ = \frac{d+2}{d} \det A. \tag{6.2}$$

*Consequently,*

$$\exp \operatorname{Jac}(G^+) \mid \operatorname{lcm}(\exp \operatorname{Jac}(G), d(d+2)). \tag{6.3}$$

*Proof.* Connectedness implies $d \geq 1$, so $G$ has a vertex outside $\{u, v\}$. The equal neighbour sets give $L_G(\mathbf{e}_u - \mathbf{e}_v) = d(\mathbf{e}_u - \mathbf{e}_v)$. Since $o \notin \{u, v\}$,

$$Ab = db, \qquad A^{-1}b = b/d, \qquad b^\top b = 2, \qquad A^+ = A + b\,b^\top.$$

Symmetry gives $b^\top A^{-1} = b^\top/d$, whence

$$(A + b\,b^\top)\left(A^{-1} - \frac{b\,b^\top}{d(d+2)}\right) = I + \frac{b\,b^\top}{d} - \frac{(d+2)b\,b^\top}{d(d+2)} = I.$$

For column vectors $x, y$, multilinearity of the determinant gives $\det(I + xy^\top) = 1 + y^\top x$: terms with two columns proportional to $x$ vanish, and the term using $y_j x$ only in column $j$ equals $y_j x_j$. It follows that

$$\det A^+ = \det A\,(1 + b^\top A^{-1} b) = \det A\,(1 + 2/d).$$

Multiplication of (6.1) by $\operatorname{lcm}(\exp \operatorname{Jac}(G), d(d+2))$ gives an integral matrix. Apply Corollary 2.4. $\square$

**Lemma 6.2.** *The graph $H_t$, $t \geq 2$, is simple, planar and biconnected, with $2t$ vertices, $4t - 3$ edges and maximum degree at most four.*

*Proof.* Its edges are the $4(t-1)$ edges between consecutive layers and the additional edge $a_1b_1$. No edge occurs twice and no edge is a loop. The first-layer degrees are three, the last-layer degrees are two, and every interior-layer degree is four. The graph contains the spanning biconnected graph $G_{2,t}$ by Proposition 4.1. Deleting a vertex therefore leaves a connected graph. Proposition 4.3 gives planarity. □

**Lemma 6.3.** *Let $A_t, C_t$ be the reduced Laplacians of $G_{2,t}, H_t$ at $b_t$, and let $b = \mathbf{e}_{a_1} - \mathbf{e}_{b_1}$ in the reduced coordinates. Then*

$$C_t^{-1} = A_t^{-1} - \tfrac{1}{8} b b^{\mathsf{T}}, \qquad 8C_t^{-1} \in \mathrm{Mat}_{2t-1}(\mathbb{Z}), \qquad \det C_t = 8^{t-1}. \tag{6.4}$$

*Proof.* In $G_{2,t}$ the vertices $a_1, b_1$ are nonadjacent, have the same two neighbours, and differ from $b_t$ because $t \geq 2$. Lemma 6.1 gives the inverse identity and $\det C_t = 2\det A_t$. Theorem 1.2 gives exponent eight for $t \geq 3$ and four for $t = 2$. Thus $8A_t^{-1}$ is integral in both cases, by Corollary 2.4. The same theorem gives $\det A_t = 2^{3t-4}$ for every $t \geq 2$. The three conclusions follow. □

**Lemma 6.4.** *For $D_i$ in* (1.4),

$$\langle [D_i], [D_j] \rangle_{H_t} = \frac{(B_{t-1})_{ij}}{8} \pmod{\mathbb{Z}}. \tag{6.5}$$

*Proof.* Write $s^{(i)} = \mathbf{e}_i - \mathbf{e}_{i+1} \in \mathbb{Z}^t$ for the layer sums of $D_i$. On $P_t$, put $z^{(j)} = \mathbf{1}_{\{1,\dots,j\}}$. Then $L_{P_t} z^{(j)} = s^{(j)}$ and

$$(s^{(i)})^{\mathsf{T}} z^{(j)} = \delta_{ij}.$$

Here $\delta_{ij}$ is the Kronecker delta, whereas $\delta_v = \deg_{P_t}(v)$. Let $d_i$ be $D_i$ with its coordinate at $b_t$ omitted; that coordinate is zero. In Lemma 3.2, the $a_v$-coordinate of a potential for $D_j$ is $z_v^{(j)}/4 + s_v^{(j)}/(4\delta_v)$. Subtracting its coordinate at $b_t$ does not change its pairing with $D_i$, since $D_i$ has total sum zero. Hence

$$d_i^{\mathsf{T}} A_t^{-1} d_j = \tfrac{1}{4}\delta_{ij} + \tfrac{1}{4}\sum_{v=1}^{t} \frac{s_v^{(i)} s_v^{(j)}}{\delta_v}.$$

Since $d_i^{\mathsf{T}} b = \delta_{i1}$, Lemma 6.3 gives

$$d_i^{\mathsf{T}} C_t^{-1} d_j = \tfrac{1}{4}\delta_{ij} + \tfrac{1}{4}\sum_{v=1}^{t} \frac{s_v^{(i)} s_v^{(j)}}{\delta_v} - \tfrac{1}{8}\delta_{i1}\delta_{j1}. \tag{6.6}$$

Put $r = t-1$. If $|i-j| \geq 2$, the supports in the sum are disjoint and both Kronecker terms vanish. If $j = i+1$, the only common layer is $i+1$, an interior vertex of $P_t$, with product of coefficients $-1$. Thus the entry is $-1/8$; symmetry covers $i = j+1$. For $i = j$, formula (6.6) gives

$$\frac{1}{4} + \frac{1}{4\delta_i} + \frac{1}{4\delta_{i+1}} - \frac{1}{8}\delta_{i1} = \begin{cases} 5/8, & r = 1, \\ 4/8, & r \geq 2,\ i < r, \\ 5/8, & r \geq 2,\ i = r. \end{cases} \tag{6.7}$$

For $r = 1$, both degrees are one. For $r \geq 2$, the case $i = 1$ has degrees $1, 2$ and the subtraction $1/8$; the indices $1 < i < r$ have degrees $2, 2$ and no subtraction; the index $i = r$ has degrees $2, 1$ and no subtraction. These cases exhaust the entries of $B_r$. □

*Proof of Theorem 1.1.* Lemma 6.2 proves the graph assertions. Lemma 6.3 and Corollary 2.4 show that $8\,\mathrm{Jac}(H_t) = 0$, so (1.6) is well defined. Put $r = t-1$ and $b_r = \det B_r$, with $b_0 = 1$. Expansion along the first row gives

$$b_1 = 5, \qquad b_r = 4b_{r-1} - b_{r-2} \quad (r \geq 2). \tag{6.8}$$

Modulo two, $b_r \equiv b_{r-2}$; hence all $b_r$ are odd. The adjugate identity implies $B_r \in \mathrm{GL}_r(\mathbb{Z}/8\mathbb{Z})$. If $\sum_i c_i[D_i] = 0$, pairing with each $[D_j]$ and applying Lemma 6.4 gives $B_r c = 0$ modulo eight. Thus $c = 0$, proving injectivity of (1.6). Its domain and codomain both have order $8^r$, by Lemmas 2.1 and 6.3; it is therefore surjective. Lemma 6.4 proves the pairing formula. □

**Corollary 6.5.** *As a finite abelian group with pairing,*

$$\operatorname{Jac}(H_t) \cong \langle 5/8\rangle^{\perp\lceil (t-1)/2\rceil} \perp \langle 7/8\rangle^{\perp\lfloor (t-1)/2\rfloor}. \tag{6.9}$$

*Proof.* Put $r = t-1$. If $r=1$, then $B_1 = (5)$ and the assertion follows. Assume $r \geq 2$ and work over $R = \mathbb{Z}/8\mathbb{Z}$. For a symmetric matrix $Q = \begin{pmatrix} A & c \\ c^{\mathsf{T}} & u \end{pmatrix}$ with $u \in R^{\times}$, set $P = \begin{pmatrix} I & 0 \\ -u^{-1}c^{\mathsf{T}} & 1 \end{pmatrix}$. Direct multiplication gives

$$\det P = 1, \qquad P^{\mathsf{T}}QP = \begin{pmatrix} A - u^{-1}cc^{\mathsf{T}} & 0 \\ 0 & u \end{pmatrix}. \tag{6.10}$$

For $B_r$, the last diagonal entry is five and $c = (0,\dots,0,-1)^{\mathsf{T}}$. Since $5^{-1} = 5$ in $R$, the last diagonal entry of the remaining block becomes $4-5=7$. All its other entries are unchanged. Inductively, the remaining block is tridiagonal with adjacent entries $-1$, all diagonal entries but the last equal to four, and last entry $u \in \{5,7\}$. Since $u^{-1} = u$, the next last entry is $4-u$, which interchanges five and seven. For a block of size one no further operation is needed. Extending each matrix $P$ by the identity on the previously separated coordinates yields an invertible congruence. There are $\lceil r/2\rceil$ diagonal entries five and $\lfloor r/2\rfloor$ diagonal entries seven, counted from the last coordinate. Apply Theorem 1.1 with $r = t-1$. ☐

**Corollary 6.6.** *For every $r \geq 1$, there exists a simple planar biconnected graph with $2r+2$ vertices, $4r+1$ edges, maximum degree at most four, and Jacobian $(\mathbb{Z}/8\mathbb{Z})^r$. In particular,* [5, arXiv version 2, Conjecture 36] *is false.*

*Proof.* The graph $H_{r+1}$ has the stated properties by Theorem 1.1. A proposed bound $b_8$ is contradicted by any integer $r > b_8$. ☐

## 7. Connected regular matroids

The cycle matroid $M(G)$ has ground set $E(G)$, independent sets the forests, and circuits the simple cycles. A matroid is connected if no partition of its ground set into two nonempty parts has every circuit contained in one part. It is regular if it is representable over every field. Choose an orientation of $G$, and let $B : \mathbb{Z}^{E(G)} \to \mathbb{Z}^{V(G)}$ have column $\mathbf{e}_v - \mathbf{e}_w$ for an edge directed from $w$ to $v$. Then

$$L_G = BB^{\mathsf{T}}. \tag{7.1}$$

**Lemma 7.1.** *The incidence matrix $B$ represents $M(G)$ over every field and is totally unimodular.*

*Proof.* The columns on a cycle have a relation with coefficients 1 or $-1$, obtained by traversing the cycle. These coefficients remain nonzero in every field. If a nonempty set of edges is a forest, a leaf row contains exactly one nonzero coefficient among its columns. In any linear relation that column therefore has coefficient zero. Deleting it and inducting on the number of edges proves independence. The empty set is independent.

Let $C$ be a square submatrix of $B$. Each column of $C$ has at most two nonzero entries. A zero column gives determinant zero. A column with one entry $\pm 1$ reduces the determinant, by expansion, to $\pm$ a smaller minor. If every column has two nonzero entries, they are $1, -1$, so the sum of the rows is zero. Induction, with empty determinant one, gives $\det C \in \{0, 1, -1\}$. ☐

For connected $G$, define its integral cycle and cut lattices by

$$\mathcal{C} = \ker(B : \mathbb{Z}^{E(G)} \to \mathbb{Z}^{V(G)}), \qquad \mathcal{B} = B^{\mathsf{T}}\mathbb{Z}^{V(G)}. \tag{7.2}$$

Then

$$\mathcal{B} = \operatorname{im}_{\mathbb{R}}(B^{\mathsf{T}}) \cap \mathbb{Z}^{E(G)}. \tag{7.3}$$

Indeed, if $B^{\mathsf{T}}x$ is integral, every edge difference of $x$ is integral. Summing differences along a path from $o$ to $v$ gives $x_v - x_o \in \mathbb{Z}$. Hence $B^{\mathsf{T}}x = B^{\mathsf{T}}(x - x_o\mathbf{1}) \in \mathcal{B}$. The other inclusion follows from the definition.

**Lemma 7.2.** *The incidence map induces an isomorphism*

$$\mathbb{Z}^{E(G)}/(\mathcal{C}+\mathcal{B}) \longrightarrow \mathrm{Div}^0(G)/L_G\mathbb{Z}^{V(G)} = \mathrm{Jac}(G). \tag{7.4}$$

*Proof.* One has $B\mathcal{C} = 0$ and $B\mathcal{B} = BB^{\mathsf{T}}\mathbb{Z}^{V(G)} = L_G\mathbb{Z}^{V(G)}$. The map is therefore well defined. For every $v$, a signed sum of the oriented edges on a path from $o$ to $v$ has incidence $\mathbf{e}_v - \mathbf{e}_o$. These differences generate $\mathrm{Div}^0(G)$, proving surjectivity. If $Bx = L_G z$ with $x, z$ integral, then $B(x - B^{\mathsf{T}}z) = 0$. Thus $x \in \mathcal{C} + \mathcal{B}$, which proves injectivity. □

If an orientation is changed, there is a diagonal sign matrix $S$ with $B' = BS$. Since $S^2 = I$, the new lattices are $S\mathcal{C}, S\mathcal{B}$, and $B'(Sx) = Bx$. Thus (7.4) commutes with changes of orientation. Its domain is the cycle–cut presentation of the regular-matroid Jacobian in [8, Section 2 and Appendix A].

**Lemma 7.3.** *For every $T, m$ in Theorem 1.2, $M(G_m(T))$ is connected.*

*Proof.* Let $E_{uv}$ be the set of the $m^2$ edges between $V_u, V_v$ for $uv \in E(T)$. Any two distinct edges of $E_{uv}$ lie on a four-cycle. For disjoint endpoints, use those four endpoints. For a common endpoint in one part, choose another vertex of that part, which exists since $m \geq 2$, and use the two distinct endpoints in the other part.

Suppose a partition $E(G_m(T)) = E_1 \sqcup E_2$ has every circuit contained in one part. The preceding four-cycles imply that each $E_{uv}$ is contained in one part. If $uv, vw \in E(T)$ are distinct, choose distinct $a, b \in V_v$, $c \in V_u$ and $d \in V_w$. The cycle $c, a, d, b, c$ meets both $E_{uv}$ and $E_{vw}$, so these sets must belong to the same part. Every two edges of a connected tree are joined by a sequence of edges in which consecutive edges share a vertex. All the sets $E_{uv}$ consequently belong to one part, and the other part is empty. If $T$ has one edge, the first paragraph already gives the same conclusion. This is the definition of connectedness. □

**Corollary 7.4.** *There are infinitely many pairwise nonisomorphic connected regular matroids with Jacobian exponent eight. Consequently,* [8, Conjecture 1.2] *is false.*

*Proof.* Take $M(G_{2,t})$, $t \geq 3$. Lemmas 7.1, 7.2 and 7.3 give regularity, the equality of Jacobians and connectedness. The exponent is eight by (1.11). The ground-set cardinalities $4(t-1)$ are distinct, so the matroids are pairwise nonisomorphic. □

**Corollary 7.5.** *For every $r \geq 1$, there exists a connected regular matroid on $4r+1$ elements with Jacobian $(\mathbb{Z}/8\mathbb{Z})^r$.*

*Proof.* Take $M(H_{r+1})$. It is regular by Lemma 7.1. In a partition with every circuit contained in one part, the edges of $G_{2,r+1}$ belong to a single part by Lemma 7.3. The added edge belongs to the triangle $a_1, b_1, a_2, a_1$, so it is in that part as well. Thus $M(H_{r+1})$ is connected. Its ground set has $4r+1$ elements by Lemma 6.2, and Lemma 7.2 and Theorem 1.1 give the group. □

## Appendix A. Integral matrices and spanning trees

**Lemma A.1.** *Let $C \in \mathrm{Mat}_q(\mathbb{Z})$ be nonsingular, with $q \geq 1$. There exist $U, V \in \mathrm{GL}_q(\mathbb{Z})$ and positive integers $s_1 \mid s_2 \mid \cdots \mid s_q$ such that*

$$UCV = \mathrm{diag}(s_1, \ldots, s_q). \tag{A.1}$$

*Consequently,*

$$\mathbb{Z}^q/C\mathbb{Z}^q \cong \bigoplus_{i=1}^{q} \mathbb{Z}/s_i\mathbb{Z}, \qquad |\mathbb{Z}^q/C\mathbb{Z}^q| = |\det C|. \tag{A.2}$$

*Proof.* For $q = 1$, a sign change proves the assertion. Assume $q \geq 2$. Among the nonzero entries of matrices integrally equivalent to $C$, choose one with least positive absolute value $a$. Row and column permutations and a sign change place $a$ at $(1, 1)$. Euclidean division shows that $a$ divides every entry in its column: otherwise a row subtraction produces a nonzero remainder strictly between zero and $a$. Subtract multiples of row one to annihilate that column below $a$. The same argument for row one, using column operations, gives $\operatorname{diag}(a, C')$. If an entry $c$ of $C'$ were not divisible by $a$, adding its row to the first row and subtracting a multiple of column one from its column would again produce a nonzero remainder less than $a$. Thus every entry of $C'$ is divisible by $a$. Nonsingularity gives $\det C' \neq 0$. Induction diagonalizes $C'$ with successive positive divisors; its entries remain divisible by $a$ under integral operations. This proves (A.1). Left multiplication by $U$ induces an isomorphism of cokernels, and $V\mathbb{Z}^q = \mathbb{Z}^q$. The cokernel of the diagonal matrix is the stated direct sum, whose order is $\prod_i s_i = |\det C|$. □

**Lemma A.2.** *Let $G$ be connected with $N \geq 2$ vertices, and let $\lambda_1, \dots, \lambda_{N-1}$ be its nonzero Laplacian eigenvalues. If $\tau(G)$ denotes the number of spanning trees, then*

$$|\operatorname{Jac}(G)| = \tau(G) = \frac{1}{N}\prod_{i=1}^{N-1}\lambda_i. \tag{A.3}$$

*Proof.* Let $B_o$ be the incidence matrix with row $o$ omitted, and set $q = N - 1$. Since $L_o = B_o B_o^{\mathsf{T}}$, the Cauchy–Binet identity gives

$$\det L_o = \sum_{\substack{F \subseteq E(G)\\ |F| = N-1}} \det(B_{o,F})^2. \tag{A.4}$$

For a $q \times |E(G)|$ matrix $X$, write $X_e$ for its column indexed by $e$. For completeness, multilinearity gives

$$\det(XX^{\mathsf{T}}) = \sum_{(e_1,\dots,e_q)\in E(G)^q} \left(\prod_{j=1}^{q} X_{j,e_j}\right) \det(X_{e_1}, \dots, X_{e_q}).$$

Repeated indices give zero. Grouping the $q!$ orders of each $q$-element subset gives the square of its determinant by the permutation formula. This proves (A.4).

If $F$ contains a cycle, its columns are dependent, so the corresponding minor vanishes. A forest on $N$ vertices with $c$ components has $N - c$ edges, by successive deletion of leaves in each nontrivial component. Thus a forest with $N - 1$ edges is a spanning tree. For a tree, choose a leaf different from $o$ and expand along its row; the only entry is 1 or $-1$. Such a leaf exists when $N \geq 2$, since the endpoints of a longest simple path are distinct leaves. Induction reduces the minor to the empty determinant one. Its square is therefore one. Consequently $\det L_o = \tau(G)$, and Lemma 2.1 gives the first equality in (A.3).

The coefficient of $z$ in $\det(zI + L_G)$ is the sum of the $N$ principal minors of size $N - 1$, hence $N\tau(G)$. The spectral theorem and (2.3) give $\det(zI + L_G) = z\prod_{i=1}^{N-1}(z + \lambda_i)$. Its coefficient of $z$ is $\prod_i \lambda_i$, proving the second equality. □

## Appendix B. A second computation of the invariant factors

**Proposition B.1.** *For every $t \geq 2$, $\operatorname{Jac}(H_t) \cong (\mathbb{Z}/8\mathbb{Z})^{t-1}$.*

*Proof.* This proof uses Lemma 6.3, not the generators or the pairing computation. Let $R_t$ be the tridiagonal matrix over $\mathbb{F}_2$ with first diagonal entry one, all other diagonal entries zero, and both adjacent diagonals one. The full Laplacian satisfies

$$L_{H_t} \bmod 2 = R_t \otimes J_2.$$

Indeed, its first diagonal layer block is $J_2$, all other diagonal blocks are zero, and each adjacent block is $J_2$. With $\epsilon_0 = 1$, $\epsilon_1 = 1$, expansion of the last row gives $\epsilon_t := \det R_t = \epsilon_{t-2}$ for $t \geq 2$. Thus $R_t$ is invertible. Formula (5.5) and Lemma 2.2 give $\mathrm{rank}_{\mathbb{F}_2}(C_t \bmod 2) = t$. Lemma 6.3 and Corollary 2.4 show that the group is annihilated by eight and has order $8^{t-1}$. In Lemma 2.6, this yields

$$a + b + c = t - 1, \qquad a + 2b + 3c = 3(t - 1).$$

Hence $2a + b = 0$. Since $a, b, c \geq 0$, one has $a = b = 0$ and $c = t - 1$. □

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