GLOBAL SECTIONS AND FIBRES OF HODGE PERIOD MAPS

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Abstract. For every closed connected oriented smooth manifold of even dimension at least four, the middle Hodge period map admits a global smooth section taking values in regular metrics with any prescribed positive volume form. On every closed connected oriented smooth manifold of dimension at least three, the Hodge inner-product map in each intermediate degree other than the middle also admits a global smooth section. In dimension four, when the intersection form has positive index one, every period fibre is smoothly path connected, also with a fixed volume form, and every smooth path of periods lifts smoothly through arbitrary compatible endpoint metrics. The lift may retain finitely many compatible prescribed lift germs and have any compatible smooth family of positive volume forms. Consequently, the inverse image of every nonempty connected open subset of the period domain is smoothly path connected. These results answer Tyurin's connected-fibre question, the chamber-path question of Kotschick and Morgan, and the prescribed-path question of Feehan and Leness. The constructions combine estimates for harmonic energies with finite-dimensional period correction, both for global realization and for deformation in a fibre.

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1. Introduction

Let M be a closed connected oriented smooth manifold of dimension n, and let $[M] \in H_n(M;\mathbb{Z})$ be its fundamental class. Write $\mathrm{Met}^\infty(M)$ for the space of smooth Riemannian metrics, with its C^∞ topology. For $g \in \mathrm{Met}^\infty(M)$, let dV_g be its volume form, let $*_g$ denote the Hodge star, and let $h^k_g(a)$ be the harmonic representative of $a \in H^k(M;\mathbb{R})$, where $0 \le k \le n$. The Hodge inner product is

$$\Gamma_k(g)(a,b) = \int_M h^k_g(a) \wedge *_g h^k_g(b), \qquad a, b \in H^k(M;\mathbb{R}).$$

We ask which inner products occur, whether realizing metrics can be chosen smoothly on the whole image, and how metrics can be joined while their Hodge inner products are prescribed. The last question will be treated in dimension four, when the intersection form has positive index one. A family of metrics over a finite-dimensional smooth manifold is smooth when its coefficients are jointly smooth in the parameter and the point of M.

1.1. **Global sections.** Suppose $n = 2m \ge 4$, and let Tor denote the torsion subgroup of an abelian group. Put

$$V_M = H^m(M;\mathbb{R}), \qquad L_M = H^m(M;\mathbb{Z})/\operatorname{Tor} H^m(M;\mathbb{Z}), \qquad Q_M(a,b) = \langle a \smile b, [M] \rangle.$$

Identify L_M with its image in V_M, fix an integral basis, and also write Q_M for the matrix of the intersection pairing. Define

$$\mathcal{D}(Q_M) = \{G = G^{\mathsf{T}} > 0 \,:\, GQ_M^{-1}G = (-1)^m Q_M\}.$$

When $V_M = 0$, this denotes a one-point space. The Hodge star identities give $\Gamma_m(g) \in \mathcal{D}(Q_M)$; see (2.1). For a positive smooth volume form ν, put

$$\mathrm{Met}^\infty_\nu(M) = \{g \in \mathrm{Met}^\infty(M) \,:\, dV_g = \nu\}.$$

A metric in this space is regular for $\Gamma_m|_{\mathrm{Met}^\infty_\nu(M)}$ if the differential onto the tangent space of $\mathcal{D}(Q_M)$ is surjective.

Theorem A (Global sections). *Let $m \geq 2$. For every positive smooth volume form ν on M, there is a smooth map*

$$s_\nu \,:\, \mathcal{D}(Q_M) \longrightarrow \mathrm{Met}^\infty_\nu(M), \qquad \Gamma_m \circ s_\nu = \mathrm{id}_{\mathcal{D}(Q_M)}.$$

Every metric in its image is regular for $\Gamma_m|_{\mathrm{Met}^\infty_\nu(M)}$.

The section has the fixed cohomology marking of M; there is no restriction on the fundamental group, torsion, or stable tangent bundle. For even m, the form Q_M is symmetric. Denote its positive and negative indices by $b^+(M)$ and $b^-(M)$. The map $G \mapsto \ker(Q_M^{-1}G - \mathrm{id})$ identifies $\mathcal{D}(Q_M)$ with the Grassmannian of maximal Q_M-positive subspaces. For odd m, Q_M is alternating and $G \mapsto Q_M^{-1}G$ identifies $\mathcal{D}(Q_M)$ with the complex structures J for which $Q_M(\cdot, J\,\cdot)$ is symmetric positive definite. A symplectic marking identifies this domain with Siegel space. Thus every compatible marked principally polarized complex torus on L_M is the Lazzeri Jacobian of a metric on M; see [54, Definition 1.2 and Notation 1.4] for this Jacobian and its period maps. For odd $m \geq 3$, Theorem A therefore determines the full Hodge-star image considered in [49, §5.3, p. 35 of arXiv v2]. Composition with s_ν realizes every smooth family of compatible forms over any finite-dimensional parameter manifold, including a noncompact one. The case of compact parameter manifolds, with prescribed smooth families of positive volume forms and regularity on the corresponding volume-form slices, is proved in [51, Corollary 7.6].

The general surjectivity problem in dimension four is stated in [56, §1, p. 1]. Scaduto proved density in every signature and surjectivity when $b^+ = 1$; reversing orientation gives the case $b^- = 1$ [56, Theorems 1.1–1.2]. Both results extend to dimensions divisible by four [56, Remarks 2.6 and 3.5]. His surjectivity theorem settles the positive-index-one assertion formulated by Katz in [35, Hypothesis 1.1] and [36, Conjecture 20.1.3], and by Hamilton in [22, Conjecture 2.1]. Shevchishin and Smirnov proved simultaneous anti-self-dual realization of the classes dual to pairwise disjoint embedded (-1)-spheres, and subsequently of those dual to pairwise disjoint embedded (-2)-spheres, when $b^+ \leq 3$ [58, Theorem 1][59, Theorem 1]. Theorem A gives a smooth right inverse on the whole compatible domain, in both middle-degree parities.

For symplectic four-manifolds with $b^+ = 1$, Li and Liu's description of the symplectic cone implies density of the period image; under smooth minimality, every positive-square class has a symplectic representative [43, Theorems 2 and 4]. Solomon gives the density argument for $\mathbb{CP}^2\#N\overline{\mathbb{CP}^2}$, $N \geq 0$, with the original attribution to Li and Liu, in [60, Theorem A.1.3 and pp. 195–196]. Gay and Kirby constructed self-dual harmonic representatives of positive-square integral classes on arbitrary closed oriented connected four-manifolds [18, Theorem 1(1),(8)]; Hamilton deduced density in positive index one [22, Proposition 1.1 and Theorem 3.1]. Katz's conformal systolic estimates motivated the image problem: continuity of the first lattice minimum makes density sufficient for those estimates [35, Remark 9.1], which are stated unconditionally in [36, Theorem 20.1.1 and Remark 20.1.4]. Scaduto's Corollary 1.3 similarly shows that the supremum of the conformal middle systole over all metrics depends only on the integral middle intersection lattice in every dimension divisible by four [56].

1.2. **Fibres and prescribed paths in dimension four.** Let X be a closed connected oriented smooth four-manifold with $b^+(X) = 1$, and write $V_X = H^2(X;\mathbb{R})$ and $Q_X(a,b) = \langle a \smile b, [X]\rangle$. We also write $a \cdot b = Q_X(a,b)$ and $a^2 = Q_X(a,a)$. Choose a component $\mathcal{P}_X$ of $\{a \in V_X \,:\, Q_X(a,a) > 0\}$ and set

$$\mathbb{H}_X = \{u \in \mathcal{P}_X \,:\, Q_X(u,u) = 1\}.$$

For $g \in \mathrm{Met}^\infty(X)$, denote by $p(g)$ the unique $u \in \mathbb{H}_X$ whose harmonic representative is self-dual. The hyperbolic distance $d = d_{\mathbb{H}}$ on $\mathbb{H}_X$ satisfies $\cosh d(u,v) = Q_X(u,v)$. For $a \in \mathcal{P}_X$, put $u_a = a/\sqrt{a^2}$. Tyurin asked whether the fibres of this period map are connected [65, p. 334].

Theorem B (Fibres in positive index one). *For every $u \in \mathbb{H}_X$, the fibre $p^{-1}(u)$ is nonempty and smoothly path connected. For every positive smooth volume form ν on X, the same holds for $p^{-1}(u) \cap \mathrm{Met}^\infty_\nu(X)$. A path joining two metrics may be chosen constant on parameter neighborhoods of its endpoints.*

Theorem A supplies a lift of a period path with the endpoint metrics selected by its section. The next theorem allows arbitrary compatible endpoint metrics. Feehan and Leness posed this question for paths contained in a chamber [16, §2.7, pp. 16–17]. Let $\gamma : [0,1] \to \mathbb{H}_X$ be smooth. A *lift germ* at t_0 is the germ of a smooth family k_t with $p(k_t) = \gamma(t)$ near t_0; at an endpoint the germ is one-sided. Finitely many specified germs are compatible if their representatives agree, after restriction, on overlaps and with any prescribed metric at the same parameter. Retaining a germ means agreeing with a representative on a sufficiently small neighborhood. A smooth family of positive volume forms ν_t is compatible if it agrees with the volumes of the retained representatives and of the prescribed endpoint metrics.

Theorem C (Prescribed paths). *Every smooth path $\gamma : [0,1] \to \mathbb{H}_X$ has a smooth lift g_t with arbitrary endpoint metrics g_0, g_1 satisfying $p(g_i) = \gamma(i)$ and with finitely many compatible prescribed lift germs. The lift may have any compatible smooth volume forms $dV_{g_t} = \nu_t$.*

For $\xi \in V_X$ with $Q_X(\xi,\xi) < 0$, put

$$W_\xi = \{u \in \mathbb{H}_X \,:\, Q_X(u,\xi) = 0\}.$$

This hypersurface is a *wall*. Fix a locally finite collection of walls: each compact subset of $\mathbb{H}_X$ meets only finitely many of them. Its *chambers* are the connected components of their complement. In the case of vanishing first Betti number, Kotschick's chamber-path argument assumes that the period map is a fibration [38, p. 432]. Kotschick and Morgan observed that this assertion was unproved and asked whether metrics over one chamber could be joined over that chamber [39, p. 434]. When the first Betti number vanishes, their Theorem 3.0.1 establishes, by a different argument, that the Donaldson invariant of a principal $SO(3)$-bundle whose second Stiefel–Whitney class has an integral lift is constant on generic metrics with periods in a fixed chamber determined by the bundle. The distinction between that theorem and the geometric path question is also recalled in [13, §1]. Theorem C gives the following answer to the geometric question, without a restriction on the first Betti number $b_1(X)$.

Corollary 1.1. *The inverse image of every nonempty connected open subset of $\mathbb{H}_X$ is smoothly path connected. The components of the space of metrics whose periods avoid the walls are exactly the inverse images of the chambers. The same statements hold with a prescribed volume form. Two periods lie in the same chamber if and only if their pairings with every defining class have the same strict signs.*

1.3. **The constructions.** For a closed connected oriented smooth manifold M^n, $n \geq 3$, and every fixed degree $1 \leq p \leq n-1$ with $2p \neq n$, the map Γ_p has a global smooth section over the full cone of positive definite forms on $H^p(M;\mathbb{R})$ (Theorem 3.1 and Corollary 13.3). In degree one, Bangert and Katz obtained metrics with harmonic forms of constant norm under a torus-submersion hypothesis [6, Construction 12.5 and Theorem 12.6]. Compact smooth families of compatible Hodge inner products, with prescribed positive volume forms, were realized in [51, Theorem 4.2]. The section here is defined on the entire positive cone. Stable norms [5, Theorem B] and positive Hodge eigenvalues

[31, Theorem 1.3] are different prescription problems; for closed one-form representatives adapted to disjoint loops, see [32, Proposition 2.1].

For $2p < n$, Section 3 constructs a smooth family of metrics $g_{G,\epsilon}$, where G is a positive definite form on $H^p(M;\mathbb{R})$ and $\epsilon > 0$, such that

$$\big(G \longmapsto \Gamma_p(g_{G,\epsilon})\big) \longrightarrow \mathrm{id}$$

in C^1 on compact subsets of the positive cone as $\epsilon \downarrow 0$ (Lemma 3.4). The construction concentrates weighted harmonic energies in disjoint framed tubes. The sums over integral directions are locally finite in G. Circle products provide the middle-degree models (Proposition 5.2 and Lemma 6.2). The transfer theorem of Section 4 carries their closed representatives and metric variations through fixed marked cap replacements, with estimates uniform on compact target sets. Sections 5 and 6 give the four-dimensional and higher-dimensional constructions, respectively. The estimates control the period error and the differentials in the chosen finite-dimensional correction directions, uniformly on compact target sets. A smooth choice of sufficiently small parameters and a finite-dimensional correction give the global section.

For the fibre theorem, fix $a \in \mathcal{P}_X$. The endpoint metrics are first joined within $p^{-1}(u_a)$ to metrics whose harmonic representatives of a are near-symplectic (Proposition 7.2). Relative positive surfaces give one connected configuration whose component classes span $H_2(X;\mathbb{R})$. The relative pencil constructions in Sections 9–10 are based on [4, §2, Theorem 4] and [3, arXiv v1, §4, Propositions 4.1–4.2]. They give two pairs of sections of a common Hermitian line bundle over X. Smooth paths of these section pairs, associated Hopf curvature forms, and fixed Thom forms then yield a smooth path of closed representatives η_t of a, $0 \le t \le 1$, with $\eta_t \wedge \eta_t \ge 0$ on X; each η_t is symplectic on one fixed connected neighborhood of the configuration. For every $g \in \mathrm{Met}^\infty(X)$ and every closed two-form η representing a, harmonic minimization gives

$$2a^2 \sinh^2 d_{\mathbb{H}}\big(p(g), u_a\big) \le \int_X \eta \wedge *_g \eta - a^2 \qquad \text{(Lemma 2.1).}$$

The polar metrics $g_{t,\delta}$, with $\delta > 0$, satisfy $\int_X \eta_t \wedge *_{g_{t,\delta}} \eta_t - a^2 \le C\delta$ for some $C > 0$ independent of t and δ, for all sufficiently small $\delta > 0$. On the fixed neighborhood they equal a smooth family of metrics k_t independent of δ. Their ellipticity and C^8 bounds there are therefore independent of δ. These local bounds, the spanning cycles, and a fixed compact neighborhood of u_a containing the periods for small δ give a positive correction radius independent of t and δ in Lemma 2.6. For sufficiently small δ, the correction gives a smooth path in $p^{-1}(u_a)$ joining the prepared endpoint metrics (Proposition 12.2). Concatenation with the preceding fibre paths joins the original metrics.

For Theorem C, apply Theorem A to X. The identification (2.5) and the map $u \mapsto \mathbb{R}u$ give a section $s : \mathbb{H}_X \to \mathrm{Met}^\infty(X)$, hence a lift $s \circ \gamma$ of the prescribed period path. Local sections give lifts through the prescribed endpoint metrics (Proposition 2.3 and Lemma 2.4). Fibre paths from Theorem B and uniform period correction join these local lifts to the reference lift while retaining the specified germs (Lemma 12.3). Conformal normalization prescribes the compatible volume forms (Lemma 2.5). Thus harmonic-energy estimates and exact period correction give both the global realizations and the relative deformation results. Section 13 contains the chamber, scalar-period, and retraction consequences.

1.4. **Conventions.** Unless a coefficient group is indicated, cohomology is real. For a bundle E, $\Gamma(E)$ denotes its smooth sections. Put $\Omega^k(M) = \Gamma(\Lambda^k T^*M)$ and $b_k(M) = \dim H^k(M;\mathbb{R})$. The smooth-family convention above also applies to differential forms and bundle maps. For a finite-dimensional real vector space V with a nondegenerate symmetric bilinear form Q, let $\mathcal{D}(V,Q)$ be the Grassmannian of maximal Q-positive subspaces, without assigned orientations; write $\mathcal{D}(V)$ when Q is understood. For a closed connected oriented smooth four-manifold X, put $V_X = H^2(X;\mathbb{R})$, $Q_X(a,b) = \langle a \smile b, [X]\rangle$ for $a, b \in V_X$, and $\mathcal{D}_X = \mathcal{D}(V_X, Q_X)$. For $g \in \mathrm{Met}^\infty(X)$, let $\Lambda^\pm_g$ be the pointwise ± 1 eigenspace bundles of $*_g$ on two-forms, and let $\mathcal{H}^\pm_g$ be the corresponding spaces of harmonic forms, identified with their

cohomology classes. The period map is $\Pi_X(g) = \mathcal{H}^+_g \in \mathcal{D}_X$; when $b^+(X) = 1$, its positive line is $\mathbb{R}p(g)$. For a two-form α, put $\alpha^\pm = (\alpha \pm *_g \alpha)/2$.

The symbol PD denotes Poincaré duality in either direction, with the indicated coefficient group. For a Hermitian line bundle $L \to M$, let $c_1(L) \in H^2(M;\mathbb{Z})$ be its first Chern class and $c_1(L)_\mathbb{R}$ its image in $H^2(M;\mathbb{R})$. If ∇ is a unitary connection on L, write F_∇ for its curvature and define the real two-form K by $F_\nabla = -2\pi i K$. Then $[K] = c_1(L)_\mathbb{R}$. On $\mathbb{CP}^1$ fix the Fubini–Study form σ with $\int_{\mathbb{CP}^1} \sigma = 1$. The symbols supp and int denote support and interior. For subsets A, B of a manifold, write $A \Subset B$ when the closure $\overline{A}$ in that manifold is compact and contained in B. Every manifold and isotopy is smooth unless stated otherwise.

2. Harmonic energy and finite-dimensional metric families

2.1. **Harmonic representatives and Gram matrices.** Let (M, g) be a closed oriented Riemannian manifold of dimension n, and write $*_g$ for its Hodge star. For $0 \le k \le n$ and $a \in H^k(M;\mathbb{R})$, let $h^k_g(a)$ be its unique harmonic representative [70, Theorem 6.11]. For smooth k-forms α, β and classes $a, b \in H^k(M;\mathbb{R})$, put

$$B_g(\alpha,\beta) = \int_M \alpha \wedge *_g \beta, \qquad E_g(\alpha) = B_g(\alpha,\alpha), \qquad B^k_g(a,b) = B_g\big(h^k_g(a), h^k_g(b)\big).$$

Harmonic representatives depend smoothly on a smooth finite-dimensional family of metrics and cohomology classes [30, Proposition 2.1 and Appendix of the author's version]. In particular, the period map is smooth on such families.

In this notation, $\Gamma_k(g) = B^k_g$. When $n = 2m$, the Hodge star preserves harmonic forms [70, §6.1(4)]; hence it defines an endomorphism $J_M(g)$ of V_M by

$$h^m_g(J_M(g)a) = *_g\, h^m_g(a).$$

Identify bilinear forms and endomorphisms with their matrices in the fixed basis of V_M, and write I for the identity matrix. The identity $*_g^2 = (-1)^m$ id on middle forms gives

$$\Gamma_m(g) = Q_M J_M(g), \qquad J_M(g)^2 = (-1)^m I, \qquad \Gamma_m(g) Q_M^{-1} \Gamma_m(g) = (-1)^m Q_M. \tag{2.1}$$

Call an endomorphism J of V_M Q_M-*compatible* if $J^2 = (-1)^m I$ and $Q_M(\cdot, J\cdot)$ is symmetric positive definite. The maps $B \mapsto Q_M^{-1}B$ and $J \mapsto Q_M J$ identify $\mathcal{D}(Q_M)$ with these endomorphisms. They satisfy $J^\mathsf{T} Q_M J = Q_M$. For even m, their $+1$ and -1 eigenspaces are mutually Q_M-orthogonal, with Q_M positive and negative definite, respectively. For odd m, they are compatible complex structures on the symplectic vector space (V_M, Q_M).

For $r \ge 1$, write $\mathrm{Sym}(r)$ and $\mathrm{Skew}(r)$ for the symmetric and skew-symmetric real $r \times r$ matrices, and $\mathrm{Sym}^+(r)$ for the positive definite cone in $\mathrm{Sym}(r)$. For odd m with $\dim V_M = 2r > 0$, fix a symplectic basis $e_1, \dots, e_r, f_1, \dots, f_r$ with $Q_M(e_i, f_j) = \delta_{ij}$. The Siegel parametrization [37, §2.1] identifies the compatible complex structures with the convex open set $\mathrm{Sym}(r) + i\,\mathrm{Sym}^+(r)$. Our coordinate $Z = C + iG$ is determined by the complex-linear isomorphism $F_J : (V_M, J) \to \mathbb{C}^r$ normalized by $F_J(e_i) = \epsilon_i$, where (ϵ_i) is the coordinate basis of $\mathbb{C}^r$: the jth column of Z is $F_J(f_j)$. Thus, for column vectors $x, y \in \mathbb{R}^r$,

$$F_J\left(\sum_i x_i e_i + \sum_i y_i f_i\right) = x + Zy.$$

The relation $F_J J = i F_J$ gives

$$J(C,G) = \begin{pmatrix} -CG^{-1} & -G - CG^{-1}C \\ G^{-1} & G^{-1}C \end{pmatrix}. \tag{2.2}$$

Let T be a finite-dimensional smooth manifold and $(g(q))_{q\in T}$ a smooth family of metrics on M. Fix $a, b \in H^k(M;\mathbb{R})$, and put $\alpha_q = h^k_{g(q)}(a)$ and $\beta_q = h^k_{g(q)}(b)$. For $v \in T_qT$, the forms $D_q\alpha_q[v]$ and $D_q\beta_q[v]$ are exact; in degree zero they vanish. Orthogonality to harmonic forms gives

$$B_{g(q)}(D_q\alpha_q[v], \beta_q) = B_{g(q)}(\alpha_q, D_q\beta_q[v]) = 0.$$

Differentiating the integral therefore yields

$$D_q(\Gamma_k \circ g)[v](a,b) = \int_M \alpha_q \wedge (D_q *_{g(q)} [v])\beta_q. \tag{2.3}$$

For the four-manifold X, fix a real basis of V_X and let Q be its intersection matrix. For $g \in \mathrm{Met}^\infty(X)$, denote the matrices of B_g^2 and $J_X(g)$ by G_g and J_g. Then (2.1) reads

$$G_g = QJ_g, \qquad J_g^2 = I, \qquad G_g Q^{-1} G_g = Q. \tag{2.4}$$

For $P \in \mathcal{D}_X$, let π_P be the Q-orthogonal projection onto P and put $J_P = 2\pi_P - I$. The map $P \mapsto G_P = QJ_P$ gives

$$\mathcal{D}_X \simeq \{G = G^\mathsf{T} > 0 \,:\, GQ^{-1}G = Q\}. \tag{2.5}$$

Its inverse is $G \mapsto \ker(Q^{-1}G - I)$: the eigenspaces of $Q^{-1}G$ are Q-orthogonal, and Q restricts to G and $-G$ on its $+1$ and -1 eigenspaces, respectively. Both maps are smooth, since $P \mapsto \pi_P$ is smooth and the positive eigenspace projection is $(I + Q^{-1}G)/2$.

Lemma 2.1. *For every $g \in \mathrm{Met}^\infty(M)$, $a \in H^k(M;\mathbb{R})$ and closed form $\alpha \in \Omega^k(M)$ with $[\alpha] = a$,*

$$B_g^k(a,a) \le E_g(\alpha). \tag{2.6}$$

If $b^+(X) = 1$, then, for every $g \in \mathrm{Met}^\infty(X)$ and $a \in V_X$,

$$B_g^2(a,a) = 2Q_X(a, p(g))^2 - Q_X(a,a). \tag{2.7}$$

For $a \in \mathcal{P}_X$ this becomes

$$B_g^2(a,a) = Q_X(a,a)\cosh\bigl(2d(u_a, p(g))\bigr). \tag{2.8}$$

Proof. Inequality (2.6) is the minimizing property in the orthogonal Hodge decomposition [70, Theorem 6.8]. For the second assertion, put $u = p(g)$ and decompose $a = Q_X(a,u)u + a^-$ with $a^- \in u^\perp$. The positive harmonic subspace is $\mathbb{R}u$. The formula [35, Lemma 3.1, equation (3.2)] therefore gives

$$B_g^2(a,a) = Q_X(a,u)^2 - Q_X(a^-,a^-) = 2Q_X(a,u)^2 - Q_X(a,a).$$

The last identity follows from $Q_X(a,u) = \sqrt{Q_X(a,a)}\cosh d(u_a, u)$. □

2.2. **The period differential.** Fix $g \in \mathrm{Met}^\infty(X)$. Write $\langle\cdot,\cdot\rangle_g$ and $|\cdot|_g$ for the pointwise inner product and norm on forms. Write $\mathrm{Sym}^2(T^*X)$ for the bundle of symmetric two-tensors and $\mathrm{Sym}^2_0(T^*X)$ for its subbundle of g-trace-free tensors. For a smooth symmetric two-tensor h, put $S = g^{-1}h$. Its action $S^{[2]}$ on two-forms is

$$(S^{[2]}\alpha)(v,w) = \alpha(Sv, w) + \alpha(v, Sw),$$

where v, w are tangent vectors at the same point. Differentiating the inner product on two-forms and the volume form gives (cf. [30, §2.4 of the author's version])

$$\dot{*}_h\alpha := D_g * [h](\alpha) = *_g \left(\tfrac{1}{2}\operatorname{tr}(S)\alpha - S^{[2]}\alpha\right). \tag{2.9}$$

Differentiation of $*_g^2 = \mathrm{id}$ gives $*_g\,\dot{*}_h + \dot{*}_h\,*_g = 0$. Thus $\dot{*}_h$ maps Λ_g^+ to Λ_g^- and Λ_g^- to Λ_g^+.

Identify $T_{\Pi_X(g)}\mathcal{D}_X$ with $\mathrm{Hom}(\mathcal{H}_g^+, \mathcal{H}_g^-)$ by graph coordinates. If $\alpha \in \mathcal{H}_g^+$ and $\beta \in \mathcal{H}_g^-$, then

$$Q_X\bigl(D\Pi_X|_g(h)[\alpha], [\beta]\bigr) = \frac{1}{2}\int_X (\dot{*}_h\alpha)\wedge\beta. \tag{2.10}$$

For small t, put $g_t = g + th$ and, in the fixed cohomology basis,

$$a_t = \tfrac{1}{2}(I + J_{g_t})[\alpha], \qquad \alpha_t = h^2_{g_t}(a_t).$$

These forms vary smoothly, are closed and self-dual, and satisfy $\alpha_0 = \alpha$. Differentiating $*_{g_t}\,\alpha_t = \alpha_t$ gives $2\dot\alpha_0^- = \dot *_h \alpha$. Since $d\dot\alpha_0 = 0$, graph coordinates and the anti-self-duality of β give

$$\begin{aligned} Q_X\big(D\Pi_X|_g(h)[\alpha],[\beta]\big) &= \int_X \dot\alpha_0 \wedge \beta = \int_X \dot\alpha_0^- \wedge \beta \\ &= \frac{1}{2}\int_X (\dot *_h \alpha)\wedge\beta. \end{aligned}$$

Pointwise, the map

$$\mathrm{Sym}^2_0(T^*_xX) \longrightarrow \mathrm{Hom}(\Lambda^+_g(x),\Lambda^-_g(x)), \qquad h \longmapsto \tfrac{1}{2}\dot *_h|_{\Lambda^+_g(x)} \tag{2.11}$$

is an isomorphism. Choose an oriented g-orthonormal eigenbasis of S, with eigenvalues $\lambda_1,\dots,\lambda_4$ and dual coframe $e^1,\dots,e^4$. Put $e^{ij} = e^i \wedge e^j$. If h is trace-free and (2.11) vanishes, its values on $e^{12}+e^{34}$, $e^{13}-e^{24}$, $e^{14}+e^{23}$ give $\lambda_1+\lambda_2 = \lambda_1+\lambda_3 = \lambda_1+\lambda_4 = 0$. Together with $\sum_i \lambda_i = 0$, these equalities give $\lambda_i = 0$ for all i. Both spaces have dimension nine. The resulting bundle isomorphism and its inverse are smooth.

Lemma 2.2. *Let ν be a positive volume form on an oriented four-manifold X. Every smooth rank-three subbundle $P \subset \Lambda^2 T^*X$ on which the wedge pairing is positive definite is Λ^+_g for a unique metric g with $dV_g = \nu$. The metric depends smoothly on P and ν, including in finite-dimensional families.*

Proof. Every positive three-plane is the self-dual subspace for a unique conformal class of Euclidean metrics [4, §7.2, p. 1106]. A volume form selects one metric in each conformal class. At a point x, the tangent space to the metrics with fixed volume is $\mathrm{Sym}^2_0(T^*_xX)$, and the differential of the positive-plane map is (2.11). The inverse function theorem therefore gives a smooth inverse. Positive rescaling gives smooth dependence on the volume form. The local metrics agree on overlaps by uniqueness, so they define a smooth metric on X, also in finite-dimensional families. □

Let $a \in V_X$ satisfy $Q_X(a,a) > 0$, and let ω_t be a smooth family of closed two-forms representing a. Suppose that $\omega_t \in \Gamma(P_t)$ for a smooth family of positive three-plane bundles P_t. For any smooth family of positive volume forms ν_t, the corresponding metrics g_t satisfy $\Lambda^+_{g_t} = P_t$ and $dV_{g_t} = \nu_t$. Since $*_{g_t}\,\omega_t = \omega_t$ and $d\omega_t = 0$, the forms ω_t are harmonic. If $b^+(X) = 1$, then $\Pi_X(g_t) = \mathbb{R}a$. If also $a \in \mathcal{P}_X$, then $p(g_t) = u_a$ for every t.

In positive index one, the period map is transverse to every wall defined by a negative integral class [16, Proposition 2.12].

Proposition 2.3. *Suppose $b^+(X) = 1$. For every metric g and every nonempty open set $U \subset X$, the differential Dp_g restricted to trace-free symmetric tensors supported in U is surjective.*

Proof. Let $u = p(g)$ and $\omega = h^2_g(u) \in \mathcal{H}^+_g$. The tangent space at u is its Q_X-orthogonal complement $u^\perp$, on which $-Q_X$ is positive definite. Differentiating $Q_X(p(g+th),p(g+th)) = 1$ and taking graph coordinates gives

$$Dp_g(h) = D\Pi_X|_g(h)[\omega].$$

If the restricted differential is not onto, choose $0 \neq b \in u^\perp$ such that $Q_X(Dp_g(h),b) = 0$ for every supported trace-free variation h. Put $\beta = h^2_g(b) \in \mathcal{H}^-_g$. For a smooth function $\chi \geq 0$ compactly supported in U, define

$$A_x(\eta) = \chi(x)\langle \eta, \omega(x)\rangle_g\, \beta(x), \qquad \eta \in \Lambda^+_g(x).$$

The bundle isomorphism (2.11) gives a smooth trace-free tensor h with $\operatorname{supp} h \subset \operatorname{supp}\chi$ and $A = \frac{1}{2}\dot *_h|_{\Lambda^+_g}$. Since β is anti-self-dual, (2.10) gives

$$0 = Q_X(Dp_g(h),b) = \int_X (A\omega)\wedge\beta = -\int_U \chi|\omega|^2_g|\beta|^2_g\, dV_g.$$

As χ is arbitrary, $|\omega|^2_g|\beta|^2_g = 0$ on U. The form ω is nonzero. Unique continuation on connected X [2, §2, Theorem 1, p. 422] therefore gives a nonempty open subset of U on which ω is nowhere zero. On that subset $\beta = 0$. The same theorem gives $\beta = 0$ on X, contrary to $b \neq 0$. □

Lemma 2.4. *Let $g \in \mathrm{Met}^\infty(X)$ and put $P = \Pi_X(g)$. Suppose that $D\Pi_X|_g$ is surjective. There are an open neighborhood V of P in $\mathcal{D}_X$ and a smooth map $s : V \to \mathrm{Met}^\infty(X)$ such that $s(P) = g$ and $\Pi_X s = \mathrm{id}_V$. If $b^+(X) = 1$, then, for any prescribed nonempty open set $U \subset X$, the section may be chosen so that $\mathrm{supp}(s(P') - g) \subset U$ for every $P' \in V$.*

Proof. Put $d = \dim \mathcal{D}_X$. If $d = 0$, take $V = \{P\}$ and $s(P) = g$. Otherwise choose symmetric tensors $h_1, \dots, h_d$ for which $D\Pi_X|_g(h_j)$ form a basis of $T_P\mathcal{D}_X$. For $q \in \mathbb{R}^d$ sufficiently small, $g(q) = g + \sum_j q_j h_j$ is a metric. The smooth map $F(q) = \Pi_X(g(q))$ has invertible derivative at zero. The inverse function theorem [55, Theorem 9.24 and equation (52)] gives a smooth local inverse F^{-1} near P. Set $s(P') = g(F^{-1}(P'))$. Then $s(P) = g$ and $\Pi_X s = \mathrm{id}_V$ after restricting to a neighborhood V of P. If $b^+(X) = 1$, Proposition 2.3 allows all the h_j to be supported in U, which gives the support assertion. □

2.3. **Prescribing the volume.** Let M^{2m} be closed and oriented, with $m \geq 1$, and let ν be a positive smooth volume form. For $g \in \mathrm{Met}^\infty(M)$, write ν / dV_g for the positive function f such that $\nu = f\, dV_g$. Set

$$N_\nu : \mathrm{Met}^\infty(M) \longrightarrow \mathrm{Met}^\infty_\nu(M), \qquad N_\nu(g) = \left(\frac{\nu}{dV_g} \right)^{1/m} g. \tag{2.12}$$

Lemma 2.5. *The map $(\nu, g) \mapsto N_\nu(g)$ is smooth on finite-dimensional smooth families and satisfies*

$$\Gamma_m \circ N_\nu = \Gamma_m, \qquad N_\nu|_{\mathrm{Met}^\infty_\nu(M)} = \mathrm{id}\,.$$

If s is a smooth local or global section of Γ_m, then $N_\nu \circ s$ is a section into $\mathrm{Met}^\infty_\nu(M)$, and every metric in its image is regular for the restricted period map. For a four-manifold X, $\Pi_X \circ N_\nu = \Pi_X$. If $D\Pi_X|_g$ is surjective, then so is $D(\Pi_X|_{\mathrm{Met}^\infty_\nu(X)})_{N_\nu(g)}$.

Proof. For a positive smooth function λ on M, one has $dV_{\lambda g} = \lambda^m dV_g$. The Hodge star on middle forms depends only on the conformal class [54, Lemma 2.2, p. 386]; hence $*_{\lambda g} = *_g$ on m-forms. Closed and coclosed middle forms are consequently the same for these two metrics. Their harmonic representatives coincide, so $\Gamma_m(\lambda g) = \Gamma_m(g)$. The formula for N_ν gives $dV_{N_\nu(g)} = \nu$, $N_\nu|_{\mathrm{Met}^\infty_\nu(M)} = \mathrm{id}$, and joint smoothness. Put $s_\nu = N_\nu \circ s$. For G in the domain of s, differentiation of $\Gamma_m \circ s_\nu = \mathrm{id}$ gives

$$D(\Gamma_m|_{\mathrm{Met}^\infty_\nu(M)})_{s_\nu(G)} \circ (Ds_\nu)_G = \mathrm{id}_{T_G\mathcal{D}(Q_M)}\,.$$

In dimension four, (2.5) identifies Γ_2 with Π_X. Suppose that $D\Pi_X|_g$ is surjective. Lemma 2.4 gives a local section of Π_X through g. Its composition with N_ν is a local section of $\Pi_X|_{\mathrm{Met}^\infty_\nu(X)}$ through $N_\nu(g)$. Differentiating at $\Pi_X(g)$ gives a right inverse to $D(\Pi_X|_{\mathrm{Met}^\infty_\nu(X)})_{N_\nu(g)}$. □

2.4. **Correction supported in a fixed open set.** Suppose that $b^+(X) = 1$. The distance on $\mathbb{H}_X$ is denoted by $d_{\mathbb{H}}$; thus $\cosh d_{\mathbb{H}}(u, v) = Q_X(u, v)$. An open subset $U \subset X$ is called *detecting* if it contains smooth singular two-cycles whose classes span $H_2(X; \mathbb{R})$. For an open set $W \subset X$, a metric k, and an integer $j \geq 0$, the $C^j(W, k)$ norm of a tensor is the maximum of the suprema on W of the pointwise k-norms of its covariant derivatives through order j, computed with the Levi-Civita connection of k. The space $H^j(W, k)$ is the corresponding L^2 Sobolev space.

Lemma 2.6. *Let $U_0 \Subset U_1 \subset X$ be connected open subsets, with U_0 detecting. Fix a metric k, an integer $\ell \geq 8$, positive constants λ, Λ, B, and a compact subset $K \subset \mathbb{H}_X$. There are $\varepsilon > 0$ and finitely many smooth symmetric tensors $V_1, \dots, V_N$, supported in U_0, with the following property. If*

$$p(g), u \in K, \qquad \lambda k \leq g \leq \Lambda k \quad \text{on } U_1, \qquad \|g\|_{C^\ell(U_1, k)} \leq B, \qquad d_{\mathbb{H}}(p(g), u) < \varepsilon,$$

then coefficients $z_j(g, u)$ can be chosen so that

$$\mathcal{R}(g, u) = g + \sum_{j=1}^{N} z_j(g, u) V_j \in \mathrm{Met}^\infty(X), \qquad p(\mathcal{R}(g, u)) = u.$$

The coefficients are smooth on every finite-dimensional smooth family of pairs (g, u) satisfying these conditions, and $\mathcal{R}(g, p(g)) = g$. No bound on g outside U_1 is required.

Proof. Put $q = b^-(X)$. If $q = 0$ or $K = \emptyset$, take $N = 0$ and $\varepsilon = 1$. Suppose that $q > 0$ and $K \neq \emptyset$. Choose linear coordinates on V_X such that

$$Q_X(x,x) = x_0^2 - x_1^2 - \cdots - x_q^2, \qquad \mathbb{H}_X = \{(\sqrt{1+|y|^2}, y) \,:\, y \in \mathbb{R}^q\}.$$

Then $P(u) = (u_1, \dots, u_q)$ defines global coordinates $P : \mathbb{H}_X \to \mathbb{R}^q$. Applying Gram–Schmidt to the coordinate tangent vectors, with the positive definite form $-Q_X$, gives a smooth frame $e_1(u), \dots, e_q(u)$ of $u^\perp$ with $Q_X(e_i(u), e_j(u)) = -\delta_{ij}$.

Choose a closed hyperbolic ball K_1 with $K \subset \operatorname{int} K_1$. Consider all metrics with period in K_1 satisfying

$$\tfrac{1}{2}\lambda k \le g \le 2\Lambda k, \qquad \|g\|_{C^\ell(U_1,k)} \le B + 1 \quad \text{on } U_1. \tag{2.13}$$

If there are no such metrics, take $N = 0$ and $\varepsilon = 1$. Write

$$u_g = p(g), \qquad \omega_g = h_g^2(u_g), \qquad \beta_{g,i} = h_g^2(e_i(u_g)).$$

These forms are respectively self-dual and anti-self-dual, and

$$\|\omega_g\|_{L^2(g)} = \|\beta_{g,i}\|_{L^2(g)} = 1. \tag{2.14}$$

The metric bounds in (2.13) convert these into uniform $L^2(U_1, k)$ bounds. The Hodge Laplacians are uniformly elliptic on relatively compact coordinate domains in U_1; their coefficients and their derivatives through order $\ell - 2$ are bounded by (2.13). The interior elliptic estimates [62, Proposition 0.5.E and equations (2.2.7), (2.2.22) of the author's revised text], at Sobolev order $\ell - 3$, give

$$\|\omega_g\|_{H^{\ell-1}(W,k)} + \sum_{i=1}^{q} \|\beta_{g,i}\|_{H^{\ell-1}(W,k)} \le C_W \qquad (W \Subset U_1),$$

where C_W depends only on W and the fixed data. Arzelà–Ascoli for the metrics and Sobolev compactness [70, Lemmas 6.22–6.23] for the forms, followed by a diagonal subsequence on a countable cover of U_1 by relatively compact coordinate balls, show that the tuples

$$T_g = (u_g, g|_{U_1}, \omega_g|_{U_1}, \beta_{g,1}|_{U_1}, \dots, \beta_{g,q}|_{U_1}) \tag{2.15}$$

have compact closure $\mathcal{T}$ for convergence in K_1, C^3_{loc} for metrics, and C^2_{loc} for forms. For $T = (u, \gamma, \omega, \beta_1, \dots, \beta_q) \in \mathcal{T}$, the metric γ is positive definite, the forms are closed, and $*_\gamma\ \omega = \omega$, $*_\gamma\ \beta_i = -\beta_i$.

Fix smooth singular two-cycles $C_1, \dots, C_m$ supported in U_0 whose classes span $H_2(X; \mathbb{R})$. Their supports are compact, so passage to the limit gives

$$\int_{C_j} \omega = \langle u, [C_j] \rangle, \qquad \int_{C_j} \beta_i = \langle e_i(u), [C_j] \rangle. \tag{2.16}$$

Consequently ω is not identically zero on U_0, and neither is $\sum_i c_i \beta_i$ for any nonzero vector $c = (c_1, \dots, c_q)$.

For a smooth symmetric tensor h supported in U_0, define

$$L_T(h) = \frac{1}{2} \sum_{i=1}^{q} DP_u(e_i(u)) \int_{U_0} \langle \dot{*}_h^\gamma \omega, \beta_i \rangle_\gamma \, dV_\gamma,$$

where $\dot{*}_h^\gamma$ is given by (2.9) with g replaced by γ. For $T = T_g$, (2.10) and $Q_X(e_i, e_j) = -\delta_{ij}$ give $L_{T_g}(h) = D(P \circ p)_g(h)$.

Suppose that a covector $\xi \in (\mathbb{R}^q)^*$ annihilates L_T. Put $c_i = \xi(DP_u(e_i(u)))$ and $\beta = \sum_i c_i \beta_i$. On a coordinate domain $O \subset U_0$, choose a smooth local frame $E_1, \dots, E_{10}$ of $\operatorname{Sym}^2 T^*X$. For every $\chi \in C_c^\infty(O)$, the tensor χE_a, extended by zero, is an allowed variation. Thus

$$\int_O \chi \langle \dot{*}_{E_a}^\gamma \omega, \beta \rangle_\gamma \, dV_\gamma = 0,$$

and continuity gives $\langle \dot{*}^{\gamma}_{E_a}\omega, \beta\rangle_\gamma = 0$ on O. Since the E_a span each symmetric tensor fibre, the pointwise isomorphism (2.11), applied to $A(\eta) = \langle \eta, \omega(x)\rangle_\gamma \beta(x)$, gives $|\omega|^2_\gamma |\beta|^2_\gamma = 0$ on U_0. By (2.16) and continuity, ω is nowhere zero on some nonempty open subset of U_0. Hence β vanishes there. The C^3 metric γ and the C^2 form β satisfy $d\beta = d(*_\gamma \beta) = 0$, so unique continuation [2, §2, Theorem 1, p. 422] gives $\beta = 0$ on connected U_1. Equation (2.16) then gives $c_i = 0$ for every i. Since $DP_u : u^\perp \to \mathbb{R}^q$ is an isomorphism, $\xi = 0$. Thus L_T is surjective.

For each $T \in \mathcal{T}$, choose q smooth tensors supported in U_0 whose images under L_T form a basis. For fixed tensors, the integrals defining L_T depend continuously on T, so their images remain a basis on a neighborhood of T. A finite subcover of $\mathcal{T}$ gives one list $V_1, \dots, V_N$. The matrices $(L_T(V_1), \dots, L_T(V_N))$ have rank q throughout $\mathcal{T}$. The least eigenvalues of their products with their transposes are continuous and positive on $\mathcal{T}$. Thus there is $c > 0$ such that

$$\mathsf{J}_g \mathsf{J}_g^{\mathsf{T}} \geq c I_q, \qquad \mathsf{J}_g = \big(D(P\circ p)_g(V_1), \dots, D(P\circ p)_g(V_N)\big) \tag{2.17}$$

for every metric satisfying (2.13) with period in K_1.

Put

$$\delta = \inf\{d_{\mathbb{H}}(v, w) : v \in K,\ w \in \mathbb{H}_X \setminus \operatorname{int} K_1\} > 0.$$

Choose $0 < \eta < \frac{1}{2}$ so that

$$\left(\frac{1+\eta}{1-\eta}\right)^2 - 1 < 2\sinh^2 \delta.$$

For $z = (z_1, \dots, z_N) \in \mathbb{R}^N$, write $|z|$ for its Euclidean norm. Choose $r > 0$ so that

$$\left\| \sum_j z_j V_j \right\|_{C^\ell(U_1,k)} < \min\{1, \eta\lambda\} \qquad (|z| < r).$$

Let $p(g) \in K$ and suppose that $\lambda k \leq g \leq \Lambda k$ and $\|g\|_{C^\ell(U_1,k)} \leq B$ on U_1. For $|z| < r$, put $g_z = g + \sum_j z_j V_j$. Then

$$(1-\eta)g \leq g_z \leq (1+\eta)g$$

on U_0, and hence on X, since $g_z = g$ outside U_0. In particular, g_z is positive definite and satisfies (2.13). Moreover,

$$E_{g_z}(\omega_g) \leq \left(\frac{1+\eta}{1-\eta}\right)^2 E_g(\omega_g) = \left(\frac{1+\eta}{1-\eta}\right)^2.$$

Lemma 2.1 gives

$$2\sinh^2 d_{\mathbb{H}}(p(g_z), p(g)) \leq \left(\frac{1+\eta}{1-\eta}\right)^2 - 1 < 2\sinh^2 \delta.$$

Thus $p(g_z) \in K_1$, and (2.17) holds throughout $|z| < r$.

Set $e = P(p(g)) - P(u)$ and consider

$$\frac{dz}{ds} = -\mathsf{J}_{g_z}^{\mathsf{T}} (\mathsf{J}_{g_z} \mathsf{J}_{g_z}^{\mathsf{T}})^{-1} e, \qquad z(0) = 0. \tag{2.18}$$

Its right-hand side is smooth for $|z| < r$, and

$$\left\| \mathsf{J}_{g_z}^{\mathsf{T}} (\mathsf{J}_{g_z} \mathsf{J}_{g_z}^{\mathsf{T}})^{-1} \right\|_{\mathrm{op}} \leq c^{-1/2}.$$

The unique maximal solution [70, Theorem 1.48] therefore satisfies $|z(s)| \leq s c^{-1/2} |e|$ for $s \geq 0$ in its interval of existence. If $|e| < r\sqrt{c}/2$, it remains in the compact ball $|z| \leq r/2$ up to time 1 and hence extends to that time. The chain rule gives

$$\frac{d}{ds} P(p(g_{z(s)})) = -e, \qquad P(p(g_{z(1)})) = P(u).$$

Since P is injective, $p(g_{z(1)}) = u$.

Let $L = \max\{1, \max_{v\in K_1} \|DP_v\|\}$, with the hyperbolic norm on $T_v\mathbb{H}_X$ and the Euclidean norm on $\mathbb{R}^q$. Since K_1 is geodesically convex, $|P(v) - P(w)| \le L d_{\mathbb{H}}(v, w)$ for $v, w \in K_1$. Take $\varepsilon = r\sqrt{c}/(2L)$ and let $z_j(g, u)$ be the components of $z(1)$. For any finite-dimensional smooth family of pairs (g, u), the vector field in (2.18) depends smoothly on the parameters. Adding zero components in the parameter directions and applying the same smooth-flow theorem gives smooth dependence of $z(1)$. If $u = p(g)$, then $e = 0$ and uniqueness gives $z(s) = 0$. □

3. Gram forms below the middle degree

Let Y^n be a closed connected oriented smooth manifold, and fix an integer k with $1 \le k < n/2$. Put $A = H^k(Y;\mathbb{R})$ and write $\Gamma_Y = \Gamma_k$ for its degree-k Hodge Gram map. Write $\mathrm{Sym}^+(A)$ for the cone of positive definite symmetric forms on A.

Theorem 3.1. *There is a smooth map*

$$\sigma_Y : \mathrm{Sym}^+(A) \longrightarrow \mathrm{Met}^\infty(Y), \qquad \Gamma_Y \circ \sigma_Y = \mathrm{id}\,.$$

The differential of Γ_Y is surjective at every metric in the image of σ_Y.

If $A = 0$, any metric gives the required map. Henceforth let $r = \dim A > 0$, and choose a $\mathbb{Z}$-basis $a_1, \dots, a_r$ of $H^k(Y;\mathbb{Z})/\mathrm{Tor}\, H^k(Y;\mathbb{Z})$, viewed as a lattice in A. The chosen basis identifies $\mathrm{Sym}^+(A)$ with $\mathcal{P} = \mathrm{Sym}^+(r)$. Norms on these finite-dimensional spaces are Euclidean norms in the fixed basis, and derivative norms are the induced operator norms.

3.1. Integral directions.

Lemma 3.2. *There is a countable locally finite cover (U_j) of $\mathcal{P}$ by relatively compact open sets, and for each j finitely many nonzero vectors $v_{j,e} \in \mathbb{Z}^r$, $1 \le e \le m_j$, with smooth nonnegative functions $a_{j,e} \in C_c^\infty(U_j)$, extended by zero to $\mathcal{P}$, such that*

$$G = \sum_{j,e} \lambda_{j,e}(G) v_{j,e} v_{j,e}^t, \qquad \lambda_{j,e} = a_{j,e}^2. \tag{3.1}$$

There are $\chi_j \in C_c^\infty(U_j;[0,1])$, also extended by zero to $\mathcal{P}$, equal to one on a neighborhood of $\bigcup_e \mathrm{supp}\, a_{j,e}$. For every compact $K \subset \mathcal{P}$ there is a constant C_K, independent of j, e, such that

$$|D\lambda_{j,e}(G)| \le C_K \sqrt{\lambda_{j,e}(G)} \quad (G \in K). \tag{3.2}$$

Only finitely many supports $\mathrm{supp}\, a_{j,e}$ *meet K.*

Proof. Fix $G_0 \in \mathcal{P}$ and put $m = r(r+1)/2$. By the positive rank-one decomposition of [12, Lemma 4.3 of the authors' manuscript], followed by congruence with $G_0^{1/2}$, there are vectors $w_1, \dots, w_m \in \mathbb{R}^r$ such that the forms $w_e w_e^t$ are a basis of $\mathrm{Sym}(r)$ and G_0 is a strictly positive linear combination of them. Choose rational vectors q_e sufficiently close to w_e. Continuity of matrix inversion gives an isomorphism

$$L_q : \mathbb{R}^m \longrightarrow \mathrm{Sym}(r), \qquad L_q(c) = \sum_e c_e q_e q_e^t, \qquad L_q^{-1}(G_0) \in (0,\infty)^m.$$

Choose an integer $d \ge 1$ with $v_e = d q_e \in \mathbb{Z}^r$. The isomorphism $L(c) = \sum_e c_e v_e v_e^t$ equals $d^2 L_q$. Thus

$$c(G) = L^{-1}(G), \qquad G = \sum_e c_e(G) v_e v_e^t,$$

and every c_e is positive on a neighborhood of G_0.

Choose a countable locally finite relatively compact refinement (U_j) of these neighborhoods and a smooth partition of unity (f_j) with $\mathrm{supp}\, f_j \subset U_j$ [70, Lemma 1.9 and Theorem 1.11]. Each f_j has compact support. Let $c_{j,e}$ denote the positive linear coefficients on U_j, and put

$$Z = \sum_j f_j^2, \qquad a_{j,e} = f_j \sqrt{c_{j,e}/Z} \quad \text{on } U_j, \qquad a_{j,e} = 0 \quad \text{off } U_j.$$

The sum defining Z is locally finite and $Z > 0$. Each $a_{j,e}$ is smooth, since its support is contained in $\operatorname{supp} f_j \Subset U_j$. Summing the local identities with weights f_j^2/Z gives (3.1). Choose χ_j equal to one near $\operatorname{supp} f_j$ by the corollary to [70, Theorem 1.11]. Local finiteness gives a common bound for $|Da_{j,e}|$ on each compact K. The identity $D\lambda_{j,e} = 2a_{j,e}Da_{j,e}$ then gives (3.2). □

Enumerate the pairs (j, e) by $i \geq 1$, and let $j(i)$ be the first index of the ith pair. Write $v_i = v_{j,e}$, and $\lambda_i = \lambda_{j,e}$ for that pair. We regard a column $v = (v_1, \dots, v_r)^t \in \mathbb{Z}^r$ as the real linear functional on A given by

$$v\left(\sum_{s=1}^{r} x_s a_s\right) = \sum_{s=1}^{r} v_s x_s \qquad (x_s \in \mathbb{R}).$$

The evaluation pairing identifies

$$H_k(Y;\mathbb{Z})/\operatorname{Tor} H_k(Y;\mathbb{Z}) \simeq \operatorname{Hom}\bigl(H^k(Y;\mathbb{Z})/\operatorname{Tor} H^k(Y;\mathbb{Z}), \mathbb{Z}\bigr)$$

by the universal coefficient theorem [23, Theorem 3.2]. For $q \geq 1$, let D^q be the closed unit disk in $\mathbb{R}^q$.

Lemma 3.3. *After multiplying each v_i by a positive integer, there are pairwise disjoint closed framed tubes $N_i \cong \Sigma_i \times D^{n-k}$ in Y, where Σ_i is a closed connected oriented smooth k-manifold, such that*

$$\langle a, [\Sigma_i] \rangle = v_i(a) \qquad (a \in A).$$

Proof. The functional v_i is represented by an integral homology class modulo torsion. Thom's theorem [63, Theorem II.4, p. 32] represents a nonzero multiple of a lift of this class by an embedded oriented submanifold with trivial normal bundle. Its codimension is $q = n - k$, and the required inequality $n < 2q$ is equivalent to $2k < n$. Reverse the orientation if the multiplier is negative. Join its components by oriented tubes along a tree of disjoint arcs whose interiors avoid the submanifold; here $k \geq 1$ and $q \geq 2$. The surgery trace in $Y \times [0,1]$ has only one-handles relative to its incoming boundary. Their attaching disks are contractible and $SO(q)$ is connected, so the given normal framings extend over the trace. Its outgoing boundary is a connected framed submanifold representing the sum of the original classes.

Suppose $N_1, \dots, N_{i-1}$ have been chosen, and let S be the next connected framed representative. By transversality and isotopy extension [28, Chapter 3, Theorem 2.4; Chapter 8, Theorem 1.3 and Exercise 5], a sufficiently small isotopy makes S disjoint from their cores, since $2k < n$. Choose pairwise disjoint extensions of the old tubular charts, with coordinates $(p, z) \in \Sigma_s \times \mathbb{R}^q$, $|z| < R_s$, where $R_s > 1$ and N_s corresponds to $|z| \leq 1$, for $1 \leq s < i$. Compactness of S gives $0 < \delta_s < 1$ such that $|z| \geq \delta_s$ on S in the sth chart. Choose smooth diffeomorphisms $f_s : [0, R_s] \to [0, R_s]$ with $f_s' > 0$, linear near zero, equal to the identity near R_s, and $f_s(\delta_s) > 1$. For $0 \leq t \leq 1$, the maps

$$(p, z) \longmapsto \left(p, \frac{(1-t)|z| + t\, f_s(|z|)}{|z|}\, z\right) \qquad (z \neq 0),$$

with value $(p, 0)$ at $z = 0$, extend by the identity outside the charts to an ambient isotopy Φ_t. Then $\Phi_1(S) \cap N_s = \emptyset$ for every $s < i$. The isotopy preserves the homology class and normal framing of S. A sufficiently small closed tube about $\Phi_1(S)$ is disjoint from all the N_s. Induction gives the sequence. □

If the multiplier for v_i is d_i, divide its corresponding amplitude $a_{j,e}$ by d_i and λ_i by d_i^2. Together with the new vector $d_i v_i$, these coefficients still satisfy (3.1). The supports and cutoffs are unchanged, and (3.2) still holds on every compact set. The family $(\operatorname{supp} \chi_{j(i)})_{i \geq 1}$ is locally finite in $\mathcal{P}$; no spatial local finiteness of (N_i) is required.

3.2. **Weighted energies.** Fix $h_0 \in \operatorname{Met}^\infty(Y)$. Choose on each Σ_i a metric of volume one, and denote its volume form by η_i. The same symbol denotes its pullback to $N_i = \Sigma_i \times D^{n-k}$. Choose a smooth metric h_i on N_i, equal to the product of this metric with the Euclidean normal metric on a smaller tube, and equal to h_0 near ∂N_i. Extend $h_i - h_0$ by zero. Choose a nonnegative $\rho_i \in C_c^\infty(\operatorname{int} D^{n-k})$ supported in

the smaller normal disk, with $\int_{D^{n-k}} \rho_i \, dz = 1$, where dz is Euclidean volume. Also write ρ_i for its pullback to N_i and zero extension to Y. For smooth k-forms on $\operatorname{int} N_i$, put

$$B_i(\alpha, \beta) = \int_{N_i} \rho_i \langle \alpha, \beta \rangle_{h_i} \, dV_{h_i}.$$

Then $B_i(\eta_i, \eta_i) = 1$. If $\alpha \in \Omega^k(Y)$ is closed and represents $a \in A$, its integral on every slice $\Sigma_i \times \{z\}$ is $v_i(a)$. Cauchy–Schwarz on this volume-one slice, followed by integration against $\rho_i \, dz$, gives

$$B_i(\alpha, \alpha) \geq v_i(a)^2. \tag{3.3}$$

For $F \in \Omega^{k-1}(\operatorname{int} N_i)$, Stokes' theorem gives

$$B_i(\eta_i, dF) = \int_{D^{n-k}} \rho_i(z) \left(\int_{\Sigma_i \times \{z\}} dF \right) dz = 0. \tag{3.4}$$

For $G \in \mathcal{P}$ and $\epsilon > 0$, define

$$h(G) = h_0 + \sum_i \chi_{j(i)}(G)(h_i - h_0), \tag{3.5}$$

$$w_{G,\epsilon} = \epsilon + \sum_i \lambda_i(G)\rho_i, \qquad g_{G,\epsilon} = w_{G,\epsilon}^{2/(n-2k)} h(G). \tag{3.6}$$

Local finiteness in $\mathcal{P}$ makes these sums finite on a neighborhood of each G. On N_i, $h(G) = (1 - \chi_{j(i)}(G))h_0 + \chi_{j(i)}(G)h_i$; outside the tubes it equals h_0. Thus $h(G)$ is positive definite, $w_{G,\epsilon} \geq \epsilon$, and the metrics depend smoothly on (G, ϵ). If $\lambda_i(G) > 0$, then $\chi_{j(i)}(G) = 1$ and $h(G) = h_i$ on N_i. On k-forms in dimension n, conformal rescaling gives $*_{w^{2/(n-2k)}h} = w \, *_h$. Hence

$$B_{g_{G,\epsilon}} = \epsilon B_{h(G)} + \sum_i \lambda_i(G) B_i. \tag{3.7}$$

For $1 \leq p \leq r$, choose a closed k-form β_p^0 representing a_p, and put $v_{ip} = v_i(a_p)$. The restriction of $\beta_p^0 - v_{ip}\eta_i$ to $\operatorname{int} N_i$ is exact: $\operatorname{int} N_i$ retracts onto the connected oriented k-manifold Σ_i, whose top real cohomology is detected by integration. The de Rham theorem [70, Theorem 5.36 and §5.37] therefore gives $F_{ip} \in \Omega^{k-1}(\operatorname{int} N_i)$ with $\beta_p^0 - v_{ip}\eta_i = dF_{ip}$. Choose $\kappa_i \in C_c^\infty(\operatorname{int} N_i; [0,1])$ equal to one near $\operatorname{supp} \rho_i$, and extend $\kappa_i F_{ip}$ by zero to Y. Put

$$\beta_p(G) = \beta_p^0 - \sum_i \chi_{j(i)}(G) d(\kappa_i F_{ip}). \tag{3.8}$$

The sum is locally finite in G. Thus the forms are jointly smooth, closed, and represent a_p. If $\lambda_i(G) > 0$, then $\beta_p(G) = v_{ip}\eta_i$ on $\operatorname{supp} \rho_i$.

Lemma 3.4. *Put $H(G, \epsilon) = \Gamma_Y(g_{G,\epsilon})$, expressed in the basis $a_1, \dots, a_r$. Write $\mathrm{id}_{\mathrm{Sym}(r)}$ for the identity endomorphism of $\mathrm{Sym}(r)$. For every compact $K \subset \mathcal{P}$ there is C_K such that, for $G \in K$ and $0 < \epsilon \leq 1$,*

$$|H(G, \epsilon) - G| \leq C_K \epsilon, \tag{3.9}$$

$$|D_G H(G, \epsilon) - \mathrm{id}_{\mathrm{Sym}(r)}| \leq C_K \sqrt{\epsilon}. \tag{3.10}$$

Proof. Choose a compact $L \subset \mathcal{P}$ with $K \subset \operatorname{int} L$, and set

$$\mathcal{I}_K = \{i : \operatorname{supp} \chi_{j(i)} \cap L \neq \emptyset\}.$$

This set is finite and contains every index contributing to the sums or their derivatives near K. The constants C_K below are independent of $G \in K$ and $0 < \epsilon \leq 1$, and may increase. The indices p, q range from 1 to r. Write $g = g_{G,\epsilon}$, $h = h(G)$, and $u_p = h_g^k(a_p)$. Smoothness on the compact set $L \times Y$ gives

$$B_h \left(\sum_p x_p \beta_p, \sum_p x_p \beta_p \right) \leq C_K |x|^2 \qquad (x \in \mathbb{R}^r).$$

Let I_r be the identity matrix. Apply (2.6) to these representatives. Together with (3.1), (3.3), and (3.7), this gives, as quadratic forms,

$$G \le H(G,\epsilon) \le G + C_K \epsilon I_r. \tag{3.11}$$

This proves (3.9).

Each $e_p = u_p - \beta_p(G)$ is exact. Harmonic orthogonality gives

$$B_g(e_p, e_p) = B_g(\beta_p, \beta_p) - H_{pp} \le C_K \epsilon. \tag{3.12}$$

Since $B_g \ge \epsilon B_h$, this implies $B_h(e_p, e_p) \le C_K$. The bound on the trial forms and $u_p = e_p + \beta_p$ then give $B_h(u_p, u_p) \le C_K$. For $i \in \mathcal{I}_K$, the fixed bilinear forms B_i satisfy

$$B_i(\gamma,\gamma) \le C_K B_h(\gamma,\gamma) \qquad \big(\gamma \in \Omega^k(Y),\ G \in K\big);$$

the constant is uniform by compactness of $L \times Y$ and finiteness of $\mathcal{I}_K$. Moreover, (3.7) gives $\lambda_i B_i(e_p, e_p) \le C_K \epsilon$. Combining these bounds and applying Cauchy–Schwarz gives, for $i \in \mathcal{I}_K$ with $\lambda_i > 0$,

$$|B_i(e_p, e_q)| \le C_K \min\{1, \epsilon/\lambda_i\} \le \frac{2C_K\epsilon}{\epsilon + \lambda_i}. \tag{3.13}$$

The last bound also holds when $\lambda_i = 0$. For $\lambda_i > 0$, exactness of e_p, e_q and (3.4) give

$$B_i(u_p, u_q) = v_{ip} v_{iq} + B_i(e_p, e_q). \tag{3.14}$$

Applying (2.3) to $g_{G,\epsilon}$ and differentiating (3.7) at fixed ϵ gives, for $V \in \mathrm{Sym}(r)$,

$$D_G H_{pq}[V] = \epsilon (D_G B_h[V])(u_p, u_q) + \sum_i D\lambda_i[V] B_i(u_p, u_q). \tag{3.15}$$

Here $D_G B_h[V]$ denotes the derivative of $G \mapsto B_{h(G)}$ with its k-form arguments fixed. The tensor $h^{-1} D_G h[V]$ is bounded by $C_K|V|$ on $L \times Y$. For fixed $\alpha, \beta \in \Omega^k(Y)$, differentiation of the metric and volume factors, followed by Cauchy–Schwarz, gives

$$|(D_G B_h[V])(\alpha,\beta)| \le C_K |V| \sqrt{B_h(\alpha,\alpha) B_h(\beta,\beta)}.$$

Thus the first term in (3.15) is bounded by $C_K \epsilon |V|$.

At a zero of λ_i, (3.2) gives $D\lambda_i = 0$. Differentiating (3.1) and applying (3.14) therefore gives

$$D_G H_{pq}[V] - V_{pq} = \epsilon (D_G B_h[V])(u_p, u_q) + \sum_{\lambda_i > 0} D\lambda_i[V] B_i(e_p, e_q).$$

By (3.2) and (3.13), each summand satisfies

$$|D\lambda_i[V] B_i(e_p, e_q)| \le C_K |V| \frac{\epsilon \sqrt{\lambda_i}}{\epsilon + \lambda_i} \le \tfrac{1}{2} C_K |V| \sqrt{\epsilon}.$$

The index set is finite, and $\epsilon \le \sqrt{\epsilon}$. This proves (3.10). □

3.3. **Correction near a prescribed value.**

Lemma 3.5. *Let B be a smooth finite-dimensional manifold and let $\mathcal{A} \subset B \times (0,\infty)$. Suppose B has an open cover $\mathcal{U}$ and numbers $\epsilon_U > 0$, $U \in \mathcal{U}$, such that $U \times (0, \epsilon_U] \subset \mathcal{A}$. There is a smooth function $t : B \to (0,\infty)$ such that $(b, t(b)) \in \mathcal{A}$ for every $b \in B$.*

Proof. Choose a locally finite smooth partition of unity $(\psi_j)_{j \in J}$ with $\operatorname{supp} \psi_j \subset U_j \in \mathcal{U}$ [70, Theorem 1.11], and put

$$t(b) = \frac{1}{2} \sum_{j \in J} \psi_j(b) \epsilon_{U_j}.$$

This function is smooth and positive. For a fixed b, the set of indices with $\psi_j(b) > 0$ is finite and nonempty. Choose j_0 in this set for which $\epsilon_{U_{j_0}}$ is maximal. Then $b \in U_{j_0}$ and

$$0 < t(b) \le \tfrac{1}{2} \epsilon_{U_{j_0}} < \epsilon_{U_{j_0}},$$

so $(b, t(b)) \in \mathcal{A}$. □

Let $d \geq 0$ be an integer. On $\mathbb{R}^d$, write $|\cdot|$ for the Euclidean norm, $\|\cdot\|$ for the induced operator norm, and I for the identity endomorphism. For $x \in \mathbb{R}^d$ and $s > 0$, put $\overline{B}_s(x) = \{y \in \mathbb{R}^d : |y - x| \leq s\}$.

Lemma 3.6. *Let B be a smooth finite-dimensional manifold, let $c : B \to \mathbb{R}^d$ and $r : B \to (0, \infty)$ be smooth, and let $f(b, v)$ be a smooth $\mathbb{R}^d$-valued map on a neighborhood of $\{(b, v) : |v| \leq r(b)\}$ in $B \times \mathbb{R}^d$. Suppose*

$$\|D_v f(b, v) - I\| \leq \tfrac{1}{4}, \qquad |f(b, 0) - c(b)| \leq r(b)/4 \quad (|v| \leq r(b)).$$

There is a unique map $z : B \to \mathbb{R}^d$ with $|z(b)| \leq r(b)$ and $f(b, z(b)) = c(b)$. It is smooth, and

$$|z(b)| \leq r(b)/3, \qquad \|(D_v f(b, z(b)))^{-1}\| \leq \tfrac{4}{3}.$$

Proof. For fixed b, the map $T_b(v) = v - f(b, v) + c(b)$ has Lipschitz constant at most $1/4$ on the complete closed ball $\overline{B}_{r(b)}(0)$ and satisfies $|T_b(v)| \leq r(b)/2$ there. The contraction theorem [55, Theorem 9.23] gives its unique fixed point $z(b)$. The identity $T_b(z(b)) = z(b)$ gives $|z(b)| \leq r(b)/4 + |z(b)|/4$, hence $|z(b)| \leq r(b)/3$. For $w \in \mathbb{R}^d$, the derivative bound gives $|D_v f(b, z(b))w| \geq 3|w|/4$, and hence the inverse bound. The implicit function theorem [70, Theorem 1.37], applied to $(b, v) \mapsto f(b, v) - c(b)$, gives smooth local solutions. Since $|z(b)| \leq r(b)/3$, these local solutions remain in the neighboring radius-r balls after restriction. Uniqueness identifies them with the fixed points there. □

Lemma 3.7. *Let $\Omega \subset \mathbb{R}^d$ be open, with $d \geq 1$, and let $F : (0, 1) \times \Omega \to \mathbb{R}^d$ be smooth. Write $F_\epsilon(x) = F(\epsilon, x)$, and suppose $F_\epsilon \to \mathrm{id}_\Omega$ in $C^1(K)$ for every compact $K \subset \Omega$ as $\epsilon \downarrow 0$. There are smooth positive functions r, ϵ on Ω, with $\epsilon < 1$, and a smooth map $z : \Omega \to \mathbb{R}^d$ such that*

$$\overline{B}_{2r(P)}(P) \subset \Omega, \qquad |z(P)| \leq r(P)/3, \qquad F_{\epsilon(P)}(P + z(P)) = P.$$

For these choices of r and ϵ, the solution in $\overline{B}_{r(P)}(0)$ is unique, and

$$\left\| \left(D_x F_{\epsilon(P)}(P + z(P)) \right)^{-1} \right\| \leq \tfrac{4}{3}.$$

Proof. Lemma 3.5 gives a smooth positive r with $2r(P) < \operatorname{dist}(P, \mathbb{R}^d \setminus \Omega)$; take $r = 1$ if $\Omega = \mathbb{R}^d$. Choose an open cover $\mathcal{U}$ of Ω by nonempty sets U whose closures are compact and contained in Ω. For $U \in \mathcal{U}$, the set

$$K_U = \{P + v : P \in \overline{U},\ |v| \leq 2r(P)\}$$

is compact in Ω. The assumed convergence gives $a_U \in (0, 1)$ such that, for $0 < e \leq a_U$,

$$\sup_{x \in K_U} \|D_x F_e(x) - I\| \leq \tfrac{1}{4}, \qquad \sup_{x \in K_U} |F_e(x) - x| \leq \tfrac{1}{4} \min_{P \in \overline{U}} r(P).$$

Let $\mathcal{A}$ be the set of pairs $(P, e) \in \Omega \times (0, 1)$ for which

$$\|D_x F_e(P + v) - I\| \leq \tfrac{1}{4} \quad (|v| \leq r(P)), \qquad |F_e(P) - P| \leq r(P)/4.$$

The preceding estimates give $U \times (0, a_U] \subset \mathcal{A}$ for every $U \in \mathcal{U}$. Lemma 3.5 therefore gives a smooth function $\epsilon : \Omega \to (0, 1)$ with $(P, \epsilon(P)) \in \mathcal{A}$. On the open set $\{(P, v) \in \Omega \times \mathbb{R}^d : |v| < 2r(P)\}$, put $f(P, v) = F_{\epsilon(P)}(P + v)$. Then

$$D_v f(P, v) = D_x F_{\epsilon(P)}(P + v).$$

Lemma 3.6, with $c(P) = P$, gives the conclusions. □

Proof of Theorem 3.1. Identify $\mathrm{Sym}(r)$ with $\mathbb{R}^{r(r+1)/2}$ by Euclidean coordinates. Lemma 3.4 gives local C^1 convergence $H(\cdot, \epsilon) \to \mathrm{id}_{\mathcal{P}}$. Apply Lemma 3.7 with $\Omega = \mathcal{P}$ and $F(e, G) = H(G, e)$, and set

$$\sigma_Y(G) = g_{G + z(G), \epsilon(G)}.$$

The metrics are jointly smooth and $\Gamma_Y(\sigma_Y(G)) = G$. Differentiation gives

$$D\Gamma_Y|_{\sigma_Y(G)} \circ D\sigma_Y|_G = \mathrm{id}_{\mathrm{Sym}(r)},$$

so the differential is surjective at every image metric. □

4. Periods under cap replacements

Throughout this section cohomology has real coefficients. A cap is a compact codimension-zero smooth submanifold with boundary. Replacements are made by fixed orientation-compatible identifications of collars of their boundaries.

4.1. **Uniform harmonic estimates.** Matrix norms are induced Euclidean operator norms. Norms of parameter derivatives are the corresponding multilinear norms. The letter I denotes an identity operator.

Lemma 4.1. *Let M be a closed oriented smooth n-manifold, let Λ be an index set, let $d, m \geq 0$ be integers, and let $H, C \geq 0$. For each $\lambda \in \Lambda$, let g_λ be a metric and $h_{1,\lambda}, \dots, h_{d,\lambda}$ smooth symmetric two-tensors with $|h_{a,\lambda}|_{g_\lambda} \leq H$ pointwise. Fix an integer $0 \leq k \leq n$ and classes $c_1, \dots, c_m \in H^k(M;\mathbb{R})$. For $q = (q_1, \dots, q_d) \in \mathbb{R}^d$, put*

$$g_\lambda(q) = g_\lambda + \sum_{a=1}^{d} q_a h_{a,\lambda}, \qquad G_\lambda(q) = \Big(B^k_{g_\lambda(q)}(c_i, c_j)\Big)_{i,j=1}^{m}$$

whenever $g_\lambda(q)$ is positive definite. Suppose $\|G_\lambda(0)\| \leq C$ for every λ. There is $r_0 = r_0(n, d, H) > 0$ such that $g_\lambda(q)$ is a metric for $|q| \leq r_0$ and, for every integer $j \geq 0$,

$$\sup_{\lambda \in \Lambda,\ |q| \leq r_0} \|D_q^j G_\lambda(q)\| \leq C_j(n, d, H, C).$$

If $n = 4$, $k = 2$, and the c_i form a basis of $V = H^2(M;\mathbb{R})$, fix a splitting $V = V_+ \oplus V_-$ orthogonal for the intersection form, positive on V_+ and negative on V_-. The graph coordinates $V_+ \to V_-$ of the periods satisfy the same derivative bounds, with additional dependence on the intersection matrix and the chosen coordinates.

Proof. Fix λ and omit it from the notation. Set $\mathcal{H} = L^2\Omega^k(M, g_\lambda)$, with scalar product $\langle\ ,\ \rangle$, and define the endomorphism of TM

$$S(q) = I + \sum_a q_a g_\lambda^{-1} h_{a,\lambda}.$$

Choose $r_0 > 0$ with $\sqrt{d}\, H r_0 < 1/2$. Since $\|S(q) - I\| \leq \sqrt{d}\, H|q|$, its eigenvalues lie in $[1/2, 3/2]$ for $|q| \leq r_0$. The bounded multiplication operator $A(q)$ on $\mathcal{H}$ defined by

$$B_{g_\lambda(q)}(\alpha, \beta) = \langle A(q)\alpha, \beta\rangle$$

is, pointwise,

$$A(q) = (\det S(q))^{1/2} \Lambda^k\big((S(q)^{-1})^*\big),$$

where the star on $S(q)^{-1}$ denotes the dual map on T^*M. The determinant, inverse, and exterior-power formulas on the stated eigenvalue interval show that $A(q)$ is smooth in operator norm and give

$$A(q) \geq cI, \qquad \|D_q^j A(q)\|_{\mathcal{H}\to\mathcal{H}} \leq K_j \quad (j \geq 0),$$

where $c = c(n) > 0$ and $K_j = K_j(n, d, H)$.

Let $E \subset \mathcal{H}$ be the closure of the smooth exact k-forms, with $E = \{0\}$ when $k = 0$, and let P_E be its orthogonal projection. The operator $L(q) = P_E A(q)|_E$ is invertible by the Lax–Milgram theorem [9, Corollary 5.8], applied to the bounded bilinear form $\langle A(q)u, v\rangle$ on E. Its coercivity gives $\|L(q)^{-1}\| \leq c^{-1}$.

Define $U(q) : \mathbb{R}^m \to \mathcal{H}$ by

$$U(q)z = \sum_{i=1}^{m} z_i h^k_{g_\lambda(q)}(c_i).$$

The smooth harmonic dependence in §2 gives smoothness of U in q. The image of $U(q) - U(0)$ lies in E, and harmonic orthogonality gives $P_E A(q) U(q) = 0$. Consequently

$$U(q) = \big(I - L(q)^{-1} P_E A(q)\big) U(0), \qquad \|U(0)\|^2 = \|G_\lambda(0)\| \leq C.$$

Thus $\|U(q)\| \le (1+c^{-1}K_0)\sqrt{C}$. Every derivative of positive order of U has image in the closed subspace E. For multiindices $\nu, \beta \in \{0,1,2,\dots\}^d$, write $|\nu| = \sum_a \nu_a$, $\partial^\nu = \partial_{q_1}^{\nu_1} \cdots \partial_{q_d}^{\nu_d}$, and $\binom{\nu}{\beta} = \prod_a \binom{\nu_a}{\beta_a}$. Write $\beta \le \nu$ when $\beta_a \le \nu_a$ for every a. Differentiating the orthogonality identity gives, for $\nu \neq 0$,

$$L(q)\partial^\nu U(q) = -P_E \sum_{\substack{\beta \le \nu \\ \beta \neq 0}} \binom{\nu}{\beta} (\partial^\beta A(q))(\partial^{\nu-\beta} U(q)).$$

Induction on $|\nu|$ and $\|L(q)^{-1}\| \le c^{-1}$ give $\|D_q^j U(q)\| \le K_j' \sqrt{C}$, with K_j' depending only on n, d, H, j. Finally,

$$G_\lambda(q) = U(q)^* A(q) U(q),$$

where $U(q)^*$ is the Hilbert adjoint. The product rule gives the asserted Gram bounds, independently of m.

In the four-dimensional case, let Q be the intersection matrix in the fixed basis and put $G = G_\lambda(q)$. Equation (2.1) gives

$$J = Q^{-1}G, \qquad G^{-1} = Q^{-1}GQ^{-1}. \tag{4.1}$$

Thus G and G^{-1} are uniformly bounded. The compatible matrices lie in a compact subset of the positive definite cone. Let $\pi_\pm : V \to V_\pm$ be the projections of the fixed splitting. By (2.5), the positive eigenspace projection is $P(G) = (I + Q^{-1}G)/2$, and its graph coordinate is

$$Z(G) = \big(\pi_- P(G)|_{V_+}\big)\big(\pi_+ P(G)|_{V_+}\big)^{-1}.$$

The image of $P(G)$ is positive and its kernel is negative. If $v \in V_+$ and $\pi_+ P(G)v = 0$, then $P(G)v \in V_- \cap \operatorname{im} P(G) = \{0\}$, and hence $v \in V_+ \cap \ker P(G) = \{0\}$. Thus $\pi_+ P(G)|_{V_+}$ is invertible. The formula for $Z(G)$ is smooth on a neighborhood of the compact set of compatible matrices. Its derivatives are bounded on this compact set; the chain rule gives the period bounds. □

Lemma 4.2. *Let $r_0, L > 0$, $M \ge 0$, and let $f : \overline{B}_{r_0}(0) \to \mathbb{R}^d$ be smooth on a neighborhood of the closed Euclidean ball. Suppose $Df(0)$ is invertible and*

$$\|Df(0)^{-1}\| \le L, \qquad \sup_{|q| \le r_0} \|D^2 f(q)\| \le M.$$

If $0 < \rho \le \min\{r_0/2, (4LM)^{-1}\}$ and $|f(0)| \le \rho/(4L)$, then f has a unique zero in $\overline{B}_\rho(0)$. That zero satisfies $|q| \le 2L|f(0)|$. For a jointly smooth finite-dimensional family $f = f_t$, the zero depends smoothly on t whenever the hypotheses hold with a common choice of r_0, L, M, ρ near each parameter. The term $(4LM)^{-1}$ is omitted when $M = 0$.

Proof. Put $A = Df(0)^{-1}$. The mean value inequality [55, Theorem 9.19], applied on the open ball $B_{r_0}(0)$ to $x \mapsto Df(x)v$ for $|v| = 1$, gives

$$\|Df(q) - Df(0)\| \le M|q|, \qquad \|ADf(q) - I\| \le LM\rho \le \tfrac{1}{4} \quad (|q| \le \rho).$$

Also $|Af(0)| \le L|f(0)| \le \rho/4$. Lemma 3.6, applied to Af with target zero and radius ρ, gives the unique zero. At that zero,

$$|q| \le |Af(0)| + \tfrac{1}{4}|q|, \qquad |q| \le \tfrac{4}{3}L|f(0)| \le 2L|f(0)|.$$

For a smooth family f_t, the matrices $Df_t(0)^{-1}$ are smooth, so the parameter assertion of the same lemma applies. □

Lemma 4.3. *Fix $g \in \mathrm{Met}^\infty(X)$, and put $\|\cdot\|_2 = \|\cdot\|_{L^2(g)}$. Let Q_X have signature (p,q), and let $a_1, \dots, a_p, b_1, \dots, b_q$ be a Q_X-orthonormal basis, the first vectors positive and the last negative. Let $e \ge 0$. Suppose closed representatives α_i of a_i and β_j of b_j satisfy*

$$\|\alpha_i^-\|_2 \le e, \qquad \|\beta_j^+\|_2 \le e.$$

For their harmonic representatives u_i, v_j,

$$\|u_i - \alpha_i\|_2 \le \sqrt{2}e, \qquad \|u_i^-\|_2 \le e,$$
$$\|v_j - \beta_j\|_2 \le \sqrt{2}e, \qquad \|v_j^+\|_2 \le e.$$

For $0 < e \le 1$, the complete harmonic Gram matrix in this basis is $I + O_{p,q}(e)$. Relative to the splitting $\mathrm{span}(a_i) \oplus \mathrm{span}(b_j)$, the graph coordinate of the period has norm $O_{p,q}(e)$. For sufficiently small e, the forms u_i^+ and v_j^- are bases of the positive and negative harmonic spaces. All estimates are uniform for a compact set $\mathcal{K}$ of such bases. If z is a fixed global graph chart, then, for sufficiently small e,

$$\|z(\Pi_X(g)) - z(\mathrm{span}(a_i))\| \le C_{\mathcal{K},z}\, e.$$

Proof. The orthogonal Hodge decomposition [70, Theorem 6.8] and the intersection normalizations give

$$\|\alpha_i - u_i\|_2^2 = \|\alpha_i\|_2^2 - \|u_i\|_2^2 = 2\big(\|\alpha_i^-\|_2^2 - \|u_i^-\|_2^2\big),$$
$$\|\beta_j - v_j\|_2^2 = \|\beta_j\|_2^2 - \|v_j\|_2^2 = 2\big(\|\beta_j^+\|_2^2 - \|v_j^+\|_2^2\big).$$

The four norm bounds follow. The pairings of the basis classes give

$$B_g(u_i, u_l) = \delta_{il} + 2B_g(u_i^-, u_l^-),$$
$$B_g(v_j, v_k) = \delta_{jk} + 2B_g(v_j^+, v_k^+),$$
$$B_g(u_i, v_j) = 2B_g(u_i^-, v_j^-).$$

Write $G_{++}, G_{+-}, G_{-+}, G_{--}$ for the corresponding blocks of the harmonic Gram matrix G. Since $\|v_j^-\|_2^2 = 1 + \|v_j^+\|_2^2 \le 1 + e^2$, Cauchy–Schwarz gives

$$\|G_{++} - I_p\| \le 2pe^2, \qquad \|G_{--} - I_q\| \le 2qe^2, \qquad \|G_{+-}\| \le 2\sqrt{pq}\, e\sqrt{1+e^2}.$$

Hence $G = I + O_{p,q}(e)$ for $e \le 1$.

In this basis the intersection matrix is $\mathrm{diag}(I_p, -I_q)$. By (2.5), the positive eigenspace projection of $J = \mathrm{diag}(I_p, -I_q)G$ is $(I + J)/2$. Its restriction to $\mathrm{span}(a_i)$ has upper block $(I_p + G_{++})/2$ and lower block $-G_{-+}/2$. Since $G_{++} \ge I_p$, the period is the graph of

$$Z = -G_{-+}(I_p + G_{++})^{-1}, \qquad \|Z\| \le \sqrt{pq}\, e\sqrt{1+e^2}.$$

This proves the period estimate. The harmonic projected forms have positive definite Gram matrices

$$\big(B_g(u_i^+, u_l^+)\big)_{i,l} = I_p + \big(B_g(u_i^-, u_l^-)\big)_{i,l} \ge I_p,$$

and

$$\big(B_g(v_j^-, v_k^-)\big)_{j,k} = I_q + \big(B_g(v_j^+, v_k^+)\big)_{j,k} \ge I_q.$$

Thus they are independent and have the dimensions of $\mathcal{H}_g^+$ and $\mathcal{H}_g^-$, respectively. For a compact set of cohomology bases, the changes to fixed coordinates and their inverses are bounded. The changes between the adapted and fixed graph charts are smooth jointly in the basis and Z for $\|Z\| < 1$. Their derivatives are bounded when the basis lies in that compact set and $\|Z\| \le 1/2$. The mean value inequality along tZ, $0 \le t \le 1$, therefore gives the last inequality. □

4.2. **Shrinking neighborhoods**. Let K be a finite subcomplex of a smooth triangulation of a manifold M without boundary. A smooth regular neighborhood of K will mean a compact codimension-zero smooth submanifold $C \subset M$ such that $K \subset \mathrm{int}\, C$ and C is a regular neighborhood of K in the sense of [27]. We write $A \Subset B$ when $\overline{A}$ is compact and contained in B. A family of sets E_δ shrinks to K if, for every open neighborhood V of K, one has $E_\delta \subset V$ for all sufficiently small $\delta > 0$.

Lemma 4.4. *Suppose* $\dim M = n \geq 3$ *and* $\dim K \leq n-2$*, and fix a metric* g *near* K*. There are constants* $\delta_0 \in (0,1)$ *and* $C > 0$ *such that, for* $0 < \delta < \delta_0$*, there are smooth functions* $0 \leq \chi_\delta \leq 1$*, equal to one near* K*, whose supports shrink to* K*, with*

$$\mathrm{vol}_g(\operatorname{supp}\chi_\delta) \longrightarrow 0, \qquad \|d\chi_\delta\|^2_{L^2(g)} \leq \frac{C}{|\log\delta|}.$$

The same functions satisfy these bounds uniformly for every compact smooth family of metrics on the same neighborhood of K*, with* C *depending on the family.*

Proof. The case $K = \emptyset$ follows by taking $\chi_\delta = 0$. Assume $K \neq \emptyset$. Write $K = \bigcup_{i=1}^N K_i$, where each K_i is compact and contained in an open set Ω_i whose closure is compact in a coordinate neighborhood in the domain of g. Each coordinate image of K_i has finite $(n-2)$-dimensional Hausdorff measure, since K has finitely many smooth simplices of dimension at most $n-2$. By [26, Theorem 2.27], with $p = 2 < n$, each image has zero 2-capacity. This means zero relative capacity in every open neighborhood [26, §2.7].

Fix an open neighborhood V of K contained in the domain of g, and let $\eta > 0$. The definition of relative capacity [26, pp. 27–28], together with comparison of g and the Euclidean metrics on the compact sets $\overline{\Omega}_i$, gives $u_i \in C_c^\infty(V \cap \Omega_i)$ with

$$0 \leq u_i \leq 1, \qquad u_i = 1 \text{ near } K_i, \qquad \|du_i\|^2_{L^2(g)} < \eta/N^2.$$

Indeed, a fixed smooth composition, zero on $(-\infty, 0]$ and one on $[1/2, \infty)$, gives this range and plateau from the smooth capacity competitors; its bounded derivative preserves arbitrarily small energy. Extend the u_i by zero and put

$$\chi = 1 - \prod_{i=1}^N (1 - u_i).$$

Then $0 \leq \chi \leq 1$, $\chi = 1$ near K, and $\operatorname{supp}\chi \Subset V$. Moreover,

$$|d\chi|^2_g \leq N \sum_{i=1}^N |du_i|^2_g, \qquad \|d\chi\|^2_{L^2(g)} < \eta.$$

Extend g from a smaller neighborhood of K to M, and put $V_\delta = \{x : \mathrm{dist}_g(x, K) < \delta\}$. For sufficiently small δ_0, these neighborhoods lie in one fixed compact subset of the original metric domain when $0 < \delta \leq \delta_0$. Take $V = V_\delta$ and $\eta = |\log\delta|^{-1}$. Since K has zero n-dimensional volume, $\mathrm{vol}_g(V_\delta) \to 0$. On this fixed compact set, every compact smooth family $\mathcal{G}$ of metrics admits a constant $a \geq 1$ such that $a^{-1}g \leq \widetilde{g} \leq ag$ for all $\widetilde{g} \in \mathcal{G}$. For every smooth function u, we have

$$d\,\mathrm{vol}_{\widetilde{g}} \leq a^{n/2}\, d\,\mathrm{vol}_g, \qquad |du|^2_{\widetilde{g}}\, d\,\mathrm{vol}_{\widetilde{g}} \leq a^{n/2+1} |du|^2_g\, d\,\mathrm{vol}_g\,.$$

These inequalities give the uniform estimates. □

Lemma 4.5. *Let* C *and* U *be smooth regular neighborhoods of* K *as in Lemma 4.4, with* $C \subset \operatorname{int} U$*. Fix a reference metric on* M*, and denote its volume and* L^2 *norm by* vol *and* $\|\cdot\|_2$*. There are ambient diffeomorphisms* F_δ *and smooth functions* $\chi_\delta, \kappa_\delta : M \to [0,1]$*, for* $0 < \delta \leq 1$*, with* $F_1 = \mathrm{id}$*, such that each* F_δ *is isotopic to the identity and supported in* $\operatorname{int} U$*. All three families are jointly smooth in* (δ, x) *for* $\delta > 0$*. The cutoffs are compactly supported in* $\operatorname{int} U$ *and, with* $C_\delta = F_\delta(C)$*,*

$$\chi_\delta = 1 \text{ near } C_\delta, \qquad \kappa_\delta = 1 \text{ near } \operatorname{supp}\chi_\delta.$$

Both supports shrink to K*, and*

$$\lim_{\varepsilon \downarrow 0} \sup_{0<\delta\leq\varepsilon} \left(\|d\chi_\delta\|_2^2 + \mathrm{vol}(\operatorname{supp}\chi_\delta) + \mathrm{vol}(\operatorname{supp}\kappa_\delta) \right) = 0.$$

The convergence is uniform for compact smooth families of metrics on a neighborhood of U*.*

Proof. If $K = \emptyset$, then $C = U = \emptyset$; take $F_\delta = \mathrm{id}$ and $\chi_\delta = \kappa_\delta = 0$. Assume $K \neq \emptyset$. Put $D_0 = C$. Choose smooth functions $u_0, v_0 : M \to [0,1]$ compactly supported in $\operatorname{int} U$, with $u_0 = 1$ near D_0 and $v_0 = 1$ near $\operatorname{supp} u_0$ [70, corollary to Theorem 1.11]. Inductively choose regular neighborhoods D_j and smooth cutoffs $u_j, v_j : M \to [0,1]$, $j \geq 1$, such that

$$D_j \Subset W_j := \{u_j = 1\}^\circ, \qquad \operatorname{supp} u_j \Subset \{v_j = 1\}^\circ, \qquad \operatorname{supp} v_j \Subset \operatorname{int} D_{j-1},$$

$$\|du_j\|_2^2 + \operatorname{vol}(\operatorname{supp} u_j) + \operatorname{vol}(\operatorname{supp} v_j) < 2^{-j}, \qquad \operatorname{supp} v_j \subset \{x : \operatorname{dist}(x,K) < 2^{-j}\}.$$

Since K has zero volume, choose an open neighborhood O_j of K compactly contained in $\operatorname{int} D_{j-1} \cap \{x : \operatorname{dist}(x,K) < 2^{-j}\}$, with volume less than 2^{-j-2}. Choose v_j supported in O_j and equal to one near K. Lemma 4.4 gives u_j supported in $\{v_j = 1\}^\circ$ with $\|du_j\|_2^2 < 2^{-j-2}$. The existence of arbitrarily small smooth regular neighborhoods [27, Theorem 1(a) and its proof] gives D_j inside $\{u_j = 1\}^\circ$. The two support volumes are at most $\operatorname{vol}(O_j)$, so the displayed inequalities follow.

Set $W_0 = \{u_0 = 1\}^\circ$. For $j \geq 0$,

$$\operatorname{supp} v_{j+1} \Subset \operatorname{int} D_j \subset W_j \subset \operatorname{supp} u_j \Subset \{v_j = 1\}^\circ.$$

Both D_j and D_{j+1} are regular neighborhoods of K in W_j. By [27, Theorem 1(b)], there is a smooth isotopy $h_t : W_j \to W_j$ with $h_0 = \mathrm{id}$ and $h_1(D_j) = D_{j+1}$. The map $(x,t) \mapsto (h_t(x), t)$ has open image in $W_j \times [0,1]$, since each h_t is an embedding between manifolds of the same dimension. The isotopy extension theorem [28, Chapter 8, Theorem 1.4], applied in W_j to the compact set D_j, gives a smooth ambient isotopy $H_j(t)$ with compact support in W_j and

$$H_j(0) = \mathrm{id}, \qquad H_j(1)(D_j) = D_{j+1}.$$

Extend $H_j(t)$ by the identity to M. Define $G_0 = \mathrm{id}$ and $G_{j+1} = H_j(1) \circ G_j$; then $G_j(C) = D_j$.

Choose smooth nondecreasing functions $a, b, c : [0,1] \to [0,1]$ whose derivatives have supports in $(0,1/3)$, $(1/3,2/3)$, and $(2/3,1)$, respectively, and which take the values zero and one at the two endpoints. For $2^{-j-1} \leq \delta \leq 2^{-j}$, put $s = 2^{j+1}(2^{-j} - \delta)$ and define

$$\begin{aligned} F_\delta &= H_j(a(s)) \circ G_j, \\ \chi_\delta &= (1 - b(s))u_j + b(s)u_{j+1}, \\ \kappa_\delta &= (1 - c(s))v_j + c(s)v_{j+1}. \end{aligned}$$

If $0 \leq s \leq 1/3$, then $C_\delta = H_j(a(s))(D_j) \Subset W_j$ and $(\chi_\delta, \kappa_\delta) = (u_j, v_j)$. If $1/3 \leq s \leq 2/3$, then $C_\delta = D_{j+1}$ and $\kappa_\delta = v_j$. Both u_j and u_{j+1} equal one near D_{j+1}, and $\operatorname{supp} \chi_\delta \subset \operatorname{supp} u_j$. If $2/3 \leq s \leq 1$, then $C_\delta = D_{j+1}$ and $\chi_\delta = u_{j+1}$; both v_j and v_{j+1} equal one near $\operatorname{supp} u_{j+1}$. Thus the two stated equalities hold. The endpoint values agree on neighborhoods of the dyadic junctions, so the families are smooth for positive δ and $F_1 = \mathrm{id}$. Each F_δ is a finite composition of time slices of these isotopies.

On the jth interval, $\operatorname{supp} \chi_\delta \subset \operatorname{supp} u_j$ and $\operatorname{supp} \kappa_\delta \subset \operatorname{supp} v_j$, and convexity gives

$$\|d\chi_\delta\|_2^2 \leq (1 - b(s))\|du_j\|_2^2 + b(s)\|du_{j+1}\|_2^2.$$

For $j \geq 1$, the sum in the statement is therefore at most 2^{1-j}. The same bound holds for all $0 < \delta \leq 2^{-j}$. The supports also lie in the 2^{-j}-neighborhood of K on this whole interval. The metric comparison in Lemma 4.4 gives uniform convergence for compact metric families. □

4.3. **Closed forms through one replacement.** Let X be a closed oriented four-manifold, and let C be a smooth regular neighborhood of a finite subcomplex K of a smooth triangulation of X, with $\dim K \leq 2$. Such a K is called a spine of C. Put $A = X \setminus \operatorname{int} C$. Let C' be a compact oriented four-manifold with boundary, and fix an orientation-compatible diffeomorphism $\varphi : \partial C' \to \partial C$, with the collar identifications stipulated at the beginning of this section. Set

$$X' = A \cup_\varphi C'.$$

Choose a smooth regular neighborhood U of K with $C \subset \operatorname{int} U$, and let

$$V \subset \ker\bigl(H^2(X;\mathbb{R}) \longrightarrow H^2(U;\mathbb{R})\bigr)$$

be a linear subspace. Write $m = \dim V$. There are closed representatives $\eta_1, \dots, \eta_m$ of a basis of V which vanish near U. Indeed, an open collar enlargement W of U deformation retracts onto U. For closed representatives θ_l of a basis, choose one-forms ξ_l on W with $d\xi_l = \theta_l|_W$ and a smooth function ρ compactly supported in W, equal to one near U [70, corollary to Theorem 1.11]. Then $\eta_l = \theta_l - d(\rho\xi_l)$ has the required properties; $\rho\xi_l$ is extended by zero outside W. Fix these representatives, and denote the zero extension of $\eta_l|_A$ over C' by η'_l. Suppose that

$$\iota : (V, Q_X|_V) \longrightarrow (H^2(X';\mathbb{R}), Q_{X'}), \qquad \iota[\eta_l] = [\eta'_l],$$

is the specified isometric embedding.

On a closed oriented four-manifold M, for a metric g, a symmetric two-tensor h, and two-forms α, β, set

$$\mathcal{B}_g(h;\alpha,\beta) = \int_M \alpha \wedge (D_g * [h])\beta. \tag{4.2}$$

Equation (2.9) and the Cauchy–Schwarz inequality give, for a numerical constant $c_4 > 0$,

$$|\mathcal{B}_g(h;\alpha,\beta)| \le c_4 \|h\|_{L^\infty(g)} \|\alpha\|_{L^2(g)} \|\beta\|_{L^2(g)}. \tag{4.3}$$

The same pointwise formula gives $D_g * [fh] = f D_g * [h]$ for every smooth function f.

Lemma 4.6. *Let B be a finite-dimensional smooth manifold, let $(g_b)_{b\in B}$ be a smooth family of metrics on X, and let $\alpha_{i,b}, \beta_{j,b}$ be finitely many smooth families of closed two-forms with classes in V. Let $h_{a,b}$ be finitely many smooth families of symmetric two-tensors. There are metrics $g'_{b,\delta}$, closed two-forms $\alpha'_{i,b,\delta}, \beta'_{j,b,\delta}$, and symmetric two-tensors $h'_{a,b,\delta}$ on the fixed X', jointly smooth for $b \in B$ and $0 < \delta \le 1$, with the following properties. The new forms represent the ι-images of the original cohomology classes. Unprimed norms and duality projections refer to g_b on X, and primed ones to $g'_{b,\delta}$ on X'. For every compact $L \subset B$, there is a function $e_L : (0,1] \to [0,\infty)$ such that*

$$\lim_{\varepsilon \downarrow 0} \sup_{0<\delta\le\varepsilon} e_L(\delta) = 0$$

and, for $b \in L$,

$$\|(\alpha'_{i,b,\delta})^-\|_2 \le \|\alpha^-_{i,b}\|_2 + e_L(\delta), \tag{4.4}$$

$$\|(\beta'_{j,b,\delta})^+\|_2 \le \|\beta^+_{j,b}\|_2 + e_L(\delta), \tag{4.5}$$

$$\|h'_{a,b,\delta}\|_{L^\infty(g'_{b,\delta})} \le \|h_{a,b}\|_{L^\infty(g_b)}. \tag{4.6}$$

For every specified tensor $h_{a,b}$ and every pair ω_b, ζ_b among the specified forms, their corresponding forms on X' satisfy

$$\left|\mathcal{B}_{g'_{b,\delta}}(h'_{a,b,\delta};\omega'_{b,\delta},\zeta'_{b,\delta}) - \mathcal{B}_{g_b}(h_{a,b};\omega_b,\zeta_b)\right| \le e_L(\delta), \qquad b \in L.$$

Proof. Fix a reference metric g_0 on X, with Green operator G_0 and codifferential d_0^*. For a closed two-form ω with $[\omega] \in V$, let $c_l(\omega)$ be the coordinates of its class in the basis $[\eta_l]$, and put

$$\tau_\omega = d_0^* G_0 \left(\omega - \sum_l c_l(\omega)\eta_l \right).$$

The Hodge identity on exact forms gives $d\tau_\omega = \omega - \sum_l c_l(\omega)\eta_l$ [70, Definition 6.9, Proposition 6.10, and Theorem 6.11]. The coefficients c_l are continuous linear coordinates of the fixed harmonic projection. The fixed operator $d_0^* G_0$ is continuous in the C^∞ topology by elliptic estimates [70, Theorem 6.8 and §§6.29,6.32]. Thus $\omega \mapsto \tau_\omega$ is continuous and linear. It preserves smooth dependence on b, and the primitives are bounded in every smooth norm on compact subsets of B.

Apply Lemma 4.5 to C, U, with reference metric g_0, and write its families as $F_\delta, \chi_\delta, \kappa_\delta$. Put $C_\delta = F_\delta(C)$ and $A_\delta = X \setminus \operatorname{int} C_\delta = F_\delta(A)$. Since F_δ is supported in $\operatorname{int} U$ and the η_l vanish near U, we have $F_\delta^* \eta_l = \eta_l$.

The fixed collars give an orientation-preserving diffeomorphism $j : N \to j(N) \subset X$ from an open neighborhood N of A in X', with $j|_A = \mathrm{id}$ and $j(N \cap C') \subset C$. Choose a smooth function $\lambda : X' \to [0,1]$ equal to one near A, with $\operatorname{supp}\lambda \Subset N$, and fix a metric g_* on X'. Define

$$g'_{b,\delta} = \lambda (F_\delta \circ j)^* g_b + (1-\lambda) g_*;$$

the first summand is extended by zero outside N. This is a positive metric, jointly smooth in (b,δ,x), and $g'_{b,\delta}|_A = F_\delta^* g_b$.

For any of the specified forms ω_b, define on X

$$\widehat{\omega}_{b,\delta} = \omega_b - d(\chi_\delta \tau_{\omega_b}) = \sum_l c_l(\omega_b)\eta_l + d\big((1-\chi_\delta)\tau_{\omega_b}\big). \tag{4.7}$$

Both $\widehat{\omega}_{b,\delta}$ and $(1-\chi_\delta)\tau_{\omega_b}$ vanish near C_δ. Pull them back by $F_\delta \circ j$ on N and extend by zero over C'; denote the resulting forms by $\omega'_{b,\delta}$ and $\sigma_{\omega,b,\delta}$. The pullbacks vanish on the fixed overlap $N \cap C'$, so these extensions are jointly smooth. Since $(F_\delta \circ j)^*\eta_l = \eta'_l|_N$ on N, equation (4.7) gives on X'

$$\omega'_{b,\delta} = \sum_l c_l(\omega_b)\eta'_l + d\sigma_{\omega,b,\delta}.$$

Hence $d\omega'_{b,\delta} = 0$ and $[\omega'_{b,\delta}] = \iota[\omega_b]$. Pull back $(1-\kappa_\delta)h_{a,b}$ by the same map on N and extend by zero over C' to define $h'_{a,b,\delta}$. These tensors are jointly smooth for the same reason.

Since $\chi_\delta \eta_l = 0$, we have on X

$$\widehat{\omega}_{b,\delta} = (1-\chi_\delta)\omega_b - d\chi_\delta \wedge \tau_{\omega_b}.$$

The new form vanishes on C', and the metric on A is $F_\delta^* g_b$. Naturality of the Hodge star and change of variables therefore give, for either sign,

$$\|(\omega'_{b,\delta})^{\pm}\|_{L^2(g'_{b,\delta})} \le \|\omega_b^{\pm}\|_{L^2(g_b)} + \|\tau_{\omega_b}\|_{L^\infty(g_b)} \|d\chi_\delta\|_{L^2(g_b)}.$$

Here the duality projections have norm one, and $|d\chi_\delta \wedge \tau_{\omega_b}| \le |d\chi_\delta|\,|\tau_{\omega_b}|$. The bound (4.6) follows from $|(1-\kappa_\delta)h_{a,b}|_{g_b} \le |h_{a,b}|_{g_b}$.

The tensor $(1-\kappa_\delta)h_{a,b}$ vanishes near $\operatorname{supp}\chi_\delta$, and $\widehat{\omega}_{b,\delta} = \omega_b$ off $\operatorname{supp}\chi_\delta$ for every specified form. Since $\kappa_\delta = 1$ on C_δ, naturality of the metric differential of the Hodge star, change of variables, and $D_{g_b} * [(1-\kappa_\delta)h_{a,b}] = (1-\kappa_\delta) D_{g_b} * [h_{a,b}]$ give

$$\begin{aligned} &\mathcal{B}_{g_b}(h_{a,b};\omega_b,\zeta_b) - \mathcal{B}_{g'_{b,\delta}}(h'_{a,b,\delta};\omega'_{b,\delta},\zeta'_{b,\delta}) \\ &\qquad = \int_X \kappa_\delta\, \omega_b \wedge (D_{g_b} * [h_{a,b}])\zeta_b. \end{aligned}$$

Fix a compact $L \subset B$. The metrics g_b, $b \in L$, are uniformly equivalent to g_0, and the given forms, tensors, and primitives have bounded pointwise norms on $L \times X$. There is therefore a constant $C_L \ge 0$, depending on g_0, the given families over L, and the fixed representatives, such that the increases in the duality norms and the absolute differences of the pairings are at most

$$e_L(\delta) = C_L\left(\|d\chi_\delta\|_{L^2(g_0)} + \operatorname{vol}_{g_0}(\operatorname{supp}\kappa_\delta)\right).$$

Lemma 4.5 gives $\sup_{0<\delta\le\varepsilon} e_L(\delta) \to 0$ as $\varepsilon \downarrow 0$. □

4.4. **The global transfer theorem.** Let $X_0, \dots, X_s$ be closed connected oriented four-manifolds. For $1 \le j \le s$, let X_j be obtained from X_{j-1} by replacing a cap C_j as in the preceding subsection. Thus C_j is a smooth regular neighborhood of a finite subcomplex of dimension at most two in a smooth triangulation of X_{j-1}. Let $V_j \subset H^2(X_j;\mathbb{R})$ be nondegenerate subspaces of common signature (p,q), with specified isometries $\iota_j : V_{j-1} \to V_j$. Suppose that each ι_j is realized by zero extension of closed representatives of a basis of V_{j-1} which vanish on a neighborhood of C_j. Assume $V_s = H^2(X_s;\mathbb{R})$, and identify all the V_j with one quadratic space (V,Q) through these isometries.

Theorem 4.7. *Suppose there is a smooth family $g_0(P)$ on X_0, for all $P \in \mathcal{D}(V,Q)$, such that V_0 is invariant under its cohomological Hodge star and the positive eigenspace on V_0 is P. Then there is a smooth map*

$$\mathfrak{s} : \mathcal{D}(V,Q) \longrightarrow \mathrm{Met}^\infty(X_s), \qquad \Pi_{X_s}(\mathfrak{s}(P)) = P.$$

Its metrics are regular. The volume form may be prescribed throughout the section.

Proof. If $pq = 0$, the target is a point, so every metric is regular and has the required period. Equation (2.12) gives the prescribed volume form. Assume $p, q > 0$, and put $d = pq$. Fix a Q-orthogonal splitting $V = P_* \oplus N_*$, positive on P_* and negative on N_*. Give these spaces the scalar products Q and $-Q$, and fix orthonormal bases. The adjoint of $A : P_* \to N_*$ for these scalar products is denoted by A^*. The matrix units $E_1, \dots, E_d$ identify $\mathrm{Hom}(P_*, N_*)$ with Euclidean $\mathbb{R}^d$. Write $A = \sum_a A_a E_a$ and $P_A = \mathrm{graph}(A)$. The graph chart

$$z : \mathcal{D}(V,Q) \longrightarrow \Omega = \{A \in \mathrm{Hom}(P_*, N_*) : \|A\|_{\mathrm{op}} < 1\}, \qquad z(P_A) = A,$$

identifies the positive Grassmannian with an open convex subset of $\mathbb{R}^d$. Put

$$R_A = (I - A^*A)^{-1/2}, \qquad S_A = (I - AA^*)^{-1/2}.$$

The columns of

$$(I, A)R_A, \qquad (A^*, I)S_A \tag{4.8}$$

give smooth Q-orthonormal frames $a_i(A)$, $b_j(A)$ of P_A and $P_A^\perp$. Their combined square basis matrix T_A and T_A^{-1} are bounded on compact subsets of Ω.

Write $g_0(A) = g_0(P_A)$, and let $\alpha_i^0(A), \beta_j^0(A)$ be the harmonic representatives of these frames. They are self-dual and anti-self-dual, respectively, by the star invariance of V_0. Set

$$h_a^0(A) = \partial_{A_a} g_0(A), \qquad L_0(A) = \left(\mathcal{B}_{g_0(A)}(h_a^0(A); \alpha_i^0(A), \beta_j^0(A))\right)_{(i,j),a}.$$

Fix an ordering of the pairs (i, j). Differentiating the two-form inner product shows that $\mathcal{B}_g(h; \alpha, \beta) = \mathcal{B}_g(h; \beta, \alpha)$. Equation (2.10), restricted to V_0, gives, for $v = (v_a) \in \mathbb{R}^d$ and $Z = \sum_a v_a E_a$,

$$L_0(A)v = \left(-2(S_A Z R_A)_{ji}\right)_{(i,j)}.$$

Indeed, the derivative of the first frame in the direction Z is $(0, Z)R_A$ modulo P_A; its pairing with the second frame has entries $-(S_A Z R_A)_{ji}$. Thus $L_0(A)$ is invertible. For a square matrix L, let $\sigma_{\min}(L)$ be its least singular value. For every relatively compact open $U \subset \Omega$, there are $H, \mu > 0$ such that

$$\|h_a^0(A)\|_{L^\infty(g_0(A))} \le H, \qquad \sigma_{\min}(L_0(A)) \ge \mu, \qquad A \in \overline{U}. \tag{4.9}$$

For $A \in \overline{U}$, consider a metric g on X_s, closed representatives α_i, β_j of $a_i(A)$, $b_j(A)$, and symmetric tensors $h_1, \dots, h_d$. Put $\widehat{L} = (\mathcal{B}_g(h_a; \alpha_i, \beta_j))_{(i,j),a}$. Suppose that

$$\|h_a\|_{L^\infty(g)} \le H, \qquad \|\alpha_i^-\|_2,\ \|\beta_j^+\|_2 \le e_0 \le 1, \qquad \|\widehat{L} - L_0(A)\| \le \mu/4,$$

where all form norms refer to g. Let u_i, v_j be their harmonic representatives. Lemma 4.3 bounds the harmonic Gram matrix G_{mov} in this moving cohomology basis. In the fixed basis it is

$$G = T_A^{-\mathsf{T}} G_{\mathrm{mov}} T_A^{-1}.$$

Hence G is bounded. Equation (4.1) bounds G^{-1} as well, so (2.5) places $P' = \Pi_{X_s}(g)$ in a compact subset of $\mathcal{D}(V,Q)$ depending only on U, p, q. This compact set applies to every $e_0 \le 1$.

The same lemma gives

$$\|\alpha_i - u_i^+\|_2,\ \|\beta_j - v_j^-\|_2 \le (1 + \sqrt{2})e_0.$$

The norms of α_i, β_j are at most $\sqrt{1 + 2e_0^2}$, and those of u_i^+, v_j^- are at most $\sqrt{1 + e_0^2}$. Put

$$\widetilde{L} = \left(\mathcal{B}_g(h_a; u_i^+, v_j^-)\right)_{(i,j),a}.$$

By (4.3), there is a constant $C'_U > 0$, depending only on U, p, q, such that $\|\widetilde{L} - \widehat{L}\| \le C'_U H e_0$. Choose $e_0 > 0$ so that $C'_U H e_0 \le \mu/4$. Then $\sigma_{\min}(\widetilde{L}) \ge \mu/2$.

Let $\pi_{P'}$ and $\pi_{P'^\perp}$ be the Q-orthogonal projections, and put $x_i = \pi_{P'} a_i(A)$ and $y_j = \pi_{P'^\perp} b_j(A)$. These are the classes of u_i^+, v_j^- and form bases of $P', P'^\perp$, respectively, by the proof of Lemma 4.3. With $T_{P'}\mathcal{D}(V,Q) = \mathrm{Hom}(P', P'^\perp)$, define the linear isomorphism

$$\mathcal{T}_{A,P'} : \mathbb{R}^d \longrightarrow \mathbb{R}^d, \qquad \mathcal{T}_{A,P'}(v) = \big(2Q(((Dz_{P'})^{-1}v)(x_i), y_j)\big)_{(i,j)}.$$

It depends smoothly on (A, P') and has a bounded norm for $A \in \overline{U}$ and P' in the compact set just obtained. For $h = (h_1, \dots, h_d)$, put

$$F_{g,h}(t) = z\left(\Pi_{X_s}\left(g + \sum_a t_a h_a\right)\right), \qquad t = (t_1, \dots, t_d) \in \mathbb{R}^d,$$

where the tensor in parentheses is positive definite. Equation (2.10) gives $\widetilde{L} = \mathcal{T}_{A,P'} DF_{g,h}(0)$. Consequently there is a constant $L_U > 0$ such that

$$\|DF_{g,h}(0)^{-1}\| \le \|\widetilde{L}^{-1}\|\,\|\mathcal{T}_{A,P'}\| \le L_U. \tag{4.10}$$

Apply Lemma 4.1 in the fixed basis of the full cohomology $H^2(X_s;\mathbb{R})$, with all the preceding data as its index set. It gives $r_U > 0$ for which the perturbed tensors are metrics and a bound $M_U \ge 1$ for $\|D^2 F_{g,h}(t)\|$ on $|t| \le r_U$. Choose

$$0 < \rho_U \le \min\{r_U/2, (4L_U M_U)^{-1}\}.$$

By decreasing e_0 if necessary, Lemma 4.3 also gives a constant $C_U \ge 1$ such that $|F_{g,h}(0) - A| \le C_U e$ whenever $\|\alpha_i^-\|_2, \|\beta_j^+\|_2 \le e \le e_0$. All these constants depend only on the source family over $\overline{U}$ and the fixed cohomology coordinates.

Choose a locally finite relatively compact open cover (U_i) of Ω and a subordinate smooth partition of unity (ψ_i) with compact supports $S_i = \operatorname{supp}\psi_i \Subset U_i$ [70, Lemma 1.9 and Theorem 1.11]. Denote the preceding constants for $\overline{U_i}$ by the corresponding subscript i. Choose smooth positive functions η, ρ such that

$$\eta(A) < \mu_i/4, \qquad \rho(A) < \rho_i \quad \text{whenever } A \in S_i. \tag{4.11}$$

Then choose a smooth positive function e such that

$$e(A) < e_{0,i}, \qquad C_i e(A) < \rho(A)/(4L_i) \quad \text{whenever } A \in S_i. \tag{4.12}$$

These choices follow from Lemma 3.5. Indeed, the S_i are closed and locally finite and cover Ω. For positive constants c_i, the function $A \mapsto \min_{i:\,A\in S_i} c_i$ is therefore positive and lower semicontinuous. It has a positive constant lower bound on a neighborhood of every point. Apply this to the bounds for η, ρ. Once ρ is fixed, its continuity and positivity give such local bounds for e as well.

At each replacement fix the basis representatives realizing ι_j. Choose a smooth regular neighborhood D_j of the spine and an open collar enlargement W_j of D_j, with $C_j \subset \operatorname{int} D_j$, inside their common zero neighborhood. Small outward collar enlargements of C_j, supported away from the spine, give such neighborhoods. In the preliminary construction of §4.3 the primitives on W_j may be zero, so these representatives and their specified zero-extension map are unchanged.

Suppose $s > 0$. At each of the s steps require the error for each form and each scalar pairing in Lemma 4.6 to be at most

$$\min\{e(A)/s,\ \eta(A)/(ds)\}.$$

After the first $j-1$ smooth positive scales have been chosen, the resulting data are smooth families on the fixed X_{j-1}. On each relatively compact parameter neighborhood the displayed positive tolerance has a positive lower bound, and Lemma 4.6 gives a scale bound in $(0,1]$ valid for all smaller positive scales. Lemma 3.5 therefore gives the next scale as a smooth positive function on Ω. Composition

preserves smoothness, and the exact cohomology assertion of Lemma 4.6 gives the specified classes at the next step.

Denote the final families by $g(A)$, $\alpha_i(A), \beta_j(A)$, and $h_a(A)$. The inequalities (4.4)–(4.5) give $\|\alpha_i^-(A)\|_2, \|\beta_j^+(A)\|_2 \le e(A)$, and (4.6) preserves the source tensor bounds. A $d \times d$ matrix with entries of absolute value at most $b \ge 0$ has operator norm at most db. Thus the final matrix satisfies $\|\widehat{L}(A) - L_0(A)\| \le \eta(A)$. If $s = 0$, take the original families; their corresponding errors are zero.

Put $h(A) = (h_1(A), \dots, h_d(A))$ and $f_A(t) = F_{g(A),h(A)}(t) - A$. For any i with $A \in S_i$, (4.10) and (4.12) give

$$\|Df_A(0)^{-1}\| \le L_i, \qquad |f_A(0)| \le C_i e(A) < \rho(A)/(4L_i).$$

Put

$$\Phi(A, t) = Df_A(0)^{-1} f_A(t).$$

For $A \in S_i$ and $|t| \le \rho(A)$,

$$\|D_t\Phi(A, t) - I\| \le L_i M_i |t| < \tfrac{1}{4}, \qquad |\Phi(A, 0)| < \rho(A)/4.$$

The map Φ is smooth on the open set where $g(A) + \sum_a t_a h_a(A)$ is positive definite. Since $\rho(A) < \rho_i \le r_i/2$ for $A \in S_i$, this set contains $\{(A, t) : |t| \le \rho(A)\}$. Lemma 3.6, with $c = 0$ and $r = \rho$, gives a smooth map $t : \Omega \to \mathbb{R}^d$ satisfying $f_A(t(A)) = 0$ and $|t(A)| \le \rho(A)/3$. The formula

$$\mathfrak{s}(P_A) = g(A) + \sum_a t_a(A) h_a(A)$$

defines a smooth section on $\mathcal{D}(V, Q)$. Differentiating $\Pi_{X_s} \circ \mathfrak{s} = \mathrm{id}$ proves regularity. For every positive smooth volume form ν on X_s, Lemma 2.5 gives the section $N_\nu \circ \mathfrak{s}$ in $\mathrm{Met}_\nu^\infty(X_s)$, whose metrics are regular for the restricted period map. □

4.5. **Transfer in higher dimensions.** For a closed oriented $2m$-manifold M, the maps Γ_m and $J_M = Q_M^{-1}\Gamma_m$ have equivalent section problems by (2.1). In the following theorem, the retained pairing may be symmetric or alternating.

Theorem 4.8. *Let $S = A \cup C$ and $T = A \cup C'$ be closed oriented smooth $2m$-manifolds, with $m \ge 3$, and fixed boundary identifications. The pieces are compact codimension-zero submanifolds, joined along their common boundary. Suppose that C is a smooth regular neighborhood of a finite subcomplex of dimension at most m in a smooth triangulation of S, or a tubular neighborhood of a closed smooth core of codimension at least two. Let $V \subset H^m(S; \mathbb{R})$ be nondegenerate for Q_S. Suppose that its classes restrict to zero on C, and that taking representatives vanishing near C and extending them by zero gives a well-defined isometric isomorphism*

$$\iota : (V, Q_S|_V) \longrightarrow (H^m(T; \mathbb{R}), Q_T).$$

If Γ_m on S has a global smooth section, then Γ_m on T has a global smooth section through regular metrics.

Proof. In the first case, let K be the given subcomplex. In the second, take for K the core with a finite smooth triangulation; C is then a smooth regular neighborhood of K [27, Lemma 4]. In either case $\dim K \le 2m - 2$. Choose a larger smooth regular neighborhood U of K, with $C \subset \operatorname{int} U$. Fix a metric g_0 on S; the norms and volumes in the following limit refer to this metric. Lemmas 4.4 and 4.5 give jointly smooth families $F_\delta, \chi_\delta, \kappa_\delta$ for $0 < \delta \le 1$, where $C_\delta = F_\delta(C)$, the cutoffs equal one on their stated neighborhoods, and

$$\|d\chi_\delta\|_2^2 + \mathrm{vol}(\operatorname{supp} \chi_\delta) + \mathrm{vol}(\operatorname{supp} \kappa_\delta) \longrightarrow 0. \tag{4.13}$$

The convergence is uniform on compact smooth metric families.

Since C and U are regular neighborhoods of K, restriction $H^m(U; \mathbb{R}) \to H^m(C; \mathbb{R})$ is an isomorphism. The construction preceding Lemma 4.6, applied in degree m, therefore gives closed forms $\eta_1, \dots, \eta_l$ representing a basis of V and vanishing near U. Let G_0 and d_0^* be the Green operator and

codifferential of g_0. For a closed m-form α with $[\alpha] \in V$, write $[\alpha] = \sum_i c_i(\alpha)[\eta_i]$. The construction in the proof of Lemma 4.6, now in degree m, gives the continuous linear operator [70, Definition 6.9, Proposition 6.10, and Theorem 6.11]

$$\mathcal{R}\alpha = d_0^* G_0 \left(\alpha - \sum_i c_i(\alpha)\eta_i \right), \qquad d\mathcal{R}\alpha = \alpha \quad \text{near } U.$$

Its continuity in smooth norms is the same elliptic estimate as in that construction. Put

$$\alpha_\delta = \alpha - d(\chi_\delta \mathcal{R}\alpha). \tag{4.14}$$

This form vanishes near C_δ and represents $[\alpha]$. On each compact smooth family there is a constant $c > 0$, independent of δ, such that

$$\|\alpha_\delta - \alpha\|^2_{L^2(g_0)} \le c \left(\mathrm{vol}_{g_0}(\operatorname{supp} \chi_\delta) + \|d\chi_\delta\|^2_{L^2(g_0)} \right) = o(1).$$

Since F_δ is isotopic to the identity, the pullback $F_\delta^* \alpha_\delta$ represents $[\alpha]$ and vanishes near C. Its zero extension to T represents $\iota[\alpha]$.

Fix a compatible star J_0 on $V^{\perp_{Q_S}}$. Both compatible-star domains have global convex coordinates. For a symmetric pairing, choose an orthogonal positive-negative splitting; maximal positive subspaces are graphs of linear maps of operator norm less than one. For an alternating pairing take the Siegel coordinates (2.2). Denote the target and source coordinate domains by Ω and Ω_S. For a compatible star J on $H^m(T;\mathbb{R})$, the extension $J \mapsto (\iota^{-1} J \iota) \oplus J_0$ gives a smooth embedding $i : \Omega \to \Omega_S$. If Ω is a point, any metric on T gives the required section. Otherwise fix a global source section $s(q)$ in these coordinates.

Choose a smooth positive function $R(P)$ on Ω with $\overline{B}_{4R(P)}(i(P)) \subset \Omega_S$. For $P \in \Omega$, $q \in B_{4R(P)}(i(P))$, and $0 < \delta \le 1$, define

$$\widehat{g}_\delta(P, q) = (1 - \kappa_\delta) s(q) + \kappa_\delta s(i(P)).$$

On a compact set of (P, q), these metrics and $s(q)$ are uniformly equivalent. For a fixed class, let u and $\widehat{u}$ be its $s(q)$-harmonic and $\widehat{g}_\delta(P, q)$-harmonic representatives, respectively, and put $v = \widehat{u} - u$. Since v is exact, harmonic orthogonality gives

$$B_{\widehat{g}_\delta}(v, v) = -(B_{\widehat{g}_\delta} - B_{s(q)})(u, v).$$

The coefficient difference is uniformly bounded and supported in $\operatorname{supp} \kappa_\delta$. The forms u are uniformly smooth on this compact family. Uniform ellipticity and Cauchy–Schwarz give a constant $c > 0$, independent of δ, such that

$$\|v\|_{L^2(g_0)} \le c \, \mathrm{vol}_{g_0}(\operatorname{supp} \kappa_\delta)^{1/2} = o(1).$$

The derivatives in q of the two Hodge stars have uniformly bounded coefficients; their difference is supported in the same shrinking set. In the fixed cohomology basis of S, put

$$H_\delta(P, q) = \Gamma_m(\widehat{g}_\delta(P, q)), \qquad H_S(q) = \Gamma_m(s(q)).$$

Equation (2.3) and the preceding estimate give, uniformly on compact sets of (P, q),

$$\|H_\delta(P, q) - H_S(q)\| + \|D_q H_\delta(P, q) - D_q H_S(q)\| \le c \, \mathrm{vol}_{g_0}(\operatorname{supp} \kappa_\delta)^{1/2}.$$

The compatible-star coordinates are smooth functions of the Gram matrices. Hence the full coordinate periods of $\widehat{g}_\delta(P, q)$ converge to q in C^1 with respect to q, uniformly on these compact sets.

Lemma 3.5 supplies a smooth positive $\delta(P) < 1$ such that these coordinate periods differ from q by at most $R(P)/4$, and their q-derivative differs from the identity by at most $1/4$, for $|q - i(P)| \le 3R(P)$. For $|q_0 - i(P)| < R(P)$, apply Lemma 3.6 with base (P, q_0), target q_0, and correction variable $q - q_0$ of radius $R(P)$. The resulting source metrics $\bar{g}(P, q_0)$ have exact full period q_0 and equal $s(i(P))$ near K, independently of q_0. They depend smoothly on (P, q_0). In the dyadic construction of Lemma 4.5,

the functions κ_δ, for δ in a compact subinterval of $(0,1]$, are convex combinations of a fixed finite family of cutoffs. Each cutoff equals one near K, so all κ_δ equal one on a common neighborhood of K. Since $\delta(P)$ is bounded away from zero on compact parameter sets, the neighborhood on which $\bar{g}$ is independent of q_0 can therefore be chosen uniformly there.

Choose a smooth positive $r(P)$ such that $\overline{B}_{2r(P)}(P) \subset \Omega$ and $i(P+z) \in B_{R(P)}(i(P))$ whenever $|z| \le 2r(P)$. Set $g(P,z) = \bar{g}(P, i(P+z))$. Its star leaves V invariant, and its retained coordinate period is $P+z$. As in the proof of Lemma 4.6, choose a fixed orientation-preserving diffeomorphism $j : N \to j(N) \subset S$, where N is an open neighborhood of A in T, with $j|_A = \mathrm{id}$ and $j(N \cap C') \subset C$. Choose a smooth function $\lambda : T \to [0,1]$ equal to one near A, with $\operatorname{supp}\lambda \Subset N$, and a fixed metric g_* on T. For $0 < \epsilon \le 1$, define

$$k_\epsilon(P,z) = \lambda (F_\epsilon \circ j)^* g(P,z) + (1-\lambda) g_*;$$

the first summand is extended by zero outside N. This is a jointly smooth metric on the fixed T, and $k_\epsilon(P,z)|_A = F_\epsilon^* g(P,z)$. Take ϵ small enough that $\operatorname{supp}\kappa_\epsilon$ lies in the neighborhood of K on which $g(P,z)$ is independent of z. Since $C_\epsilon \subset \operatorname{supp}\chi_\epsilon \subset \operatorname{supp}\kappa_\epsilon$, the metric on C' and on a neighborhood of the pulled-back cutoff supports is independent of z.

For a fixed class $a \in V$, put $\alpha = h^m_{g(P,z)}(a)$, $\beta = *_{g(P,z)}\,\alpha$, and $E = \|\alpha\|^2_{L^2(g(P,z))}$. Both forms are closed and both classes lie in V. Apply (4.14) to them. Denote their pullbacks and zero extensions to T again by $\alpha_\epsilon, \beta_\epsilon$, and compute norms on T with $k_\epsilon(P,z)$. Then

$$\|\alpha_\epsilon\|_2^2 = E + o(1), \qquad \|\beta_\epsilon\|_2^2 = E + o(1), \qquad \int_T \alpha_\epsilon \wedge \beta_\epsilon = E. \tag{4.15}$$

The last identity follows from the isometry ι; the error estimates are uniform on compact sets of (P,z). If $a \neq 0$, E is bounded away from zero on each such set. In this case, the target harmonic representative u_ϵ of ιa satisfies

$$\frac{E^2}{\|\beta_\epsilon\|_2^2} \le \|u_\epsilon\|_2^2 \le \|\alpha_\epsilon\|_2^2, \qquad \|u_\epsilon - \alpha_\epsilon\|_2^2 = \|\alpha_\epsilon\|_2^2 - \|u_\epsilon\|_2^2 = o(1).$$

The first inequality follows from Cauchy–Schwarz and the exact pairing in (4.15); the second and the identity follow from harmonic minimization and orthogonality. For $a = 0$ the harmonic representatives vanish. Apply these estimates to a basis and its pairwise sums. The target Gram matrices converge to the retained source Gram matrices.

The derivative in z of the target Hodge star is zero on the new cap and on a surrounding neighborhood containing the cutoff supports. On the remaining common region it is the pullback of the derivative of the source star. Write $\|\cdot\|_{\mathrm{op},k_\epsilon}$ for the pointwise operator norm on m-forms induced by $k_\epsilon(P,z)$. On each compact parameter set there is a constant $c > 0$, independent of sufficiently small ϵ, such that

$$\|D_z *_{k_\epsilon(P,z)} [w]\|_{\mathrm{op},k_\epsilon} \le c|w|$$

for every coordinate vector w. Where this derivative is nonzero, the cutoff forms equal the source forms under the same identification. For fixed $a, b \in V$, write $\alpha_\epsilon(a), \alpha_\epsilon(b)$ for the corresponding cutoff forms and $u_\epsilon(a), u_\epsilon(b)$ for their harmonic representatives on T. Then

$$\int_T \alpha_\epsilon(a) \wedge D_z *_{k_\epsilon} [w] \alpha_\epsilon(b) = \int_S h^m_g(a) \wedge D_z *_g [w] h^m_g(b).$$

By (2.3) and the operator bound, on each compact parameter set there is a constant $c_{a,b} > 0$, independent of sufficiently small ϵ, such that

$$\begin{aligned} &|D_z(\Gamma_m \circ k_\epsilon)[w](\iota a, \iota b) - D_z(\Gamma_m \circ g)[w](a,b)| \\ &\quad \le c_{a,b}|w|\big(\|u_\epsilon(a) - \alpha_\epsilon(a)\|_2 + \|u_\epsilon(b) - \alpha_\epsilon(b)\|_2\big) = o(1)|w|. \end{aligned}$$

Applying this to the basis $[\eta_1], \dots, [\eta_l]$ gives convergence of the first z-derivatives of the Gram matrices. Smoothness of the compatible-star coordinates then gives convergence of the target coordinate periods to $P+z$ in C^1 with respect to z, uniformly on compact parameter sets.

With the family $g(P,z)$ fixed, choose a smooth positive $\epsilon(P)<1$ by Lemma 3.5 such that supp $\kappa_{\epsilon(P)}$ lies in the stated neighborhood and the coordinate period $f(P,z)$ of $k_{\epsilon(P)}(P,z)$ satisfies

$$\|D_z f(P,z)-I\|\le \tfrac14, \qquad |f(P,0)-P|\le r(P)/4 \quad (|z|\le r(P)).$$

Lemma 3.6 gives a smooth map $P\mapsto z(P)$ satisfying

$$f(P,z(P))=P, \qquad |z(P)|\le r(P)/3.$$

The family $k_{\epsilon(P)}(P,z(P))$ is a global smooth section in the chosen coordinates. Differentiating its section identity gives regularity. □

Corollary 4.9. *For closed oriented $2m$-manifolds, $m\ge 3$, the existence of a global smooth middle Gram section is preserved in both directions by surgery on a framed j-sphere, $1\le j\le m-2$, and by surgery on a framed $(m-1)$-sphere with nonzero real homology class. A section on $Z\#(S^m\times S^m)$ gives a section on Z. Every such section can be chosen through regular metrics.*

Proof. For a subcritical surgery, the cap cores are S^j and S^{2m-j-1}. Their codimensions are at least two, and both cap groups H^m and both boundary groups H^{m-1}, H^m vanish. Mayer–Vietoris [23, §3.1, pp. 203–204] identifies the middle pairings on the common exterior, and all middle classes restrict to zero on both caps. Theorem 4.8 applies in both directions.

For an essential $(m-1)$-sphere, let $x\in H_{m-1}(X;\mathbb{R})$ be its nonzero class. Write the incoming and outgoing caps as $C=S^{m-1}\times D^{m+1}$ and $C'=D^m\times S^m$, and their common exterior as A. Excision and the Thom isomorphism [23, Corollary 4D.9 and Theorem 4D.10] give the following part of the exact sequence of (X,A):

$$0\longrightarrow H^m(X;\mathbb{R})\longrightarrow H^m(A;\mathbb{R})\longrightarrow H^{m+1}(X,A;\mathbb{R})=\mathbb{R}\xrightarrow{1\mapsto \mathrm{PD}(x)} H^{m+1}(X;\mathbb{R}).$$

The relative generator is chosen so that its image is $\mathrm{PD}(x)$ [47, Problem 11-C]. The last arrow is injective. Thus $H^m(X;\mathbb{R})\simeq H^m(A;\mathbb{R})$, and the boundary restriction of this group is zero because its classes extend over C, whose H^m vanishes. A class in $H^{m-1}(X;\mathbb{R})$ evaluating nontrivially on x restricts to a generator of $H^{m-1}(\partial C;\mathbb{R})$, so the restriction from $H^{m-1}(A;\mathbb{R})$ onto that group is surjective. Mayer–Vietoris for $X'=A\cup C'$ now identifies $H^m(X';\mathbb{R})$ with $H^m(A;\mathbb{R})$ and shows that its restriction to C' is zero: the map $H^m(C';\mathbb{R})\to H^m(\partial C';\mathbb{R})$ is an isomorphism. Representatives supported on the common exterior preserve the pairing. Theorem 4.8 applies in both directions. Let $x_{\mathbb{Z}}\in H_{m-1}(X;\mathbb{Z})$ be the integral class of the sphere. If its image in $H_{m-1}(X;\mathbb{Z})/\operatorname{Tor} H_{m-1}(X;\mathbb{Z})$ is primitive, evaluation on $x_{\mathbb{Z}}$ is onto $\mathbb{Z}$, and $\mathrm{PD}(x_{\mathbb{Z}})$ generates a direct summand [23, Theorems 3.2 and 3.30]. The same sequences over $\mathbb{Z}$ therefore identify the middle cohomology groups. By Poincaré duality, integral middle homology and its intersection pairing are preserved as well.

For $Z\#(S^m\times S^m)$, retain the orthogonal subspace $H^m(Z;\mathbb{R})$. Its classes vanish on the punctured sphere product, which has a finite geometric spine of dimension m. Replacement by a ball identifies the retained pairing with that of Z. Theorem 4.8 gives the final assertion. □

5. The marked four-manifold

Put

$$H=\begin{pmatrix}0&1\\1&0\end{pmatrix}, \qquad Z_r=\#_{i=1}^{r}(S^2\times S^2) \quad (r\ge 1),$$

with the product orientations on the summands. Write mH for the orthogonal direct sum of m copies of H. The Poincaré duals of the ordered pairs of coordinate spheres determine the fixed integral marking $H^2(Z_r;\mathbb{Z})\cong rH$.

5.1. **A product with prescribed periods.**

Lemma 5.1. *For $r \geq 1$ put $d = r(r-1)/2$ and $N = r + d$. There is a closed connected oriented smooth three-manifold Y_r with $H_1(Y_r;\mathbb{Z}) \cong \mathbb{Z}^N$ and an integral basis*

$$a_1, \dots, a_r, \quad c_{ij} \quad (1 \leq i < j \leq r)$$

of $H^1(Y_r;\mathbb{Z})$ whose only nonzero basis triple products, up to alternation, are

$$\langle c_{ij} \smile a_i \smile a_j, [Y_r] \rangle = 1. \tag{5.1}$$

Its fundamental group has N generators whose homology classes are an integral basis.

Proof. Apply Sullivan's realization of alternating integral trilinear forms [61, assertion (i), pp. 275–276] to (5.1). His construction has torsion-free first homology and a Heegaard splitting of genus N, as stated at the end of the construction on p. 276. Its surface gluing maps may be taken smooth. A genus-N Heegaard splitting supplies N generators of the fundamental group. Their images generate the free abelian group $H_1(Y_r;\mathbb{Z})$ of rank N, and hence form an integral basis. □

Fix $r \geq 1$ and a manifold Y_r as in Lemma 5.1. Order the classes c_{ij} and write the resulting basis as $a_1, \dots, a_N$. Since $H_1(Y_r;\mathbb{Z})$ is free, integral Poincaré duality [23, Theorem 3.30 and Proposition 3.38] gives a dual basis $b_1, \dots, b_N$ of $H^2(Y_r;\mathbb{Z})$, characterized by

$$\langle a_A \smile b_C, [Y_r] \rangle = \delta_{AC} \qquad (1 \leq A, C \leq N).$$

On $S^1 = \mathbb{R}/\mathbb{Z}$ take the form dt of integral one. Orient $M_r = S^1 \times Y_r$ by dt followed by the orientation of Y_r. Pullbacks from the two factors are omitted from the notation. Put

$$e_A = [dt] \smile a_A, \qquad f_A = b_A, \qquad L = \operatorname{span}_{\mathbb{Z}}\{e_i, f_i \,:\, 1 \leq i \leq r\}.$$

The Künneth formula [23, §3.2, p. 216] gives the integral basis $e_1, f_1, \dots, e_N, f_N$ of $H^2(M_r;\mathbb{Z})$, with

$$e_A \cdot e_C = f_A \cdot f_C = 0, \qquad e_A \cdot f_C = \delta_{AC}.$$

Thus the intersection form is NH, and L is a primitive orthogonal summand isometric to rH.

Every maximal positive plane of $L_{\mathbb{R}} = L \otimes_{\mathbb{Z}} \mathbb{R}$ has a unique expression

$$P_M = \operatorname{span}_{\mathbb{R}} \left(e_i + \sum_{j=1}^{r} M_{ji} f_j \,:\, 1 \leq i \leq r \right), \qquad M = S + B, \quad S = S^t > 0, \quad B = -B^t. \tag{5.2}$$

Indeed, projection onto $\operatorname{span}_{\mathbb{R}}\{e_i \,:\, 1 \leq i \leq r\}$ along $\operatorname{span}_{\mathbb{R}}\{f_i \,:\, 1 \leq i \leq r\}$ is injective on a positive plane, since its kernel is isotropic. It is an isomorphism by dimension, and the induced form on the graph is $M + M^t = 2S$.

Proposition 5.2. *There is a smooth family of metrics g_M on M_r, for all $M = S + B$ in* (5.2)*, such that the cohomological Hodge star preserves $L_{\mathbb{R}}$ and its restriction to $L_{\mathbb{R}}$ has $+1$-eigenspace P_M.*

Proof. Let I_d and 0_d denote the identity and zero matrices of order d; when $d = 0$, the corresponding blocks are omitted. Theorem 3.1, applied to Y_r with $n = 3$ and $k = 1$, gives a smooth family h_S whose first-cohomology Gram matrix in the basis $a_1, \dots, a_N$ is

$$\widehat{S} = \operatorname{diag}(S, I_d).$$

Choose closed one-forms γ_{ij} on Y_r with $[\gamma_{ij}] = c_{ij}$, and put

$$\theta_B = -\sum_{i<j} B_{ij}\gamma_{ij}, \qquad \tau_B = dt + \theta_B, \qquad g_M = \tau_B^{\otimes 2} + h_S.$$

The coframe change from dt to τ_B is invertible and preserves the product orientation. Thus g_M is a metric, smooth in M. Let η_A be the h_S-harmonic representative of a_A. In the oriented orthogonal splitting $\mathbb{R}\tau_B \oplus T^*Y_r$, the Hodge star satisfies

$$*_{g_M}(\tau_B \wedge \eta_A) = *_{h_S}\eta_A, \qquad *_{g_M}(*_{h_S}\eta_A) = \tau_B \wedge \eta_A.$$

Since $d\tau_B = d\eta_A = d(*_{h_S}\,\eta_A) = 0$, the forms

$$\omega_A^{\pm} = \tau_B \wedge \eta_A \pm *_{h_S}\,\eta_A$$

are closed and satisfy $*_{g_M}\,\omega_A^{\pm} = \pm\omega_A^{\pm}$. They are therefore harmonic.

Put $\widehat{B} = \operatorname{diag}(B, 0_d)$. By (5.1),

$$\int_{Y_r} \eta_C \wedge \theta_B \wedge \eta_A = -\sum_{i<j} B_{ij}\langle a_C \smile c_{ij} \smile a_A, [Y_r]\rangle = \widehat{B}_{CA},$$

$$\int_{Y_r} \eta_C \wedge *_{h_S}\,\eta_A = \widehat{S}_{CA}.$$

The dual basis then gives

$$[\omega_A^{\pm}] = e_A + \sum_{C=1}^{N} (\widehat{B}_{CA} \pm \widehat{S}_{CA}) f_C.$$

For each sign these N classes are independent, since their e-components form a basis. They are therefore bases of the full positive and negative eigenspaces, whose dimensions are N. For $1 \le i \le r$ they belong to $L_{\mathbb{R}}$, and

$$[\omega_i^+] - [\omega_i^-] = 2\sum_{j=1}^{r} S_{ji} f_j.$$

Since S is invertible, these $2r$ classes span $L_{\mathbb{R}}$. Hence

$$L_{\mathbb{R}} = \operatorname{span}_{\mathbb{R}}\{[\omega_i^+] \,:\, 1 \le i \le r\} \oplus \operatorname{span}_{\mathbb{R}}\{[\omega_i^-] \,:\, 1 \le i \le r\}.$$

The two summands have eigenvalues $+1$ and -1, respectively, and the first is P_M. □

5.2. **Circle surgeries and hyperbolic forms.**

Lemma 5.3. *Let M be a closed connected oriented smooth four-manifold, and let an embedded oriented circle γ have primitive nonzero class in $H_1(M;\mathbb{Z})/\operatorname{Tor} H_1(M;\mathbb{Z})$. Surgery on any framing of its oriented normal three-plane bundle preserves the integral second homology and its intersection form. The identifications are represented by cycles in the common exterior and, over $\mathbb{R}$, by cutoff followed by zero extension of closed two-forms.*

Proof. Let $U \cong S^1 \times D^3$ be the framed tube, put $E = M \setminus \operatorname{int} U$, and let $M' = E \cup (D^2 \times S^2)$ be the surgered manifold. In this proof homology has integral coefficients. Excision [23, Theorem 2.20] and the exact sequence of (M, E) give

$$H_3(M) \longrightarrow \mathbb{Z} \longrightarrow H_2(E) \longrightarrow H_2(M) \longrightarrow 0, \qquad H_1(E) \cong H_1(M).$$

The first homomorphism is intersection with $[\gamma]$ [23, Theorem 3.44]. There is $\xi \in H^1(M;\mathbb{Z})$ with $\langle \xi, [\gamma]\rangle = 1$; its Poincaré dual maps to a generator. Thus $H_2(E) \to H_2(M)$ is an isomorphism. For (M', E) the sequence is

$$0 \longrightarrow H_2(E) \longrightarrow H_2(M') \longrightarrow \mathbb{Z} \longrightarrow H_1(E).$$

The last homomorphism sends a generator to $\pm[\gamma]$ under $H_1(E) \cong H_1(M)$, and is injective. Hence $H_2(E) \to H_2(M')$ is also an isomorphism. Cycles in the interior of E have the same intersection numbers in M and M'.

Since $H^2(U;\mathbb{R}) = 0$, the cutoff construction preceding Lemma 4.6 gives closed representatives vanishing on a neighborhood of U. Their zero extensions to M' have the same integrals on cycles in E, and hence induce the dual cohomology identification. □

Theorem 5.4. *The period map of the marked manifold Z_r has a global smooth section for every $r \ge 1$.*

Proof. Fix $r \geq 1$ and take $M_r = S^1 \times Y_r$, N and L as in §5.1. Choose disjoint embedded oriented circles representing, up to conjugacy, the N generators of $\pi_1(Y_r)$ in Lemma 5.1 and the additional S^1 generator. Their homology classes form a basis of $H_1(M_r;\mathbb{Z}) \cong \mathbb{Z}^{N+1}$. Their oriented normal three-plane bundles are trivial. Fix framings and perform circle surgeries.

By relative transversality, loops and homotopies of loops can be made disjoint from the circles, since $2+1<4$. Their complement deformation retracts onto the tube exterior, so removing the tubes does not change the fundamental group. Van Kampen's theorem [23, Theorem 1.20] identifies the group after surgery with the quotient by the normal closure of these generators; it is therefore trivial. At each step the next homology class is primitive and nonzero in the quotient by the preceding basis elements. Lemma 5.3 gives a simply connected four-manifold W with marked integral form NH and the same marked summand L.

The manifolds W and Z_N are closed, oriented, smooth and simply connected, with isometric integral intersection forms. Wall's Theorems 2 and 3 [67, p. 141] give an orientation-preserving diffeomorphism

$$W\#_k(S^2\times S^2) \cong Z_{N+k} \tag{5.3}$$

for some $k \geq 0$. Increase k so that $k \geq 1$; then $N+k \geq 2$. Extend the marking of W by the standard markings of the added summands. In the decomposition $Z_{N+k} = Z_{N+k-1}\#(S^2\times S^2)$, the first factor is simply connected and has indefinite form. Wall's realization theorem [66, Theorem 2, pp. 131 and 136] therefore realizes every integral isometry of Z_{N+k} by an orientation-preserving diffeomorphism. Compose the diffeomorphism in (5.3) with one inducing the required change of marking, and denote the resulting diffeomorphism by F. Its pullback identifies the first r standard hyperbolic summands, with their markings, with L.

Each stabilization replaces a ball by a punctured $S^2\times S^2$. Put $m = N+k-r \geq 1$. In the fixed decomposition $Z_{N+k} = Z_r\#Z_m$, let

$$C = Z_m \setminus \operatorname{int} B^4.$$

Choose a smooth triangulation of S^2 containing a closed disk D as a subcomplex, with an interior vertex p to which D collapses. Write $\searrow$ for a finite simplicial collapse. In each hyperbolic summand, put

$$R = (S^2\times D)\cup(D\times S^2).$$

The relative product collapses give

$$R \searrow (S^2\times\{p\})\cup(D\times S^2) \searrow (S^2\times\{p\})\cup(\{p\}\times S^2).$$

The disk-bundle collapses are relative to $D\times D$ and $D\times\{p\}$, respectively. The closed complement of R is $D^2\times D^2$, hence a ball after rounding corners. Take boundary connected sums of these neighborhoods along product tubes about disjoint arcs joining the sphere pairs in a tree. Choose the arcs radial in the disk fibres near their ends and triangulate the tubes compatibly. The product collapses, relative to these radial segments, and the transverse-disk collapses of the tubes give a finite collapse onto the sphere pairs and the joining arcs. After rounding corners, C is therefore a smooth regular neighborhood of this finite two-dimensional complex [27, definition and Theorem 1, pp. 525–526]. The coordinate spheres in the last m summands generate $H_2(C;\mathbb{R})$ and pair trivially with $L_{\mathbb{R}}$. Thus $L_{\mathbb{R}}|_C = 0$. Replace C by the ball with the boundary identification of the fixed connected-sum decomposition. In $W\#_k(S^2\times S^2)$ this is the replacement of $F^{-1}(C)$, with the boundary identification transported by F. The resulting manifold is the originally marked Z_r.

All circles, framings, diffeomorphisms and caps are fixed. Write $X_0,\dots,X_s$ for the successive manifolds, with $X_0 = M_r$ and $X_s = Z_r$, and let $V_j \subset H^2(X_j;\mathbb{R})$ be the transported copy of $L_{\mathbb{R}}$. Each V_j has signature (r,r). The removed circle tubes and stabilization balls have spines of dimensions one and zero, respectively [27, Lemma 4, pp. 527–528]; the final cap has the two-dimensional spine above. The classes in V_{j-1} restrict to zero on the cap removed at the jth step: the circle tubes and balls have zero second cohomology, and the last restriction vanishes by $L_{\mathbb{R}}|_C = 0$. A collar enlargement of

each cap has the same homotopy type. The construction preceding Lemma 4.6 therefore gives closed representatives vanishing near that cap.

For a circle surgery, zero extension induces the isometry of Lemma 5.3; for a stabilization or the final replacement, it induces the inclusion or projection onto the retained summands. These maps identify V_{j-1} with V_j, and $V_s = H^2(Z_r;\mathbb{R})$ with its fixed marking. Under the graph coordinates (5.2), Proposition 5.2 gives a smooth family $g_0(P)$ on X_0, for all $P \in \mathcal{D}(L_{\mathbb{R}})$, whose cohomological Hodge star preserves V_0 and has $+1$-eigenspace P on V_0. Theorem 4.7, applied to the X_j and V_j, gives a smooth section of Π_{Z_r}. □

5.3. **Odd forms.** We give $\mathbb{CP}^2$ its complex orientation and denote the oppositely oriented manifold by $\overline{\mathbb{CP}}^2$. For $e \in \mathbb{Z}$, let $D(e)$ denote the closed disk bundle of the oriented real rank-two bundle over the oriented sphere S^2 with Euler number e.

Lemma 5.5. *There are specified cap replacements*

$$S^2 \times S^2 \longrightarrow \mathbb{CP}^2, \qquad S^2 \times S^2 \longrightarrow \overline{\mathbb{CP}}^2$$

which remove $D(-4)$ and $D(4)$, respectively, and insert rational homology balls.

Proof. Choose a smooth map $f : S^2 \to S^2$ of degree -2. Its graph is a section of the trivial sphere bundle; its oriented normal bundle is f^*TS^2 and has Euler number -4. The fibrewise antipodal graph is disjoint from it and has normal Euler number 4. The fibrewise closed hemispheres centred at the two sections give a decomposition along their common boundary:

$$S^2 \times S^2 = D(-4) \cup D(4).$$

Let $C \subset \mathbb{CP}^2$ be a smooth conic, and let νC be a closed tubular neighborhood. Since $[C] = 2[\mathbb{CP}^1]$, we have $\nu C \cong D(4)$. Its complement $B = \mathbb{CP}^2 \setminus \operatorname{int} \nu C$ is a rational homology ball [17, arXiv version, Lemma 2.1, second construction, with $p = 2$].

Fix an orientation-preserving identification of the surviving $D(4)$ with νC, and glue B by the boundary identification inherited from $\mathbb{CP}^2$. The result is $\mathbb{CP}^2$. Reverse orientation for the other replacement, identifying the reversed source with $S^2 \times S^2$ by $(x, y) \mapsto (x, -y)$. These boundary identifications are fixed throughout. □

Theorem 5.6. *For integers $p, q \geq 0$ with $p + q > 0$, the period map of*

$$U_{p,q} = \#_p \mathbb{CP}^2 \#_q \overline{\mathbb{CP}}^2$$

has a global smooth section.

Proof. If $pq = 0$, the period domain is a point, and any metric gives a section. Suppose $p, q > 0$, and put $n = p+q$. In the ith summand of Z_n, choose the cap C_i and its replacement B_i from Lemma 5.5, with $C_i = D(-4)$ for $i \leq p$ and $C_i = D(4)$ for $i > p$. Choose the connected-sum balls in the complements of these caps. Let X_j be the manifold after the first j replacements, so that $X_0 = Z_n$ and $X_n = U_{p,q}$.

Let $\Sigma_i \subset C_i$ be the zero-section, and put $v_i = \mathrm{PD}[\Sigma_i] \in H^2(Z_n;\mathbb{R})$ for $1 \leq i \leq n$. These classes are pairwise orthogonal, with $v_i^2 = -4$ for $i \leq p$ and $v_i^2 = 4$ for $i > p$. Thus

$$V = \big(\operatorname{span}_{\mathbb{R}}\{v_1, \dots, v_n\}\big)^{\perp}$$

has signature (p, q), and $V^{\perp}$ has signature (q, p). For $a \in H^2(Z_n;\mathbb{R})$, the restriction $a|_{C_i}$ is determined by its evaluation on Σ_i, namely $a \cdot v_i$. Hence $V|_{C_i} = 0$ for every i.

Put $A = Z_n \setminus \bigcup_{i=1}^n \operatorname{int} C_i$. Each ∂C_i is a rational homology sphere [17, arXiv version, §2], so $H^1(\partial C_i;\mathbb{R}) = H^2(\partial C_i;\mathbb{R}) = 0$. Also $H^2(B_i;\mathbb{R}) = 0$. The Mayer–Vietoris sequence [23, §3.1, pp. 203–204] therefore gives restriction isomorphisms

$$H^2(X_j;\mathbb{R}) \cong H^2(A;\mathbb{R}) \oplus \bigoplus_{i=j+1}^{n} H^2(C_i;\mathbb{R}) \qquad (0 \leq j \leq n).$$

Let V_j be the inverse image of $H^2(A;\mathbb{R}) \oplus 0$ under this isomorphism. Then $V_0 = V$ and $V_n = H^2(U_{p,q};\mathbb{R})$.

Disjoint tubular enlargements of the C_i retract onto them. The cutoff construction preceding Lemma 4.6 gives closed representatives of a basis of V which vanish on neighborhoods of all the C_i. Their restrictions to A extend by zero to every X_j and represent the corresponding basis of V_j. The integrals of their wedge products are all taken over A. Thus the resulting identifications $V_{j-1} \cong V_j$ are isometries. Each C_i is a smooth regular neighborhood of its zero-section [27, Lemma 4].

Let $s_Z : \mathcal{D}_{Z_n} \to \mathrm{Met}^\infty(Z_n)$ be the section of Theorem 5.4, and fix $P_0 \in \mathcal{D}(V^\perp)$. For $P \in \mathcal{D}(V)$, the orthogonal sum $P \oplus P_0$ is positive of dimension $p+q=n$, hence belongs to $\mathcal{D}_{Z_n}$. The smooth map $P \mapsto P \oplus P_0$ therefore defines a smooth family $g_0(P) = s_Z(P \oplus P_0)$. Its cohomological Hodge star has eigenspaces $P \oplus P_0$ and $(P \oplus P_0)^\perp$ for the eigenvalues $+1$ and -1, respectively. Since

$$V = P \oplus \big(V \cap (P \oplus P_0)^\perp\big),$$

this star preserves V and has positive eigenspace P there. Theorem 4.7, applied to $X_0, \dots, X_n$ and $V_0, \dots, V_n$, gives a smooth section of $\Pi_{U_{p,q}}$. □

5.4. **Simply connected manifolds.**

Theorem 5.7. *The period map of every closed connected simply connected oriented smooth four-manifold has a global smooth section.*

Proof. Let X be such a manifold, with intersection indices (p,q). If $pq=0$, the period domain is a point, and any metric gives a section. Suppose $p,q>0$, and put $X_0 = X\#\mathbb{CP}^2\#(S^2\times S^2)$. Since X_0 is simply connected, $H^2(X_0;\mathbb{Z})$ is free and its intersection form is unimodular [23, Theorem 3.2 and Proposition 3.38]. This form is odd and indefinite, with indices $(p+2, q+1)$. By [57, Chapter V, Section 2.2, Theorem 4], it is isometric to the integral intersection form of $U_{p+2,q+1}$.

Wall's stable diffeomorphism theorem [67, (4.1), p. 147] gives, for some integer $\ell \geq 0$, an orientation-preserving diffeomorphism

$$X_0 \#_\ell (S^2\times S^2) \cong U_{p+2,q+1}\#_\ell (S^2\times S^2).$$

Wall's connected-sum identity [66, Lemma 1, p. 133, and Corollary 1 to Theorem 1, p. 136] is

$$\mathbb{CP}^2\#(S^2\times S^2) \cong 2\mathbb{CP}^2\#\overline{\mathbb{CP}}^2.$$

Put $k=\ell+1$, $A = \mathbb{CP}^2\#_k(S^2\times S^2)$ and $M = X\#A$. Applying this identity to the ℓ stabilizing summands of $U_{p+2,q+1}$ gives an orientation-preserving diffeomorphism

$$F : M \longrightarrow U_{p+k+1,q+k}. \tag{5.4}$$

Fix such an F and the connected-sum decomposition $M = X\#A$.

Let $C = A \setminus \operatorname{int} B^4$ be the cap in this decomposition. The standard projective line in $\mathbb{CP}^2$ has a tubular neighborhood $D(1)$ whose closed complement is a four-ball [66, proof of Lemma 1, p. 133]. The neighborhood $D(1)$ is a smooth regular neighborhood [27, Lemma 4]. Form its boundary connected sum with k copies of the coordinate-sphere neighborhood constructed in the proof of Theorem 5.4. The result is C, a smooth regular neighborhood of the finite two-dimensional complex consisting of the projective line, the coordinate-sphere pairs and a tree of joining arcs in a compatible smooth triangulation.

The connected-sum decomposition gives the orthogonal splitting

$$H^2(M;\mathbb{R}) = V \oplus W, \qquad V = H^2(X;\mathbb{R}), \quad W = H^2(A;\mathbb{R}),$$

where the two factors are identified with their images in $H^2(M;\mathbb{R})$. Their signatures are (p,q) and $(k+1,k)$. The restriction of V to C is zero, hence also to a collar enlargement of C. The cutoff construction preceding Lemma 4.6 gives closed representatives of a basis of V vanishing near C. Replace C by the original ball of X, with the boundary identification of the fixed connected-sum decomposition. Their zero extensions represent the original classes in $H^2(X;\mathbb{R})$, since restriction

to the punctured X is an isomorphism in degree two. The resulting identification $V \cong H^2(X;\mathbb{R})$ is therefore the given isometry.

Put $U = U_{p+k+1,q+k}$, and let $s_U : \mathcal{D}_U \to \mathrm{Met}^\infty(U)$ be the section of Theorem 5.6. Fix a maximal positive subspace $P_0 \subset W$. Thus $\dim P_0 = k+1$. For $P \in \mathcal{D}_X$, the orthogonal sum $P \oplus P_0$ is positive of dimension $p+k+1$. Since F preserves orientation, $(F^{-1})^*(P \oplus P_0) \in \mathcal{D}_U$. Put

$$g_0(P) = F^* s_U\big((F^{-1})^*(P \oplus P_0)\big).$$

The outer pullback is that of metrics; the inner one is that of cohomology subspaces. This family is smooth, and naturality of the Hodge star gives $\Pi_M(g_0(P)) = P \oplus P_0$. As in the last paragraph of the proof of Theorem 5.6, the cohomological Hodge star of $g_0(P)$ preserves V and has positive eigenspace P there. Theorem 4.7, applied to the single replacement $M \to X$ with the specified isometry, gives the required section. □

5.5. **The original fundamental group.**

Proof of Theorem A in dimension four. Let X be the given closed connected oriented smooth four-manifold, and fix a positive smooth volume form ν on X. By (2.5), it suffices to construct a smooth section of $\Pi_X|_{\mathrm{Met}^\infty_\nu(X)}$. Put $p = b^+(X)$ and $q = b^-(X)$. If $pq = 0$, choose any metric and apply (2.12). Its period is the sole point of $\mathcal{D}_X$, whose tangent space is zero. The differential is therefore surjective. Suppose now that $p, q > 0$.

The manifold X has the homotopy type of a finite CW complex [28, Chapter 6, §1, Theorem 1.2, and §4, Theorem 4.1]; hence $\pi_1(X)$ is finitely generated. Choose normal generators $g_1, \dots, g_m$, allowing $m = 0$ when $\pi_1(X) = 1$. By [28, Chapter 5, §1, Lemma 1.5; Chapter 2, §2, Theorem 2.13], these classes have disjoint smooth embedded circle representatives, up to conjugacy. Orient the circles. Their oriented normal three-plane bundles are trivial [25, Proposition 1.14 and the discussion on p. 25]. Fix disjoint framed closed tubes $T_a \cong S^1 \times D^3$, disjoint tubular enlargements, and boundary collars. Put

$$E = X \smallsetminus \mathrm{int}\Big(\coprod_{a=1}^{m} T_a\Big),$$

and replace each T_a by $C_a = D^2 \times S^2$ along the framed boundary identification. Denote the resulting closed oriented smooth four-manifold by W. Since each ∂T_a is connected, a tube meets only one component of E. The connectedness of X therefore implies that of E and W. Write $\langle\!\langle g_1, \dots, g_m \rangle\!\rangle$ for their normal closure in $\pi_1(X)$. The isomorphisms $\pi_1(\partial T_a) \to \pi_1(T_a)$ and the equalities $\pi_1(C_a) = 1$ give, by van Kampen's theorem applied successively to collar enlargements [23, Theorem 1.20],

$$\pi_1(E) \cong \pi_1(X), \qquad \pi_1(W) \cong \pi_1(X)/\langle\!\langle g_1, \dots, g_m \rangle\!\rangle = 1.$$

For $0 \le i \le m$, put

$$X_i = E \cup \Big(\coprod_{a=1}^{i} T_a\Big) \cup \Big(\coprod_{a=i+1}^{m} C_a\Big).$$

All gluings have the fixed boundary identifications and collars; on the T_a these are the original ones. Thus $X_0 = W$ and $X_m = X$. The passage from X_{i-1} to X_i replaces C_i by T_i. Each C_i is a closed tubular neighborhood of $\{0\} \times S^2$, hence a smooth regular neighborhood of a finite two-dimensional subcomplex in a compatible smooth triangulation [27, Lemma 4, pp. 527–528].

Put $r = \dim H^2(X;\mathbb{R})$. The tubular enlargements retract onto circles, so their second real cohomology groups vanish. The cutoff construction preceding Lemma 4.6 gives fixed closed two-forms $\beta_1, \dots, \beta_r$ representing a basis of $H^2(X;\mathbb{R})$ and vanishing on neighborhoods of all the T_a. Extend their restrictions to E by zero to each X_i, and denote the resulting smooth closed forms by $\beta_l^{(i)}$. Define the linear maps

$$j_i : H^2(X;\mathbb{R}) \longrightarrow H^2(X_i;\mathbb{R}), \qquad j_i[\beta_l] = [\beta_l^{(i)}] \quad (0 \le i \le m).$$

All wedge integrals are taken over the same oriented exterior, so

$$Q_{X_i}(j_i a, j_i b) = Q_X(a, b) \qquad (a, b \in H^2(X;\mathbb{R})).$$

Nondegeneracy of Q_X implies that each j_i is injective and that $V_i = j_i H^2(X;\mathbb{R})$ is nondegenerate of signature (p, q). For $1 \le i \le m$, the isometry

$$\iota_i = j_i \circ j_{i-1}^{-1} \,:\, V_{i-1} \longrightarrow V_i$$

is realized by zero extension of the representatives $\beta_l^{(i-1)}$, which vanish near C_i. For the forms constructed in Lemma 4.6, the identity following (4.7) gives the cohomology map ι_i. Since $\beta_l^{(m)} = \beta_l$, one has $j_m = \mathrm{id}$ and $V_m = H^2(X;\mathbb{R})$.

Let $s_W : \mathcal{D}_W \to \mathrm{Met}^\infty(W)$ be the section of Theorem 5.7. Fix a maximal positive plane $P_0 \subset V_0^\perp$, where orthogonality is with respect to Q_W. The orthogonal decomposition $H^2(W;\mathbb{R}) = V_0 \oplus V_0^\perp$ shows that $j_0 P \oplus P_0 \in \mathcal{D}_W$ for every $P \in \mathcal{D}_X$. Put

$$g_0(P) = s_W(j_0 P \oplus P_0).$$

This family is smooth. The orthogonal decomposition in the last paragraph of the proof of Theorem 5.6 shows that its cohomological Hodge star preserves V_0 and has positive eigenspace $j_0 P$ there. Theorem 4.7, applied to the X_i, the V_i, and the ι_i, therefore gives a smooth section $s : \mathcal{D}_X \to \mathrm{Met}^\infty(X)$ with $\Pi_X \circ s = \mathrm{id}_{\mathcal{D}_X}$. By (2.12) and Lemma 2.5, the map $s_\nu = N_\nu \circ s$ is a smooth section in $\mathrm{Met}_\nu^\infty(X)$, and all its values are regular for $\Pi_X|_{\mathrm{Met}_\nu^\infty(X)}$. □

6. Middle periods in higher dimensions

Throughout this section, $m \ge 3$. Stable bundle identifications include a specified stable isomorphism. For an oriented stable rank-$2m$ bundle η on a space K, a map $f : X^{2m} \to K$ together with $TX \simeq_{\mathrm{st}} f^*\eta$ specifies a normal K-structure; its stable normal bundle is the pullback of $-\eta$. On a bordism $F : W^{2m+1} \to K$, the corresponding identification is

$$TW \simeq_{\mathrm{st}} F^*\eta \oplus \mathbb{R}.$$

Write $b_j(K) = \dim H_j(K;\mathbb{Q})$ when these groups are finite-dimensional. Let $\mathrm{SO}(r)$ be the special orthogonal group and $\mathrm{SO} = \varinjlim_r \mathrm{SO}(r)$.

6.1. Cup products and circle metrics.

Put $p = m - 1 \ge 2$. For $d \ge 0$, let F_d denote the free group of rank d, with $F_0 = \{1\}$.

Lemma 6.1. *For each $r \ge 1$ there is a closed connected stably framed $(2p+1)$-manifold Y with free integral homology and*

$$H_p(Y;\mathbb{Z}) = H_{p+1}(Y;\mathbb{Z}) = \mathbb{Z}^r, \qquad H_j(Y;\mathbb{Z}) = 0 \quad (2 \le j \le p-1).$$

Its degree-p homology has a spherical basis. If p is even, then $\pi_1(Y) = F_{r(r+1)/2}$ and an integral basis $a_i \in H^p(Y;\mathbb{Z})$ makes

$$H^1(Y;\mathbb{R}) \longrightarrow \mathrm{Sym}(r), \qquad [\theta] \longmapsto \left(\int_Y \theta a_i a_j \right)_{ij} \tag{6.1}$$

an isomorphism. If p is odd, then $\pi_1(Y) = F_{r(r-1)/2}$ and such a basis makes

$$H^1(Y;\mathbb{R}) \longrightarrow \mathrm{Skew}(r), \qquad [\theta] \longmapsto \left(\int_Y a_i \theta a_j \right)_{ij} \tag{6.2}$$

an isomorphism.

Proof. Let W be a closed connected stably framed $(2p+1)$-manifold. For the map $W \to \{*\}$ and the trivial bundle over the point, Wall's framed-surgery theorem [69, Theorem 1.1 and its corollary] gives an embedded representative of every spherical degree-p class, with a normal framing for which the specified stable framing extends over the surgery trace; here $2p < 2p+1$. Finitely many such representatives can be chosen disjoint by general position [28, Chapter 2, §2, Theorem 2.13].

Suppose that $H_p(W;\mathbb{Z})$ and $H_{p+1}(W;\mathbb{Z})$ are free. Let W' result from surgery on a primitive nonzero class $x \in H_p(W;\mathbb{Z})$, and let T be its trace. Both relative handles have index $p+1$. The boundary map $H_{p+1}(T, W;\mathbb{Z}) = \mathbb{Z} \to H_p(W;\mathbb{Z})$ sends 1 to x. It is injective with free cokernel, so $H_{p+1}(W;\mathbb{Z}) \simeq H_{p+1}(T;\mathbb{Z})$ and $H_p(T;\mathbb{Z}) \simeq H_p(W;\mathbb{Z})/\mathbb{Z}x$. Under the first identification, the map $H_{p+1}(T;\mathbb{Z}) \to H_{p+1}(T, W';\mathbb{Z}) = \mathbb{Z}$ is intersection with x, by Poincaré–Lefschetz duality [23, Theorem 3.43]. It is onto by integral Poincaré duality [23, Theorem 3.30 and Corollary 3.39]. The other relative sequence therefore identifies $H_p(W';\mathbb{Z})$ with $H_p(W;\mathbb{Z})/\mathbb{Z}x$ and $H_{p+1}(W';\mathbb{Z})$ with the annihilator of x in $H_{p+1}(W;\mathbb{Z})$. All other homology is unchanged, and freeness is preserved. The classes in $H^p(W;\mathbb{Z})$ extending over T are exactly those vanishing on x; their extensions restrict isomorphically to $H^p(W';\mathbb{Z})$. Every degree-one class extends.

Suppose first that p is even. Put $N = r(r+1)/2$, and let $e_1, \dots, e_r$ be the coordinate basis of $\mathbb{Z}^r$. Let $v_1, \dots, v_N$ enumerate the vectors

$$e_1, \dots, e_r, \qquad e_i + e_j \quad (i < j).$$

Write $v_e = \sum_i v_{ei} e_i$. Their rank-one matrices form a basis of $\mathrm{Sym}(r)$. Start with

$$Y_0 = \#_{e=1}^{N} (S^1 \times S^p \times S^p).$$

Each summand bounds $S^1 \times S^p \times D^{p+1}$; boundary connected sums give a stable framing. In summand e, choose integral cohomology generators θ_e, x_e, y_e of degrees $1, p, p$ with $\int \theta_e x_e y_e = 1$, and put

$$a_i = \sum_e v_{ei}(x_e + y_e).$$

The unit-vector summands exhibit an identity minor, so the lattice spanned by the a_i is primitive. Its homological annihilator is a primitive free summand of rank $2N - r$. All degree-p classes are spherical, since the factor spheres give a spherical basis. Perform the framed surgeries just described on a basis of this annihilator. At every stage the remaining attaching classes are primitive in the quotient. Denote the resulting manifold by Y. The extensions of the classes a_i restrict to an integral basis of $H^p(Y;\mathbb{Z})$, since their span is the full annihilator of the subgroup removed from $H_p(Y_0;\mathbb{Z})$. The resulting manifold has the stated free homology and fundamental group, since the handles have index at least three from either side. The classes a_i and θ_e extend over the oriented trace. Their triple products are unchanged; mixed connected-sum products vanish. Hence

$$\int_Y \theta_e a_i a_j = 2 v_{ei} v_{ej}.$$

These matrices form a real basis of $\mathrm{Sym}(r)$.

Suppose next that p is odd. For $r = 1$ take $Y = S^p \times S^{p+1}$. For $r \geq 2$ put $N = r(r-1)/2$ and start with $\#_{i<j}(S^1 \times S^p \times S^p)$. Write its integral cohomology generators as $\theta_{ij}, x_{ij}, y_{ij}$, oriented by $\int \theta_{ij} x_{ij} y_{ij} = 1$, and put

$$a_i = \sum_{j>i} x_{ij} + \sum_{j<i} y_{ji}.$$

The coordinates $x_{12}, y_{12}, y_{13}, \dots, y_{1r}$ give an identity minor. Perform surgeries on a primitive basis of the homological annihilator as before. The surviving triple product for θ_{ij} has entry -1 in position (i, j), entry 1 in position (j, i), and zero elsewhere; the sign follows from $a_i \theta_{ij} = -\theta_{ij} a_i$. These matrices form a basis of $\mathrm{Skew}(r)$.

In either construction, choose lifts of a basis of the surviving degree-p homology before surgery. They are spherical and can be represented disjointly from the attaching spheres, again because $2p < 2p+1$. They give the required spherical basis afterwards. □

Let Y^{2m-1} be closed, connected and oriented, with $\dim H^{m-1}(Y;\mathbb{R}) = \dim H^m(Y;\mathbb{R}) = r$. Choose dual bases $a_i \in H^{m-1}(Y;\mathbb{R})$ and $b_i \in H^m(Y;\mathbb{R})$, with $\int_Y a_i b_j = \delta_{ij}$. Orient $X = S^1 \times Y$ by dt followed by the orientation of Y, where $S^1 = \mathbb{R}/\mathbb{Z}$ and $\int_{S^1} dt = 1$. Pullbacks from the two factors are understood. If $r = 0$, then $H^m(X;\mathbb{R}) = 0$ and the period space is a point; assume henceforth that $r \geq 1$. For a metric h and a closed one-form θ on Y, set

$$g_{\theta,h} = (dt + \theta)^2 + h, \qquad G = \Gamma_{m-1}(h). \tag{6.3}$$

Put $e_i = b_i$ and $f_i = (-1)^m[dt]a_i$, and write $e(v) = \sum_i v_i e_i$ and $f(v) = \sum_i v_i f_i$ for $v \in \mathbb{R}^r$. These classes form a basis of $H^m(X;\mathbb{R})$ with $\int_X e_i f_j = \delta_{ij}$ and $\int_X e_i e_j = \int_X f_i f_j = 0$.

Lemma 6.2. *If m is odd, the marked Siegel period of* (6.3)*, with the convention of* (2.2)*, is*

$$Z = C(\theta) + iG, \qquad C(\theta)_{ij} = \int_Y \theta a_i a_j. \tag{6.4}$$

If m is even, the pairing is hyperbolic, and the positive eigenspace is

$$P_{G,B} = \{e((G+B)v) + f(v) \,:\, v \in \mathbb{R}^r\}, \qquad B_{ij} = \int_Y a_i \theta a_j. \tag{6.5}$$

The parameters $G > 0$ symmetric and B skew describe every maximal positive subspace exactly once.

Proof. Put $g = g_{\theta,h}$, and let α_i, β_i be the h-harmonic representatives of a_i, b_i. Duality gives

$$*_h\, \alpha_i = \sum_j G_{ji}\beta_j, \qquad *_h\, \beta_i = \sum_j (G^{-1})_{ji}\alpha_j.$$

Put $u = dt + \theta$ and $\zeta_i = u \wedge \alpha_i$. Since $du = 0$, the product-star formulas are valid locally and give

$$*_g\, \zeta_i = *_h\, \alpha_i, \qquad *_g\, \beta_i = (-1)^m u \wedge *_h\, \beta_i. \tag{6.6}$$

These forms and their stars are closed, so all are harmonic.

Write J_g for the cohomological Hodge star. For odd m one has $[\zeta_i] = -f_i + \sum_j C_{ji} e_j$ and $J_g[\zeta_i] = \sum_j G_{ji} e_j$. Let $F : (H^m(X;\mathbb{R}), J_g) \to \mathbb{C}^r$ be complex linear, taking the Lagrangian basis e_i to the standard real basis. The latter identity gives

$$i\left(-F(f_i) + \sum_j C_{ji} F(e_j)\right) = \sum_j G_{ji} F(e_j).$$

Thus $F(f_i) = \sum_j (C_{ji} + iG_{ji})F(e_j)$, which proves (6.4) with its stated marking.

For even m, the classes satisfy $[\zeta_j] = f_j + \sum_i B_{ij} e_i$, and $B^t = -B$. The forms $\zeta_j + \sum_i G_{ij}\beta_i$ are self-dual by (6.6), giving (6.5). With e_i and f_i interchanged and $S = G$, (5.2) gives the bijection between maximal positive subspaces and pairs (G, B) with $G = G^t > 0$ and $B = -B^t$. □

6.2. **Middle representatives and geometric spines.**

Lemma 6.3. *Let N^{2m} be closed, simply connected and oriented, with $m \geq 4$. Every integral middle homology class has a nonzero even multiple represented by an embedded closed connected simply connected oriented m-manifold.*

Proof. Let $a \in H_m(N;\mathbb{Z})$. Thom's representability theorem [63, Theorem II.29, p. 55], applied to $2a$, gives an embedded oriented representative of ℓa for some $\ell \in 2\mathbb{Z} \setminus \{0\}$. If this representative is empty, replace it by a small sphere bounding an $(m+1)$-disk in a coordinate ball. Joining its components by tubes gives a connected representative $j : B \hookrightarrow N$.

Choose finitely many disjoint embedded circles normally generating $\pi_1(B)$. Their normal bundles in B are oriented bundles over S^1, hence trivial. Choose framings and let T be the oriented surgery trace, with incoming boundary B and outgoing boundary B'. The handles have index two from B and index $m - 1 \geq 3$ from B'. Thus B' is connected and $\pi_1(B') \cong \pi_1(T) = 0$. Since $\pi_1(N) = 0$, the map j

extends to a map $F : T \to N$. Choose a smooth map $f : B' \to N$ homotopic to $F|_{B'}$. The oriented boundary of T gives

$$f_*[B'] = j_*[B] = \ell a \qquad \text{in } H_m(N;\mathbb{Z}).$$

Haefliger's embedding theorem [21, existence theorem (a), p. 47], with connectivity parameter zero, applies to f: its map on fundamental groups is surjective, the ambient dimension is $2m$, and $m > 2$. A homotopic embedding therefore represents ℓa. □

Lemma 6.4. *Let N be a closed simply connected oriented smooth six-manifold. For every $a \in H_3(N;\mathbb{Z})$, the class $2a$ is represented by an embedded S^3 with trivial normal bundle.*

Proof. The degree-three Hurewicz homomorphism $h_3 : \pi_3(N) \to H_3(N;\mathbb{Z})$ is onto, by Whitehead's exact sequence [71, Theorem 1, p. 55]. Choose a smooth map $f : S^3 \to N$ with $f_*[S^3] = 2a$. Haefliger's embedding theorem [21, existence theorem (a), p. 47], with connectivity parameter zero, makes f homotopic to an embedding: the source is connected, its map on fundamental groups is surjective, and $6 = 2 \cdot 3$ with $3 > 2$. The oriented rank-three normal bundle of this embedding is trivial, since $\pi_2 \mathrm{SO}(3) = 0$ [25, Proposition 1.14 and p. 75]. □

Lemma 6.5. *Let P_0 be a smooth regular neighborhood of a finite geometric spine K of dimension d in a smooth n-manifold. Attach finitely many disjoint exterior two-handles to P_0 and round corners, obtaining P. Then P is a smooth regular neighborhood of a finite geometric spine of dimension at most $\max\{d,2\}$. If P_0 has a handle decomposition with indices at most d, then P has one with indices at most $\max\{d,2\}$.*

Proof. Write the handles and their attaching regions and cores as

$$H_i = D^2 \times D^{n-2}, \qquad A_i = S^1 \times D^{n-2}, \qquad D_i = D^2 \times \{0\}.$$

Let $a_i : A_i \to \partial P_0$ be the attaching embeddings and put $L_i = a_i(S^1 \times \{0\})$. Choose product triangulations in which the A_i and D_i are subcomplexes. Whitehead's boundary-extension theorem [47, §20, p. 240], applied first to

$$R = \partial P_0 \setminus \bigcup_i a_i(\operatorname{int} A_i)$$

and then to P_0, gives a smooth triangulation $\tau_1 : \mathcal{T}_1 \to P_0$ in which the given attaching maps are piecewise linear. Let $\tau_0 : \mathcal{T}_0 \to P_0$ be a smooth triangulation in which P_0 collapses to K. The uniqueness part of the same theorem gives a piecewise linear homeomorphism $\alpha : \mathcal{T}_0 \to \mathcal{T}_1$. Set

$$K_1 = \tau_1\big(\alpha(\tau_0^{-1}(K))\big).$$

Then $K_1 \subset \operatorname{int} P_0$, $\dim K_1 = d$, and the regular-neighborhood pair is transported by α. Thus, after a compatible subdivision, P_0 collapses to K_1.

By the regular-neighborhood mapping-cylinder theorem [10, Theorem 3.5, p. 230], identify P_0, relative to K_1 and its boundary, with the mapping cylinder of a piecewise linear map

$$r : \partial P_0 \longrightarrow K_1.$$

After a common subdivision, r is simplicial and the L_i are subcomplexes. Let $C_i \subset P_0$ be the mapping cylinder of $r|_{L_i} : L_i \to r(L_i)$. It has dimension at most two and meets K_1 in $r(L_i)$.

After a product subdivision, H_i collapses relative to A_i onto $A_i \cup D_i$. Thus the union of P_0 and all handles first collapses onto $P_0 \cup_i D_i$. The collapse in the proof of [10, Proposition 3.15, p. 233], applied to the simplices of ∂P_0 outside $\bigcup_i L_i$ in decreasing dimension, collapses P_0 onto $K_1 \cup_i C_i$ while fixing every L_i. It therefore extends over the handle cores. The resulting spine is

$$K' = K_1 \cup \bigcup_i C_i \cup \bigcup_i D_i.$$

It is an embedded finite polyhedron of dimension at most $\max\{d,2\}$, contained in the interior of the handle union. The piecewise regular smoothing in the proof of [27, Theorem 1(a)], supported

away from K', rounds the corners and transports the triangulation and its finite collapse. The boundary-extension theorem gives a compatible ambient smooth triangulation. Thus P is a smooth regular neighborhood of K'. The smooth regular-neighborhood theorem also gives arbitrarily small ambient-isotopic regular neighborhoods of K' [27, Theorem 1(a)–(c)]. The handle assertion follows by appending the given two-handles to the handle decomposition of P_0. □

Proposition 6.6. *Let N^{2m} be closed, simply connected and oriented, with $m \geq 3$ and $b = \dim H_m(N;\mathbb{Q}) > 0$. There is a compact connected codimension-zero submanifold $P \subset N$ such that:*

(1) P is a smooth regular neighborhood of a finite geometric spine with a compatible smooth triangulation and dimension at most m, and has a handle decomposition with indices at most m;

(2) $P \simeq K = \bigvee_{i=1}^{b} B_i$, where the B_i are embedded closed connected simply connected oriented m-manifolds; the equivalence $K \to P$ restricts on each B_i to a map homotopic to its inclusion, with paths joining the chosen basepoints; when $m = 3$, the B_i are 3-spheres with trivial normal bundle;

(3) $H_m(P;\mathbb{Z}) = \mathbb{Z}^b$, $H_{m-1}(P;\mathbb{Z}) = 0$, and inclusion induces $H_m(P;\mathbb{Q}) \cong H_m(N;\mathbb{Q})$;

(4) the middle intersection matrix Λ in the basis $[B_i]$ is nonsingular and its diagonal entries are divisible by four;

(5) ∂P is connected and $H^{m-1}(\partial P;\mathbb{R}) = H^m(\partial P;\mathbb{R}) = 0$.

Proof. Choose integral classes whose images form a rational basis of $H_m(N;\mathbb{Q})$. For $m \geq 4$, apply Lemma 6.3; for $m = 3$, apply Lemma 6.4. In both cases their nonzero even multiples have embedded simply connected representatives B_i. Put the representatives in transverse position with pairwise point intersections and no triple intersections. Join the connected components of their union by disjoint arcs. The union K_0 of these pieces and arcs is a finite geometric polyhedron of dimension m in a compatible smooth triangulation. At each intersection point the two manifold pieces are transverse coordinate planes. The arcs may be chosen so that, near each endpoint, an arc is a coordinate half-axis normal to the coordinate m-plane of its manifold piece. These local models admit the simplex-thickening construction of a smooth regular neighborhood P_0 [27, Theorem 1(a)], with handles of index at most m [68, Part IV, Chapter 7, Proposition 7.2, p. 30]. Let Γ be the graph with one vertex for each B_i and one edge for each intersection point or joining arc, and put $s = \dim H_1(\Gamma;\mathbb{Q})$. Paths from the attachment points to basepoints in the B_i, followed by the contraction of a spanning tree of Γ, give

$$P_0 \simeq \bigvee_i B_i \vee \bigvee_{a=1}^{s} S^1.$$

Put $E_0 = N \setminus \operatorname{int} P_0$. General position makes $N \setminus K_0$ connected, since $1 + m < 2m$. A loop in this complement bounds a disk in N, and relative general position moves that disk off K_0, since $2 + m < 2m$. The mapping-cylinder description of P_0 identifies $P_0 \setminus K_0$ with $\partial P_0 \times (0,1]$. Hence $N \setminus K_0$ retracts onto E_0, which is simply connected. The absolute handles of P_0 have index at most m; hence the relative handles of $(P_0, \partial P_0)$ have index at least $m \geq 3$. Consequently ∂P_0 is connected and $\pi_1(\partial P_0) \to \pi_1(P_0)$ is an isomorphism. Choose disjoint embedded boundary circles representing a free basis of the graph factor. Their nullhomotopies in E_0 can be represented by disjoint properly embedded disks, since $4 < 2m$. The normal bundles of these disks are trivial; adjoining their normal neighborhoods attaches exterior two-handles. Attaching the two-cells along a free graph basis contracts the graph factor. The resulting equivalence $K = \bigvee_i B_i \to P$ restricts, up to homotopy and the chosen basepoint paths, to the original inclusion on every B_i. Lemma 6.5 gives its finite geometric spine and the required handle bound.

Each B_i has top homology $\mathbb{Z}$ and $H_{m-1}(B_i;\mathbb{Z}) = 0$, by simple connectivity and Poincaré duality. This proves the homology assertions. The represented classes are even multiples of a rational middle basis, so their intersection matrix is nonsingular and its diagonal entries are divisible by four.

Poincaré–Lefschetz duality identifies $H^m(P, \partial P;\mathbb{R}) \to H^m(P;\mathbb{R})$ with this nonsingular intersection map. The adjacent groups $H^{m-1}(P;\mathbb{R})$ and $H^{m+1}(P, \partial P;\mathbb{R}) = H_{m-1}(P;\mathbb{R})$ vanish. The pair sequence therefore gives both asserted boundary vanishings. The dual handles for $(P, \partial P)$ have indices at least m; thus inclusion of the boundary induces an isomorphism on components, and ∂P is connected. □

Fix this equivalence $K \to P$ and homotopies of its restrictions to the inclusions of the B_i. Pull back the stable tangent bundle of P to a stable oriented rank-$2m$ bundle η on K. Let ν_i be the oriented normal bundle of $B_i \subset P$. The chosen homotopies and the tangent–normal splitting give

$$\eta|_{B_i} \simeq_{\mathrm{st}} TB_i \oplus \nu_i. \tag{6.7}$$

Writing e for the Euler class, one has $\langle e(\nu_i), [B_i]\rangle = \Lambda_{ii}$.

Lemma 6.7. *For every i, there is an oriented rank-m bundle ξ_i on B_i, with $e(\xi_i) = 0$ and a specified stable isomorphism $\xi_i \simeq_{\mathrm{st}} \nu_i$. Its sphere bundle has a section.*

Proof. Suppose first that m is even. Let $[TS^m] \in \pi_{m-1}\,\mathrm{SO}(m)$ be the clutching class of the tangent bundle of the oriented sphere S^m. Trivialize ν_i over an oriented disk in B_i and change its clutching map on the boundary of that disk by

$$-\frac{\Lambda_{ii}}{2}[TS^m] \in \pi_{m-1}\,\mathrm{SO}(m).$$

The coefficient is integral. Let ξ_i be the resulting bundle. Since TS^m is stably trivial and has Euler number two, this changes the Euler number by $-\Lambda_{ii}$ and leaves the stable class unchanged. Fix a nullhomotopy of the stabilized clutching change; it gives a specified stable isomorphism $\xi_i \simeq_{\mathrm{st}} \nu_i$, equal to the original identification outside the disk. Thus ξ_i has zero Euler class, since $H^m(B_i;\mathbb{Z}) = \mathbb{Z}$. If m is odd, the intersection pairing is alternating and $\Lambda_{ii} = 0$; take $\xi_i = \nu_i$ with the identity stable isomorphism. The same top-cohomology calculation gives $e(\xi_i) = 0$.

Choose a bundle metric on ξ_i and denote its unit sphere bundle by $S(\xi_i)$. The primary obstruction to a section is its Euler class [47, Theorem 12.5, p. 147]. The fiber S^{m-1} is $(m-2)$-connected and the base B_i has dimension m, so there are no higher obstructions. Hence $S(\xi_i)$ has a section, which may be chosen smooth. □

6.3. **A circle source with prescribed stable tangent bundle.** For $r \geq 1$, put

$$d_{m,r} = \begin{cases} r(r+1)/2, & m \text{ odd}, \\ r(r-1)/2, & m \text{ even}. \end{cases}$$

Proposition 6.8. *Let $m \geq 3$ and let $K = \bigvee_{i=1}^{b} B_i$, where the B_i are closed connected simply connected oriented m-manifolds. Let η be an oriented stable rank-$2m$ bundle on K. Suppose that oriented rank-m bundles ξ_i have zero Euler class and specified stable isomorphisms*

$$TB_i \oplus \xi_i \simeq_{\mathrm{st}} \eta|_{B_i}.$$

For every $r \geq \max\{1, b\}$, there is a closed connected oriented $(2m-1)$-manifold Y and a map $q : Y \to K$ such that:

(1) $TY \oplus \mathbb{R} \simeq_{\mathrm{st}} q^*\eta$;

(2) $\pi_1(Y) = F_{d_{m,r}}$,

$$H_j(Y;\mathbb{Z}) = \bigoplus_i H_j(B_i;\mathbb{Z}) \quad (2 \leq j \leq m-2), \qquad H_{m-1}(Y;\mathbb{Z}) = H_m(Y;\mathbb{Z}) = \mathbb{Z}^r;$$

(3) there are integral bases $a_i \in H^{m-1}(Y;\mathbb{Z})$, $b_i \in H^m(Y;\mathbb{Z})$, with $\int_Y a_i b_j = \delta_{ij}$, for which the map

$$H^1(Y;\mathbb{R}) \longrightarrow \begin{cases} \mathrm{Sym}(r), & m \text{ odd}, \\ \mathrm{Skew}(r), & m \text{ even} \end{cases}$$

is an isomorphism, sending $[\theta]$ to the matrix $\left(\int_Y \theta a_i a_j\right)_{ij}$ for odd m and $\left(\int_Y a_i \theta a_j\right)_{ij}$ for even m;

(4) q_* is onto on π_j, for $1 \leq j \leq m-1$, and on $H_m(-;\mathbb{Z})$.

For $X = S^1 \times Y$ and $f = q \circ \mathrm{pr}_Y$, the induced stable isomorphism $TX \simeq_{\mathrm{st}} f^*\eta$ can be chosen to bound over K, and

$$H_m(X;\mathbb{Z}) = \mathbb{Z}^{2r}, \qquad H_m(X;\mathbb{Z}) \longrightarrow H_m(K;\mathbb{Z}) = \mathbb{Z}^b \quad \text{is onto.}$$

Proof. Put $p = m-1$. Fix an oriented stable trivialization of η at the wedge point. For $i \le b$, choose a bundle metric on ξ_i and put $E_i = S(\xi_i)$, with projection $\pi_i : E_i \to B_i$. Since $\dim B_i = m$ and $e(\xi_i) = 0$, there is a section $s_i : B_i \to E_i$ [47, Theorem 12.5]. For $b < i \le r$, put $E_i = S^p \times S^m$, with a stable framing and constant map to K. For $i \le b$, the tangent identity is

$$TE_i \oplus \mathbb{R} \cong \pi_i^*(TB_i \oplus \xi_i) \simeq_{\mathrm{st}} \pi_i^*(\eta|_{B_i}).$$

All the E_i are simply connected, since $m \ge 3$.

The integral Gysin sequence and Poincaré duality [23, §4.D, pp. 438, 444; Theorem 3.30], together with $H_{m-1}(B_i;\mathbb{Z}) = H_1(B_i;\mathbb{Z}) = 0$, give

$$H_j(E_i;\mathbb{Z}) = H_j(B_i;\mathbb{Z}) \quad (j \le m-2), \qquad H_p(E_i;\mathbb{Z}) = \mathbb{Z}[v_i], \qquad H_m(E_i;\mathbb{Z}) = \mathbb{Z}[t_i] \quad (i \le b).$$

Here $t_i = (s_i)_*[B_i]$, and v_i is the fibre class, oriented so that its intersection with t_i is 1. For $i > b$, take the factor classes v_i, t_i with the same intersection normalization. Integral Poincaré duality makes $H^p(E_i;\mathbb{Z})$ and $H^m(E_i;\mathbb{Z})$ free cyclic for every i. Let $c_i \in H^p(E_i;\mathbb{Z})$ be the evaluation-dual generator, with $c_i(v_i) = 1$ [23, Proposition 3.38].

Let Y_0 be the stably framed manifold of Lemma 6.1. Choose its spherical basis $u_i \in H_p(Y_0;\mathbb{Z})$, the intersection-dual basis $y_i \in H_m(Y_0;\mathbb{Z})$, and the evaluation-dual basis $a_i^0 \in H^p(Y_0;\mathbb{Z})$. The cup map remains an isomorphism for this basis, since a change of basis acts on its matrices by invertible congruence. Form

$$\widehat{Y} = Y_0 \# E_1 \# \cdots \# E_r.$$

In degrees $1, \dots, 2m-2$, regard the homology and cohomology classes of each summand as classes on $\widehat{Y}$. Choose the gluing balls disjoint from the sections and the sphere representatives. Homotope each projection, relative to its section, to the wedge point on the gluing balls. The resulting maps, constant on Y_0, the product summands and the necks, define $\widehat{q} : \widehat{Y} \to K$. Transport the specified stable isomorphisms along these homotopies. On the connected-sum bordism their trivializations on the attaching balls extend across the 1-handles: the oriented stable frame group is connected. Thus

$$T\widehat{Y} \oplus \mathbb{R} \simeq_{\mathrm{st}} \widehat{q}^*\eta.$$

Join the basepoints of the sections to a point in the Y_0 summand by paths. Their images in K are nullhomotopic loops, since K is simply connected. The sections consequently give a map $s : K \to \widehat{Y}$ with $\widehat{q} \circ s \simeq \mathrm{id}_K$.

Put

$$L = \langle v_i - u_i \,:\, 1 \le i \le r \rangle \subset H_p(\widehat{Y};\mathbb{Z}).$$

This subgroup is primitive of rank r: its quotient has basis the images of the u_i, with $v_i = u_i$. Choose the differences of the corresponding sphere representatives in $\pi_p(\widehat{Y})$. Their images under $\widehat{q}_*$ vanish; choose nullhomotopies. Choose an oriented bundle ν on K and a stable trivialization of $\eta \oplus \nu$ [25, Proposition 1.4]. The specified tangential isomorphism gives a stable trivialization of $T\widehat{Y} \oplus \widehat{q}^*\nu$. Wall's theorem [69, Theorem 1.1 and its corollary], applied to these relative homotopy classes and $2p < 2p+1$, gives framed embedded representatives for which the map and the specified stable structure extend over surgery. They can be chosen disjoint by general position [28, Chapter 2, §2, Theorem 2.13]. Let T be the resulting trace, Y its outgoing boundary, $F : T \to K$ the extended map, and $q = F|_Y$. Then

$$TT \simeq_{\mathrm{st}} F^*\eta, \qquad TY \oplus \mathbb{R} \simeq_{\mathrm{st}} q^*\eta.$$

Both relative handle decompositions of T have index m. The integral handle calculation in the proof of Lemma 6.1 applies to L, since both middle homology groups of $\widehat{Y}$ are free and L is primitive.

Thus homology below degree p is unchanged, and $\pi_1(Y) = \pi_1(\widehat{Y}) = \pi_1(Y_0) = F_{d_{m,r}}$. Let $L^\perp \subset H_m(\widehat{Y};\mathbb{Z})$ denote the annihilator of L under the integral intersection pairing. Then

$$H_p(Y;\mathbb{Z}) = H_p(\widehat{Y};\mathbb{Z})/L, \qquad H_m(Y;\mathbb{Z}) = L^\perp.$$

The first group has basis $[u_i] = [v_i]$. Since $u_i \cdot y_j = v_i \cdot t_j = \delta_{ij}$ and the cross-summand pairings vanish, the second has basis $y_i + t_i$. The class $y_i + t_i$ maps to $[B_i]$ for $i \le b$ and to zero for $i > b$. Hence $q_* : H_m(Y;\mathbb{Z}) \to H_m(K;\mathbb{Z})$ is onto.

The subgroup of $H^p(\widehat{Y};\mathbb{Z})$ extending over T has basis $a_i^0 + c_i$; let a_i be the restrictions of their extensions to Y. They evaluate as δ_{ij} on the basis $[u_j]$. By integral Poincaré duality, $H^p(Y;\mathbb{Z})$ and $H^m(Y;\mathbb{Z})$ are free of rank r. Thus the a_i form an integral basis, and the perfect integral cup pairing supplies a basis b_i with $\int_Y a_i b_j = \delta_{ij}$.

Restriction from T identifies $H^1(Y;\mathbb{R})$ with $H^1(\widehat{Y};\mathbb{R}) = H^1(Y_0;\mathbb{R})$. Products of classes extending over T have equal evaluations on its two oriented ends. All mixed positive-degree products between connected-sum summands vanish. Hence, under this identification,

$$\int_Y \theta a_i a_j = \int_{Y_0} \theta a_i^0 a_j^0 \quad (m \text{ odd}), \qquad \int_Y a_i \theta a_j = \int_{Y_0} a_i^0 \theta a_j^0 \quad (m \text{ even}).$$

Lemma 6.1 gives the cup-map isomorphisms.

By relative cellular approximation [23, Corollary 4.12], each boundary inclusion into T is an isomorphism on π_j for $1 \le j \le m-2$ and is onto on π_{m-1}. Since $\widehat{q}$ has a homotopy section, F_* is onto in these degrees. Every such class in $\pi_j(T)$ lifts to Y, so $q_* : \pi_j(Y) \to \pi_j(K)$ is onto for $1 \le j \le m-1$. The integral Künneth theorem [23, Theorem 3B.6] gives

$$H_m(S^1 \times Y;\mathbb{Z}) = H_m(Y;\mathbb{Z}) \oplus \big(H_1(S^1;\mathbb{Z}) \otimes H_{m-1}(Y;\mathbb{Z})\big) = \mathbb{Z}^{2r}.$$

The first summand maps onto $H_m(K;\mathbb{Z})$. Define $\bar{f} : D^2 \times Y \to K$ by $\bar{f} = q \circ \mathrm{pr}_Y$. Then

$$T(D^2 \times Y) \simeq_{\mathrm{st}} \bar{f}^*\eta \oplus \mathbb{R}.$$

Give $S^1 \times Y$ the induced boundary stable structure. It is then nullbordant over K. □

6.4. **Surgery over the tangential space and comparison with the double.**

Lemma 6.9. *Let K be a simply connected finite CW complex of dimension at most m, where $m \ge 3$, and let η be an oriented stable rank-$2m$ bundle on K. Suppose that*

$$H_{m-1}(K;\mathbb{Z}) = 0, \qquad H_m(K;\mathbb{Z}) \text{ is free.}$$

Let X^{2m} be closed, connected and oriented, with a map $f : X \to K$ and a specified stable identification $TX \simeq_{\mathrm{st}} f^*\eta$. Suppose that $H_m(X;\mathbb{Z})$ is free, that $H_m(f)$ is onto, and that $\pi_j(f)$ is onto for $2 \le j \le m-1$.

A finite sequence of framed surgeries over K gives a simply connected manifold X' and an m-connected map $f' : X' \to K$. Every surgery preserves integral middle homology, its intersection form, and its map to $H_m(K;\mathbb{Z})$. The surgeries have dimensions at most $m-2$, except for final surgeries on $(m-1)$-spheres representing primitive nonzero integral homology classes. The specified stable structure extends over all their traces.

Proof. Choose an oriented bundle ν on K and a stable trivialization of $\eta \oplus \nu$ [25, Proposition 1.4, p. 13]. The specified tangential identification gives a stable trivialization of $TX \oplus f^*\nu$. Wall's theorem [69, Theorem 1.1 and its corollary, pp. 10–11] then gives a compatible framed surgery for every element of $\ker \pi_j(f)$ with a chosen nullhomotopy in K, for $1 \le j \le m-1$: one has $j \le 2m-2$ and $2j < 2m$. The map to K and the specified stable structure extend over the trace. Finite families of the same dimension have disjoint representatives by general position [28, Chapter 2, §2, Theorem 2.13].

First perform such surgeries on finitely many loops normally generating $\pi_1(X)$. Their images in K are nullhomotopic. The trace has incoming handles of index two and outgoing handles of index $2m-1$; hence the outgoing manifold is simply connected [23, Proposition 1.26 and Corollary 4.12].

For $j = 2, \dots, m-2$, in increasing order, perform surgery on a finite generating set of the kernel in degree j. The groups in question are finitely generated: the current manifold and K are simply connected and have finite CW type, so Serre's theorem applies [24, Theorem 1.7, p. 14]. The incoming $(j+1)$-cells quotient π_j by the attaching subgroup and preserve the smaller homotopy groups [23, Lemma 4.38].

For every surgery in these steps, the outgoing relative handle index is $2m - j \geq m + 2$. Thus the outgoing inclusion into the trace induces isomorphisms on homotopy through degree $m-1$ [23, Corollary 4.12]. Every class in $\pi_i(K)$, $2 \leq i \leq m-1$, that lifts to the incoming boundary therefore lifts to the outgoing boundary. The incoming relative indices are at most $m-1$, so both boundary inclusions induce isomorphisms on integral H_m. Poincaré–Lefschetz duality [23, Theorem 3.43] identifies the intersection pairings, and the isomorphisms commute with the maps to $H_m(K;\mathbb{Z})$. Denote the resulting manifold and map by M and $g : M \to K$. Then g is $(m-1)$-connected and $H_m(g)$ is onto. For $m = 3$, only the circle surgeries occur in these steps.

Let C_g be the mapping cylinder of g, with M its domain end. The pair (C_g, M) is $(m-1)$-connected and both spaces are simply connected. Relative Hurewicz [23, Theorem 4.37] gives

$$\pi_m(C_g, M) \cong H_m(C_g, M; \mathbb{Z}).$$

Since $H_m(g)$ is onto and $H_{m-1}(K;\mathbb{Z}) = 0$, the homology exact sequence gives

$$H_m(C_g, M; \mathbb{Z}) \xrightarrow{\cong} H_{m-1}(M; \mathbb{Z}).$$

The homomorphism $H_m(K;\mathbb{Z}) \to H_m(C_g, M;\mathbb{Z})$ is zero. Naturality of Hurewicz implies that $\pi_m(K) \to \pi_m(C_g, M)$ is zero. The homotopy exact sequence therefore gives

$$\ker\{\pi_{m-1}(M) \to \pi_{m-1}(K)\} \cong \pi_m(C_g, M) \cong H_{m-1}(M;\mathbb{Z}). \tag{6.8}$$

The resulting map from the kernel to homology is ordinary Hurewicz.

The group $H_m(M;\mathbb{Z})$ is free. The universal coefficient theorem and Poincaré duality [23, Theorems 3.2 and 3.30] give

$$H_{m-1}(M;\mathbb{Z}) \cong H^{m+1}(M;\mathbb{Z}) \cong \mathrm{Hom}(H_{m+1}(M;\mathbb{Z}), \mathbb{Z}).$$

It is therefore free of finite rank, say s. Choose elements of the kernel in (6.8) whose Hurewicz images $x_1, \dots, x_s$ form an integral basis, and choose their nullhomotopies in K. Wall's theorem gives disjoint compatibly framed representatives. Let M_k result from surgery on the first k representatives, with $M_0 = M$ and $M_s = X'$. The incoming and outgoing relative handle indices are m and $m+1$. Their homology exact sequences give

$$H_{m-1}(M_k;\mathbb{Z}) \cong H_{m-1}(M;\mathbb{Z})/\langle x_1, \dots, x_k \rangle.$$

Each remaining attaching class is consequently primitive and nonzero over $\mathbb{R}$. The integral calculation in the proof of Corollary 4.9 preserves H_m and its intersection pairing at each surgery.

Let T be the trace of these last surgeries and $G : T \to K$ the extended map. In the exact sequence

$$0 \longrightarrow H_m(M;\mathbb{Z}) \longrightarrow H_m(T;\mathbb{Z}) \longrightarrow \mathbb{Z}^s \xrightarrow{\partial} H_{m-1}(M;\mathbb{Z}),$$

the map ∂ sends the handle generators to $x_1, \dots, x_s$ and is an isomorphism. Hence $H_m(M;\mathbb{Z}) \to H_m(T;\mathbb{Z})$ is an isomorphism. The outgoing inclusion is onto on H_m, since $H_m(T, X';\mathbb{Z}) = 0$, and is an isomorphism between free groups of the same rank. These identifications commute with $H_m(G)$; in particular, $H_m(f')$ is onto for $f' = G|_{X'}$.

The incoming m-cells quotient $\pi_{m-1}(M)$ by the kernel in (6.8), and the outgoing inclusion is an isomorphism on homotopy below degree m [23, Corollary 4.12 and Lemma 4.38]. Thus $\pi_j(f')$ is an isomorphism for $j < m$. Since $H_{m-1}(X';\mathbb{Z}) = 0$ and $H_m(f')$ is onto, the homology exact sequence gives $H_m(C_{f'}, X';\mathbb{Z}) = 0$, where $C_{f'}$ is the mapping cylinder of f'. Relative Hurewicz gives $\pi_m(C_{f'}, X') = 0$. Hence f' is m-connected. □

Proposition 6.10. *Let $m > 2$, let K be simply connected, and let η be an oriented stable bundle on K. Suppose closed simply connected oriented $2m$-manifolds X_0, X_1 admit m-connected maps $f_i : X_i \to K$ and specified stable identifications $TX_i \simeq_{\mathrm{st}} f_i^*\eta$. Suppose their resulting normal K-structures are bordant and $\chi(X_0) = \chi(X_1)$. If m is odd, or if m is even and $X_1 \cong X_1' \# (S^m \times S^m)$, then X_0 and X_1 are orientation-preservingly diffeomorphic.*

Proof. Let $F : W \to K$ be the given normal bordism. Relative CW approximation [23, Proposition 4.17, with $n = 0$], applied to the mapping-cylinder pair of F, gives a CW complex K_c containing W and a weak homotopy equivalence $u : K_c \to K$ with $u|_W = F$. Pulling back η along u preserves the specified normal structures, and the boundary maps into K_c remain m-connected. Thus we may take K to be a CW complex.

Let BSO and BO denote the classifying spaces for stable oriented real bundles and stable real bundles, respectively. Apply the Moore–Postnikov factorization [23, Theorem 4.71, p. 414] to the classifying map of $-\eta$:

$$K \xrightarrow{a} \mathcal{B} \xrightarrow{p} \mathrm{BSO}.$$

At degree m, the map a is m-connected, while p induces isomorphisms on π_j for $j > m$ and an injection on π_m. In particular, $\mathcal{B}$ is simply connected. Let $\bar{p}$ be the composite with BSO $\to$ BO. Since this latter map induces isomorphisms on homotopy in degrees at least two, the relative homotopy groups of $\bar{p}$ vanish in degrees at least $m + 1$. After fibration replacement, $\bar{p}$ is a common normal $(m-1)$-type and the maps af_i are normal $(m-1)$-smoothings [40, pp. 708 and 711–712]. The given normal bordism induces a bordism of these smoothings.

Kreck's Theorem D [40, p. 709] now applies: the manifolds are closed and simply connected, have equal Euler characteristic and bordant normal $(m-1)$-smoothings, and $m > 2$. For even m, the required $S^m \times S^m$ summand is present. The theorem gives a diffeomorphism. Its proof passes through an s-cobordism with the specified normal structures [40, Theorem 4 and Corollary 4, pp. 735 and 741]. Since these structures factor through BSO, the diffeomorphism preserves the given orientations. □

For a compact oriented P with boundary, put $D(P) = \partial(P \times [0,1])$, with corners smoothed. It consists of P and $-P$ joined along their boundary.

Proposition 6.11. *Let P, K, η be obtained from Proposition 6.6, and let $m \geq 3$. Apply Lemma 6.7 and Proposition 6.8 with $r = b + 1$ to form $f : X = S^1 \times Y \to K$. Apply Lemma 6.9, obtaining an m-connected map $f' : X' \to K$. Then there is an orientation-preserving diffeomorphism*

$$X' \cong D(P) \# (S^m \times S^m), \qquad D(P) = \partial(P \times [0,1]).$$

Proof. The source map is onto on π_j for $2 \leq j \leq m-1$ and on integral H_m, and $H_m(X;\mathbb{Z}) = \mathbb{Z}^{2b+2}$. The wedge K is a simply connected finite CW complex of dimension m, with $H_{m-1}(K;\mathbb{Z}) = 0$ and $H_m(K;\mathbb{Z}) = \mathbb{Z}^b$. Thus Lemma 6.9 applies and gives

$$H_m(X';\mathbb{Z}) = \mathbb{Z}^{2b+2}.$$

The specified normal K-structure on X' bounds, by the filling $D^2 \times Y$ of Proposition 6.8 and the surgery traces.

Choose a homotopy inverse $\rho : P \to K$ to the fixed equivalence $K \to P$, together with a stable identification $TP \simeq_{\mathrm{st}} \rho^*\eta$ induced by the inverse homotopy. Put $V = P \times [0,1]$ and $\bar{h} = \rho \circ \mathrm{pr}_P : V \to K$. The absolute handles of V have indices at most m, and their duals relative to $\partial V = D(P)$ have indices at least $m + 1$ [28, Chapter 6, §3]. Hence $D(P) \hookrightarrow V$ is m-connected [23, Corollary 4.12], and so is $h = \bar{h}|_{D(P)}$. The identity

$$TV \simeq_{\mathrm{st}} \bar{h}^*\eta \oplus \mathbb{R}$$

induces the stable tangential identification on $D(P)$.

Let $Q = S^m \times S^m$ and $Z = D(P) \# Q$. Map Q constantly to the wedge point of K, with the stable structure induced by a stable framing of $S^m \times D^{m+1}$. Homotope the two boundary maps to be constant

on the gluing balls. Their stable structures extend over the connected-sum bordism, after choosing compatible oriented trivializations on these balls. This gives a normal K-structure on Z which bounds, using V, $S^m \times D^{m+1}$ and the connected-sum bordism. Its map $h_Z : Z \to K$ is homotopic to $h \circ c$, where $c : Z \to D(P)$ is the connected-sum collapse. The manifolds Z and $D(P)$ are simply connected. Mayer–Vietoris gives that $H_j(c;\mathbb{Z})$ is an isomorphism for $j < m$ and a surjection for $j = m$, since Q is $(m-1)$-connected. Relative Hurewicz therefore makes c, and hence h_Z, m-connected.

Put $S = \sum_{j<m}(-1)^j b_j(K)$. The m-connected map f' induces homology isomorphisms below degree m. Poincaré duality gives

$$\chi(X') = 2S + (-1)^m(2b + 2).$$

Since ∂P is closed and odd-dimensional, $\chi(\partial P) = 0$. Consequently

$$\chi(D(P)) = 2\chi(P) = 2S + 2(-1)^m b, \qquad \chi(Z) = \chi(D(P)) + 2(-1)^m = \chi(X').$$

The manifolds X' and Z have bordant normal K-structures and m-connected maps to K. For even m, Z has the required sphere-product summand. Proposition 6.10 gives the stated orientation-preserving diffeomorphism. □

6.5. **The global middle section.**

Proof of Theorem A for $m \geq 3$. Fix a positive smooth volume form ν on M. If $H^m(M;\mathbb{R}) = 0$, its period domain is a point. For any metric g_0, the constant map with value $N_\nu(g_0)$ is the required section by Lemma 2.5. Suppose that $b_m(M) > 0$. Choose finitely many disjoint embedded circles normally generating $\pi_1(M)$. Their oriented normal bundles are trivial [25, Proposition 1.14 and p. 25]. Fix framings and perform surgery on these circles. The traces have incoming handle index two and outgoing handle index $2m - 1 > 2$, so the resulting manifold N is simply connected [23, Proposition 1.26]. Since $1 \leq m - 2$, Corollary 4.9 preserves the middle pairing and transfers sections in both directions. Put $b = b_m(N)$ and choose $P \subset N$ as in Proposition 6.6, with its fixed equivalence $K \to P$ and stable bundle η. Apply Proposition 6.8 with $r = b + 1$ and the bundles of Lemma 6.7. Denote its map by $q : Y \to K$, and put

$$X = S^1 \times Y, \qquad f = q \circ \mathrm{pr}_Y : X \to K.$$

Choose the dual bases a_i, b_i, $1 \leq i \leq r$, of Proposition 6.8. Theorem 3.1, applied in degree $m - 1$ on Y^{2m-1}, gives a smooth family h_G, $G \in \mathrm{Sym}^+(r)$, with $\Gamma_{m-1}(h_G) = G$. The required inequality is $2(m-1) < 2m - 1$. For odd m, the cup isomorphism and fixed closed representatives of a basis of $H^1(Y;\mathbb{R})$ give a linear map $C \mapsto \theta_C$ from $\mathrm{Sym}(r)$ to closed one-forms, with

$$\int_Y \theta_C a_i a_j = C_{ij}.$$

The metrics $(dt + \theta_C)^2 + h_G$ have Siegel period $C + iG$ by Lemma 6.2. For even m, choose in the same way a linear map $B \mapsto \theta_B$ from $\mathrm{Skew}(r)$ to closed one-forms, with $\int_Y a_i \theta_B a_j = B_{ij}$. The same lemma gives every maximal positive subspace as the period of $(dt + \theta_B)^2 + h_G$. These families are smooth in the global coordinates (2.2) and (6.5). By (2.1), X has a global smooth middle Gram section.

Apply Lemma 6.9 to f with the specified stable structure of Proposition 6.8, and let X' be the outgoing manifold. Its surgeries have dimensions $1 \leq j \leq m - 2$, followed by surgeries on $(m-1)$-spheres representing primitive integral homology classes with nonzero images in real homology. Corollary 4.9 transfers the section to X'. Proposition 6.11 gives an orientation-preserving diffeomorphism $X' \cong D(P)\#(S^m \times S^m)$. The connected-sum conclusion of Corollary 4.9 then gives a global section on $D(P)$.

Let $E = N \setminus \mathrm{int}\, P$. The two boundary vanishings in Proposition 6.6 and Mayer–Vietoris [23, §3.1, pp. 203–204], applied to collar enlargements of P and E, give

$$H^m(N;\mathbb{R}) \cong H^m(P;\mathbb{R}) \oplus H^m(E;\mathbb{R}).$$

Since $\dim H^m(P;\mathbb{R}) = b_m(N)$, one has $H^m(E;\mathbb{R}) = 0$. Put $W = H^m(P,\partial P;\mathbb{R})$. The exact sequence of the pair gives an isomorphism $c : W \to H^m(P;\mathbb{R})$. For $Z = D(P)$ or N, let

$$j_Z : W \longrightarrow H^m(Z;\mathbb{R}), \qquad r_Z : H^m(Z;\mathbb{R}) \longrightarrow H^m(P;\mathbb{R})$$

be extension by zero from P and restriction to P, respectively. Then $r_Z j_Z = c$. Thus $j_{D(P)}$ is injective; since r_N is an isomorphism, so is j_N. For $u, v \in W$, choose closed representatives α, β compactly supported in $\operatorname{int} P$. The orientations give

$$Q_{D(P)}(j_{D(P)}u, j_{D(P)}v) = \int_P \alpha \wedge \beta = Q_N(j_N u, j_N v).$$

Since Q_N is nondegenerate, the subspace $V = j_{D(P)}(W)$ is nondegenerate and

$$\iota = j_N \circ j_{D(P)}^{-1} : (V, Q_{D(P)}|_V) \longrightarrow (H^m(N;\mathbb{R}), Q_N)$$

is an isometric isomorphism. These representatives vanish near the second half $-P$ of the double. Injectivity of $j_{D(P)}$ shows that their zero extension to N is independent of the representative and induces ι.

The manifold $-P$ is a smooth regular neighborhood of a finite geometric spine of dimension at most m. Theorem 4.8 therefore applies with source $D(P) = P \cup (-P)$, target $N = P \cup E$, retained subspace V and isometry ι. It gives a global smooth section on N. The reverse direction of Corollary 4.9, applied to the initial circle surgeries, gives a section on M. For each fixed oriented diffeomorphism in this construction, transport the target Gram form by its induced cohomology isometry and pull back the chosen metric. The resulting section is therefore expressed in the original marking of M.

Denote this section by $s : \mathcal{D}(Q_M) \to \mathrm{Met}^\infty(M)$, and put $s_\nu = N_\nu \circ s$. Lemma 2.5 gives $\Gamma_m \circ s_\nu = \mathrm{id}$, the prescribed volume form and regularity, including smooth dependence on finite-dimensional smooth families of volume forms. □

7. Near-symplectic forms and positive surfaces

7.1. **Transversality with fixed period.** Assume $b^+(X) = 1$, let $a \in \mathcal{P}_X$, and put

$$\mathcal{F}_a = p^{-1}(u_a), \qquad \omega_g = h_g^2(a) \quad (g \in \mathcal{F}_a).$$

A closed two-form ω is *near-symplectic* if $\omega \wedge \omega > 0$ off its zero set and, at each zero x, the derivative $D\omega_x : T_xX \longrightarrow \Lambda^2 T_x^* X$ has rank three and positive definite image for the wedge pairing. Here $D\omega_x = (\nabla\omega)_x$ for any connection; at a zero this derivative is independent of the connection. For a self-dual harmonic form, near-symplecticity is equivalent to transversality to the zero section of Λ_g^+: indeed, $\omega \wedge \omega = |\omega|_g^2 dV_g$ and $\operatorname{im} D\omega_x \subset \Lambda^+_{g,x}$ at every zero. Honda's transversality theorems for self-dual harmonic two-forms on four-manifolds allow the cohomology class to vary [30, Author's version, Theorem 1.1, pp. 1–2, and Theorem 2.24, p. 16].

Let $\mathcal{K} \subset \mathcal{F}_a$ be compact, and fix a metric k. The eigenvalues of g relative to k have uniform positive lower and upper bounds on $\mathcal{K}$, and $\|g\|_{C^8(X,k)}$ is uniformly bounded there. Apply Lemma 2.6 with $U_0 = U_1 = X$, $\ell = 8$, constants for which its bounds are strict on $\mathcal{K}$, and a closed period ball centered at u_a. It gives, on a neighborhood $\mathcal{U}$ of $\mathcal{K}$, a map

$$\mathcal{R}_a(g) = \mathcal{R}(g, u_a) \in \mathcal{F}_a$$

which is smooth on finite-dimensional families and fixes $\mathcal{U} \cap \mathcal{F}_a$. For $g \in \mathcal{U} \cap \mathcal{F}_a$ and a smooth symmetric tensor v, put

$$L_g v = \left.\frac{d}{ds}\right|_{s=0} \mathcal{R}_a(g + sv).$$

Differentiating $p \circ \mathcal{R}_a = u_a$ and $\mathcal{R}(g, p(g)) = g$ gives

$$Dp_g \circ L_g = 0, \qquad L_g v = v \quad (v \in \ker Dp_g).$$

For smooth families $g_t \in \mathcal{U} \cap \mathcal{F}_a$ and symmetric tensors v_t, the family $L_{g_t} v_t$ is smooth: it is the derivative at $s = 0$ of the smooth family $\mathcal{R}_a(g_t + sv_t)$.

Lemma 7.1. *Let $g \in \mathcal{F}_a$ and $x \in X$ satisfy $\omega_g(x) = 0$. The values $\frac{d}{dt}\big|_{t=0} \omega_{g_t}(x)$, for smooth curves $g_t \in \mathcal{F}_a$ with $g_0 = g$, span $\Lambda^+_{g,x}$.*

Proof. All forms and distributions in this proof are real, and all operators refer to g. Write $*=*_g$ and I for the identity operator, let d^* be the $L^2(g)$ adjoint of d, and put $\Delta = dd^* + d^*d$ and $P^\pm = (I \pm *)/2$. Let $\mathsf{H}^\pm$ be the $L^2(g)$ projections onto $\mathcal{H}^\pm_g$, and let G be the Green operator in each degree, zero on harmonic forms. By [70, Definition 6.9 and Proposition 6.10, p. 225], G is self-adjoint, commutes with d, d^* and $*$, and satisfies on two-forms

$$\Delta G = G\Delta = I - \mathsf{H}^+ - \mathsf{H}^-.$$

Since $* \, dd^* \, *= d^*d$ on two-forms,

$$2P^+ dd^* G\beta = \beta - \mathsf{H}^+\beta, \qquad 2P^- dd^* G\alpha = \alpha - \mathsf{H}^-\alpha \tag{7.1}$$

for self-dual β and anti-self-dual α, respectively. Distributional pairings are induced by the $L^2(g)$ pairing. The operators and these identities extend to distributions by duality.

Put $\omega = \omega_g$. Since $a \neq 0$, choose a nonempty open set $W \Subset X \setminus \{x\}$ on which ω is nowhere zero. Write $\Omega^-_c(W)$ for the smooth anti-self-dual two-forms compactly supported in W. For $\alpha \in \Omega^-_c(W)$, the bundle isomorphism (2.11) gives a smooth trace-free tensor v_α supported in W such that

$$\tfrac{1}{2} D_g * [v_\alpha]\omega = \alpha. \tag{7.2}$$

Indeed, take the homomorphism on $\Lambda^+_g|_W$ to be $\zeta \mapsto \langle \zeta, \omega\rangle_g \alpha/|\omega|^2_g$ and extend it by zero. Since $a = \sqrt{Q_X(a,a)}\, u_a$, equation (2.10) gives

$$\sqrt{Q_X(a,a)}\, Dp_g(v_\alpha) = [\mathsf{H}^-\alpha].$$

Thus $Dp_g(v_\alpha) = 0$ if and only if $\mathsf{H}^-\alpha = 0$. When $\mathsf{H}^-\alpha = 0$, take $\mathcal{R}_a$ above with $\mathcal{K} = \{g\}$ and set $g_t = \mathcal{R}_a(g + tv_\alpha)$. Then $g_t \in \mathcal{F}_a$, $g_0 = g$ and $\dot g_0 = v_\alpha$. For $\dot\omega = \frac{d}{dt}\big|_0 \omega_{g_t}$, differentiation of $*_{g_t} \omega_{g_t} = \omega_{g_t}$ and $[\omega_{g_t}] = a$ gives $\dot\omega^- = \alpha$ and $[\dot\omega] = 0$. Hence

$$\dot\omega = 2dd^*G\alpha. \tag{7.3}$$

The right-hand side is exact and has anti-self-dual part α by (7.1). Their difference is therefore exact, closed and self-dual, hence harmonic and zero. For any smooth curve through g in $\mathcal{F}_a$, differentiation of self-duality at the zero x also gives $\dot\omega(x) \in \Lambda^+_{g,x}$.

Suppose $v \in \Lambda^+_{g,x}$ annihilates the values in (7.3) with $\mathsf{H}^-\alpha = 0$. Put $\beta = v\delta_x$, where $\langle \beta, \phi\rangle = \langle v, \phi(x)\rangle_g$ for every smooth test two-form ϕ, and set

$$T = 2P^- dd^* G\beta.$$

Since dd^*G is self-adjoint,

$$\langle \alpha, T\rangle = 0 \quad \text{for all } \alpha \in \Omega^-_c(W) \text{ with } \mathsf{H}^-\alpha = 0.$$

Thus $\alpha \mapsto \langle \alpha, T\rangle$ factors through the image of $\alpha \mapsto \mathsf{H}^-\alpha$. Extend the induced linear functional to the finite-dimensional space $\mathcal{H}^-_g$, and represent it by $\eta \in \mathcal{H}^-_g$ under the $L^2(g)$ inner product. Thus $\langle \alpha, T - \eta\rangle = 0$ for every $\alpha \in \Omega^-_c(W)$. Since $T - \eta$ is anti-self-dual, it vanishes on W.

The first identity in (7.1) gives

$$T = 2dd^*G\beta - \beta + \mathsf{H}^+\beta, \qquad dT = -d\beta.$$

On $X \setminus \{x\}$, the distribution T is closed and anti-self-dual, so $dT = d^*T = 0$ and $\Delta T = 0$. Elliptic regularity [15, Theorem 15.23, pp. 206–207], applied to the Hodge Laplacian on two-forms, makes T smooth there. The form $T - \eta$ is closed, coclosed and zero on W. Since $X \setminus \{x\}$ is connected, unique continuation [2, §2, Theorem 1, pp. 421–422] gives $T = \eta$ on $X \setminus \{x\}$.

Put $S = T - \eta$. Then $\operatorname{supp} S \subset \{x\}$, $* \, S = -S$, and $dS = -d(v\delta_x)$. Choose oriented g-normal coordinates centered at x. By [15, Theorem 4.19, p. 53], applied to each component, S is a finite sum

of derivatives of δ_x with real two-form coefficients. If derivatives of highest order $m \geq 1$ occur, write their sum as $\sum_{|I|=m} c_I \partial^I \delta_x$, where I is a multi-index, and put

$$A(\xi) = \sum_{|I|=m} c_I \xi^I \quad (\xi \in T^*_x X).$$

Multiplication by $*$ has highest-order coefficients $*_{g,x}\ c_I$; derivatives of its coefficients contribute only lower orders. Since $d(v\delta_x)$ has order at most one, the terms of orders m and $m+1$ in $*\,S = -S$ and $dS = -d(v\delta_x)$ give

$$*_{g,x}\ A(\xi) = -A(\xi), \qquad \xi \wedge A(\xi) = 0.$$

For $\xi \neq 0$, write $A(\xi) = \xi \wedge \theta$ with a one-form θ. Then

$$0 = A(\xi) \wedge A(\xi) = -|A(\xi)|^2_{g,x} dV_g(x),$$

so A is zero, contrary to the choice of m. Consequently $S = c\delta_x$ for some $c \in \Lambda^-_{g,x}$. Now $d((c+v)\delta_x) = 0$. Its first-order terms give $\xi \wedge (c+v) = 0$ for every $\xi \in T^*_x X$, hence $c + v = 0$. The two eigenspaces of $*_{g,x}$ intersect only at zero, so $v = 0$. □

Proposition 7.2. *Every metric in $\mathcal{F}_a$ can be joined, inside any prescribed C^∞ neighborhood in $\mathcal{F}_a$, to a metric whose harmonic representative of a is near-symplectic. More generally, a smooth path $g_t \in \mathcal{F}_a$, stationary near its endpoints and with near-symplectic endpoint forms, admits arbitrarily small C^∞ perturbations $\widetilde{g}_t \in \mathcal{F}_a$, fixed on smaller endpoint collars, such that the section $(x,t) \mapsto h^2_{\widetilde{g}_t}(a)(x)$ of the rank-three bundle with fibre $\Lambda^+_{\widetilde{g}_t,x}$ over $X \times [0,1]$ is transverse to its zero section.*

Proof. Let J be a point or $[0,1]$, and write g_t for the given family. Choose $\mathcal{R}_a$ and L_{g_t} as above for $\mathcal{K} = \{g_t : t \in J\}$. If $J = [0,1]$, choose $0 < \varepsilon < 1/4$ so that g_t is constant on $[0, 2\varepsilon]$ and $[1-2\varepsilon, 1]$, and put

$$C = [0,\varepsilon] \cup [1-\varepsilon, 1], \qquad K = [2\varepsilon, 1-2\varepsilon].$$

If J is a point, put $C = \emptyset$ and $K = J$.

At a zero of ω_{g_t} with $t \in K$, Lemma 7.1 gives three tensors in $\ker Dp_{g_t}$ whose harmonic variations span $\Lambda^+_{g_t,x}$. Extend such a tensor to a constant tensor family v near the chosen parameter, replace it by $L_{g_t} v$, and multiply by a parameter cutoff equal to one there and zero near C. This gives a smooth family in $\ker Dp_{g_t}$ with the prescribed value. The spanning condition is open, and the zero set over K is compact. Hence finitely many such families $v_1(t), \ldots, v_M(t)$ suffice. Outside K the original form is transverse in the X directions, since it equals an endpoint form. If the zero set over K is empty, take $M = 0$.

For $\lambda = (\lambda_1, \ldots, \lambda_M) \in \mathbb{R}^M$ sufficiently close to zero, uniformly in t, define

$$\widetilde{g}_{\lambda,t} = \mathcal{R}_a \left(g_t + \sum_{i=1}^{M} \lambda_i v_i(t) \right).$$

These are metrics in $\mathcal{F}_a$, smooth in (λ, t), with

$$\widetilde{g}_{0,t} = g_t, \qquad \partial_{\lambda_i} \widetilde{g}_{\lambda,t}\big|_{\lambda=0} = v_i(t).$$

On C, all v_i vanish, so $\widetilde{g}_{\lambda,t} = g_t$.

Let $E \to X \times J$ have fibre $E_{(x,t)} = \Lambda^+_{g_t,x}$. For small λ, the projection $(1 + *_{g_t})/2$ restricts to an isomorphism from $\Lambda^+_{\widetilde{g}_{\lambda,t},x}$ to $E_{(x,t)}$, uniformly on $X \times J$. Define the sections

$$s_\lambda(x,t) = \tfrac{1}{2}(1 + *_{g_t}) h^2_{\widetilde{g}_{\lambda,t}}(a)(x) \quad \text{of } E.$$

At every zero with $\lambda = 0$, the derivatives in λ and X span the fibre by the preceding construction. Compactness of $X \times J$ and openness of surjectivity imply that $(\lambda, x, t) \mapsto s_\lambda(x,t)$ is transverse to the zero section on a smaller parameter ball. Parametric transversality [28, Chapter 3, §2, Theorem 2.7, pp. 79–80], applied to the zero section in the total space of E over X if J is a point and over $X \times (0,1)$

otherwise, gives arbitrarily small λ for which s_λ is transverse. In the interval case the fixed endpoint collars are already transverse. The bundle isomorphisms preserve zeros and transversality, so the same conclusions hold for the harmonic forms.

The family $\widetilde{g}_{\lambda,t}$ converges to g_t in C^∞ as $\lambda \to 0$. If J is a point, the curve $s \mapsto \widetilde{g}_{s\lambda}$, $0 \le s \le 1$, lies in $\mathcal{F}_a$ and, for sufficiently small λ, in any prescribed neighborhood of the given metric. A smooth reparametrization makes it stationary near its endpoints. □

7.2. **Common integral classes.** Let ω be a closed two-form on X. An oriented embedded surface $S \subset X$ is *ω-positive* if $\omega|_{TS}$ is a positive area form. An *orthogonal positive configuration* for ω is a finite union of connected closed embedded ω-positive surfaces, disjoint from its zero set, with positive transverse pairwise intersections, no triple points, and ω-orthogonal tangent planes at each intersection. For two members S, S' meeting at x, orthogonality means $\omega_x(T_xS, T_xS') = 0$. An integral class and its image in real cohomology are denoted by the same symbol when the coefficient group is unambiguous.

For a symplectic form ω, the class $[\omega]$ is a positive linear combination of Poincaré duals of symplectic surfaces with positive transverse intersections [52, arXiv:1011.6358v1, Theorem 2].

Proposition 7.3. *Let X be a closed oriented four-manifold with $b^+(X) = 1$, and let ω_0, ω_1 be near-symplectic forms with $[\omega_0] = [\omega_1] = a$ and $a^2 > 0$. Put $n = b_2(X)$. For every neighborhood U of $\mathbb{R}_{>0}a$ in*

$$\{v \in H^2(X;\mathbb{R}) \,:\, v^2 > 0\}/\mathbb{R}_{>0},$$

there are classes $d_1, \dots, d_n \in H^2(X;\mathbb{Z})$ and positive real numbers $\lambda_1, \dots, \lambda_n$ such that

$$H^2(X;\mathbb{R}) = \bigoplus_{j=1}^{n} \mathbb{R}d_j, \qquad a = \sum_{j=1}^{n} \lambda_j d_j, \qquad d_j^2 > 0, \quad \mathbb{R}_{>0}d_j \in U.$$

For each $i = 0, 1$, there is an orthogonal positive configuration $D_{i1} \cup \cdots \cup D_{in}$ for ω_i satisfying

$$[D_{ij}] = \mathrm{PD}(d_j) \in H_2(X;\mathbb{Z}) \qquad (1 \le j \le n).$$

7.3. **A finite restriction lemma.**

Lemma 7.4. *Let Ω be a near-symplectic form with the standard local models of* [4, §1, p. 1046] *along $\Gamma = \Omega^{-1}(0)$. Let $N > 0$, and let L be a Hermitian line bundle with $c_1(L)_{\mathbb{R}} = N[\Omega]$, equipped with a unitary connection of curvature $-2\pi i N\Omega$ and holonomy -1 around each component of Γ. Let $C_1, \dots, C_r \subset X \setminus \Gamma$ be closed embedded Ω-positive surfaces with positive orthogonal crossings and no triple points. Put $P = \bigcup_{j<l}(C_j \cap C_l)$. Suppose that J is Ω-compatible on $X \setminus \Gamma$ and preserves every TC_j, and that a closed two-form ω agrees with Ω near Γ and tames J on $X \setminus \Gamma$.*

For every sufficiently large odd integer k, there is a smooth closed embedded surface $S \subset X \setminus \Gamma$ satisfying

$$[S] = \mathrm{PD}(kc_1(L)) \in H_2(X;\mathbb{Z}).$$

It is positive for both Ω and ω, meets every C_j transversely and positively, and is disjoint from P. If $b^+(X) = 1$, then S is connected.

Proof. Choose nested tubular neighborhoods $V_0 \Subset V$ of Γ, disjoint from the C_j, with $\omega = \Omega$ on V. Choose a relatively compact open set $W \subset X \setminus \overline{V}$ containing the C_j. Interpolation of metrics and the compatible polar construction [11, Proposition 3.4] give an Ω-compatible structure J^0 on $X \setminus \Gamma$, equal to the standard model structure on $V_0 \setminus \Gamma$ and to J outside V. It is ω-tame, since the forms agree on V. A conformal multiple of $\Omega(\cdot, J^0\cdot)$ extends to a smooth background metric which is Euclidean in the zero models. The structures J_k on $X \setminus \Gamma$ of [4, §3.1], constructed with this metric, are Ω-compatible and equal to J outside V.

Write $L^k = L^{\otimes k}$, with the induced connection. Put

$$q_k = 2\pi N k, \qquad \varepsilon_k = q_k^{-1/3}, \qquad g_k = q_k\Omega(\cdot, J_k\cdot).$$

The curvature of L^k is $-iq_k\Omega$. The local model is homogeneous of degree three; dilation by $q_k^{1/3}$ gives the standard form, with circle period $2\pi\varepsilon_k^{-1}$. Write $x = (x_0, x_1, x_2)$ and t for the scaled normal and circle coordinates. The holonomy is $(-1)^k$. Let $K_k \subset \operatorname{int} Z_{0,k} \subset X \setminus \Gamma$ be the complements of the tubes $|x| < 10$ and $|x| < 1$, respectively, in these scaled model coordinates. For odd k, the coordinate and localization estimates H1–H2 of [4, Proposition 2 and §6.1] hold with error $O(\varepsilon_k)$. Choose the localized model sections with cutoffs supported in V, and ordinary symplectic peak sections on the complement of a smaller tube containing the region where J_k varies [4, §6.1, pp. 1093–1094]. Covariant norms below are taken in g_k and the Hermitian metric of L^k; ∂ and $\bar\partial$ denote the complex-linear and complex-antilinear parts with respect to J_k. For a section s of L^k, a transversality bound $\eta > 0$ means that at every point either $|s| \geq \eta$ or ∇s has a right inverse of norm at most η^{-1}. If $\Gamma = \emptyset$, take $V = \emptyset$, $J_k = J$, $K_k = Z_{0,k} = X$ and $u_k = 0$.

For $\Gamma \neq \emptyset$, let F^{I} be the doubly periodic map of [4, Lemma 17], regarded as a map $\mathbb{C}/(2\mathbb{Z}+2\pi i\mathbb{Z}) \to \mathbb{CP}^1$. It maps the imaginary-axis circle onto $E = i\mathbb{R} \cup \{\infty\}$. In each scaled zero model put

$$Q(x) = x_0^2 - \tfrac{1}{2}(x_1^2 + x_2^2), \qquad z = \varepsilon_k^{-1} Q + i\varepsilon_k t.$$

On $1 \leq |x| \leq 10$, for sufficiently large k, [4, equations (7)–(8) and Lemma 1] give

$$|dQ|_{g_k} = \varepsilon_k, \qquad |dt|_{g_k} = \varepsilon_k^{-1}, \qquad \langle dQ, dt\rangle_{g_k} = 0, \qquad dz \circ J_k = i\, dz.$$

Thus z is J_k-holomorphic and $|dz|_{g_k}$ is a fixed positive constant. Choose $p \in \mathbb{C}$ outside E and the branch values. The derivative of F^{I} has a positive lower bound over a sufficiently small fixed neighborhood of p, by compactness of the elliptic quotient. Choose the branch disks disjoint from the imaginary-axis circle and with images disjoint from this neighborhood; their inverse images under z are the branch tubes. Choose the parameter $\alpha > 0$ in Stages II–IV of [4, §6.2]. On $|x| \leq 10$ the corrections are uniformly $O(\alpha)$ in C^0, and on $1 \leq |x| \leq 10$ they are $O(\alpha)$ in C^1 away from the branch tubes. For sufficiently small fixed α, the resulting model pair $(\sigma_{0,k}, \sigma_{1,k})$, cut off inside V as in [4, §6.2, p. 1100], has quotient $F_k = \sigma_{1,k}/\sigma_{0,k}$ satisfying

$$|\partial F_k| - |\bar\partial F_k| \geq a_0 > 0$$

near its p-level on $Z_{0,k} \setminus K_k$; this level avoids the branch tubes. Here the target has its spherical metric. All constants are independent of sufficiently large odd k.

Set $u_k = \sigma_{1,k} - p\sigma_{0,k}$. On $Z_{0,k} \setminus K_k$, the lower bound for $|\sigma_{0,k}|^2 + |\sigma_{1,k}|^2$ in [4, Hypothesis H3 and Proposition 12] implies a lower bound for $|\sigma_{0,k}|$ when $|u_k|$ is small. Together with the derivative bounds in H3 and

$$\nabla u_k = \sigma_{0,k}\, dF_k + u_k \sigma_{0,k}^{-1} \nabla \sigma_{0,k},$$

this gives constants $b, \eta_0 > 0$ such that

$$|u_k| \leq b \quad \Longrightarrow \quad |\partial u_k| - |\bar\partial u_k| \geq \eta_0 \qquad \text{on } Z_{0,k} \setminus K_k.$$

In $|x| < 1$, the zeros avoid the branch tubes and are levels of the model functions Q, t. They are symplectic and disjoint from Γ. These conclusions hold in both parity models [4, §6.3].

Apply the scalar local estimated transversality theorem [3, arXiv version, Proposition 4.2]. The summation and separation bounds H2 give a global perturbation by the argument of [3, arXiv version, Proposition 4.1], as in the zeroth-jet part of [4, §2, proof of Theorem 4]. The resulting section s_k has a transversality bound $\eta > 0$ on K_k and $\|\bar\partial s_k\|_{C^1(g_k; K_k)} = O(\varepsilon_k)$. The perturbation $s_k - u_k$ is supported in $Z_{0,k}$ and has arbitrarily small fixed $C^2(g_k)$ norm. Choose its $C^1(g_k)$ norm at most $\min(b/2, \eta_0/4)$ when $\Gamma \neq \emptyset$. Then

$$|s_k| \leq b/2 \quad \Longrightarrow \quad |\partial s_k| - |\bar\partial s_k| \geq \eta_0/2 \qquad \text{on } Z_{0,k} \setminus K_k.$$

On W all model contributions vanish. The summed estimates for ordinary peak sections give a constant B, independent of k, such that

$$\|s_k\|_{C^2(g_k; W)} \leq B, \qquad \|\bar\partial s_k\|_{C^1(g_k; W)} \leq B k^{-1/2}.$$

Since $g_k|_W = 2\pi k\, N\Omega(\cdot, J\cdot)$, the section $s_k|_W$ satisfies the hypotheses of the relative Donaldson–Auroux theorem [48, §1, p. 657] for the almost Kähler manifold $(W, N\Omega, J)$ and the line bundle $L|_W$.

Choose $\chi \in C_c^\infty(W)$ equal to one near $\bigcup_j C_j$. Apply that theorem successively to $C_1, \dots, C_r$, with rank one and $m_{\max} = 2$, and multiply each resulting perturbation t_k by χ. For a prescribed amplitude $\delta > 0$, its estimates imply

$$\|\chi t_k\|_{C^2(g_k)} + k^{1/2}\|\bar\partial(\chi t_k)\|_{C^1(g_k)} \le A\delta,$$

with A independent of k and δ. Indeed, the derivatives of the fixed cutoff in the scaled metric are $O(k^{-1/2})$ and $O(k^{-1})$. At each application choose $A\delta$ smaller than one quarter of every current transversality bound, and replace the retained bounds by half their values. If $P \neq \emptyset$, apply the same theorem to the zero-dimensional submanifold P, with the same amplitude condition. Since its tangent spaces are zero, the quantitative conclusion gives a constant $\eta_P > 0$ with $|s_k| \ge \eta_P$ on P. All these perturbations are supported in W. There are finitely many steps, so one threshold for odd k satisfies all the estimates. The final section still satisfies

$$\|\bar\partial s_k\|_{C^1(g_k;K_k)} = O(\varepsilon_k), \qquad \|\bar\partial s_k\|_{C^1(g_k;W)} = O(k^{-1/2}),$$

since the sum of the added $\bar\partial$ terms is $O(k^{-1/2})$ in $C^1(g_k)$ and $k^{-1/2} = O(\varepsilon_k)$. The annular and inner estimates are unchanged.

Let $S = s_k^{-1}(0)$ for the final section. Orient its normal bundle $\nu S = TX|_S/TS$ by the isomorphism $ds_k : \nu S \to L^k|_S$ and the complex orientation of L^k, and give S the induced orientation. On K_k its tangent planes differ from J_k-complex lines by $O(\varepsilon_k)/\eta'$, where $\eta' > 0$ is the final ambient transversality bound. Hence they are Ω-positive. On the compact set $X \setminus V$, $J_k = J$ and $\omega(v, Jv)$ has a positive lower bound on the unit sphere of any fixed metric. The same tangent-plane estimate gives ω-positivity there. On V the two forms agree; the preceding estimate on $K_k \cap V$ and the annular and inner model estimates give both positivities. These estimates show that the induced orientation of S is its Ω-positive orientation. At a zero of $s_k|_{C_j}$, its derivative has the form $\xi\, dz + \zeta\, d\bar z$ in oriented complex coordinates, with $|\xi| > |\zeta|$ for large k. Its real determinant $|\xi|^2 - |\zeta|^2$ is positive. Thus the intersections with C_j are transverse and positive.

Near each zero circle the inner corrections vanish. For each fixed k, as $x \to 0$, the quotient $F_k(x,t)$ tends uniformly in t to $F^{\mathrm{I}}(i\varepsilon_k t) \in E$. Since $p \notin E$ and the model sections have no common zero there, s_k is nonzero on $T \setminus \Gamma$ for a sufficiently small tube $T \simeq S^1 \times D^3$ about that circle. The bundle $L^k|_T$ is trivial. Since

$$H^1(T;\mathbb{Z}) \longrightarrow H^1(\partial T;\mathbb{Z})$$

is an isomorphism, the nonvanishing boundary section extends over T; a collar adjustment makes it agree with the given section near ∂T. The resulting smooth global section has the same zero set and normal orientation. In particular S is closed. The integral Thom-class formula and the equality of the Euler and first Chern classes of a complex line bundle [25, §3.2, p. 91, and Proposition 3.13(c)] give

$$[S] = \mathrm{PD}(kc_1(L)).$$

The class $kc_1(L)$ has positive square, so $S \neq \emptyset$. If $S = \coprod_\nu S_\nu$, distinct components have zero intersection, whence

$$[S_\nu]^2 = kN \int_{S_\nu} \Omega > 0.$$

Two components would give two orthogonal positive directions in the intersection form. Thus $b^+(X) = 1$ implies that S is connected. □

Proof of Proposition 7.3. Put $V = H^2(X;\mathbb{R})$, with rational structure given by $H^2(X;\mathbb{Q})$, and let $\mathcal{P}$ be the component of $\{v \in V : v^2 > 0\}$ containing a. Choose a neighborhood W of a in $\mathcal{P}$ whose rays belong to U.

Choose $\ell \in V^*$ with $\ell(a) = 1$. For $n > 1$, choose small open angular cones $G_1, \dots, G_n \subset \ker \ell$ about the vertex directions of a simplex containing zero in its interior. The cones may be chosen so that any $w_j \in G_j$ have rank $n-1$ and their space of relations is generated by a vector $(\rho_1, \dots, \rho_n)$ with every $\rho_j > 0$. These properties persist under small angular variations of the vertex directions, since their maximal minors are nonzero; positive radial rescaling preserves the signs of the relation. Rational density permits successive rational choices $b_j \in W$, arbitrarily close to a, with

$$\ell(b_j) > 0, \qquad w_j = \frac{b_j}{\ell(b_j)} - a \in G_j.$$

The distance permitted at stage j may depend on the preceding choices. For $\sum_j \rho_j w_j = 0$, set

$$\beta_j = \frac{\rho_j}{\ell(b_j) \sum_l \rho_l}. \qquad \text{Then} \qquad a = \sum_j \beta_j b_j, \quad \beta_j > 0.$$

If $\sum_j u_j b_j = 0$, then $\sum_j u_j \ell(b_j) = 0$ and $\sum_j u_j \ell(b_j) w_j = 0$. The one-dimensional relation space and $\sum_j \rho_j > 0$ force every u_j to vanish. Thus the b_j form a basis. For $n = 1$, choose a rational $b_1 \in W$ on the positive ray through a and define $\beta_1 > 0$ by $a = \beta_1 b_1$.

Honda's normalization, as stated in [4, §1, p. 1046], gives, for each $i = 0, 1$, a homeomorphism ϕ_i supported in arbitrarily small zero tubes of ω_i, smooth off their cores. Its pullback $\phi_i^* \omega_i$, initially defined away from the zero circles, extends by the standard models across them. Each ϕ_i acts trivially on $H^2(X; \mathbb{Z})$. Indeed, for the union T_i of its supporting tubes, $H^2(T_i; \mathbb{Z}) = 0$ and $H^1(T_i; \mathbb{Z}) \to H^1(\partial T_i; \mathbb{Z})$ is onto. The Mayer–Vietoris sequence makes restriction from $H^2(X; \mathbb{Z})$ to the complement injective, and the homeomorphism is the identity there. Surfaces for $\phi_i^* \omega_i$ can therefore be pushed forward by ϕ_i with their integral classes and positivity preserved. They are smooth because they avoid the cores. Denote the normalized forms again by ω_0, ω_1.

Suppose that the first $j-1$ surfaces have been constructed for each form ω_i, with coordinate-plane germs in Darboux charts at their crossings. For each i, choose zero tubes disjoint from these surfaces and balls about their crossings, with pairwise disjoint slightly larger neighborhoods. Choose linear maps $\eta_i : V \to \Omega^2(X)$, $i = 0, 1$, assigning closed representatives which vanish on the tubes and balls. Such maps are obtained by choosing representatives of a basis and subtracting differentials of cutoff primitives in the larger neighborhoods, whose second real cohomology is zero. They are bounded in C^m for every fixed integer $m \geq 0$.

Choose b_j as above, sufficiently close to a, and put

$$\widehat{\Omega}_{ij} = \omega_i + \eta_i(b_j - a).$$

These forms represent b_j, agree with ω_i in the zero tubes and crossing charts, and are positive on all previously constructed surfaces. They are near-symplectic, since their zero models are unchanged and nondegeneracy persists on the compact complement.

Fix an ω_i-compatible structure J_i^0 preserving the previous tangent planes, equal to the product structure in the crossing charts, and put $g_i = \omega_i(\cdot, J_i^0 \cdot)$. For a sufficiently close form Ω equal to ω_i in those charts and near the zeros, let P_Ω be the projection onto TC along $(TC)^{\perp_\Omega}$ on each previous surface C. The metric

$$h_\Omega(v, w) = g_i(P_\Omega v, P_\Omega w) + g_i((1 - P_\Omega)v, (1 - P_\Omega)w)$$

makes this splitting orthogonal, tends to g_i as $\Omega \to \omega_i$, and equals g_i in the crossing charts. Extend $h_\Omega - g_i$ by fixed tubular cutoffs away from these charts. The extended metric remains positive and close to g_i. The compatible polar construction [11, Proposition 3.1, its following remark, and Proposition 3.4] then gives an Ω-compatible J preserving every TC. Smooth dependence permits $\|J - J_i^0\|_{g_i} < 1/2$ on the compact support of the changes, so that

$$\omega_i(v, Jv) \geq \tfrac{1}{2}|v|_{g_i}^2.$$

Outside that set $J = J_i^0$. In particular ω_i tames J.

Let q_j be a positive integer with $q_j b_j$ integral. For each positive multiple N of q_j, fix one integral lift c of $N b_j$ and a Hermitian line bundle L with $c_1(L) = c$ [25, Proposition 3.10]. The form $-2\pi i N\widehat{\Omega}_{ij}$ differs from the curvature of any unitary connection on L by an exact imaginary two-form. For each i, adding an imaginary one-form therefore gives a connection with curvature $-2\pi i N\widehat{\Omega}_{ij}$. In a zero tube, let t have period τ and let $\chi(r)$ equal one near the core and zero near the boundary, where r is the radial coordinate. Adding $-ih\chi(r)\,dt$ to a connection multiplies its core holonomy by $\exp(ih\tau)$. A choice $|h| \le \pi/\tau$ makes this holonomy -1. Summing the real forms $h\chi(r)\,dt$ gives α_i with uniformly bounded C^{m+1} norm, independently of N and of the initial connections. Changing the connection by $-i\alpha_i$ gives curvature $-2\pi i N\Omega_{ij}$, where

$$\Omega_{ij} = \widehat{\Omega}_{ij} + \frac{d\alpha_i}{2\pi N}.$$

The correction is exact, is $O(N^{-1})$ in C^m, and is supported in fixed annuli disjoint from the surfaces and zero cores. Choose a multiple N_j of q_j sufficiently large for both forms, and fix one integral lift c_j of $N_j b_j$ and one line bundle L_j with $c_1(L_j) = c_j$. Make the preceding connection choices for this bundle. Then Ω_{ij} has all the preceding positivity properties, and the preceding metric construction gives an Ω_{ij}-compatible structure J_{ij} tamed by ω_i and preserving all previous tangent planes.

For each $i = 0, 1$, apply Lemma 7.4 to L_j, with model tubes inside the region where $d\alpha_i = 0$ and $\Omega_{ij} = \omega_i$. Choose one odd integer k_j above both thresholds, and put

$$d_j = k_j c_j, \qquad (d_j)_{\mathbb{R}} = k_j N_j b_j.$$

The resulting connected surfaces represent the same integral class $\mathrm{PD}(d_j)$, are positive for ω_i, meet the previous surfaces positively, and avoid their crossings. Denote them by D_{ij}.

At each new crossing apply the local graph construction of [50, arXiv:1601.06136v3, Lemma 6, pp. 8–9]. In a Darboux chart with the older disk as a coordinate plane, it isotopes the new disk through ω_i-positive graphs to the complementary coordinate plane near the crossing, keeping the older disk and the exterior fixed. Choose these balls disjoint and containing no third sheet. The isotopies preserve connectedness and integral classes, and their relative ambient extensions preserve all previous crossings. The resulting surfaces form an orthogonal positive configuration.

Set $\lambda_j = \beta_j/(k_j N_j)$. Then $\lambda_j > 0$ and $a = \sum_j \lambda_j d_j$. Positive rescaling preserves the basis property, membership in $\mathcal{P}$, and the rays in U. The basis property and nondegeneracy of the intersection form imply that $Q = (d_j \cdot d_l)$ is nonsingular. The signature is $(1, n-1)$, so two vectors in $\mathcal{P}$ have positive pairing. Thus every entry of Q is positive. For $\lambda = (\lambda_1, \dots, \lambda_n)^{\mathsf{T}}$, every entry of $Q\lambda$ is positive. □

8. Exact handles along prescribed arcs

Let S be an oriented surface in an oriented four-manifold X. A cord for S is a smooth embedding $\gamma : [0,1] \to X$ such that $\gamma^{-1}(S) = \{0,1\}$, the endpoints lie in the interior of S, and $\dot\gamma(i) \notin T_{\gamma(i)}S$ for $i = 0, 1$. An oriented handle along γ replaces two small disks in S by the lateral annulus of an embedded disk tube with core γ, with corners rounded and orientation extending that of S. The core is considered up to isotopy through cords for S, relative to its endpoints in the indicated neighborhood. All isotopies are stationary outside that neighborhood.

Lemma 8.1. *Let ω be a closed two-form on an oriented four-manifold X, and let $P \subset \Lambda^2 T^*X$ be a smooth positive three-plane bundle containing ω. Let B be a symplectic coordinate ball for ω, disjoint from a prescribed compact subset. There is a smooth exact deformation of ω, supported in a compact subset of B, through sections of smooth positive three-plane bundles which start at P and remain fixed outside B. In B the forms have no zeros before one birth parameter and one isolated nontransverse zero at that parameter. Afterwards their zero set in B is one circle, along which they are transverse to the zero section of the corresponding positive three-plane bundle. Near the birth, the family is a positive constant multiple of the polynomial family defined in the proof.*

Proof. Give (t,x,y,z) the orientation $\mu = dt\wedge dx\wedge dy\wedge dz$. For a real parameter s, put

$$R = t^2+x^2, \qquad B_0 = y^2+z^2, \qquad A = R - s,$$

and abbreviate

$$u = dt\wedge dx,\quad v = dy\wedge dz,\quad p = dt\wedge dy,\quad q = dz\wedge dx,\quad r = dt\wedge dz,\quad w = dx\wedge dy.$$

Let J_0 be the standard complex structure given by $J_0\partial_t = \partial_x$, $J_0\partial_y = \partial_z$, and put

$$d^c\phi = -\phi_x dt + \phi_t dx - \phi_z dy + \phi_y dz.$$

Fix $0 < \varepsilon \le 1/100$ and define

$$\omega_s = 3\varepsilon A(u+v) + y\{2p + 2(1-3\varepsilon x)q + 6\varepsilon tr\} - 2z(r+w),$$
$$\eta_s = \omega_s + \varepsilon dd^c(AB_0 + B_0^2), \qquad \alpha = 3u+7v.$$

The form ω_s is the Luttinger–Simpson birth model [41, §5, Deformation 1, equation (1), p. 301]. Differentiation gives $d\omega_s = 0$ and

$$\eta_s = \eta_0 - \varepsilon s\alpha. \tag{8.1}$$

Moreover $\eta_s = AF_1 + yF_2 + zF_3$, where

$$F_1 = \varepsilon(3u+7v),$$
$$F_2 = (2+4\varepsilon x)p + (2-10\varepsilon x)q + 10\varepsilon tr - 4\varepsilon tw + 4\varepsilon yu + 16\varepsilon yv,$$
$$F_3 = (-2+4\varepsilon x)r + (-2-4\varepsilon x)w - 4\varepsilon tp + 4\varepsilon tq + 4\varepsilon zu + 16\varepsilon zv.$$

All coordinate norms are Euclidean; in particular, (u,v,p,q,r,w) is orthonormal. Regard $E = (F_1/\varepsilon, F_2, F_3)$ and $E_0 = (3u+7v, 2(p+q), -2(r+w))$ as maps from $\mathbb{R}^3$ to two-forms. On the coordinate box $[-1,1]^4$,

$$\|E - E_0\|_{\mathrm{op}}^2 \le \varepsilon^2(148R + 272B_0) \le 840\varepsilon^2, \qquad \|E_0\|_{\mathrm{op}} = \sqrt{58} < 8.$$

The wedge pairing, divided by μ, has norm one in this basis. The wedge Gram matrix of E_0 is $\mathrm{diag}(42,8,8)$, and its change under $E_0 \mapsto E$ has norm at most $480\varepsilon + 900\varepsilon^2 < 8$. Thus $P_F = \mathrm{span}(F_1,F_2,F_3)$ is positive, and

$$C_s := \eta_s^{-1}(0) = \{t^2+x^2 = s,\ y = z = 0\}$$

within the coordinate box. For $0 < s < 1$, the normal derivative along C_s maps the radial unit vector in the (t,x)-plane, ∂_y, and ∂_z to $2\sqrt{s}F_1, F_2, F_3$, respectively. It is therefore an isomorphism onto $P_F|_{C_s}$. At $s = 0$ the only zero is the origin and the spatial derivative has rank two; for $s < 0$ there is no zero.

For a tangent vector V, put $\zeta(V) = (dt(V) + i\,dx(V), dy(V) + i\,dz(V))^T$. Then $\eta_s(V, J_0V) = \zeta(V)^* H_s \zeta(V)$, where

$$H_s = \varepsilon\begin{pmatrix} 3A+4B_0 & 7ty + 4xz + i(7xy - 4tz) \\ 7ty + 4xz - i(7xy - 4tz) & 7A + 16B_0 \end{pmatrix}.$$

Its determinants at $s = 0$ and $s = -\delta < 0$ are

$$\det H_0 = \varepsilon^2(21R^2 + 27Ry^2 + 60Rz^2 + 64B_0^2),$$
$$\det H_{-\delta} = \det H_0 + \varepsilon^2(42\delta R + 21\delta^2 + 76\delta B_0).$$

The diagonal entries are positive away from the origin at $s = 0$ and everywhere at $s = -\delta$. Hence η_0 tames J_0 away from the origin and vanishes there, whereas $\eta_{-\delta}$ tames J_0 everywhere.

Choose a Darboux ball $D \Subset B$ centered at the origin in these coordinates, contained in the unit box, with $\omega = C\alpha$ for some $C > 0$. Put $Q = (3R + 7B_0)/4$, so $dd^cQ = \alpha$. Fix $0 < a_1 < a_2 < a_3$ with $V_j := \{Q < a_j\} \Subset D$ for $j = 1,2,3$. Choose smooth functions $\chi \in C_c^\infty(D)$ and $\chi_1 \in C_c^\infty(V_2)$, equal

to one near $\overline{V}_3$ and $\overline{V}_1$, respectively. Choose a smooth convex nondecreasing function $\psi : \mathbb{R} \to \mathbb{R}$ which is zero on $(-\infty, a_2]$ and has derivative one near a_3. Let ϑ be the radial primitive of η_0 on D, so $d\vartheta = \eta_0$, and set $\lambda_\alpha = d^c Q$. The remaining parameters $\delta, \theta > 0$ are chosen after these functions and domains have been fixed.

On the compact annulus $K = \overline{V}_2 \setminus V_1$, put

$$m = \min_{\substack{\xi \in K \\ |V|=1}} (\eta_0)_\xi(V, J_0 V) > 0, \qquad N = \|\alpha\|_{C^0(K)} + 2\|d(\chi_1 \lambda_\alpha)\|_{C^0(K)}.$$

Choose $\delta > 0$ such that $\delta < \min(1, 4a_1/3)$ and $\varepsilon\delta N < m/2$. Then choose $\theta > 0$ such that

$$\theta \|d(\chi\vartheta)\|_{C^0(D)} < 3C/2, \qquad \theta\varepsilon\delta < C.$$

The path

$$\omega + \tau\theta d(\chi\vartheta), \qquad 0 \le \tau \le 1,$$

is fixed outside D. On D, its value on $(V, J_0 V)$ is at least $(3C - \theta\|d(\chi\vartheta)\|_{C^0(D)})|V|^2 > 0$ for $V \neq 0$. Its endpoint equals $\theta\eta_0 + C\alpha$ on V_3.

On V_3 continue by the path

$$\theta\eta_0 + \theta\varepsilon\delta\alpha + (C - \theta\varepsilon\delta) dd^c\big((1-\tau)Q + \tau\psi(Q)\big), \qquad 0 \le \tau \le 1.$$

The identity

$$dd^c\psi(Q) = \psi'(Q)\alpha + \psi''(Q) dQ \wedge d^c Q$$

shows that the potential term is J_0-semipositive. The sum of the first two terms is $\theta\eta_{-\delta}$ and tames J_0. The path is therefore symplectic on V_3. Its difference from its initial value is the exterior derivative of $\tau(C - \theta\varepsilon\delta)d^c(\psi(Q) - Q)$. This one-form vanishes near ∂V_3, so it extends by zero. Denote the resulting global endpoint by $\overline{\omega}$; it equals $\theta\eta_{-\delta}$ on V_2.

Finally, set

$$\widetilde{\omega}_s = \overline{\omega} - \theta\varepsilon(s + \delta) d(\chi_1 \lambda_\alpha), \qquad -\delta \le s \le \delta.$$

On V_1 this is $\theta\eta_s$. On K,

$$\|\theta^{-1}\widetilde{\omega}_s - \eta_0\|_{C^0(K)} \le \varepsilon\delta N < m/2,$$

so $\widetilde{\omega}_s$ tames J_0 there. Outside V_2 it equals $\overline{\omega}$. Since $Q = 3s/4$ on C_s, every new zero lies in V_1. Thus these exact paths have precisely the stated zeros. Reparametrize them to be constant near their ends, with s affine and strictly increasing near zero, and denote the concatenated path by $(\Omega_\tau)_{0 \le \tau \le 1}$.

By Lemma 2.2, choose a metric g with $\Lambda^+_g = P$. Write $\Omega_\tau = a_\tau + b_\tau$ in $P \oplus \Lambda^-_g$. For the norms induced by g, the positive three-planes containing Ω_τ are the graphs of maps in the convex set

$$\{L : P \longrightarrow \Lambda^-_g : La_\tau = b_\tau,\ \|L\|_{\mathrm{op}} < 1\};$$

see [4, §7.2, p. 1106]. Where $\Omega_\tau \neq 0$, one such map is

$$L_\tau = \frac{b_\tau \otimes a_\tau^\flat}{|a_\tau|^2}, \qquad a_\tau^\flat = \langle a_\tau, \cdot \rangle_g.$$

It is zero wherever $\Omega_\tau = \omega$, and hence extends by zero across the unchanged neighborhood of the original zero set. On V_1 during the last path, let L_F be the graph map of P_F. Choose a smooth function ρ on $X \times [0,1]$, with values in $[0,1]$, equal to one near all new zeros and supported in this model region, away from the initial parameter. Extend ρL_F by zero outside this region. Away from the new zeros, take the graph of $\rho L_F + (1-\rho)L_\tau$, and extend it by P_F near those zeros, where $\rho = 1$. Convexity gives a smooth positive bundle containing Ω_τ. At $\tau = 0$ and outside D it is P. □

Proposition 8.2. *Let ω be a closed two-form on an oriented four-manifold X, contained in a prescribed smooth positive three-plane bundle P, and let S be a compact embedded ω-positive oriented surface. Let γ be a cord for S whose image is disjoint from the zero set of ω. For any neighborhood U of $\gamma([0,1])$, there is*

a smooth deformation $\omega_s = \omega + d\beta_s$, $0 \le s \le 1$, where $\beta_0 = 0$ and the one-forms β_s are supported in one compact subset of U, together with smooth positive three-plane bundles P_s such that $P_0 = P$, $\omega_s \in \Gamma(P_s)$, and $P_s = P$ outside U. The form ω_1 is positive on a surface obtained by attaching one oriented handle to S along γ.

The surface is unchanged outside U. Exactly one transverse zero circle is born, and its birth point is the only new nontransverse zero. The surface changes through positive isotopies and one quadratic cone surgery replacing two disks by an annulus, with vertex at a regular point of the new zero circle. At the singular parameter the surface is positive away from the vertex. All original zero germs and all other prescribed surfaces disjoint from U are retained.

8.1. **Exact linearization at a regular zero.**

Lemma 8.3. *Let ω be a smooth closed two-form on an oriented four-manifold X, contained in a prescribed smooth positive three-plane bundle P. Suppose that, in coordinates (t, n), with $n = (n_1, n_2, n_3)$, its zero set is $n = 0$ and its vanishing there is transverse as a section of P. Let its first-order Taylor polynomial at $(0,0)$ be the closed linear form $L = \sum_{j=1}^{3} n_j A_j$, where the constant two-forms A_j span a positive three-plane.*

There are a smooth one-form α, supported in a compact subset of a sufficiently small coordinate neighborhood U of $(0,0)$, and a smooth family of positive three-plane bundles P_v, $0 \le v \le 1$, such that

$$P_0 = P, \qquad P_v = P \text{ outside } U, \qquad \omega_v := \omega - v\, d\alpha \in \Gamma(P_v).$$

The forms ω_v have the same zero set as ω and remain transverse along the given arc. On a smaller neighborhood of $(0,0)$,

$$\omega_1 = L, \qquad P_1 = \operatorname{span}(A_1, A_2, A_3).$$

Proof. Choose a product coordinate neighborhood U of $(0,0)$, with normal factor an open Euclidean ball centered at 0. Since $\partial_{n_j}\omega(0,0) = A_j$, the forms A_j are a basis of $P_{(0,0)}$. Choose a smooth frame (e_1, e_2, e_3) of P with $e_i(0,0) = A_i$, and write $\omega = \sum_i f_i e_i$. Since $f_i(t,0) = 0$, Hadamard's lemma gives

$$f_i(t,n) = \sum_j M_{ij}(t,n) n_j, \qquad M_{ij}(t,n) = \int_0^1 \partial_{n_j} f_i(t, an)\, da.$$

The Taylor hypothesis gives $M(0,0) = I$. After shrinking the neighborhood, the forms

$$B_j = \sum_i M_{ij} e_i$$

are a frame of P, with $\omega = \sum_j n_j B_j$ and $B_j(0,0) = A_j$. Put $C_j = B_j - A_j$. In the Euclidean coordinate norms,

$$|C_j(t,n)| \le C(|t| + |n|).$$

Set $W = \omega - L$, $h_a(t,n) = (t, an)$ for $0 \le a \le 1$, and $\mathcal{R} = \sum_j n_j \partial_{n_j}$. The form W is closed and $h_0^* W = 0$. The normal homotopy formula [53, Lemma 3.1, proof, Step 2, p. 245] gives

$$\beta = \int_0^1 a^{-1} h_a^*(\iota_{\mathcal{R}} W)\, da, \qquad d\beta = W.$$

Since $W = \sum_j n_j C_j$ and $h_a^* n_j = a n_j$, one has

$$\beta = \sum_j n_j b_j, \qquad b_j = \int_0^1 h_a^*(\iota_{\mathcal{R}} C_j)\, da.$$

These integrands are smooth at $a = 0$. Moreover,

$$|b_j(t,n)| \le C|n|(|t| + |n|), \qquad |\beta(t,n)| \le C|n|^2(|t| + |n|).$$

For $0 < r_n \le r_t$ with $\{|t| < r_t,\ |n| < r_n\} \Subset U$, choose a smooth product cutoff $\chi \in C_c^\infty(U)$, equal to one near the origin and supported in this product, with $0 \le \chi \le 1$ and

$$|d\chi| \le C(r_t^{-1} + r_n^{-1}).$$

Here and below the constants are independent of v, r_t, r_n. For $0 \le v \le 1$, put

$$B_{j,v} = (1 - v\chi)B_j + v\chi A_j - v\, d\chi \wedge b_j.$$

Then

$$\sum_j n_j B_{j,v} = \omega - v\, d(\chi\beta).$$

On supp χ,

$$|d\chi \wedge b_j| \le C(r_t^{-1} + r_n^{-1}) r_n (r_t + r_n) \le 2C(r_t + r_n),$$
$$|B_{j,v} - A_j| \le C'(r_t + r_n).$$

The wedge Gram matrix of (A_j) is positive definite. Thus, for sufficiently small r_t, the forms $(B_{j,v})$ are a positive frame for every v; outside supp χ they equal (B_j).

Take $\alpha = \chi\beta$, extended by zero. The bundles $P_v = \operatorname{span}(B_{1,v}, B_{2,v}, B_{3,v})$ equal P at $v = 0$ and near the boundary of U, and hence extend by P outside U. They contain ω_v. Linear independence gives

$$\omega_v(t, n) = 0 \iff n = 0, \qquad \partial_{n_j}\omega_v(t, 0) = B_{j,v}(t, 0).$$

The normal derivative is therefore an isomorphism onto P_v. Where $\chi = 1$, one has $B_{j,1} = A_j$ and $\omega_1 = L$. □

For the form η_s in Lemma 8.1, take $s = \delta$ as in its proof. Near the zero point $(t, x, y, z) = (0, \sqrt{\delta}, 0, 0)$, the coordinates

$$x = \sqrt{\delta - t^2} + u, \qquad |t| < \sqrt{\delta},$$

preserve orientation and send the local arc of the zero circle to $u = y = z = 0$. The first-order Taylor polynomial at the new origin is

$$\begin{aligned} L = {} & u\{6\varepsilon\sqrt{\delta}\,\mathrm{d}t \wedge \mathrm{d}u + 14\varepsilon\sqrt{\delta}\,\mathrm{d}y \wedge \mathrm{d}z\} \\ & + y\{(2 + 4\varepsilon\sqrt{\delta})\mathrm{d}t \wedge \mathrm{d}y + (2 - 10\varepsilon\sqrt{\delta})\mathrm{d}z \wedge \mathrm{d}u\} \\ & + z\{(-2 + 4\varepsilon\sqrt{\delta})\mathrm{d}t \wedge \mathrm{d}z + (-2 - 4\varepsilon\sqrt{\delta})\mathrm{d}u \wedge \mathrm{d}y\}. \end{aligned}$$

Put $k = \varepsilon\sqrt{\delta}$. The choices $0 < \varepsilon \le 1/100$ and $0 < \delta < 1$ give $0 < k < 1/100$. Relative to $dt \wedge du \wedge dy \wedge dz$, the wedge Gram matrix of the three coefficient forms is

$$\operatorname{diag}(168k^2,\ 8 - 24k - 80k^2,\ 8 - 32k^2) > 0.$$

Moreover,

$$dL = \{14k + (2 - 10k) + (-2 - 4k)\}\, du \wedge dy \wedge dz = 0.$$

Lemma 8.3 therefore applies. For a positive constant multiple of η_δ, multiply the deformation and its compactly supported primitive by the same constant.

8.2. **Two disks and an arbitrarily thin annulus.** The restriction of L to the hyperplane $t = 0$ is

$$\beta = au\,\mathrm{d}y \wedge \mathrm{d}z + b\,y\,\mathrm{d}z \wedge \mathrm{d}u - cz\,\mathrm{d}u \wedge \mathrm{d}y, \qquad a, b, c > 0, \quad c = a + b.$$

In the displayed birth model, $a = 14\varepsilon\sqrt{\delta}$, $b = 2 - 10\varepsilon\sqrt{\delta}$ and $c = 2 + 4\varepsilon\sqrt{\delta}$. Fix $\ell > 0$ such that the segment $\{t = u = y = 0,\ |z| \le 2\ell\}$ lies in the linear coordinate neighborhood. For all sufficiently small $r > 0$, put $L_r = \sqrt{\ell^2 + r^2}$; the cylinder

$$\{t = 0,\ u^2 + y^2 \le r^2,\ |z| \le L_r\}$$

then lies in that neighborhood.

Put $R_0(z) = z^2 - \ell^2$. Choose a smooth nondecreasing function $m_r : \mathbb{R} \to \mathbb{R}$ such that

$$m_r(q) = r^2/4 \quad (q \le r^2/8), \qquad m_r(q) = q \quad (q \ge r^2/2), \qquad m_r(q) > 0.$$

One may take the convolution of $q \mapsto \max(q, r^2/4)$ with an even nonnegative smooth function of integral one supported in $(-r^2/8, r^2/8)$. Set $R_1(z) = m_r(R_0(z))$. For $0 \le \lambda \le 1$, define

$$R_\lambda(z) = (1-\lambda)R_0(z) + \lambda R_1(z), \qquad F_\lambda(u, y, z) = u^2 + y^2 - R_\lambda(z),$$

and

$$A_\lambda = \{t = 0,\ F_\lambda(u, y, z) = 0,\ |z| \le L_r\}.$$

Orient the regular part of A_λ by the outward normal to $\{F_\lambda \le 0\}$ in the oriented space $(\{t = 0\}, du \wedge dy \wedge dz)$. For $|z| \le L_r$, both $R_0(z)$ and $R_1(z)$ are at most r^2, so A_λ lies in the indicated cylinder. If $\ell^2 + r^2/2 \le z^2 \le \ell^2 + r^2$, then $R_1(z) = R_0(z)$. Thus all A_λ have the same oriented boundary collars, with boundary circles $u^2 + y^2 = r^2$, $z = \pm L_r$.

The derivative is

$$R'_\lambda(z) = 2z\big(1 - \lambda + \lambda m'_r(z^2 - \ell^2)\big).$$

A singular point of $F_\lambda = 0$ must satisfy $u = y = 0$ and $R_\lambda(z) = R'_\lambda(z) = 0$. For $\lambda < 1$, the factor in parentheses is positive, so $z = 0$. For $\lambda = 1$, the inequality $R_1 > 0$ excludes singular points. Since $R_\lambda(0) = -(1-\lambda)\ell^2 + \lambda r^2/4$, the only singular point is $(u, y, z) = (0, 0, 0)$, at

$$\lambda_* = \frac{\ell^2}{\ell^2 + r^2/4}.$$

For $\lambda < \lambda_*$, the function R_λ has one positive zero h_λ, and $R'_\lambda > 0$ on $[h_\lambda, L_r]$. Inverting its restriction to this interval gives two smooth graph disks over $u^2 + y^2 \le r^2$. At $\lambda = 0$ they are

$$z = \pm\sqrt{\ell^2 + u^2 + y^2}, \qquad u^2 + y^2 \le r^2.$$

For $\lambda > \lambda_*$, the function R_λ is positive on $[-L_r, L_r]$, so A_λ is an annulus. Near $z = 0$, one has $R_1 = r^2/4$; hence the germ of A_{λ_*} at the origin is the quadratic cone

$$u^2 + y^2 = (1 - \lambda_*)z^2.$$

Put $D = \ell^2 + r^2/4$. Near the singular point the set $\{(\lambda, u, y, z) : F_\lambda(u, y, z) = 0\}$ is the graph

$$\lambda - \lambda_* = \frac{u^2 + y^2 - (1-\lambda_*)z^2}{D - z^2}.$$

Its projection to λ has a nondegenerate critical point of index one. This is the local transition from two disks to an annulus through an indefinite quadratic fibre [4, §1, p. 1045].

Where $R_\lambda > 0$, put $u = \sqrt{R_\lambda}\cos\theta$ and $y = \sqrt{R_\lambda}\sin\theta$, with $\theta \in \mathbb{R}/(2\pi\mathbb{Z})$. The chosen orientation is $-dz \wedge d\theta$, and

$$\beta|_{A_\lambda} = -\left\{(a\cos^2\theta + b\sin^2\theta)R_\lambda + \frac{c}{2}zR'_\lambda(z)\right\}\mathrm{d}z \wedge \mathrm{d}\theta. \tag{8.2}$$

The first term in braces is positive, and the second is nonnegative because $zR'_\lambda(z) \ge 0$. At a regular cap center $p = (0, 0, 0, z_0)$, one has $z_0 \ne 0$ and $R'_\lambda(z_0) \ne 0$, so (u, y) are smooth coordinates. The orientation there is $-\operatorname{sign}(z_0)\, du \wedge dy$, and

$$(\beta|_{A_\lambda})_p = -cz_0(du \wedge dy)_p.$$

This value is positive for the stated orientation. Thus every regular slice is L-positive, and the singular slice is positive away from its vertex. The form L and the chosen positive three-plane bundle remain fixed.

For the two graph disks at $\lambda = 0$,

$$0 \le |z| - \ell = \frac{u^2 + y^2}{\sqrt{\ell^2 + u^2 + y^2} + \ell} \le \frac{u^2 + y^2}{2\ell}.$$

Consequently, for any representatives of their germs at $(0, 0, 0, \pm\ell)$, all sufficiently small $r > 0$ place the whole disks and neighborhoods of their boundary circles inside those representatives.

8.3. **Hamiltonian transport of the attaching germs.** We take $\iota_{X_H}\omega = -dH$. Let $I = [0,1]$. A smooth isotopy $\gamma_s : I \to M$, $0 \le s \le 1$, of embedded arcs in a symplectic manifold (M,ω) has a Hamiltonian extension: the arcs are isotropic and $H^1(I;\mathbb{R}) = 0$ [64, Proposition 2, p. 13407]. Suppose that the isotopy is stationary on an open interval $J \subset (0,1)$, and fix a closed interval $J_0 \Subset J$. Along $\gamma_s(I)$, put $V_s(\gamma_s(x)) = \partial_s\gamma_s(x)$ and $\alpha_s = -\iota_{V_s}\omega$. Choose $h_s : I \to \mathbb{R}$ with $dh_s = \gamma_s^*\alpha_s$ and $h_s|_J = 0$. The compatible first jet

$$H_s \circ \gamma_s = h_s, \qquad (dH_s)_{\gamma_s(x)} = (\alpha_s)_{\gamma_s(x)}$$

extends smoothly in s in tubular coordinates. Since this jet is zero on J, the extension can be zero on a neighborhood of $\gamma_0(J_0)$. A cutoff gives Hamiltonians supported in one compact subset of any prescribed neighborhood of the trace. The same construction applies to a closed two-form symplectic outside a closed set Z, provided that $\bigcup_s \gamma_s(I \setminus \operatorname{int} J_0)$ is disjoint from Z: the Hamiltonians and their vector fields are zero near Z, and the vector fields extend by zero there.

Consider two positive symplectic surface germs in a symplectic four-manifold, at an endpoint of an arc, with tangent planes symplectically orthogonal to the arc tangent. Extend the arc slightly past the endpoint. A symplectic frame along the arc gives a tubular chart in which the full two-forms agree on the arc. The relative Moser theorem [11, Theorem 1.8, p. 90], applied after restricting to a shorter arc, gives coordinates (q,p,U,V), centered at the endpoint, in which the arc is the q-axis and

$$\omega = dp \wedge dq + dU \wedge dV.$$

Every symplectic two-plane in $(\mathbb{R}\partial_q)^\omega = \{dp = 0\}$ is of the form $p = 0$, $q = \ell(U,V)$ for a linear form ℓ. For the planes defined by ℓ_0 and ℓ_1, the time-one flow of $p(\ell_1 - \ell_0)$ carries the first to the second and fixes the q-axis pointwise. A cutoff equal to one near the origin gives the same germ. After this Hamiltonian adjustment and a linear symplectic change of coordinates fixing that axis, the two surface germs are graphs

$$q = Q_i(U,V), \qquad p = P_i(U,V), \qquad P_i, Q_i = O(\rho^2), \quad \rho^2 = U^2 + V^2, \quad i = 0,1.$$

Set $Q_s = (1-s)Q_0 + sQ_1$ and $P_s = (1-s)P_0 + sP_1$. On a sufficiently small disk, uniformly for $0 \le s \le 1$,

$$\sigma_s := 1 + P_{s,U}Q_{s,V} - P_{s,V}Q_{s,U} = 1 + O(\rho^2) > 0.$$

Thus the graph has area form $\sigma_s\, dU \wedge dV$. Dots denote differentiation in s. Put

$$H_s = \frac{-\dot P_s(q - Q_s) + \dot Q_s(p - P_s)}{\sigma_s}.$$

On the graph,

$$d(q - Q_s)(X_{H_s}) = \dot Q_s, \qquad d(p - P_s)(X_{H_s}) = \dot P_s.$$

Consequently $\partial_s + X_{H_s}$ is tangent to the total graph family. The coefficients of $q - Q_s$ and $p - P_s$ vanish to second order at $(U,V) = 0$, so $H_s = dH_s = 0$ on the q-axis. Multiply by a spatial cutoff equal to one near the origin. On the graphs, the (U,V)-component of the uncut vector field is $O(\rho^3)$, uniformly in s; hence trajectories starting in a sufficiently small graph disk remain in the uncut region for $0 \le s \le 1$. Its flow identifies the two surface germs and fixes the arc pointwise.

Proof of Proposition 8.2. Put $Z = \omega^{-1}(0)$. Choose a regular neighborhood $N \Subset U \setminus Z$ of the cord, with $S \cap N$ consisting of two endpoint disks. Since ω is a nonzero section of a positive three-plane bundle on $\overline{N}$, it is symplectic there. At each endpoint write the cord tangent as $v_T + v_N$ in $TS \oplus (TS)^\omega$. The vector v_N is nonzero, and $v_N + (1-s)v_T$ remains transverse to S. An isotopy of the cord, relative to its endpoints and supported in small endpoint neighborhoods, therefore makes its endpoint tangents lie in $(TS)^\omega$. The core in the proposition is unchanged up to the stipulated isotopy of cords.

Choose a Darboux ball $B \Subset N \setminus S$ meeting the cord in a single central subarc, with coordinates adapted to it. Choose the compact model and cutoffs of Lemma 8.1 in a smaller constant-form chart,

and then choose δ so small that a translation by $\sqrt{\delta}$ stays inside B. Apply the lemma with the translated model coordinates in which the cord has equation

$$t = y = 0, \qquad x = \sqrt{\delta}.$$

The initial constant form is invariant under this translation. The final zero circle $C = \{t^2 + x^2 = \delta,\ y = z = 0\}$ meets the cord only at $(0, \sqrt{\delta}, 0, 0)$. All changes of the form are supported away from S.

Apply Lemma 8.3 near this intersection. The coordinate change $x = \sqrt{\delta - t^2} + u$ sends the selected cord to the z-axis. Denote the final form and positive three-plane bundle by $\bar{\omega}$ and $\bar{P}$. Let L be the resulting linear form, including its positive birth factor. Choose $\ell > 0$ such that $\{|z| \le 2\ell\}$ on this axis lies in the region where $\bar{\omega} = L$.

Choose increasing embeddings $f_s : I \to I$, with $f_0 = \mathrm{id}$, which are the identity on a central interval and carry the endpoints at $s = 1$ to the points with coordinates $z = \pm\ell$. They may be obtained by choosing one such f_1 and putting $f_s = (1 - s)\mathrm{id} + sf_1$. The fixed central interval is chosen strictly inside the interval between these two points and contains the intersection with C in its interior. Choose a closed interval J_0 inside this fixed interval, with the intersection point in its interior. The trace of $I \setminus \operatorname{int} J_0$ under $\gamma \circ f_s$ is compact and disjoint from $Z \cup C$. The relative construction above gives a Hamiltonian isotopy for $\bar{\omega}$, supported in a compact subset of $N \setminus C$, which compresses the cord to the segment $|z| \le \ell$. Its velocity is tangent to the cord, so the Hamiltonian may have value zero on the whole cord.

The endpoint tangent planes of the transported surface remain symplectically orthogonal to the cord. Apply the preceding germ construction in disjoint neighborhoods of the two endpoints, contained in $N \setminus C$, to identify its surface germs with the two graph germs of §8.2. These isotopies fix the cord pointwise. Let Φ be the composition of the compression and both endpoint corrections, and put $S' = \Phi(S)$. Then

$$\Phi^*\bar{\omega} = \bar{\omega}, \qquad \Phi(\gamma(I)) = \{t = u = y = 0,\ |z| \le \ell\}.$$

The interior of this segment is disjoint from S'.

Choose small neighborhoods of its endpoints in which S' equals the two model graphs. The part of S' outside these neighborhoods is compact and disjoint from the closed segment, so their distance is positive for any fixed auxiliary metric. The cylinders

$$C_r = \{t = 0,\ u^2 + y^2 \le r^2,\ |z| \le \sqrt{\ell^2 + r^2}\}$$

converge to that segment as $r \to 0$. The final estimate of §8.2 places both whole graph disks and their boundary neighborhoods in the chosen endpoint neighborhoods for every sufficiently small r. Choose such an r with C_r contained in the linear region and disjoint from the remaining part of S'. Thus $S' \cap C_r$ consists exactly of the two model disks; their common collars extend into the unchanged surface.

Choose the convolution profile of §8.2 and perform the family A_λ inside C_r. The common collars make the replacement smooth away from the unique cone vertex. Equation (8.2) and the cap-center calculation prove positivity at every regular point. To identify the handle, consider the embedded disk tube

$$H = \{t = 0,\ u^2 + y^2 \le r^2/4,\ |z| \le \sqrt{\ell^2 + u^2 + y^2}\}.$$

Its core is $|z| \le \ell$, and its two cap faces are subdisks of A_0. Replacing these cap interiors by its lateral boundary gives the profile

$$u^2 + y^2 = \max(z^2 - \ell^2, r^2/4).$$

The convolution defining m_r rounds its two attachment corners and gives exactly A_1, relative to the outer collars. Orient H by $du \wedge dy \wedge dz$. Its cap faces have the orientations opposite to those of A_0; its lateral boundary has the orientation $-dz \wedge d\theta$ of §8.2. This is therefore an oriented handle attachment.

Apply Φ^{-1} to the modified surface, through the inverse Hamiltonian isotopy. The form $\bar{\omega}$ and the bundle $\bar{P}$ remain fixed. The resulting handle is $\Phi^{-1}(H)$, with the same corner rounding, and its core is the adjusted cord. The resulting surface agrees with S outside N. The local argument of [34, Lemma 3.3 of the accepted manuscript], in a regular neighborhood of the cord with its two oriented endpoint disks, gives uniqueness of the orientation-compatible handle up to ambient isotopy supported in that neighborhood.

The birth and linearization primitives are supported in one compact subset of B. Concatenate their paths, adding the final birth primitive to the linearization primitive, and keep the resulting primitive constant during the surface isotopies and surgery. This gives $\omega_s = \omega + d\beta_s$ with $\beta_0 = 0$ and a common compact support in U. Concatenate the corresponding positive-plane paths and then keep $\bar{P}$ fixed; thus $P_0 = P$, $\omega_s \in \Gamma(P_s)$ and $P_s = P$ outside U. The lemmas give exactly one new zero circle, with a single nontransverse birth point. Linearization preserves that circle; all Hamiltonian supports avoid it. The original zero germs and all prescribed surfaces disjoint from U are unchanged. Smooth reparametrizations, constant near the joining times and with positive derivative at the birth and cone parameters, give the stated deformation. □

9. Isotopies of positive configurations

A stabilization of an oriented surface is an oriented handle attachment along a cord, as defined in §8. The cord may belong to any isotopy class.

Lemma 9.1. *Let $D = \bigcup_j S_j \subset X$ be an orthogonal positive configuration for two closed two-forms ω_0, ω_1 which are symplectic and induce the same orientation on a neighborhood of D. Suppose that $\int_{S_j} \omega_0 = \int_{S_j} \omega_1$ for every j. There is an ambient isotopy Φ_t, supported in any prescribed neighborhood of D, such that $\Phi_t(S_j) = S_j$ for every j and $t \in [0,1]$, and $\Phi_1^* \omega_1 = \omega_0$ near D.*

Proof. Fix a neighborhood U of D. After each pullback below, denote the second form again by ω_1.

At a crossing, straighten the two branches to the coordinate planes in $\mathbb{R}^2 \times \mathbb{R}^2$. For either form, choose area coordinates on each plane. Its restrictions then coincide with those of the product area form, and orthogonality gives equality of the full tensors at the origin. The convex interpolation is symplectic on a sufficiently small product neighborhood. The relative Darboux theorem for a pair [46, Theorem 2.2.3] therefore gives product Darboux coordinates adapted to both branches. The transition F between the two charts preserves each coordinate plane and its orientation. The germs $z \mapsto s^{-1}F(sz)$, $0 < s \leq 1$, extend smoothly at $s = 0$ to dF_0; the latter belongs to the connected group $\mathrm{GL}^+(2,\mathbb{R}) \times \mathrm{GL}^+(2,\mathbb{R})$. Thus F is isotopic to the identity through germs preserving both planes. Extend the generating fields by cutoffs in disjoint crossing neighborhoods contained in U. On smaller neighborhoods their flows coincide with the original germs. After these pullbacks, the forms agree on crossing balls.

On each S_j, the difference of the restricted area forms has integral zero and vanishes near the crossing disks. It has a primitive vanishing on smaller disks: integration identifies the compactly supported second de Rham cohomology of the punctured connected surface with $\mathbb{R}$ [8, Corollary 5.8, p. 47]. The convex interpolation consists of positive area forms. Moser's theorem [11, Theorem 1.7 and its surface example], with this primitive, gives an isotopy of S_j fixed on those disks. Extend its vector field in tubular neighborhoods disjoint from the other components outside the crossing balls, and cut it off in U. The pulled-back forms now have the same restriction to every TS_j and still agree on smaller crossing balls.

Put $N_j = TX|_{S_j}/TS_j$. For $i = 0, 1$, let $q_{ij} : N_j \to TX|_{S_j}$ be the ω_i-orthogonal splitting of the quotient map, and let $\eta_{ij} = q_{ij}^* \omega_i$ be the induced fiber area form. The common ambient orientation and the common orientation of S_j imply $\eta_{1j} = f_j \eta_{0j}$ for a smooth function $f_j > 0$. For $0 \leq t \leq 1$, set

$$q_{j,t} = (1-t)q_{0j} + t q_{1j}, \qquad a_{j,t} = 1 - t + t f_j^{-1/2},$$
$$L_{j,t}(v + q_{0j}n) = v + q_{j,t}(a_{j,t}n), \qquad v \in TS_j,\ n \in N_j.$$

The maps $L_{j,t}$ are bundle automorphisms fixing TS_j pointwise, with $L_{j,0} = 1$ and $L_{j,1}^* \omega_1 = \omega_0$ along S_j. They are the identity near the crossing disks. Let exp be the exponential map of an auxiliary Riemannian metric. On a sufficiently small common neighborhood of the zero section, the maps

$$e_{j,t}(n) = \exp_x\bigl(q_{j,t}(a_{j,t} n)\bigr), \qquad x \in S_j, \quad n \in (N_j)_x,$$

are tubular embeddings, independent of t over the crossing disks. The germs $e_{j,t} \circ e_{j,0}^{-1}$ fix S_j and have differential $L_{j,t}$ there. The ambient tubular-neighborhood isotopy theorem [28, Chapter 8, §1, Theorem 1.8 and its proof] extends them to ambient isotopies supported in U. The extending vector fields may be chosen zero on smaller crossing balls, with disjoint tubular supports elsewhere. Their flows fix D pointwise. After the time-one pullbacks, the two forms agree as bilinear forms on $TX|_D$.

By [20, Lemma 1 and its proof], there is a local isotopy ψ_t, fixing D pointwise, with $\psi_1^* \omega_1 = \omega_0$ near D. In the crossing balls, choose product tubular coordinates. The radial contractions in that proof preserve smaller crossing balls, where the forms agree; the generating vector fields vanish there. On a common neighborhood of D contained in U, multiply these fields by a cutoff compactly supported in that neighborhood and equal to one near D. Their flows fix D and smaller crossing balls. A sufficiently small neighborhood of D has all its flow tracks in the region where the cutoff is one. There the time-one pullback is ω_0. Composing these isotopies proves the assertion. □

Lemma 9.2. *Let W be a compact connected oriented four-manifold with boundary, and let $F_0, F_1 \subset W$ be connected properly embedded oriented surfaces. Suppose that they agree on a collar of their common boundary L and that the closed difference cycle satisfies*

$$[F_0 - F_1] = 0 \quad \text{in } H_2(W; \mathbb{Z}).$$

After finitely many oriented handle attachments in the interior of W, the resulting surfaces are ambiently isotopic relative to a smaller boundary collar. The numbers of handles need not be equal when the initial genera differ.

Proof. The image of $[F_0 - F_1]$ under $H_2(W; \mathbb{Z}) \to H_2(W, L; \mathbb{Z})$ is $[F_0] - [F_1]$. Thus these relative classes agree. By [33, Proposition 2.13], finitely many oriented one-handle stabilizations in the interior give surfaces F_0^+, F_1^+ and an ambient isotopy, fixed on ∂W, carrying F_0^+ to F_1^+. The one-handles in that proposition are embedded three-dimensional one-handles [33, Definition 2.9(2)].

Suppose first that $L \neq \emptyset$. Choose an ambient collar adapted to the common collar of F_0^+ and F_1^+ [28, Chapter 4, §6, Theorem 6.2], and a tubular neighborhood of L in ∂W. Write $r \geq 0$ for the collar coordinate. The surfaces of the isotopy are transverse to ∂W. Projection to $L \times [0, \varepsilon)$ is therefore locally invertible along L on each surface. Compactness of $L \times [0, 1]$ gives a uniform choice of ε; compactness of the track outside a collar in the initial surface excludes any other sheets there. The surfaces are thus graphs of smooth sections $u_t(p, r)$ of the normal bundle of L in ∂W, pulled back to $L \times [0, \varepsilon)$, with $u_t(p, 0) = 0$. After decreasing ε, their endpoint sections satisfy $u_0 = u_1 = 0$. Choose a smooth function $\chi : [0, \varepsilon) \to [0, 1]$ which is zero near 0 and one near ε. Replace $u_t(p, r)$ by $\chi(r) u_t(p, r)$. These graphs join the unmodified surfaces where $\chi = 1$ and have the same endpoints. They form a smooth family constant on a smaller collar. On its total space over $[0, 1]$, lift ∂_t to a vector field equal to ∂_t on this product collar. Integrating that field gives parametrizations stationary there. If $L = \emptyset$, the compact surface track lies in the interior and is disjoint from a sufficiently small boundary collar.

Double W and F_0^+ along the stationary collar, and extend the surface isotopy by the identity on the reflected copy of F_0^+. The isotopy extension theorem [28, Chapter 8, §1, Theorem 1.3] gives an extending time-dependent vector field on the double. Multiply it by a cutoff equal to one on the support of the surface velocity and zero on a smaller collar of the seam. Its flow preserves each half and restricts to the required ambient isotopy of W. If $\partial W = \emptyset$, isotopy extension applies directly. □

9.1. **Integral rim translations by positive point pushes.** Let $A = \bigcup_{j=1}^{r} A_j$ be a connected orthogonal positive configuration for a closed two-form Ω such that $\Omega \wedge \Omega > 0$ near A. For a sufficiently small closed plumbing neighborhood E of A, put

$$W = X \setminus \operatorname{int} E, \qquad Y = \partial E.$$

The exterior W is connected. Indeed, a path in X with endpoints in W can be perturbed off the surfaces and their finitely many crossings, and then moved into W along the radial collars of the plumbing.

Lemma 9.3. *Let $B \subset X$ be a compact connected oriented embedded surface, with $\partial B \cap A = \emptyset$. Suppose that B is positive for Ω near $B \cap A$, that all these intersections are positive and transverse, and that none is a crossing of A. Assume $B \cap A_j \neq \emptyset$ for every j.*

After an isotopy of B supported near $B \cap A$ and through Ω-positive disks there, there are nested closed plumbings $E \subset \operatorname{int} E^+$, with $E^+ \cap \partial B = \emptyset$, for which the following holds. Put $F = B \cap W$. For every

$$\kappa \in \ker\bigl(H_2(W;\mathbb{Z}) \longrightarrow H_2(X;\mathbb{Z})\bigr),$$

a finite sequence of Hamiltonian point pushes carries B to a surface B' such that, for $F' = B' \cap W$,

$$[F' - F] = \kappa.$$

Each Hamiltonian isotopy preserves Ω, E, and every A_j, is the identity near the crossings of A and outside E^+, and at its endpoint restores the normal fibre disks $B \cap E$ and their boundary collars. In particular, F' and F have the same boundary collar. If B is positive for Ω everywhere, every transported surface is positive for Ω.

Proof. All homology and cohomology groups in this proof have integral coefficients. For any sufficiently small plumbing, the exact sequence of (X, W), excision, and Poincaré–Lefschetz duality [23, Theorems 2.20 and 3.43] give

$$H_3(X) \longrightarrow H_3(E, Y) \xrightarrow{\partial} H_2(W) \longrightarrow H_2(X), \qquad H_3(E, Y) \cong H^1(E).$$

Here ∂ is the boundary homomorphism into $H_2(Y)$ followed by inclusion in W. Let Γ be the graph with one vertex for each A_j and one edge for each crossing. The disk-bundle pieces of E meet in contractible crossing neighborhoods and have no triple overlaps. The cohomology Mayer–Vietoris sequence [23, §3.1, pp. 203–204] therefore gives

$$0 \longrightarrow H^1(\Gamma) \longrightarrow H^1(E) \longrightarrow \bigoplus_j H^1(A_j) \longrightarrow 0.$$

Delete small crossing disks from A_j and denote the resulting surface by A_j°. For an oriented simple closed curve $\gamma \subset \operatorname{int} A_j^\circ$, let P_γ be its normal disk bundle in E, oriented as a relative three-cycle in (E, Y), and let $T_\gamma = \partial P_\gamma \subset Y$. Thus T_γ is the normal circle bundle over γ. Choose $2g_j$ curves whose intersection functionals form an integral basis of $H^1(A_j)$, where g_j is the genus of A_j. The duals of their relative cycles P_γ restrict to this basis.

Also choose one boundary-parallel curve for each crossing disk. Its dual restricts to zero on every $H_1(A_k)$. Orient an edge of Γ and take the puncture curve at one of its ends. A loop in E following an oriented graph cycle meets the corresponding P_γ with intersection number equal, up to the choice of orientation, to the coefficient of that edge. Indeed, the loop crosses this normal disk bundle precisely when it passes from that vertex piece to the crossing neighborhood. Choose a spanning tree of Γ. The puncture curves for its non-tree edges pair with the corresponding fundamental graph cycles as the integral identity matrix, after choices of signs. Their duals therefore form a basis of the subgroup $H^1(\Gamma)$. Together with the handle curves, they generate $H_3(E, Y)$. Consequently the $[T_\gamma]$ generate the required kernel.

Call the points of $B \cap A$ marked points. Choose all the generator curves above disjoint from this finite set, and choose $p_j \in B \cap A_j$ for each j. Join p_j to each of the finitely many curves on A_j° by a path avoiding the other marked points. Traversing the path, the curve, and the reverse path gives a based loop η with the same homology class as γ. Smooth the parametrization, keeping the base point fixed. Choose a compact connected subsurface $K_j \subset \operatorname{int} A_j^\circ$, with nonempty boundary, containing the marked points, these curves, and these paths in its interior. If A_j has no crossing, remove an additional disk disjoint from this finite set of curves and points.

Let $\pi_j : N_j \to A_j$ be the symplectic normal bundle, and put $\sigma_j = \Omega|_{A_j}$. Choose a Hermitian metric on N_j compatible with its fibre area form and with the product charts at the crossings. Write ρ_j for its fibre radius and $s_j = \rho_j^2/2$. Since $H^2(K_j;\mathbb{Z}) = 0$, the complex line bundle $N_j|_{K_j}$ is trivial [25, Proposition 3.10]. For a local angular coordinate θ_j of period 2π, let α_j be a real connection one-form on the unit-circle bundle, pulled back radially to $N_j \setminus A_j$ and normalized by $\alpha_j(\partial_{\theta_j}) = 1$. Choose $\alpha_j = d\theta_j$ in a unitary trivialization over a neighborhood of K_j and choose the product connection on smaller crossing disks. These prescriptions are on disjoint closed sets and extend to a connection on N_j. The forms

$$\pi_j^*\sigma_j + d(s_j\alpha_j)$$

are smooth along the zero sections and agree in the product crossing charts. They define a symplectic form on a sufficiently small plumbing. Choose the tubular identifications to have the symplectic normal differential along A and to agree with those charts. The resulting form agrees with Ω on $TX|_A$ and on the crossing neighborhoods. The relative Moser construction in the proof of Lemma 9.1 identifies the two forms, fixing A and smaller crossing neighborhoods. In the resulting normal coordinates, over K_j,

$$\Omega = \pi_j^*\sigma_j + ds_j \wedge d\theta_j.$$

In disjoint balls at $B \cap A$, the positive graph deformation [50, arXiv version, Lemma 6] replaces B near each intersection by the prescribed normal fibre disk. The form and A remain fixed. Choose $E \subset \operatorname{int} E^+$ sufficiently small that $B \cap E^+$ consists precisely of these disks, $E^+ \cap \partial B = \emptyset$, and both plumbings have constant normal radii over the K_j.

Fix one of the based loops η on K_j. The isotropic isotopy extension theorem [64, Proposition 2], applied to the moving point in $\operatorname{int} K_j \setminus ((B \cap A_j) \setminus \{p_j\})$, gives a Hamiltonian h_t whose flow ψ_t satisfies $\psi_t(p_j) = \eta(t)$; a point is isotropic and has zero first homology. A cutoff equal to one near the compact point trace makes h_t compactly supported in this punctured surface. In particular, $\psi_1(p_j) = p_j$, and the other marked points are fixed.

Choose a smooth normal cutoff χ, equal to one on E and on a neighborhood of its boundary over K_j, and zero near ∂E^+. Extend

$$H_t = \chi(s_j)\pi_j^* h_t$$

by zero outside $N_j|_{K_j}$. Let X_{H_t} be its Hamiltonian vector field, defined by $\iota_{X_{H_t}}\Omega = -dH_t$, and let Φ_t be its flow. Since H_t is independent of θ_j, this flow preserves s_j. Write z for the fibre coordinate in the chosen unitary trivialization. Where $\chi = 1$, the product formula gives

$$\Phi_t(x,z) = (\psi_t(x), z).$$

Thus Φ_t preserves E, every A_k, and their crossing charts. At time one it fixes $B \cap E$ and its boundary collars pointwise. The support lies in E^+, where Ω is symplectic. Extension by zero preserves the global closed form, since $\iota_{X_{H_t}}\Omega = -dH_t$ and hence $\mathcal{L}_{X_{H_t}}\Omega = 0$.

The trace of F under Φ_t is a three-chain in W. Its boundary consists of $\Phi_1(F) - F$ and the boundary traces. All boundary circles except the one over p_j are fixed, including those in ∂B. In the product coordinates the remaining trace is the normal circle over η; it represents $[T_\gamma]$. Therefore

$$[\Phi_1(F) - F] = \pm[T_\gamma] \quad \text{in } H_2(W).$$

Reversing η gives the opposite sign. Every class in the kernel is an integral sum of these generators. Successive point pushes restore the same disks and collars, so their differences add. Hamiltonian invariance of Ω preserves positivity. □

Proposition 9.4. *Let ω_0, ω_1 be closed two-forms in the same real class a, carried by prescribed positive three-plane bundles P_0, P_1, respectively. Suppose that their zero sets are transverse and that each carries a connected positive configuration of connected closed embedded oriented surfaces*

$$D^i = D_1^i \cup \cdots \cup D_N^i, \qquad i = 0, 1.$$

Assume $[D_j^0] = [D_j^1] \in H_2(X;\mathbb{Z})$ for every j, that all intersections of distinct components are transverse and positive, that there are no triple points, and that $[D_k^i] \cdot [D_\ell^i] > 0$ for $k < \ell$.

There are smooth paths of closed forms in the class a, with smooth positive three-plane bundles extending the prescribed initial bundles, whose final forms carry the same embedded positive configuration after finitely many stabilizations of its components. The final form germs may be chosen equal on a plumbing of that configuration. All final zeros are transverse.

Proof. The closed surfaces D_1^0, D_1^1 have the same integral homology class. By [7, arXiv version, Theorem 5], finitely many oriented handle attachments make them ambiently isotopic. Choose the cord endpoints away from all component intersections. General position, relative to these endpoints, makes each cord disjoint from the other components and from the zero circles. Proposition 8.2, with sufficiently small supports, realizes the attachments through closed forms in a and positive three-plane bundles, leaving every other component and every previous zero germ unchanged. Let Φ_t carry the second stabilized surface to the first. Transport its configuration by Φ_t and its form and positive bundle by $(\Phi_t^{-1})^*$. The first components then coincide.

Suppose that the first $j-1$ components coincide and that their union A is orthogonal for both forms. This holds for $j = 2$. Their areas agree, since both forms represent a. They are disjoint from the zero sets, and the positive three-plane condition gives the same ambient orientation. Lemma 9.1 gives an isotopy Ψ_t, preserving each component of A, such that $\Psi_1^*\omega_1 = \omega_0$ near A. Pull back the second form and its positive bundle by Ψ_t, and replace its components by their inverse images under Ψ_t. Denote the common form germ by Ω and the next components by $B_i = D_j^i$.

For each component A_k, the sets $B_0 \cap A_k$ and $B_1 \cap A_k$ have the same positive cardinality, namely $[D_j^0] \cdot [A_k] = [D_j^1] \cdot [A_k]$. On the punctured A_k, choose an isotopy of the finite point set $B_1 \cap A_k$ to $B_0 \cap A_k$. Proposition 2 of [64] gives a Hamiltonian extension, since a finite set is isotropic and has zero first homology. Choose its support away from the crossings of A. Extend the Hamiltonian to a symplectic normal neighborhood [11, Theorem 1.11], constant on the fibres near the zero section and cut off farther out. Its flow preserves Ω and every retained component. Transport all unmatched components of the second configuration and its positive bundle by these Hamiltonian maps.

For this common marked set, choose the finite generator curves, based loops, and a common normal model for Ω as in the proof of Lemma 9.3. At each common intersection point, a sufficiently small ball contains only the retained branch and B_i in the ith configuration. The positive graph deformation [50, arXiv version, Lemma 6], with Ω fixed, replaces each of the two germs by the same prescribed normal fibre disk. It fixes A and all other components. Thus $A \cup B_i$ is orthogonal at these crossings. Choose the nested constant-radius plumbings $E \subset \operatorname{int} E^+$ from that construction sufficiently small for both B_i, so that $B_0 \cap E = B_1 \cap E$ consists of these common disks. Both plumbings lie in the common form neighborhood. The surfaces

$$F_i = B_i \cap W, \qquad W = X \setminus \operatorname{int} E,$$

are connected properly embedded surfaces with the same boundary collar: each is obtained from a connected closed surface by deleting finitely many disks. Their common caps in E give

$$[F_1 - F_0] \in \ker\bigl(H_2(W;\mathbb{Z}) \longrightarrow H_2(X;\mathbb{Z})\bigr).$$

Apply Lemma 9.3 to B_1 and the current global form ω_1, with the plumbings just chosen and $\kappa = [F_0 - F_1]$. Its hypotheses hold because B_1 meets every component of A positively. The Hamiltonian point pushes leave the common disks unchanged at their endpoints and give $[F_1 - F_0] = 0$ in $H_2(W;\mathbb{Z})$ after replacing F_1. Transport all remaining components and the positive bundle by the same maps. Lemma 9.2 now gives interior handle attachments and an ambient isotopy fixed on a boundary collar. Choose its cords, by general position, away from the unmatched components and zero circles, and apply Proposition 8.2 with supports disjoint from E and those components. Extend the final isotopy by the identity on E, and transport the second form, its positive bundle, and its whole configuration. The jth components then coincide. Their crossings with A have the common normal fibre germs, so the enlarged retained configuration is orthogonal for both forms.

Every local graph deformation was positive and supported in a ball containing only the two specified branches. Every ambient transport preserves all pairwise intersection signs and the absence of triple points. Every handle attachment has support disjoint from the other components. An oriented handle attachment changes its surface cycle by a boundary. Hence all component classes remain unchanged in $H_2(X;\mathbb{Z})$. These facts preserve the hypotheses at each induction step. There are finitely many steps and finitely many attachments at each step. Proposition 8.2 gives transverse final zeros; ambient pullbacks preserve their transversality. All form changes are exact deformations or pullbacks by isotopies, and hence remain in a. The positive bundles extend the initial bundles by Proposition 8.2 and by transport. The final common configuration is orthogonal and disjoint from both zero sets. A last application of Lemma 9.1, with the same pullback and inverse-image convention, identifies the final form germs. Reparametrize the finite paths to be stationary at their joining parameters. □

9.2. **Matching the prequantum connections.**

Lemma 9.5. *Let $D = \bigcup_{j=1}^{r} S_j \subset X$ be an orthogonal positive configuration for a symplectic form Ω defined near D. Let Ω_0, Ω_1 be closed two-forms on X, in the same cohomology class and equal to Ω near D. Let $L \to X$ be a Hermitian line bundle with unitary connections ∇_0, ∇_1 of curvatures $F_{\nabla_i} = -2\pi i\kappa\Omega_i$, where $\kappa > 0$. For every neighborhood O of D, there is an ambient isotopy ψ_t, $0 \le t \le 1$, supported in O, preserving every S_j setwise and fixing crossing charts, such that $\psi_t^*\Omega_1 = \Omega$ on one neighborhood of D for all t. The pullback of ∇_1 by a unitary lift of ψ_1 is gauge equivalent to ∇_0 near D. The isotopy preserves the global cohomology classes of all transported closed forms. Its support can avoid any prescribed closed set disjoint from D.*

Proof. Choose an open neighborhood U of D with $\overline{U} \subset O$ on which $\Omega_0 = \Omega_1 = \Omega$. The difference of two unitary connections on $L|_U$ is an imaginary-valued one-form [11, §6.3, pp. 166–168]. Write

$$\nabla_1 - \nabla_0 = -2\pi i\beta, \qquad d\beta = 0, \qquad \theta_0 = -\kappa^{-1}\beta,$$

where β is real. The equality $d\beta = 0$ follows from $F_{\nabla_1} = F_{\nabla_0}$ on U.

For each crossing point p, choose balls $B_p^- \Subset B_p \Subset U$, with the B_p pairwise disjoint. By the Poincaré lemma [70, Theorem 4.18], $\theta_0 = dh_p$ on B_p. Choose $\chi_p \in C_c^\infty(B_p)$ equal to one near $\overline{B_p^-}$, and put

$$\theta_1 = \theta_0 - d\Big(\sum_p \chi_p h_p\Big).$$

The products $\chi_p h_p$ are extended by zero outside B_p. Thus θ_1 is closed and vanishes near each $\overline{B_p^-}$.

Let $N_j = (TS_j)^\Omega$ be the symplectic normal bundle of S_j, and put $K_j = \operatorname{supp}(\theta_1|_{N_j}) \subset S_j$. The compact sets K_j are disjoint from the other components and from the closed balls $\overline{B_p^-}$. Choose pairwise disjoint tubular neighborhoods $T_j \Subset U$ of neighborhoods of K_j in S_j, avoiding those balls and the other components. In normal coordinates $(x,v) \in N_j$, choose $\rho_j \in C_c^\infty(T_j)$ equal to one near K_j and set

$$F_j(x,v) = \rho_j(x,v)(\theta_1)_x(v).$$

Extend F_j by zero to X. Along S_j,

$$F_j = 0, \qquad dF_j|_{TS_j} = 0, \qquad dF_j|_{N_j} = \theta_1|_{N_j}.$$

Consequently, the function

$$f = -\sum_p \chi_p h_p - \sum_j F_j \in C_c^\infty(U;\mathbb{R})$$

and the one-form $\theta = \theta_0 + df$ satisfy

$$d\theta = 0, \qquad \theta|_{N_j} = 0, \qquad \theta = 0 \text{ on each } B_p^-.$$

Regard f as a smooth function on X by extension by zero.

Define V on U by $\iota_V\Omega = \theta$. The equality $\theta|_{N_j} = 0$ gives $V|_{S_j} \in (N_j)^{\Omega} = TS_j$, and $V = 0$ on each B_p^-. Moreover, $\theta(V) = \Omega(V, V) = 0$, so Cartan's formula gives

$$\mathcal{L}_V\Omega = d\theta = 0, \qquad \mathcal{L}_V\theta = d\big(\theta(V)\big) = 0.$$

Choose $\chi \in C_c^\infty(U)$ equal to one on an open neighborhood U_0 of D, and extend $W = \chi V$ by zero to X. Its flow ψ_t is defined for all t, preserves each S_j setwise, and fixes the balls B_p^-. Continuity of $(t, x) \mapsto \psi_t(x)$ and compactness of $[0, 1] \times D$ give a neighborhood U_1 of D such that $\psi_t(U_1) \subset U_0$ for $0 \le t \le 1$. On U_1,

$$\psi_t^*\Omega_1 = \Omega, \qquad \psi_t^*\theta = \theta \quad (0 \le t \le 1).$$

Let $\widehat{\psi}_t : L \to L$ be the unitary lift given by ∇_1-parallel transport along $s \mapsto \psi_s(x)$, $0 \le s \le t$, and put $\nabla_{1,t} = \widehat{\psi}_t^*\nabla_1$. Since $W = V$ along the trajectories from U_1, the formula for the flow of a horizontal lift [1, proof of Theorem 2.2, pp. 2511–2512] gives

$$\nabla_{1,t} - \nabla_1 = -2\pi i\kappa t\theta \quad \text{on } U_1.$$

Hence

$$\nabla_{1,1} - \nabla_0 = -2\pi i(\beta + \kappa\theta) = -2\pi i\kappa\, df.$$

The unitary gauge $g = e^{2\pi i\kappa f}$, equal to one outside U, therefore satisfies $g^{-1}\nabla_{1,1}g = \nabla_0$ on U_1. Since ψ_t is isotopic to the identity, it induces the identity on cohomology. Given a closed set K disjoint from D, choose U with $\overline{U} \subset O \setminus K$. □

10. Common pencils and Thom forms

Lemma 10.1. *Suppose ω_0, ω_1 are near-symplectic forms in the same class a, with the same symplectic germ on an orthogonal positive configuration $D = \bigcup_{j=1}^n S_j$. Put $d_j = \mathrm{PD}[S_j] \in H^2(X; \mathbb{Z})$, and suppose*

$$d_j^2 > 0, \qquad a = \sum_j \lambda_j d_j, \qquad \lambda_j > 0.$$

There are closed Thom forms ρ_j^0 representing d_j, positive numbers μ_j, and a positive rational class b such that

$$\Omega_i = \omega_i - \sum_j \mu_j\rho_j^0, \qquad [\Omega_i] = b, \qquad a = b + \sum_j \mu_j d_j.$$

The forms Ω_i are near-symplectic with the original zero germs and have the same symplectic germ near D. The forms ρ_j^0 are simultaneous inflation forms for both Ω_i:

$$(\rho_j^0)^2 \ge 0, \qquad \Omega_i \wedge \rho_j^0 \ge 0, \qquad \rho_j^0 \wedge \rho_l^0 \ge 0 \quad (j \ne l).$$

Their supports can be chosen in any prescribed neighborhood of D.

Proof. Write ω for the common symplectic form near D and put $\sigma_j = \omega|_{S_j}$. The normal-form construction of [44, arXiv version, §2, proof of Lemma 1.1 and Remark 2.1(ii)] gives tubular neighborhoods $\pi_j : T_j \to S_j$ and circle connections α_j of integral one on each oriented circle fiber, pulled back radially to $T_j \setminus S_j$. For fiber radii r_j, put $s_j = \pi r_j^2$. Then

$$H_j = \pi_j^*\sigma_j, \qquad d\alpha_j = -c_jH_j, \qquad \omega = H_j + d(s_j\alpha_j).$$

Here $c_j \ge 0$ is a smooth function on S_j with $\int_{S_j} c_j\sigma_j = d_j^2$; functions on S_j are identified with their pullbacks. Distinct tubes overlap only in product crossing charts. Each component of an overlap is a product of disks, saturated by the fibers of both tube projections. There both curvature densities vanish and the two connections are the angular forms in the respective normal factors. All tubes lie in the prescribed neighborhood and in the common symplectic region. Put

$$V_j = ds_j \wedge \alpha_j, \qquad A_j = 1 - s_jc_j.$$

The forms V_j extend smoothly over S_j, and $H_j^2 = V_j^2 = 0$, $H_j \wedge V_j > 0$. Shrink the tubes so that $A_j \geq \frac{1}{2}$.

Choose smooth profiles $h_j : [0,\infty) \to [0,1]$ with $h_j(0) = 1$, $h_j' \leq 0$, and support contained in the radial interval of T_j. Put $B_j = -h_j' \geq 0$ and

$$\rho_j^0 = -d\big(h_j(s_j)\alpha_j\big) = h_j c_j H_j + B_j V_j.$$

The expression on the right extends smoothly over S_j and by zero outside T_j. It is closed, and its integral on each oriented normal fiber is $-\int_0^\infty h_j'(s)\,ds = 1$. Hence it is a Thom form and $[\rho_j^0] = d_j$. Moreover,

$$(\rho_j^0)^2 = 2h_j c_j B_j\, H_j \wedge V_j \geq 0.$$

At a crossing of S_j and S_l, the product model gives

$$H_j = V_l, \qquad H_l = V_j, \qquad \omega = V_j + V_l, \qquad \rho_j^0 = B_j V_j, \qquad \rho_l^0 = B_l V_l.$$

Thus $\rho_j^0 \wedge \rho_l^0 = B_j B_l V_j \wedge V_l \geq 0$ there; outside these charts the supports are disjoint.

Take positive μ_j satisfying

$$\mu_j \|c_j\|_\infty < \tfrac{1}{4}, \qquad \mu_j \|B_j\|_\infty < \tfrac{1}{2}.$$

Away from the crossing overlaps, the modified common form is

$$\omega - \mu_j \rho_j^0 = (A_j - \mu_j h_j c_j) H_j + (1 - \mu_j B_j) V_j.$$

Both coefficients are positive, since $0 \leq h_j \leq 1$. Its wedge product with ρ_j^0 is

$$\big((A_j - \mu_j h_j c_j) B_j + (1 - \mu_j B_j) h_j c_j\big) H_j \wedge V_j \geq 0.$$

At a crossing the simultaneous subtraction is

$$\omega - \mu_j \rho_j^0 - \mu_l \rho_l^0 = (1 - \mu_j B_j) V_j + (1 - \mu_l B_l) V_l.$$

Its square is positive, and its wedge product with ρ_j^0 is $(1 - \mu_l B_l) B_j V_j \wedge V_l \geq 0$; the formula with j, l interchanged gives the other inequality. There are no triple overlaps. Consequently the forms Ω_i are symplectic on the supports and satisfy all the stated wedge inequalities. They agree near D and equal ω_i near their zero sets.

The same Thom forms retain their normal models for Ω_i. Set

$$\widetilde{s}_j = s_j + \mu_j h_j(s_j) - \mu_j, \qquad \sigma_j' = \sigma_j - \sum_l \mu_l\, \rho_l^0|_{S_j}.$$

Then $\widetilde{s}_j(0) = 0$ and $d\widetilde{s}_j/ds_j = 1 - \mu_j B_j > 0$. The quotient $\widetilde{s}_j / s_j$ extends smoothly with positive value $1 - \mu_j B_j(0)$ at zero, so the radial change extends smoothly over the zero section. Away from the crossing disks, $\sigma_j' = (1 - \mu_j c_j)\sigma_j$; on those disks $c_j = 0$. For $l \neq j$, the product models give $\rho_l^0|_{T_j} = \pi_j^*(\rho_l^0|_{S_j})$. Thus, on T_j,

$$\Omega_i = \pi_j^* \sigma_j' + d(\widetilde{s}_j \alpha_j), \qquad d\alpha_j = -\frac{c_j}{1 - \mu_j c_j}\, \pi_j^* \sigma_j'.$$

The form σ_j' is positive, the new curvature density is nonnegative, and the profile $h_j(s_j(\widetilde{s}_j))$ is nonincreasing, with value one at zero. At a crossing the two radial coordinate changes act on separate factors.

Choose, as in [45, pp. 279–280], rational numbers q_j with $0 < q_j < \lambda_j$ sufficiently close to λ_j that $\mu_j = \lambda_j - q_j$ satisfy the preceding bounds, and put $b = \sum_j q_j d_j$. Then $[\Omega_i] = b$ and $a = b + \sum_j \mu_j d_j$. Positive intersections give $d_j d_l \geq 0$ for $j \neq l$, so

$$b^2 = \sum_j q_j^2 d_j^2 + 2 \sum_{j<l} q_j q_l\, d_j d_l > 0.$$

The same calculation for the coefficients $(1-t)\lambda_j + t q_j$, $0 \leq t \leq 1$, places b in the same component of the positive cone as a. □

A *singular Lefschetz pencil* on X consists of a finite set $B \subset X$, a closed embedded one-manifold $\Gamma \subset X \setminus B$, and a smooth map $f : X \setminus B \to \mathbb{CP}^1$ with the following local models in oriented source and target coordinates, complex for the first two models, [4, Definition 2, pp. 1045–1046]. At a base point it is $[z_0 : z_1]$. At each of finitely many isolated critical points in $X \setminus (B \cup \Gamma)$ it is $z_1^2 + z_2^2$. Along Γ it is

$$(t, x_1, x_2, x_3) \longmapsto (t, x_1^2 + x_2^2 - x_3^2), \qquad \Gamma = \{x_1 = x_2 = x_3 = 0\}.$$

The map is a submersion elsewhere. Its *compactified fiber* over y is $F_y = f^{-1}(y) \cup B$. For a regular value y, this is a closed smooth surface, oriented by the orientations of X and $\mathbb{CP}^1$. In the base-point chart, if $y = [u : v]$, its tangent line at the base point is $\{vz_0 - uz_1 = 0\}$. A pencil is *compatible* with a near-symplectic form ω if $\Gamma = \omega^{-1}(0)$ and $\omega|_{TF_y} > 0$ for every regular value y, including at the base points.

A *positive simple branched covering* between closed connected oriented surfaces is a smooth surjective map which, in oriented local coordinates, has model $z \mapsto z^2$ at each ramification point and is an orientation-preserving local diffeomorphism elsewhere, with distinct critical values.

Proposition 10.2. *Let X be a closed oriented four-manifold, and let Ω_0, Ω_1 be near-symplectic forms in the same positive rational class b. Suppose that they are symplectic near an orthogonal positive configuration $D = \bigcup_j S_j$, with no triple points, and that their germs along D have been identified by ambient isotopies. Put $Z_i = \Omega_i^{-1}(0)$. Fix an integer $N_0 > 0$, a class $c_0 \in H^2(X; \mathbb{Z})$ with $(c_0)_{\mathbb{R}} = N_0 b$, and a Hermitian line bundle L_0 with $c_1(L_0) = c_0$.*

There are cohomologous near-symplectic deformations supported near $Z_i \cup D$ such that, denoting their final forms again by Ω_i, the following hold.

(1) For some integer $m > 0$, the bundle $L = L_0^{\otimes m}$ has unitary connections ∇_i of curvature $-2\pi i N \Omega_i$, where $N = mN_0$.

(2) For a common sufficiently large odd integer k, there are smooth section pairs $s^i = (s_0^i, s_1^i)$ of $L^{\otimes k} \oplus L^{\otimes k}$ defining singular Lefschetz pencils $f_i = [s_0^i : s_1^i]$.

(3) On a neighborhood U of D, the forms, connections, and section pairs agree. The common pair is nonvanishing, and its projective map is a submersion with symplectic fibres. Its restriction to each S_j is a positive simple branched covering. No crossing of D is a branch point.

At each basepoint, Ω_i is positive on every complex line in the coordinates given by the section pair. The induced complex structure is tamed on a smaller basepoint neighborhood. Moreover, $\Omega_i \wedge f_i^* \sigma \geq 0$ off the basepoints. All basepoints are simple and positive. Their number is $(kN)^2 b^2$ at both endpoints, and can be made at least two.

Suppose, in addition, that the initial forms satisfy

$$\Omega_i = \omega_i - \sum_j \mu_j \rho_{j,i}^0,$$

where ω_i are near-symplectic forms in a common class $a \in H^2(X; \mathbb{R})$, $\mu_j > 0$, and $\rho_{j,i}^0$ are closed Thom forms of S_j, supported near D and away from the chosen neighborhoods of Z_i. Let $P_i \subset \Lambda^2 T^* X$ be prescribed smooth positive rank-three subbundles containing ω_i. Then the deformations above induce paths of near-symplectic forms ω_i in a, together with smooth paths of positive rank-three subbundles containing them and starting at P_i.

Proof. Forms and connections. Choose disjoint tubes about the components of Z_i, away from D. By [4, Proposition 1 and §7.2, p. 1106], Ω_i is contained in a smooth positive rank-three bundle Q_i. In the last assertion of the proposition, choose $Q_i = P_i$ near Z_i: there $\Omega_i = \omega_i$, and the convex extension in the cited proof is relative to a smaller tube. When P_i is prescribed, choose the Honda and holonomy deformations below with support in the region $Q_i = P_i$. Lemma 2.2 gives a metric with $\Lambda^+ = Q_i$. Thus Ω_i is self-dual and harmonic; its zeros are transverse by near-symplecticity. Honda's theorem [29, §4, Theorem 7 and its proof, in the author's version] therefore gives a cohomologous deformation,

supported in these tubes, to the standard even or odd model, after a choice of orientation of each zero circle.

Write (θ, x), with $x \in \mathbb{R}^3$, for Honda's normal coordinates, and let Ω_{std} denote the chosen standard model. On a smaller tube the family is $\beta_t = (1-t)\Omega_i + t\Omega_{\mathrm{std}}$. In the eigenbundle coordinates, write the transverse coefficient of Ω_i at $x = 0$ as $L_+(\theta) \oplus \lambda_-(\theta)$, with $L_+(\theta) > 0$ on the two-dimensional positive eigenspace and $\lambda_-(\theta) < 0$ on the negative eigenline. The transverse coefficient of β_t has blocks $(1-t)L_+(\theta) + t\,\mathrm{id}$ and $(1-t)\lambda_-(\theta) - 2t$. These blocks are invertible, and their image is the positive plane Q_i at the core. Trivialize the two-form bundle along each normal fibre, preserving the wedge pairing and making Q_i constant in that fibre. Hadamard's formula gives

$$\beta_t(\theta, x) = \sum_{\ell=1}^{3} x_\ell B_{\ell,t}(\theta, x), \qquad B_{\ell,t}(\theta, x) = \int_0^1 \partial_{x_\ell} \beta_t(\theta, ux)\, du.$$

At $x = 0$ the $B_{\ell,t}$ form the positive transverse frame. Their spans remain positive and of rank three on a uniform smaller tube, by compactness in (θ, t); at $t = 0$ they equal Q_i. In the odd model the formula is equivariant on the double cover and hence descends. On the annuli in Honda's construction the forms have positive square. Fix a positive volume form and a wedge-orthogonal splitting $\Lambda^2 T^*X = E^+ \oplus E^-$ into positive and negative rank-three subbundles, and write $\beta_t = \beta_t^+ + \beta_t^-$ in this splitting. The positive graphs containing β_t are the graphs of maps $A : E^+ \to E^-$ satisfying

$$A\beta_t^+ = \beta_t^-, \qquad \|A\| < 1.$$

Here the operator norm is induced by the positive forms on E^+ and E^- obtained from the wedge pairing and its negative. The convex graph construction of [4, §7.2] extends the positive subbundles containing β_t relative to the inner tube, the exterior, and $t = 0$. Thus they form a smooth family with the prescribed initial bundle on the zero tubes.

Choose a sufficiently large multiple $N = mN_0$ and put $L = L_0^{\otimes m}$. Since $c_1(L)_{\mathbb{R}} = Nb$, choose unitary connections with curvature $-2\pi i N\Omega_i$. On each zero tube let θ have period one and let χ equal one on an inner tube and zero outside the chosen tube. For a suitable $h \in [-1/2, 1/2]$, the replacements

$$\nabla_i \longmapsto \nabla_i - 2\pi i h\chi\, d\theta, \qquad \Omega_i \longmapsto \Omega_i + \frac{h}{N}\, d(\chi\, d\theta)$$

give holonomy -1 along the zero circle. The change of form is exact, vanishes on the inner tube, and is $O(N^{-1})$ on the symplectic annulus. The straight deformation is symplectic there for all sufficiently large N. One N suffices for the finitely many annuli at both endpoints, and the core models are unchanged. The same relative graph construction gives positive bundles along this annular deformation.

Apply Lemma 9.5 in the common symplectic neighborhood of D, with support disjoint from the zero tubes. Its isotopy transports the forms, bundle, and Thom forms together. After its unitary lift and a gauge transformation, the two connections agree on a smaller neighborhood of D. Write Ω for the common form there. Choose an Ω-compatible almost-complex structure J preserving each TS_j. Orthogonality gives such choices in the crossing charts, and the contractibility of the compatible choices gives an extension over a neighborhood of D. Extend J to an Ω_i-compatible structure J_i on $X \setminus Z_i$, equal to the standard model structure on smaller zero tubes. Let $J_{i,k}$ be the structures of [4, §3]; they agree with J_i outside the zero tubes and with J near D. Put

$$q_k = 2\pi N k, \qquad \epsilon_k = q_k^{-1/3}, \qquad g_{i,k} = q_k \Omega_i(\cdot, J_{i,k}\cdot).$$

For odd k, the holonomy of $L^{\otimes k}$ along each zero circle is $(-1)^k = -1$. The rescaling in the proof of Lemma 7.4 gives the coordinate and localization estimates of [4, §3.1 and §6.1], with error $O(\epsilon_k)$. On the common symplectic region near D the error is $O(k^{-1/2})$, with N fixed. Scaled covariant norms are taken with respect to $g_{i,k}$, the induced Hermitian connection on $L^{\otimes k}$, and the metric on $\mathbb{CP}^1$ associated with σ. The symbols ∂ and $\bar\partial$ denote the complex-linear and complex-antilinear parts for $J_{i,k}$ and the standard complex structure on the target. The Hermitian metric, connection, and almost-complex structure are common near D for every k.

Common sections. In a closed symplectic four-manifold with rational symplectic class, a prescribed symplectic surface is a positive branched multisection of a sufficiently high-degree Lefschetz pencil [14, arXiv version, Proposition 2.9 and Definition 2.7]. For the present configuration choose compact neighborhoods

$$D \Subset V_0 \Subset V_1 \Subset V_2 \Subset V_3 \Subset V_4$$

inside the common symplectic region, disjoint from the zero tubes. These neighborhoods and the cutoffs separating them are independent of k. Put $g(v, w) = \Omega(v, Jw)$ and $g_k = q_k g$ on this region; thus $g_{0,k} = g_{1,k} = g_k$ there. For every such cutoff χ and every integer $q \geq 0$, $\|\nabla^q \chi\|_{g_k} = O(k^{-q/2})$. The Leibniz rule therefore preserves the approximately holomorphic estimates under multiplication by χ. Gaussian bounds are preserved as well; at a fixed positive distance from the center the tails are exponentially small in k.

Let $K_{i,k}$ and $W_{i,k}$ be the sets denoted by K and Z_0 in [4, §2] for endpoint i. If $Z_i = \emptyset$, take $K_{i,k} = W_{i,k} = X$ and the model pair to be zero. The model pairs of [4, §6.2, Proposition 12] have support in the zero-model neighborhoods, chosen disjoint from V_4. Both pairs are consequently zero on V_4. For sufficiently large k,

$$V_4 \subset \operatorname{int} K_{0,k} \cap \operatorname{int} K_{1,k}.$$

Perturbations supported in V_4 leave the model pairs on $W_{i,k} \setminus K_{i,k}$ unchanged. Their fixed-region approximately holomorphic bounds are $O(k^{-1/2}) = O(\epsilon_k)$, so the uniform constants in Hypothesis H3 of [4] may be enlarged independently of k.

Choose one system of Gaussian peak sections from the common connection and J, with centers covering V_3 and cutoffs supported in the interior of V_4. The distance from V_3 to the cutoff region is fixed and positive. For large k the scaled balls about these centers lie where the cutoff is one. Multiplication by coordinate monomials of degree at most two gives uniformly bounded frames for values and holomorphic jets through order two, with Gaussian decay [3, arXiv version, §4, proof of Theorem 1.1]. The local perturbation and finite-color estimates of [3, arXiv version, Propositions 4.1–4.2] apply on V_3: discarding centers outside a covering of V_3 only decreases the Gaussian sums. Their total scaled C^3 norm may be made arbitrarily small, with a positive quantitative transversality constant independent of sufficiently large k.

Apply these estimates first to the pair and to the projective complex derivative away from its basepoint neighborhoods, as in [4, §2, proof of Theorem 4] and [3, arXiv version, §5, projective maps]. The pair is quantitatively transverse to zero on V_3; outside small scaled balls about its zeros, the projective complex derivative is quantitatively transverse to zero. The estimates for the pair apply on the remaining balls.

Write $s = (s_0, s_1)$ for the common pair and $f = [s_0 : s_1]$ wherever $s \neq 0$. For each S_j, impose successively

(1) a positive lower bound for the norm of the pair;
(2) a positive lower bound for $|\partial f|$, where $\partial f \in (T^{1,0}X)^* \otimes f^* T\mathbb{CP}^1$;
(3) quantitative transversality of $\partial(f|_{S_j})$ to zero.

The first two zero strata have complex codimension two, whereas S_j has complex dimension one; quantitative transversality therefore gives positive lower bounds for $|s|$ and $|\partial f|$ on S_j. The third zero stratum has complex codimension one. Its quantitative transversality gives a uniformly nonzero tangential complex Hessian at each zero.

These restricted perturbations are restrictions of ambient sections. Since S_j is J-holomorphic, its parametrization in scaled coordinates differs from its tangent complex line by $O(k^{-1/2})$ in each fixed derivative norm. Restricting the polynomial peak sections gives bounded frames for the tangential jets; the normal-coordinate monomials give the full derivative components in the second condition. The defining jet functions retain the approximately holomorphic estimates. Thus [3, arXiv version, Proposition 4.2 and the jet-frame argument in §4] applies on each complex one-dimensional restricted domain. The ambient Gaussian estimates give the finite-color estimates for centers on S_j.

At a crossing choose scaled coordinates (z, w) in which the two tangent planes are the coordinate axes, and a denominator component s_0 bounded away from zero. Let τ be a peak section centered at the crossing, with value one in the chosen unitary trivialization. Adding $(\lambda z + \mu w)\tau$ to s_1 leaves the pair's value there unchanged and varies the two restricted derivatives by

$$(\partial_z f, \partial_w f) \longmapsto (\partial_z f + c\lambda, \partial_w f + c\mu), \qquad c = \tau/s_0 \neq 0.$$

Both $|c|$ and $|c|^{-1}$ have uniform bounds. In any fixed coefficient ball, the coefficients may therefore be chosen a uniform positive distance from the two affine hyperplanes where one derivative vanishes. The crossings are finite and their scaled balls are disjoint for large k. Choose each successive total scaled C^3 perturbation smaller than the openness bounds for all preceding conditions. There are finitely many surfaces and crossings; all final lower bounds and transversality constants are positive and independent of k. The same sections and coefficients at both endpoints give equal pairs on V_4, with these conditions on V_3 and D, and leave the zero-circle models unchanged.

Sections on the complements and pencil singularities. Choose a fixed cutoff ϑ equal to zero near V_0 and one outside V_1. At endpoint i, multiply the localized sections of [4, Hypothesis H2] by ϑ, retaining centers covering $K_{i,k} \setminus V_2$. Their scaled balls lie where $\vartheta = 1$ for large k, since $V_1 \Subset V_2$, so their jet frames retain their lower bounds. The cutoff preserves the derivative and Gaussian estimates. Its transition lies in the fixed common symplectic region; near $W_{i,k} \setminus K_{i,k}$ it is identically one.

The retained systems satisfy the localization, summation, and separation bounds of H2, with chart balls covering $K_{i,k} \setminus V_2$. The common perturbations vanish on $W_{i,k} \setminus K_{i,k}$ and have uniform approximately holomorphic bounds on $K_{i,k}$. Let $\kappa_{1,i}$, $\kappa_{2,i}$, $\kappa_{3,i}$ be the H3 constants at endpoint i. Choose the coefficient bound $c > 0$ so that the endpoint perturbations preserve a positive lower bound for the norm of the pair on $W_{i,k} \setminus K_{i,k}$, $\kappa_{1,i}/2$-transversality of its projective complex derivative there, and every condition on V_3. Let $B_i c$ bound the scaled C^3 change of the projective map on $W_{i,k} \setminus K_{i,k}$, and let $\nu_{0,i}$ be the threshold in [4, proof of Theorem 4, pp. 1053–1055] for these uniform bounds and the constants $\kappa_{1,i}/2$, $\kappa_{3,i}/2$. Decrease c so that

$$2B_i c \leq \kappa_{3,i}/2, \qquad B_i c \leq \nu_{0,i}/2, \qquad \epsilon_k \kappa_{2,i} \leq \nu_{0,i}/2.$$

The last inequality holds after increasing k. The finite-color argument in the cited proof now gives the pencil transversality conditions on $K_{i,k} \setminus V_2$, while openness retains them on V_3. These sets cover $K_{i,k}$. On V_3, the pair remains quantitatively transverse to zero, and its projective complex derivative remains quantitatively transverse to zero outside the basepoint neighborhoods. The two independent perturbations vanish near V_0, so the pairs remain equal there.

The full derivative bound on D and the uniform second derivative bounds separate the complex critical points from D by a positive scaled distance. The lower bound for $|s|$ on D gives the same separation for basepoints. The local Lefschetz modifications in [4, proof of Theorem 4, pp. 1054–1055] can therefore be supported away from D; the original model collars also lie away from D. In a projective denominator chart they are modifications of the numerator section.

At a basepoint trivialize $L^{\otimes k}$ and take scaled approximately holomorphic coordinates $z \in \mathbb{C}^2$ centered there, complex-linear for the chosen compatible structure at the center. The pair has expansion

$$s(z) = Az + B\bar{z} + O(|z|^2), \qquad \|A^{-1}\| \leq C_1, \qquad \|B\| \leq C_2\epsilon_k.$$

The inverse bound follows from pair transversality and the small antilinear derivative. A cutoff on a ball of scaled radius r replaces s by Az on an inner ball, with C^1 change at most $C_3(r + \epsilon_k)$. Put $\delta = r + \epsilon_k$. For the modified pair $\widehat{s}$,

$$|\widehat{s} - Az| \leq C\delta|z|, \qquad |d\widehat{s} - A| \leq C\delta.$$

The covector $\ell = \widehat{s}_0 d\widehat{s}_1 - \widehat{s}_1 d\widehat{s}_0$ therefore satisfies

$$|\ell^{1,0}| \geq c_1|z|, \qquad |\ell^{0,1}| \leq C\delta|z|$$

on the punctured ball: for Az it equals $\det(A)(z_0\,dz_1 - z_1\,dz_0)$, and its error is $O(\delta|z|)$. Fix r small, then take k large. The real kernel is a small perturbation of a complex line and is Ω_i-positive. The first error bound excludes new common zeros. On the inner ball the coefficient coordinates of the section pair are Az; their complex structure agrees with the compatible structure at the center and is tamed on a smaller ball by continuity. Its complex lines are the tangent planes of the compactified fibres at the basepoint. All these balls may be chosen disjoint from D.

Restrictions to the surfaces. Put $\tau_k = k^{-1/2}$ for the error on the fixed common symplectic region. All norms in this paragraph are scaled norms. The lower bound for $|s|$ on D gives denominator charts with uniform C^3 bounds for f and C^1 bounds $O(\tau_k)$ for $\bar\partial f$. At a zero q of $\partial(f|_{S_j})$, make a constant unitary change of components so that s_0 is bounded away from zero on its scaled ball, and fix the target coordinate $f = s_1/s_0$ on that ball. Choose an intrinsic complex coordinate z centered at q and normalized there in the scaled metric, and write $F(z) = (s_1/s_0)|_{S_j}(z)$. Norms in the one-variable Taylor and cutoff estimates are Euclidean coordinate norms, uniformly equivalent to the scaled norms. Then

$$F(z) = c + a_q z^2 + E(z), \qquad |a_q| \geq \eta > 0,$$

with uniform bounds

$$|E(z)| \leq C(|z|^3 + \tau_k|z|), \qquad |dE(z)| \leq C(|z|^2 + \tau_k).$$

Indeed, the complex-linear first derivative vanishes at q; the antilinear first derivative and the mixed quadratic coefficients are $O(\tau_k)$. Quantitative transversality and this bound give the lower bound for the pure quadratic coefficient.

Quantitative transversality and the uniform third derivative bound give a positive scaled separation of these zeros. Indeed, quantitative transversality gives a uniformly bounded inverse for the real derivative of the local representative of ∂F wherever its norm is sufficiently small. The quantitative inverse function theorem then gives

$$\operatorname{dist}(p, \{\partial F = 0\}) \leq C\eta^{-1}|\partial F(p)|$$

whenever $|\partial F(p)|$ is below a fixed positive constant. Consequently, for every sufficiently small fixed $r > 0$, $|\partial F| \geq c_*\eta r$ outside the radius-r discs about these zeros, where $c_* > 0$ is independent of k. The crossing derivative bounds separate these discs from the crossings. Compactness of the fixed configuration and its transverse crossings give disjoint ambient balls about them, lying in V_0 and meeting no other sheet of D. Their radii may also be smaller than the separation from ambient critical and basepoint neighborhoods.

Fix such a scaled radius r, independent of k. Let $\chi(z/r)$ equal one on $|z| \leq r/2$ and zero outside the radius-r disc, and set

$$\widehat{F}(z) = c + a_q z^2 + (1 - \chi(z/r))E(z).$$

On the transition annulus the least singular value of $d(c + a_q z^2)$ is at least ηr, and

$$\left|d\widehat{F} - d(c + a_q z^2)\right| \leq C(r^2 + \tau_k), \qquad |\bar\partial F| \leq C\tau_k$$

for a constant C independent of k and small r. If

$$Cr \leq \eta/8, \qquad C\tau_k \leq \eta r/8, \qquad C\tau_k \leq c_*\eta r/2,$$

then the transition has positive Jacobian, and the inner disc has the positive simple branch model. Outside the discs, $|\bar\partial F| < |\partial F|$, so there are no other critical points of the restriction.

In a smooth tubular chart (z, v) for S_j, extend the correction by

$$\Delta(z, v) = -\psi(v/r)\chi(z/r)E(z),$$

where ψ is one near zero and supported in a fixed small normal disc. For the ambient derivative estimates, take g_k on the source and the Euclidean metric in this target chart. The value of Δ is

$O(r^3+\tau_k r)$ and its full scaled C^1 norm is $O(r^2+\tau_k)$. On the correcting ball the full complex derivative of f has a uniform positive lower bound $\lambda>0$. Increase C so that $\|\Delta\|_{C^1}\le C(r^2+\tau_k)$ and $|\bar\partial f|\le C\tau_k$. Choose r and then an odd k so that the three preceding inequalities and $2C(r^2+\tau_k)+C\tau_k\le\lambda/2$ hold. Then

$$|\partial(f+\Delta)|-|\bar\partial(f+\Delta)|\ge\lambda/2>0.$$

Thus the corrected map is a submersion with symplectic fibres. In this denominator chart the section correction is

$$s_1\longmapsto s_1+s_0\Delta,\qquad s_0\longmapsto s_0.$$

Make the same correction to both section pairs. The balls are disjoint, so the uniform bound is independent of their number.

Arbitrarily small independent constant shifts on the inner branch discs, extended with cutoffs in the same balls, make all branch values distinct and disjoint from the finite set of crossing images. Their scaled C^1 norms are bounded by a fixed multiple of the shift size divided by r. Choose them below the preceding submersion bounds and the positive Jacobian bounds on the cutoff annuli, identically at both endpoints. Let $\mathcal{O}(1)\to\mathbb{CP}^1$ denote the hyperplane line bundle. The nonvanishing pair on S_j gives

$$(f|_{S_j})^*\mathcal{O}(1)\simeq L^{\otimes k}|_{S_j},\qquad \deg(f|_{S_j})=kN\int_{S_j}\Omega_i>0.$$

Each restriction is therefore a positive simple branched covering of the sphere. The crossing neighborhoods are unchanged.

If the model section pairs were initially defined only off the zero circles, their lifts extend smoothly over the tubes without changing the projective map. The local fold map is smooth across the core. A zero tube is an oriented rank-three disc bundle over S^1, hence is diffeomorphic to $S^1\times D^3$, also in the odd model. Both its pullback of $\mathcal{O}(1)$ and $L^{\otimes k}$ are trivial. Two nonvanishing lifts on the outer collar differ by a map to $\mathbb{C}^*$, and

$$H^1(S^1\times D^3;\mathbb{Z})\longrightarrow H^1(S^1\times S^2;\mathbb{Z})$$

is an isomorphism. The classification of maps to $\mathbb{C}^*\simeq K(\mathbb{Z},1)$ [23, Theorem 4.57, pp. 393–394], followed by a collar homotopy and relative smoothing, extends this scalar without zeros and with its prescribed outer value. The resulting section pair is smooth on the full tube. Nothing changes near D.

For the final odd k, choose $U\subset V_0$ small enough to avoid all basepoints, isolated critical points, and the supports of independent modifications. The pairs and connections agree on U, and its projective map has symplectic fibres. Every common zero has local index $+1$ in its section coordinates. The Euler zero-count theorem [8, Theorem 11.17] and [25, Proposition 3.13(b,c)] give

$$\#\{s^i=0\}=\langle e(L^{\otimes k}\oplus L^{\otimes k}),[X]\rangle=\langle c_1(L^{\otimes k})^2,[X]\rangle=(kN)^2b^2.$$

Since $b^2>0$, this number is at least two for large k. At every regular point of every fibre, Ω_i is positive on its oriented tangent plane. At a fold point $\Omega_i=0$; at an isolated critical point $df_i=0$. These three cases give $\Omega_i\wedge f_i^*\sigma\ge0$ off the basepoints.

Finally, in the last assertion, each change near Z_i is exact and occurs where the Thom forms vanish. There $\omega_i=\Omega_i$, so the same deformation and the relative positive bundles above apply to ω_i. Every change near D is a pullback by an ambient isotopy; pull back ω_i, all $\rho^0_{j,i}$, and their positive bundle together. Thus the identity $\omega_i=\Omega_i+\sum_j\mu_j\rho^0_{j,i}$ persists, $[\omega_i]=a$, and the positive bundles extend P_i. Section modifications do not change these forms. □

Lemma 10.3. *Let Ω be closed and symplectic on a neighborhood U of an orthogonal positive configuration $D=\bigcup_j S_j$, with $S_j^2>0$. Let $f:U\to\mathbb{CP}^1$ be a submersion such that $\Omega\wedge f^*\sigma\ge0$. Suppose that each restriction $f|_{S_j}$ is a positive simple branched covering and that no crossing is a branch point. Every neighborhood of D contained in U supports closed Thom forms ρ_j satisfying*

$$\rho_j^2\ge0,\qquad \rho_j\wedge\rho_l\ge0\quad(j\ne l),\qquad \rho_j\wedge\Omega\ge0,\qquad \rho_j\wedge f^*\sigma\ge0.$$

For every $t > 0$ and every choice of $\mu_j > 0$, the form $t f^*\sigma + \sum_j \mu_j \rho_j$ has positive square on a neighborhood of D.

Proof. Put $P = f^*\sigma$ and let $\pi_j : N_j \to S_j$ be the Ω-orthogonal normal bundle $N_j = (TS_j)^{\Omega}$, oriented by $TX|_{S_j} = TS_j \oplus N_j$ and the given orientations of X and S_j. Fix product Darboux coordinates at the crossings, as in the proof of Lemma 9.1. At a branch point $q \in S_j$, choose positive coordinates z on S_j and ζ on the target with $\zeta \circ f|_{S_j} = z^2$, and put $F = \zeta \circ f$. Since $\ker df_q = T_q S_j$, the map $dF_q|_{N_{j,q}}$ is an isomorphism. The inequality $\Omega \wedge P \geq 0$ implies that it preserves orientation. Extend z with $dz|_{N_j} = 0$ along S_j, and put $w = F - z^2$. Along S_j one has

$$dw|_{TS_j} = 0, \qquad dw|_{N_j} = dF|_{N_j}.$$

After shrinking the chart, (z, w) are positive coordinates and $F = z^2 + w$. With fibre coordinate $v \mapsto dF(v)$ on N_j, the corresponding tubular map has normal differential equal to the inclusion of N_j in $TX|_{S_j}$.

Extend the product metrics in these branch and crossing charts to Riemannian metrics for which $TS_j \perp N_j$ along S_j. The normal exponential maps give tubular maps with this same normal differential, agreeing with the prescribed maps on smaller discs; see [28, Chapter 4, §6, Exercise 3]. Choose the tubes inside the prescribed neighborhood of D, with distinct tubes meeting only in the product crossing charts. Identify each tube with its disc bundle in N_j.

Choose orientation-compatible complex structures on S_j and on the fibres of N_j, agreeing with the branch and product coordinates. Choose Hermitian metrics which are constant in the product trivializations at the crossings. In a branch chart, take squared norm $s_j = k(z)|w|^2$, where $k(z) = e^{-\varepsilon|z|^2}$ and $\varepsilon > 0$ is small. Prescribe the Chern connection there and the flat product connection at the crossings. Their curvature densities extend to smooth functions $c_j \geq 0$ such that

$$\int_{S_j} c_j \, \Omega|_{S_j} = \langle c_1(N_j), [S_j] \rangle = S_j^2,$$

with $c_j > 0$ away from the crossing discs and $c_j = 0$ on smaller crossing discs. Indeed, for small ε, the prescribed positive densities have total integral less than S_j^2. The curvature criterion for circle connections [11, §6.3 and §6.6, footnote 81] gives a unitary connection whose normalized angular form has curvature $-c_j \Omega|_{S_j}$. On each prescribed disc its difference from the local connection is the pullback of a closed one-form $d\phi$. Subtracting $d(\chi\phi)$, with $\chi = 1$ on a smaller disc and supported in the original disc, gives agreement there without changing the curvature. Shrink the branch and crossing charts to these regions of agreement, and restrict the tubes so that all their overlaps lie in the smaller crossing charts.

Write α_j for the resulting angular connection form, normalized to have integral one on each positively oriented fibre circle and extended radially to $N_j \setminus S_j$. Put

$$s_j = |v|_j^2, \qquad H_j = \pi_j^*(\Omega|_{S_j}), \qquad V_j = ds_j \wedge \alpha_j, \qquad d\alpha_j = -c_j H_j,$$

where c_j also denotes its pullback by π_j. As in the proof of Lemma 10.1, V_j extends smoothly over S_j, and

$$H_j^2 = V_j^2 = 0, \qquad H_j \wedge V_j > 0.$$

In a branch chart put $d^c = i(\bar\partial - \partial)$. For $w \neq 0$, the Chern connection satisfies $4\pi s_j \alpha_j = d^c s_j$. Writing $\eta = dw + w\,\partial \log k$, one obtains

$$V_j = \frac{i}{2\pi s_j} \partial s_j \wedge \bar\partial s_j = \frac{ik}{2\pi} \eta \wedge \bar\eta.$$

The last expression is smooth at $w = 0$. Thus H_j, V_j, and P are semipositive forms of type $(1,1)$ in the branch chart; for P, this follows from $F = z^2 + w$ and positivity of σ in the oriented target coordinate ζ.

For a two-form Q on the jth tube, put

$$A_{j,Q} = \frac{H_j \wedge Q}{H_j \wedge V_j}, \qquad B_{j,Q} = \frac{V_j \wedge Q}{H_j \wedge V_j}.$$

On the zero section, V_j is a positive normal area form and $\Omega|_{TS_j} = H_j|_{TS_j}$; hence $B_{j,\Omega} = 1$ there. On the compact complement in S_j of smaller branch discs, $f|_{S_j}$ has positive Jacobian, so $B_{j,P} > 0$ there. After shrinking the tubes, compactness gives constants $b > 0$, $M \geq 0$, and $C_0 \geq 0$ such that

$$B_{j,Q} \geq b, \qquad |A_{j,Q}| \leq M, \qquad c_j \leq C_0,$$

for $Q = \Omega$ throughout each tube, and for $Q = P$ on the tubes over these compact complements.

Choose $C > 0$ with $Cb > C_0 M$. For sufficiently small $R_j > 0$, let $\chi_j : [0,\infty) \to [0,1]$ be smooth and nonincreasing, equal to one near zero, positive for $s < R_j^2$, and zero for $s \geq R_j^2$. Put

$$h_j(s) = e^{-Cs}\chi_j(s), \qquad \rho_j = -d\big(h_j(s_j)\alpha_j\big) = h_j c_j H_j - h_j' V_j.$$

Here h_j and h_j' in the last expression are evaluated at s_j. The angular Thom-form construction in the proof of Lemma 10.1 gives a closed extension over S_j and by zero outside the tube, with normal-fibre integral one. Moreover, $-h_j' \geq Ch_j$, $h_j'(0) = -C$, and

$$\rho_j^2 = -2h_j h_j' c_j \, H_j \wedge V_j \geq 0.$$

For $Q = \Omega$ throughout each tube, and for $Q = P$ over the compact complements, the preceding bounds give

$$\frac{\rho_j \wedge Q}{H_j \wedge V_j} = h_j c_j A_{j,Q} - h_j' B_{j,Q} \geq h_j(Cb - C_0 M) \geq 0.$$

In the branch charts, ρ_j and P are semipositive $(1,1)$ forms, so $\rho_j \wedge P \geq 0$ there as well. At a crossing of S_j and S_l, the constant product metrics and flat product connections give $\rho_j = -h_j' V_j$ and $\rho_l = -h_l' V_l$, where V_j, V_l are positive area forms in the complementary normal factors. Hence $\rho_j \wedge \rho_l = h_j' h_l' V_j \wedge V_l \geq 0$. Outside these charts the supports are disjoint.

On S_j outside the smaller branch discs, the strict estimate above gives $P \wedge \rho_j > 0$. On those discs, $c_j > 0$ and $h_j'(0) = -C < 0$ give $\rho_j^2 > 0$. Since $P^2 = 0$, for $t > 0$ and $\mu_j > 0$ one has along D

$$\left(tP + \sum_j \mu_j \rho_j\right)^2 = 2t \sum_j \mu_j P \wedge \rho_j + \sum_j \mu_j^2 \rho_j^2 + 2 \sum_{j<l} \mu_j \mu_l \rho_j \wedge \rho_l > 0.$$

The same inequality holds on a neighborhood of the compact set D. □

Lemma 10.4. *With the notation and hypotheses of Lemma 10.3, suppose that Ω is defined and closed on X. Let m be a positive integer. For $a = 1, \dots, m$, let $(\rho_{j,a}^0)_j$ be a collection of closed Thom forms supported in U, given by the symplectic normal models in the proof of Lemma 10.1, with nonnegative normal curvature and product data at the crossings. Every neighborhood $W \subset U$ of D supports one collection $(\rho_j^1)_j$ as in Lemma 10.3 such that, for each a, there are smooth paths of closed Thom forms $\rho_{j,a}^u$, $0 \leq u \leq 1$, supported in U, with*

$$\rho_{j,a}^{u=0} = \rho_{j,a}^0, \qquad \rho_{j,a}^{u=1} = \rho_j^1,$$

and

$$(\rho_{j,a}^u)^2 \geq 0, \qquad \Omega \wedge \rho_{j,a}^u \geq 0, \qquad \rho_{j,a}^u \wedge \rho_{l,a}^u \geq 0 \quad (j \neq l).$$

For fixed $\mu_j > 0$, each path $\Omega + \sum_j \mu_j \rho_{j,a}^u$ is symplectic on the union of the supports, equals Ω elsewhere, and has cohomology class $[\Omega] + \sum_j \mu_j \,\mathrm{PD}[S_j]$.

Proof. For parametric inflation with prescribed endpoints, see [45, Lemma 5.1.4(ii) and its proof]. Write $\pi_j : N_j = (TS_j)^\Omega \to S_j$, with the normal orientation of Lemma 10.3, and put $H_j = \pi_j^*(\Omega|_{S_j})$. Fix one initial collection and omit a for the moment. The assumed symplectic normal models consist of

tubular maps τ_j^0 fixing S_j and having normal differential equal to the inclusion, fibre metrics g_j^0, and angular connection forms α_j^0 of fibre integral one. With $s_j^0 = |v|^2_{g_j^0}$, they satisfy

$$(\tau_j^0)^*\Omega = H_j + d(s_j^0\alpha_j^0), \qquad d\alpha_j^0 = -c_j^0 H_j, \qquad c_j^0 \geq 0,$$

$$(\tau_j^0)^*\rho_j^0 = -d\big(h_j^0(s_j^0)\alpha_j^0\big), \qquad h_j^0(0) = 1, \quad h_j^0 \geq 0, \quad (h_j^0)' \leq 0.$$

The profiles are smooth and compactly supported inside their radial intervals. Distinct tubes meet only in product crossing charts, where the metrics are constant and the connections are the flat product connections. Put $V_j^0 = ds_j^0 \wedge \alpha_j^0$. Then $H_j \wedge V_j^0 > 0$ and

$$(\tau_j^0)^*\Omega = (1 - s_j^0 c_j^0)H_j + V_j^0, \qquad 1 - s_j^0 c_j^0 > 0.$$

The last inequality follows from nondegeneracy on each model neighborhood and its value at the zero section. In particular, $\Omega^2 > 0$ on these tubes.

Fix product Darboux charts in W at the crossings, with squared norm $s = \pi|w|^2$ and angular form $(2\pi)^{-1} d\arg w$ on each normal factor with coordinate w. Normalize the initial product charts in the same way. In the construction of Lemma 10.3, choose tubular maps τ_j^1 with images in W, fibre metrics g_j^1, and connections α_j^1, with the prescribed product data in these charts and $d\alpha_j^1 = -c_j^1 H_j$, where $c_j^1 \geq 0$. Leave the profiles h_j^1 undetermined. These terminal choices are independent of a.

The initial product charts may be made equal to the chosen ones by Ω-symplectic isotopies preserving each component of D. Indeed, let F be the transition germ at a crossing. It preserves each axis, and its derivative belongs to the connected group $\mathrm{Sp}(2,\mathbb{R}) \times \mathrm{Sp}(2,\mathbb{R})$ [11, §3.2, p. 112]. For the ambient dilations $D_r(z,w) = (rz, rw)$, the symplectic germs $D_r^{-1} F D_r$ extend at $r = 0$ to dF_0. Together with a path in this linear group they give a symplectic germ isotopy preserving each axis. Its vector field is locally Hamiltonian [11, §5.1, p. 140]; write its Hamiltonian as K_u. Tangency to either axis means that dK_u vanishes in its normal directions. A cutoff $\chi(|z|^2, |w|^2)$, equal to one near the origin and supported in the chart, has the same property. Consequently the Hamiltonian χK_u extends by zero and its flow preserves both axes, agreeing with the germ isotopy on a smaller chart. The crossing charts are disjoint, so these extensions may be performed simultaneously, with support in U. Writing ϕ_u for this isotopy, with $\phi_0 = \mathrm{id}$, transport all the initial forms by $(\phi_u^{-1})^*$. This gives paths starting at the prescribed forms and preserving their three wedge inequalities. After the terminal map $\phi = \phi_1$, the transported tubular map on the fixed bundle N_j is

$$(y, v) \longmapsto \phi\big(\tau_j^0\big(x, (d\phi_x|_{N_{j,x}})^{-1} v\big)\big), \qquad x = (\phi|_{S_j})^{-1}(y).$$

It again fixes S_j and has normal differential equal to the inclusion. Transport the metrics and connections by the same bundle isomorphism. After restricting the crossing charts, all these data agree there with the chosen terminal product data. Retain the superscript 0 for the transported data.

Hirsch's tubular neighborhood theorem and its canonical homotopy [28, Chapter 4, §5, Theorem 5.3 and its proof] give paths on sufficiently small disc bundles. Explicitly, put $F_j = (\tau_j^0)^{-1}\tau_j^1$ and let $\delta_u(v) = uv$ be fibre dilation. Then

$$\tau_{j,u} = \tau_j^0 \circ \delta_u^{-1} \circ F_j \circ \delta_u \quad (u > 0), \qquad \tau_{j,0} = \tau_j^0.$$

The transition F_j is the identity in the fixed crossing charts and has identity differential along the zero section. Thus this homotopy is fixed in those charts and every $\tau_{j,u}$ has normal differential equal to the inclusion.

Put $g_{j,u} = (1-u)g_j^0 + u g_j^1$. Let $A_{j,u}$ be the positive g_j^0-self-adjoint endomorphism with $g_{j,u}(v,w) = g_j^0(A_{j,u}v, w)$, and put $Q_{j,u} = A_{j,u}^{-1/2}$. This is an orientation-preserving isometry from g_j^0 to $g_{j,u}$ and identifies their unit circle bundles. On these bundles define $\alpha_{j,u}$ by

$$Q_{j,u}^*\alpha_{j,u} = (1-u)\alpha_j^0 + u Q_{j,1}^*\alpha_j^1.$$

Connections on a fixed circle bundle form an affine space [11, §6.3, pp. 166–167]. Thus

$$d\alpha_{j,u} = -c_{j,u}H_j, \qquad c_{j,u} = (1-u)c_j^0 + u c_j^1 \geq 0.$$

The metrics and connections remain fixed in the crossing charts. Extend $\alpha_{j,u}$ radially and put $s_{j,u} = |v|^2_{g_{j,u}}$ and $V_{j,u} = ds_{j,u} \wedge \alpha_{j,u}$. As in Lemma 10.1, the forms $V_{j,u}$ extend smoothly across the zero section, and $H_j^2 = V_{j,u}^2 = 0$, $H_j \wedge V_{j,u} > 0$.

Write $\Omega_{j,u} = \tau_{j,u}^* \Omega$ and set

$$p_{j,u} = \frac{H_j \wedge \Omega_{j,u}}{H_j \wedge V_{j,u}}, \qquad q_{j,u} = \frac{V_{j,u} \wedge \Omega_{j,u}}{H_j \wedge V_{j,u}}.$$

On the zero section, $q_{j,u} = 1$, whereas $p_{j,u} > 0$ because $\Omega|_{N_j}$ is positive and the normal differential is the inclusion. Compactness of $S_j \times [0,1]$ gives a common neighborhood where both functions are positive. The metrics $g_{j,u}$ are uniformly equivalent there. We may therefore choose $\varepsilon > 0$ so that all discs $s_{j,u} \le \varepsilon$ lie in this neighborhood and their images lie in U. Shrink ε so that distinct images meet only in the fixed crossing charts. Otherwise intersections for radii tending to zero would, by compactness, converge to an intersection of two components outside those charts. Make these choices for every initial collection and take the minimum of the finitely many permissible values of ε. Now choose the profiles h_j^1 in Lemma 10.3, all supported in $[0, \varepsilon)$.

Extend each h_j^0 by zero to $[0, \infty)$ and choose $b_j \in (0,1]$ so that $\widehat{h}_j^0(s) = h_j^0(s/b_j)$ is supported in $[0, \varepsilon)$. Before moving the tubular maps, put

$$k_{j,t} = (1-t)h_j^0 + t\widehat{h}_j^0, \qquad (\tau_j^0)^* \eta_j^t = -d\big(k_{j,t}(s_j^0)\alpha_j^0\big), \qquad 0 \le t \le 1,$$

and extend η_j^t by zero. The normal data are fixed, so

$$\eta_j^t = (1-t)\rho_j^0 + t\eta_j^1, \qquad \operatorname{supp} \eta_j^1 \subset \tau_j^0\big(\{s_j^0 < \varepsilon\}\big) \subset U.$$

Consequently $\operatorname{supp} \eta_j^t \subset \operatorname{supp} \rho_j^0 \cup \operatorname{supp} \eta_j^1 \subset U$. The profiles satisfy $k_{j,t}(0) = 1$, $k_{j,t} \ge 0$, and $k'_{j,t} \le 0$. Hence

$$\begin{aligned}(\tau_j^0)^*\big((\eta_j^t)^2\big) &= -2c_j^0 k_{j,t} k'_{j,t}\, H_j \wedge V_j^0 \ge 0,\\ (\tau_j^0)^*(\Omega \wedge \eta_j^t) &= \big[c_j^0 k_{j,t} - (1 - s_j^0 c_j^0)k'_{j,t}\big] H_j \wedge V_j^0 \ge 0,\end{aligned}$$

where the profiles and their derivatives are evaluated at s_j^0. Distinct forms have nonnegative mixed product in the transported initial product charts, and disjoint supports elsewhere.

Set $h_{j,u} = (1-u)\widehat{h}_j^0 + uh_j^1$. In the moving tube define $\rho_{j,u}$ by

$$\tau_{j,u}^* \rho_{j,u} = -d\big(h_{j,u}(s_{j,u})\alpha_{j,u}\big) = h_{j,u} c_{j,u} H_j - h'_{j,u} V_{j,u},$$

where the profiles and their derivatives are evaluated at $s_{j,u}$. Extend $\rho_{j,u}$ by zero. The profiles have value one at zero, are nonnegative and nonincreasing, and vanish on a common neighborhood of ε. The Thom-form construction of Lemma 10.1 therefore gives a smooth family of closed forms in class $\mathrm{PD}[S_j]$. Moreover,

$$\begin{aligned}\tau_{j,u}^*(\rho_{j,u}^2) &= -2h_{j,u}h'_{j,u}c_{j,u}\, H_j \wedge V_{j,u} \ge 0,\\ \frac{\tau_{j,u}^*(\Omega \wedge \rho_{j,u})}{H_j \wedge V_{j,u}} &= h_{j,u}c_{j,u}p_{j,u} - h'_{j,u}q_{j,u} \ge 0.\end{aligned}$$

At a crossing the forms are $-h'_{j,u}V_j$ and $-h'_{l,u}V_l$, where V_j, V_l are the fixed positive area forms in the complementary normal factors. Their product is nonnegative; outside the crossing charts the supports are disjoint.

Reparametrize the three stages to be constant near their endpoints and concatenate them. Their initial forms are the prescribed $\rho_{j,a}^0$, and their final forms are the same ρ_j^1 for every a. All supports lie in the component neighborhoods where $\Omega^2 > 0$. Hence, for $\mu_j > 0$,

$$\left(\Omega + \sum_j \mu_j \rho_{j,a}^u\right)^2 = \Omega^2 + 2\sum_j \mu_j \Omega \wedge \rho_{j,a}^u + \sum_j \mu_j^2 (\rho_{j,a}^u)^2 + 2\sum_{j<l} \mu_j \mu_l \rho_{j,a}^u \wedge \rho_{l,a}^u > 0$$

on their union. Closedness and the Thom classes give the remaining assertions. □

Proposition 10.5. *Let $b^+(X) = 1$ and let $a \in H^2(X;\mathbb{R})$ satisfy $Q_X(a,a) > 0$. For $i = 0, 1$, let ω_i be a near-symplectic representative of a, and let $P_i \subset \Lambda^2 T^*X$ be a smooth positive three-plane bundle containing ω_i. There are smooth paths of closed representatives of a from ω_i to forms $\Omega_i + R$, contained in smooth paths of positive three-plane bundles starting at P_i, with the following properties.*

(1) There are connected closed embedded oriented surfaces $S_1, \dots, S_n$, where $n = b_2(X)$, whose union $D = \bigcup_{j=1}^n S_j$ is connected. The integral classes $d_j = \mathrm{PD}[S_j] \in H^2(X;\mathbb{Z})$ form a basis after realification. There are $\lambda_j, \mu_j > 0$ such that

$$a = \sum_j \lambda_j d_j, \qquad b = a - \sum_j \mu_j d_j \in H^2(X;\mathbb{Q}), \qquad Q_X(b,b) > 0.$$

All $Q_X(d_j, d_l)$ are positive.

(2) The forms Ω_i are near-symplectic representatives of b, equal to a symplectic form Ω on an open neighborhood U of a compact plumbing E of D. The configuration D is orthogonal and positive for Ω. Both $X \setminus \operatorname{int} E$ and $X \setminus E$ are connected.

(3) One Hermitian line bundle $\mathcal{L}$ with $c_1(\mathcal{L})_{\mathbb{R}} = \kappa b$, $\kappa > 0$, has unitary connections ∇_i of curvature $-2\pi i\kappa\Omega_i$ and smooth section pairs $s^i = (s^i_0, s^i_1)$ of $\mathcal{L} \oplus \mathcal{L}$. Both connections and pairs agree on U, and the common pair is nonzero there. The projective maps $f_i = [s^i_0 : s^i_1]$ are singular Lefschetz pencils with positive simple basepoints. Their common number is $\kappa^2 Q_X(b,b) \geq 2$.

(4) The common map $f : U \to \mathbb{CP}^1$ is a submersion. Every restriction $f|_{S_j}$ is a positive simple branched covering, with no branch at a crossing. At each endpoint $\Omega_i \wedge f_i^*\sigma \geq 0$ off its basepoints. At every basepoint Ω_i is positive on all complex lines in the coordinates defined by s^i, or equivalently strictly tames their complex structure.

(5) There are closed Thom forms ρ_j with compact supports in $\operatorname{int} E$, representing d_j, such that

$$\rho_j^2 \geq 0, \quad \rho_j \wedge \rho_l \geq 0 \ (j \neq l), \quad \rho_j \wedge \Omega \geq 0, \quad \rho_j \wedge f^*\sigma \geq 0.$$

One has $R = \sum_j \mu_j \rho_j$, and

$$\left(\kappa^{-1} f^*\sigma + R\right)^2 > 0$$

on a neighborhood of D.

Proof. Choose the rays in Proposition 7.3 in the component of the positive cone containing a. It gives the classes d_j, the coefficients λ_j, and configurations at both endpoints with the same integral component classes. Since Q_X has signature $(1, n-1)$, $Q_X(d_j, d_l) > 0$ for all j, l. Thus distinct components meet, and each configuration is connected; for $n = 1$ its sole component is connected. Proposition 9.4 therefore applies with the prescribed bundles P_i. Write ω_i' and P_i' for its final forms and bundles. The final common configuration D is orthogonal and positive for both forms, which agree near D. Its integral component classes are unchanged.

Apply Lemma 10.1 in this common neighborhood, with the closures of the normal models disjoint from both zero sets. It gives $\mu_j > 0$, a positive rational class b, and

$$\omega_i' = \Omega_i + \sum_j \mu_j \rho_j^0, \qquad [\Omega_i] = b, \qquad a = b + \sum_j \mu_j d_j.$$

Choose an integer $N_0 > 0$ and $c_0 \in H^2(X;\mathbb{Z})$ with $(c_0)_{\mathbb{R}} = N_0 b$, and a Hermitian line bundle L_0 with $c_1(L_0) = c_0$ [25, Proposition 3.10]. Apply Proposition 10.2, including its last assertion, with the bundles P_i' and zero tubes disjoint from these normal models. That assertion gives paths in class a and positive three-plane bundles extending P_i'. Let $m > 0$ be the integer supplied by that proposition, and let k be a common sufficiently large odd integer. Put

$$\mathcal{L} = L_0^{\otimes mk}, \qquad \kappa = mkN_0.$$

Choose k so large that $\kappa^2 Q_X(b,b) \geq 2$. The tensor-power connections and section pairs supplied by that proposition have the properties in (3) and (4).

Retain the notation Ω_i for the resulting forms and write the current class-a endpoints as

$$\Omega_i + \sum_j \mu_j \rho_{j,i}^0.$$

The old normal models have been transported with Ω_i. Choose a neighborhood V of D with compact closure in the common nonvanishing submersion region of Proposition 10.2. In particular, $\Omega_0 = \Omega_1$ there, and their connections and section pairs agree there.

In the old normal model of S_j at endpoint i, let $h_{j,i}$ be the profile, extended by zero to $[0,\infty)$, and let $s_{j,i}$ be the squared fibre radius. Choose $r_{j,i} > 0$ so that the closed disc bundle $\{s_{j,i} \le r_{j,i}\}$ lies in this model and its tubular image lies in V. Choose $\delta_{j,i} \in (0,1]$ so that

$$\widehat{h}_{j,i}(s) = h_{j,i}(s/\delta_{j,i}) \quad \text{has support in } [0, r_{j,i}).$$

On each fixed normal model, interpolate the profiles by

$$(1-t)h_{j,i} + t\widehat{h}_{j,i}, \qquad 0 \le t \le 1,$$

as in the proof of Lemma 10.4. This gives closed Thom forms $\eta_{j,i}^t$ from $\rho_{j,i}^0$ to forms $\widehat{\rho}_{j,i}$ supported in these smaller discs. They represent d_j and satisfy

$$(\eta_{j,i}^t)^2 \ge 0, \qquad \Omega_i \wedge \eta_{j,i}^t \ge 0, \qquad \eta_{j,i}^t \wedge \eta_{l,i}^t \ge 0 \quad (j \ne l).$$

Restrict each old normal model to the smaller disc bundle. Its entire image now lies in V, so it is a symplectic normal model for the globally defined form Ω_0 at both endpoints.

Choose a closed regular plumbing E of D with $E \subset V$. The path argument at the beginning of §9.1 shows that $X \setminus \operatorname{int} E$ is connected. Its interior $X \setminus E$ is connected by a boundary collar. Apply Lemma 10.4 with global form Ω_0, neighborhood V, final support neighborhood $\operatorname{int} E$, and the two collections $(\widehat{\rho}_{j,i})_j$. Its hypotheses concerning f hold by Proposition 10.2, and $S_j^2 = d_j^2 > 0$. It gives one collection $(\rho_j)_j$ as in Lemma 10.3, supported in $\operatorname{int} E$, and paths $\rho_{j,i}^u$ from $\widehat{\rho}_{j,i}$ to ρ_j. All these paths are supported in V. Since $\Omega_0 = \Omega_1$ on V, their three wedge inequalities hold with either background form.

Fix $i \in \{0,1\}$. For either of the two families of Thom forms just constructed at endpoint i, write θ_j for its members at a fixed parameter. Then

$$\left[\Omega_i + \sum_j \mu_j \theta_j\right] = b + \sum_j \mu_j d_j = a, \qquad \left(\Omega_i + \sum_j \mu_j \theta_j\right)^2 \ge \Omega_i^2 > 0$$

on the union of their supports. Outside that union the form equals Ω_i. The supports of these two paths lie in one compact set where Ω_i is symplectic: compact subsets of the old model neighborhoods together with $\overline{V}$. The relative positive-plane construction in the proof of Lemma 8.1 therefore extends the bundles already obtained, keeping them fixed near the current zero sets and at the initial parameters. These paths end at $\Omega_i + R$, where $R = \sum_j \mu_j \rho_j$.

Lemma 10.3, with $t = \kappa^{-1} > 0$, gives all the inequalities in (5), including strict positivity near D. Choose U with $E \subset U \Subset V$ and restrict the common form, connections, and section pair to U. Reparametrize the successive paths to be stationary near their endpoints and concatenate them. The resulting paths start at the prescribed forms and positive bundles. □

11. Paths of sections and curvature forms

Let X be a closed connected oriented smooth four-manifold, and let $L \to X$ be a Hermitian line bundle. At a zero x of a smooth section s of $L \oplus L$, differentiation in a local trivialization defines the intrinsic linear map

$$D_x s \,:\, T_x X \longrightarrow (L \oplus L)_x.$$

The zero is *positive and simple* if this map is an orientation-preserving real isomorphism. The target has its complex orientation.

11.1. **A change along an arc.** Let $F \to M$ be an oriented real bundle of rank four on an oriented five-manifold, possibly with boundary. For a smooth section S transverse to zero on M and on ∂M, put $Z(S) = S^{-1}(0)$. Orient $Z(S)$ by the exact sequence

$$0 \longrightarrow TZ(S) \longrightarrow TM|_{Z(S)} \xrightarrow{DS} F|_{Z(S)} \longrightarrow 0. \tag{11.1}$$

Thus a tangent vector v to $Z(S)$ is positive if (n_1, n_2, n_3, n_4, v) is positive whenever $DS(n_1), \dots, DS(n_4)$ is a positive basis of F.

Lemma 11.1. *Let S be as above, and let $\gamma : [-1,1] \hookrightarrow \operatorname{int} M$ be a smooth embedded arc with endpoints on distinct components of $Z(S)$. Assume that*

$$\gamma((-1,1)) \cap Z(S) = \emptyset, \qquad \dot\gamma(\pm 1) \notin T_{\gamma(\pm 1)} Z(S).$$

For every open neighborhood W of $\gamma([-1,1])$, there is a smooth homotopy of S, fixed outside a compact subset of $W \cap \operatorname{int} M$, whose final section is transverse to zero and whose final zero set is the oriented band sum along γ. If one of the two components is a circle and the other an interval, the resulting component is an interval.

Proof. Fix W and extend γ to a slightly larger open interval, with image in $W \cap \operatorname{int} M$. On this extended arc put

$$a(u) = \frac{S(\gamma(u))}{1-u^2} \qquad (u \neq \pm 1).$$

This section extends smoothly across the two endpoints, with

$$a(-1) = \tfrac{1}{2} D_{\gamma(-1)} S\big(\dot\gamma(-1)\big), \qquad a(1) = -\tfrac{1}{2} D_{\gamma(1)} S\big(\dot\gamma(1)\big).$$

These vectors are nonzero by the non-tangency assumption. After shortening the extension, a has no zeros. Since the arc is contractible, choose an oriented frame of F along it with fourth vector a, and extend this frame to a neighborhood. Write $S_1, \dots, S_4$ for the coefficients of S in this frame.

Near $\gamma(-1)$ set

$$u = -\sqrt{1 - S_4}, \qquad w_j = S_j \quad (1 \le j \le 3),$$

and take the positive square root near $\gamma(1)$. These functions agree with the chosen parameter and with $w = 0$ along the extended arc, where $w = (w_1, w_2, w_3)$. Their differentials are independent at the two endpoints, since DS is surjective there. Complete them to coordinates (u, v, w), with $v = 0$ on the arc and v a coordinate on the zero interval. Then

$$S = (w_1, w_2, w_3, 1 - u^2). \tag{11.2}$$

Choose v positively oriented on the zero interval at $u = -1$ and negatively oriented at $u = 1$. Both coordinate systems are positively oriented: for the volume form $du \wedge dv \wedge dw_1 \wedge dw_2 \wedge dw_3$, the positive zero tangent in (11.2) is $-\operatorname{sgn}(u)\partial_v$.

The endpoint charts define tubular germs of the extended open arc. Their oriented normal frames extend over the arc, since $\mathrm{GL}^+(4, \mathbb{R})$ is connected. Apply the extension of tubular germs [28, Chapter 4, §6, Exercise 3], with the closed subset equal to two short closed end subarcs. It gives a coordinate tube $T \subset W \cap \operatorname{int} M$ agreeing with the endpoint charts on smaller neighborhoods.

Put $\widehat{S} = (w_1, w_2, w_3, 1 - u^2)$ on T. On a compact middle subarc, $S_4 = 1 - u^2 > 0$. Shrink T there so that $S_4 > 0$ throughout its middle part. Near the attaching intervals $S = \widehat{S}$ already. Choose a product tube $T_0 \Subset T$ containing $\gamma([-1,1])$ and a function $\zeta \in C_c^\infty(T, [0,1])$ equal to one near $\overline{T_0}$. The homotopy

$$S + \theta\zeta(\widehat{S} - S), \qquad 0 \le \theta \le 1,$$

is extended by S outside T. Where $S \neq \widehat{S}$, both fourth coefficients are positive. Hence the zero set and its germs are unchanged. Denote the final section by $\overline{S}$; it equals $\widehat{S}$ near $\overline{T_0}$.

Choose $\eta > 0$ so that $[-1-2\eta, 1+2\eta]$ lies in the u range of T_0. Choose smooth cutoffs $0 \le \chi, \chi_w \le 1$ compactly supported in its u and w ranges, with $\chi = 1$ for $|u| \le 1+\eta$ and $\chi_w = 1$ near $w = 0$. Choose a smooth even nonnegative function ρ compactly supported in its v range, with

$$\rho(0) = 1, \qquad \rho''(0) < 0, \qquad \rho'(v) < 0 \quad \text{if } v > 0 \text{ and } \rho(v) > 0.$$

For example, rescale the function $\exp(-v^2/(1-v^2))$ on $|v| < 1$, extended by zero. On T_0 put

$$S_\lambda = \big(w_1, w_2, w_3, 1 - u^2 - \lambda\chi(u)\rho(v)\chi_w(w)\big), \qquad 0 \le \lambda \le 2, \tag{11.3}$$

and put $S_\lambda = \overline{S}$ outside T_0. The perturbation has compact support in T_0. At a zero in T_0, one has $w = 0$ and $1 - u^2 = \lambda\chi(u)\rho(v) \ge 0$. Hence $|u| \le 1$, and both cutoffs are one near that zero. The derivative in w has rank three. The remaining derivatives are

$$-2u, \qquad -\lambda\rho'(v).$$

They vanish at a zero only at $(u, v, \lambda) = (0, 0, 1)$.

Let $p : M \times [0,2] \to M$ be the projection, and let $\mathcal{S}(x, \lambda) = S_\lambda(x)$ be the corresponding section of p^*F. At the exceptional point its fourth coefficient has λ derivative -1, so $\mathcal{S}$ is transverse to zero. Write $\Sigma = \mathcal{S}^{-1}(0)$. Near that point its equations are

$$w = 0, \qquad \lambda = \frac{1-u^2}{\rho(v)}.$$

The Hessian of the latter function at $(0,0)$ is

$$\begin{pmatrix} -2 & 0 \\ 0 & -\rho''(0) \end{pmatrix}.$$

Thus the parameter projection on Σ has one critical point, of index one. In particular, S_2 is transverse to zero.

For $\lambda < 1$, the two zero branches in T_0 are $u = \pm\sqrt{1 - \lambda\rho(v)}$, $w = 0$. Let $v_* > 0$ be the unique number with $\rho(v_*) = 1/2$. At $\lambda = 2$, there are no zeros for $|v| < v_*$. The two branches join smoothly at each of $(u, v) = (0, \pm v_*)$, since $\rho'(\pm v_*) \ne 0$. They form one arc joining the two lower ends and one joining the two upper ends. At a regular zero, the positive tangent is

$$\lambda\rho'(v)\partial_u - 2u\partial_v.$$

The reconnection therefore respects the original orientations. If one original component is a circle and the other an interval, their complements after removing the attaching intervals consist of one arc and two arcs, respectively. The two new connectors join these three arcs into one interval.

The derivative of $\mathcal{S}$ induces the normal isomorphism

$$\big(T(M \times [0,2])|_\Sigma\big)/T\Sigma \longrightarrow (p^*F)|_\Sigma.$$

On every open set where the homotopy is fixed, this is the pullback of the isomorphism induced by DS. Finally, reparametrize the normalization and the band homotopy to be stationary near their endpoints and concatenate them. □

11.2. **Relative paths of actual sections.**

Proposition 11.2. *Let $E \subset X$ be compact with $X \setminus E$ connected. Let $s^0, s^1 \in \Gamma(L \oplus L)$ agree and be nonzero on a neighborhood of E. Suppose they have the same positive number N of zeros, all positive and simple. There is a smooth path s_t from s^0 to s^1, stationary near its endpoints and equal to the common section on a fixed neighborhood of E, such that every s_t has exactly N positive simple zeros. Its total zero set is a disjoint union of N graphs over $[0,1]$. Near each graph there are smoothly varying unitary frames e_t of L and orientation-preserving coordinates $(z_1, z_2) \in \mathbb{C}^2$ in which*

$$s_t = (z_1 e_t, z_2 e_t). \tag{11.4}$$

The coordinate balls may have one fixed sufficiently small radius.

Proof. On $M = X \times [0,1]$, oriented with the interval coordinate last, interpolate the actual sections, fixing their common germ near E and their endpoint collars. Relative transversality on the interior, with local section variations supported away from these fixed sets, gives a section S of the pullback of $L \oplus L$ transverse to zero [28, Chapter 3, §2, Theorem 2.7]. Its zero set is a compact oriented one-manifold. On the endpoint collars its orientation is the positive interval direction, since the vertical derivatives of s^0 and s^1 preserve orientation. Every interval component therefore runs from the lower boundary to the upper boundary. There are exactly N such intervals and finitely many circle components.

The set $(X \setminus E) \times (0,1)$ is connected. An embedded arc in it can join a circle component to one of the interval components, with interior disjoint from the zero set and endpoint velocities not tangent to it. To obtain it, join short normal segments by a path and apply relative transversality: intersections of two curves in a five-manifold have codimension five and parameter dimension two. Lemma 11.1 absorbs the circle into the interval. There are finitely many circles, so finitely many such changes remove them all. Their supports are compact and disjoint from $E \times [0,1]$ and the endpoint collars. After shrinking the original neighborhood of E, the section is still fixed on that neighborhood.

Parametrize the remaining oriented zero arcs by

$$j_i(r) = (h_i(r), p_i(r)), \qquad 0 \le r \le 1, \qquad 1 \le i \le N,$$

with $p_i(r) = r$ on smaller endpoint collars. Small relative perturbations $\widetilde{h}_i$ make the graphs

$$j_i'(r) = (\widetilde{h}_i(r), r) \tag{11.5}$$

pairwise disjoint. For $i \neq j$, coincidence at the same r has parameter dimension one and codimension four in $X \times X$; independent variations of the two maps give the required transversality. Each graph is embedded because its last coordinate is r. The perturbations and a small homotopy to them remain in $X \setminus E$.

Interpolate the last coordinate between $p_i(r)$ and r, and the first coordinate from h_i to $\widetilde{h}_i$ by the preceding small homotopy. This gives a homotopy from the disjoint union of the j_i to that of the j_i', relative to the endpoint collars. It can be made an isotopy relative to its ends (cf. [28, Chapter 8, §1, Exercise 10]). Indeed, for a homotopy $J_i(r,\lambda)$, the failure of immersion $\partial_r J_i = 0$ has codimension five and parameter dimension two. Coincidence

$$J_i(r,\lambda) = J_j(r',\lambda), \qquad (i,r) \neq (j,r'),$$

has codimension five and parameter dimension three. Local perturbations vary the first derivative arbitrarily wherever the homotopy is not fixed. On the fixed sets the curves are already embedded. Parametric transversality therefore makes every curve an immersion. Compactness then excludes coincidence in a neighborhood of the diagonal. For a pair of distinct source points, both evaluations can be varied independently if neither point is fixed; if exactly one is fixed, the other evaluation can be varied arbitrarily. Pairs of fixed points already have distinct images. Apply parametric transversality on the compact complement of a sufficiently small diagonal neighborhood. The second perturbation can be chosen C^1-small, preserving immersion and injectivity near the diagonal. All perturbations have compact support in $(X \setminus E) \times (0,1)$.

Double M and the zero arcs along their stationary endpoint collars, and extend the isotopy by the identity on the second half. The doubled zero set is a compact submanifold of the boundaryless double. Isotopy extension [28, Chapter 8, §1, Theorem 1.3 and its proof] applies. Its extending vector field may be cut off in a compact neighborhood of the moving trace, inside $(X \setminus E) \times (0,1)$. Restriction to M gives an ambient isotopy Φ_λ supported away from the fixed sets, with $\Phi_1(j_i(r)) = j_i'(r)$. Fix a unitary connection on the pullback of L. Parallel transport along $\lambda \mapsto \Phi_\lambda(y)$ gives a unitary map

$$T_\lambda(y) \,:\, L_y \longrightarrow L_{\Phi_\lambda(y)}.$$

Here the fibres are those of the pullback bundle on M. Define a new actual section by

$$S'(\Phi_1(y)) = (T_1(y) \oplus T_1(y))S(y).$$

It is transverse, has the prescribed germs, and its zeros are the graphs (11.5). The ambient isotopy and the unitary bundle maps preserve orientations. Hence the induced orientation on each zero graph is its increasing r orientation.

Set $s_t(x) = S'(x,t)$. At a zero, the kernel of DS' is tangent to its graph and projects positively onto the interval. Its intersection with T_xX is zero, so $D_x s_t$ is an isomorphism. If $n_1, \dots, n_4$ are its inverse images of a positive target basis, and v is the positive zero tangent, then (11.1) makes $(n_1, \dots, n_4, v)$ positive. Since $dt(v) > 0$, the basis $(n_1, \dots, n_4)$ is positive in T_xX. Thus all zeros in every slice are positive and simple.

The pullback line bundle is trivial near each zero graph, because that neighborhood retracts onto an interval. Choose a unitary frame stationary near the two ends. The two complex coefficients of s_t have invertible positive vertical derivative. The inverse function theorem with parameter makes them coordinates as in (11.4). Compactness of the finitely many graphs gives one sufficiently small radius on which all these coordinate maps are defined and their balls are pairwise disjoint and disjoint from a fixed neighborhood of E. □

11.3. **An exact extension of the Hopf connection.** All exterior derivatives are taken on X. For radial extensions near a pencil basepoint, cf. [19, pp. 194–195] and [4, §7.1, Step 4].

Lemma 11.3. *Let $s_t = (s_{t,1}, s_{t,2})$, $0 \le t \le 1$, be a smooth path of sections of $L \oplus L$ whose zeros are positive and simple and form disjoint graphs. Let $E \subset X$ be compact, and suppose s_t equals a fixed nonzero section near E. Put*

$$f_t = [s_{t,1} : s_{t,2}] : X \setminus s_t^{-1}(0) \longrightarrow \mathbb{CP}^1,$$

and let f denote the common restriction of these maps near E. There is a smooth family of unitary connections on L, with real curvature forms K_t, such that

$$[K_t] = c_1(L)_{\mathbb{R}}, \qquad K_t \wedge K_t \ge 0, \qquad K_t = f^*\sigma \quad \text{near } E. \tag{11.6}$$

The connections may be chosen stationary near each endpoint at which s_t is stationary.

Suppose in addition that, at an endpoint $i \in \{0,1\}$, a closed two-form Ω_i has nonnegative square, satisfies $\Omega_i \wedge f_i^*\sigma \ge 0$ off the zeros, and is positive on every complex line in the section coordinates at each zero. The connections may moreover be chosen so that

$$\Omega_i \wedge K_i \ge 0 \quad \text{on } X. \tag{11.7}$$

Proof. On the complement of the zeros, write $s_t = (z_1 e, z_2 e)$ in a unitary frame e. Put $z = (z_1, z_2)$ and

$$\beta_e = \frac{\operatorname{Im}(\bar z_1\, dz_1 + \bar z_2\, dz_2)}{2\pi(|z_1|^2 + |z_2|^2)}. \tag{11.8}$$

If $e' = e^{2\pi i\phi} e$, then $z' = e^{-2\pi i\phi} z$ and $\beta_{e'} = \beta_e - d\phi$. These are precisely the transition rules for the potentials of $d - 2\pi i\beta_e$ on the actual line bundle L. On $z_1 \ne 0$, with $w = z_2/z_1$,

$$d\beta_e = \frac{i}{2\pi} \frac{dw \wedge d\bar w}{(1+|w|^2)^2} = f_t^*\sigma.$$

Its square is zero.

By the inverse function argument at the end of the proof of Proposition 11.2, choose moving coordinate balls of a common radius $R > 0$. In a chosen unitary frame on each ball let $q = |z_1|^2 + |z_2|^2$. Choose a smooth nondecreasing function $\chi : [0, R^2] \to [0,1]$, equal to cq near zero for some $c > 0$ and equal to one near R^2. Replace the potential in that frame by

$$A = \chi(q)\beta_e.$$

It extends smoothly over the origin, since there it is $\frac{c}{2\pi} \operatorname{Im} \sum_{j=1}^2 \bar z_j\, dz_j$. Near the boundary it is the original potential, so it gives a connection on L. The prescription is made in the chosen frame; in every other frame its potential is obtained by the connection transition rule.

The curvature is invariant under constant unitary changes of the coordinates. At $z = (r, 0)$, with $z_j = x_j + iy_j$ and $q = r^2 > 0$, it is

$$dA = \frac{\chi'(q)}{\pi}\, dx_1 \wedge dy_1 + \frac{\chi(q)}{\pi q}\, dx_2 \wedge dy_2. \tag{11.9}$$

Both coefficients are nonnegative. At the origin the curvature is $\frac{c}{\pi}(dx_1 \wedge dy_1 + dx_2 \wedge dy_2)$. Thus the curvature on each ball is a nonnegative real $(1,1)$-form in orientation-preserving coordinates, and its square is nonnegative. Off the balls it is $f_t^*\sigma$, whose square is zero. Hence $K_t^2 \geq 0$ on X. Since these are curvatures of connections on L, the Chern–Weil formula gives $[K_t] = c_1(L)_{\mathbb{R}}$ [11, p. 180, footnote 81]. The balls miss a fixed neighborhood of E, where the common actual section makes the connection, and therefore its curvature, independent of t.

Frames and coordinate balls over the zero graphs can be chosen smoothly and stationary near stationary ends. A single sufficiently small radius and a single function χ therefore give a smooth family of connections with the required endpoint behavior.

Finally, at an endpoint zero the hypothesis on Ω_i means that its $(1,1)$ part is positive definite. Positivity persists throughout a sufficiently small coordinate ball: the unit complex lines over the closure of a smaller ball form a compact set. The wedge product of this positive $(1,1)$ part with (11.9) is nonnegative, while the $(2,0) + (0,2)$ part has zero wedge product with a $(1,1)$-form in complex dimension two. Hence $\Omega_i \wedge K_i \geq 0$ on the balls. Off them the given inequality with $f_i^*\sigma$ applies. □

11.4. **A path in a fixed real class.**

Proposition 11.4. *Let $a, b \in H^2(X;\mathbb{R})$ and let $D \subset \operatorname{int} E \subset X$, where D is compact and connected, E is compact, and $X \setminus E$ is connected. Suppose the following data are given:*

(1) closed two-forms Ω_0, Ω_1 representing b, with $\Omega_i^2 \geq 0$ for $i = 0, 1$, equal to one symplectic form Ω near E;

(2) a Hermitian line bundle L with $c_1(L)_{\mathbb{R}} = \kappa b$, $\kappa > 0$, and section pairs s^0, s^1 as in Proposition 11.2, with the endpoint positivity hypotheses of Lemma 11.3. Let f be their common projective map near E, and put $T = \kappa^{-1} f^*\sigma$;

(3) closed two-forms $\rho_1, \dots, \rho_m$ supported in $\operatorname{int} E$, positive numbers $\mu_1, \dots, \mu_m$, and $R = \sum_j \mu_j \rho_j$, such that $[R] = a - b$ and, on E,

$$\rho_j^2 \geq 0, \qquad \rho_j \wedge \rho_l \geq 0, \qquad \rho_j \wedge \Omega \geq 0, \qquad \rho_j \wedge T \geq 0. \tag{11.10}$$

Suppose also that $\Omega \wedge T \geq 0$ near E and $(T + R)^2 > 0$ near D. There is a smooth path of closed forms η_t in the real class a, stationary near its endpoints, such that

$$\eta_i = \Omega_i + R, \qquad \eta_t^2 \geq 0 \quad \text{on } X,$$

and η_t is symplectic on one fixed connected neighborhood of D for every t.

Proof. Proposition 11.2 gives a path of actual sections fixed near E. Lemma 11.3 gives curvatures K_t with

$$[K_t] = \kappa b, \qquad K_t^2 \geq 0, \qquad \kappa^{-1} K_t = T \quad \text{near } E, \qquad \Omega_i \wedge K_i \geq 0.$$

Take a smooth function $v : [0,1] \to [0,1]$ equal to zero on smaller endpoint collars and to one outside larger endpoint collars. Choose these larger collars inside the stationary collars of the section path and of K_t. Define

$$H_t = \begin{cases} (1 - v(t))\Omega_0 + v(t)\kappa^{-1} K_0, & t \text{ in the first collar,} \\ \kappa^{-1} K_t, & t \text{ outside the endpoint collars,} \\ (1 - v(t))\Omega_1 + v(t)\kappa^{-1} K_1, & t \text{ in the last collar.} \end{cases}$$

The definitions agree on overlaps. Each H_t is closed, $[H_t] = b$, and $H_t^2 \geq 0$. Near E,

$$H_t = (1 - v(t))\Omega + v(t)T.$$

Set $\eta_t = H_t + R$. It is closed, represents a, and has the stated endpoints. Outside $\operatorname{supp} R$ its square is $H_t^2 \geq 0$. On E, all products in (11.10) and $\Omega \wedge T$ are nonnegative. Since $T^2 = 0$, expansion gives

$$\eta_t^2 \geq (1 - v(t))^2 \Omega^2 + v(t)^2 (T + R)^2. \tag{11.11}$$

Indeed, the difference is

$$2v(1-v)\Omega \wedge T + 2(1-v)\Omega \wedge R + 2v(1-v)T \wedge R + (1-v^2)R^2,$$

which is nonnegative. If $D = \emptyset$, take the empty neighborhood. Assume $D \neq \emptyset$. On D the right-hand side of (11.11) is strictly positive for every $v \in [0,1]$. The continuous positive function obtained by dividing it by a fixed positive volume form has a positive minimum on $D \times [0,1]$. It remains positive on an open neighborhood V of D with $V \subset \operatorname{int} E$. The component of V containing D is the required connected neighborhood. □

12. Exact fibres and prescribed paths

12.1. The determinant-normalized polar metric.

Lemma 12.1. *Let k be a metric and η a two-form on an oriented four-manifold. Define the k-skew endomorphism A by $\eta(v, w) = k(Av, w)$. For $\delta > 0$ put*

$$S_\delta = (-A^2 + \delta^2 I)^{1/2}, \qquad G_\delta = (\det S_\delta)^{-1/4} k(S_\delta \cdot, \cdot). \tag{12.1}$$

This is a smooth metric with volume form dV_k. It equals k wherever η is k-self-dual, including its zeros. If $\eta^2 \geq 0$, then pointwise

$$0 \leq |\eta|^2_{G_\delta} dV_{G_\delta} - \eta \wedge \eta \leq \sqrt{2}\delta |\eta|_k \, dV_k. \tag{12.2}$$

Where $\eta^2 > 0$, the same formula at $\delta = 0$ is smooth and makes η self-dual.

Proof. The matrix $-A^2 + \delta^2 I$ is positive definite. Its positive square root is smooth in all its entries. The determinant of the endomorphism $(\det S_\delta)^{-1/4} S_\delta$ is one, which proves the volume assertion. At a point choose an oriented k-orthonormal coframe with

$$\eta = a_1 e^{12} + a_2 e^{34}, \qquad s_j = \sqrt{a_j^2 + \delta^2}.$$

The eigenvalues of S_δ are s_1, s_1, s_2, s_2, so

$$|\eta|^2_{G_\delta} dV_{G_\delta} = \left(a_1^2 \frac{s_2}{s_1} + a_2^2 \frac{s_1}{s_2} \right) dV_k. \tag{12.3}$$

The arithmetic-geometric mean inequality gives the lower bound $2|a_1 a_2|$. Since $a_j^2 / s_j \leq |a_j|$ and $s_j \leq |a_j| + \delta$, the same expression is at most $2|a_1 a_2| + \delta(|a_1| + |a_2|)$. Now $\eta^2 = 2 a_1 a_2 \, dV_k$ and $\eta^2 \geq 0$, so $a_1 a_2 = |a_1 a_2|$; also $|a_1| + |a_2| \leq \sqrt{2}|\eta|_k$. This proves (12.2).

If η is self-dual then $a_1 = a_2$, so S_δ is scalar and (12.1) equals k. If $\eta^2 > 0$, then A is invertible and G_0 is smooth. The polar construction [11, Proposition 3.1 and the following remark, pp. 108–109] makes $k(S_0 \cdot, \cdot)$ an η-compatible metric. Since $\eta^2 > 0$, η is self-dual for this metric in the given orientation, hence also for its conformal multiple G_0. □

Proposition 12.2. *Assume $b^+(X) = 1$. Let $a \in \mathcal{P}_X$ and let η_t, $0 \leq t \leq 1$, be a smooth path of closed representatives of a, constant near its endpoints. Suppose $\eta_t^2 \geq 0$ on X and that all η_t are symplectic on one connected neighborhood of a compact connected set D containing two-cycles spanning $H_2(X; \mathbb{R})$. If k_0, k_1 are metrics for which η_0, η_1 are self-dual, then k_0 and k_1 are joined by a smooth path in $\mathcal{F}_a$, constant near its endpoints.*

Proof. Choose connected open neighborhoods

$$D \subset U_0 \Subset U_1 \Subset U_2$$

on which every η_t is symplectic. Their existence follows from the given fixed neighborhood and the connectedness and compactness of D. Let $\bar{k}_t$ be a smooth metric path from k_0 to k_1, stationary on the collars where η_t is stationary. A cutoff interpolation of the two metrics supplies it. On U_2, apply (12.1) with $\delta = 0$, $k = \bar{k}_t$, and $\eta = \eta_t$; denote the result by $\widehat{k}_t$. It makes η_t self-dual there and equals the given endpoint metric on each endpoint collar. Choose a smooth spatial cutoff $0 \le \zeta \le 1$ equal to one on U_1 and supported in U_2, and set

$$k_t = \zeta \widehat{k}_t + (1 - \zeta)\bar{k}_t.$$

Outside U_2 this means $k_t = \bar{k}_t$. Convexity of positive definite forms makes k_t a smooth global metric path, equal to the prescribed endpoint metrics and making η_t self-dual on U_1.

Apply (12.1) globally to (η_t, k_t) with $\delta > 0$, and write the resulting metric as $G_{t,\delta}$. It equals k_t on U_1 and on the endpoint collars. Integrating (12.2) gives

$$E_{G_{t,\delta}}(\eta_t) \le Q_X(a,a) + C\delta, \qquad C = \sqrt{2} \sup_{0\le t\le 1} \int_X |\eta_t|_{k_t}\, dV_{k_t} < \infty.$$

Lemma 2.1 therefore yields

$$\sinh^2 d_{\mathbb{H}}(p(G_{t,\delta}), u_a) \le \frac{C\delta}{2Q_X(a,a)}. \tag{12.4}$$

Fix a compact neighborhood K of u_a in $\mathbb{H}_X$. For all sufficiently small δ, the periods lie in K. On U_1 the metrics are the fixed smooth family k_t; hence one background metric, two ellipticity bounds, and one C^8 bound apply to every $G_{t,\delta}$, independently of δ. Moreover U_0 is detecting. Lemma 2.6 gives one correction radius $\varepsilon > 0$ for all these metrics. Choose $\delta > 0$ so small that $p(G_{t,\delta}) \in K$ for every t and

$$\operatorname{arsinh} \sqrt{\frac{C\delta}{2Q_X(a,a)}} < \varepsilon$$

and put

$$g_t = \mathcal{R}(G_{t,\delta}, u_a).$$

Then $p(g_t) = u_a$. At each endpoint collar, $G_{t,\delta} = k_i$ and $p(k_i) = u_a$, because η_i is a closed self-dual form in class a. The correction is zero there. Thus g_t has the prescribed endpoints and is stationary on their collars. □

12.2. **The fibres.**

Proof of Theorem B. Nonemptiness, with any prescribed positive volume form, follows from Theorem A. Fix $a \in \mathcal{P}_X$ and $g_0, g_1 \in \mathcal{F}_a$. If $b^-(X) = 0$, the target consists of one point, and convex interpolation joins g_0 to g_1 in $\mathcal{F}_a$. Assume now that $b^-(X) > 0$. For $i = 0, 1$, Proposition 7.2 gives a smooth path $\gamma_i : [0,1] \to \mathcal{F}_a$ from g_i to a metric $\widehat{g}_i$ whose harmonic representative $\omega_i = h^2_{\widehat{g}_i}(a)$ is near-symplectic. Put $P_i = \Lambda^+_{\widehat{g}_i}$.

Apply Proposition 10.5 to (ω_i, P_i), and retain its notation. By Lemma 2.2, each positive three-plane bundle path, with the fixed volume form $dV_{\widehat{g}_i}$, determines a smooth metric path α_i starting at $\widehat{g}_i$. The corresponding forms are closed self-dual representatives of a, so α_i lies in $\mathcal{F}_a$. Put $k_i = \alpha_i(1)$; then $\Omega_i + R$ is self-dual for k_i.

By Proposition 10.5(1)–(2), $D = \bigcup_j S_j$ is compact and connected, the classes $[S_j]$ span $H_2(X;\mathbb{R})$, and $X \setminus E$ is connected. Moreover, $D \subset \operatorname{int} E$, and the near-symplectic forms Ω_i represent b and equal a symplectic form Ω on a neighborhood U of E. Item (3) gives a Hermitian line bundle $\mathcal{L}$ with $c_1(\mathcal{L})_{\mathbb{R}} = \kappa b$, $\kappa > 0$, and section pairs s^0, s^1 that agree and are nonzero on U; each has $\kappa^2 Q_X(b,b) \ge 2$ positive simple zeros. Let f be the common restriction to U of their projective maps $f_i = [s^i_0 : s^i_1]$, and put $T = \kappa^{-1} f^* \sigma$. Item (4) gives the endpoint positivity hypotheses of Lemma 11.3 and, on U,

$$\Omega \wedge T = \kappa^{-1} \Omega_i \wedge f_i^* \sigma \ge 0.$$

By items (1) and (5), $R = \sum_j \mu_j \rho_j$ represents $a - b$, the forms ρ_j have compact support in $\operatorname{int} E$ and satisfy (11.10), and $(T + R)^2 > 0$ near D. Proposition 11.4 therefore gives a smooth path η_t of closed representatives of a, constant near its endpoints, with

$$\eta_i = \Omega_i + R, \qquad \eta_t^2 \geq 0 \quad \text{on } X,$$

symplectic on one fixed connected neighborhood of D. Proposition 12.2 gives a smooth path β in $\mathcal{F}_a$ from k_0 to k_1. After smooth reparametrizations making each constituent path constant near both endpoints, concatenate γ_0, α_0, β, and the reverses of α_1 and γ_1. This gives a smooth path from g_0 to g_1 in $\mathcal{F}_a$.

In either case the path may be chosen constant near both endpoints, by a smooth reparametrization in the case $b^-(X) = 0$. If $dV_{g_0} = dV_{g_1} = \nu$, composition with N_ν from (2.12) gives a path in $\mathcal{F}_a \cap \operatorname{Met}_\nu^\infty(X)$ with the same endpoints and stationary endpoint neighborhoods, by Lemma 2.5. □

12.3. **Gluing local lifts.** For a surjective smooth submersion between finite-dimensional manifolds with connected fibres, every smooth map from a compact one-dimensional manifold to the base admits a smooth lift [42, Lemma 2.2, p. 490].

Lemma 12.3. *Let $\gamma : (-\tau, \tau) \to \mathbb{H}_X$ be smooth, and let k_t, l_t be two smooth lifts of γ. There are $0 < \tau' < \tau$ and a smooth lift on $(-\tau', \tau')$ agreeing with k_t near $-\tau'$ and with l_t near τ'.*

Proof. By Theorem B, choose a smooth path h_s in the fibre over $\gamma(0)$ from k_0 to l_0, constant near $s = 0, 1$. Choose a smooth function $\chi : [0,1] \to [0,1]$ equal to zero and one on these endpoint neighborhoods, respectively, and set

$$B(s,t) = h_s + (1 - \chi(s))(k_t - k_0) + \chi(s)(l_t - l_0).$$

Since $B(s,0) = h_s$, positivity and compactness of $X \times [0,1]$ give $0 < \delta < \tau$ such that $B(s,t)$ is a metric for $0 \leq s \leq 1$ and $|t| \leq \delta$. Fix a background metric g_*. This compact smooth family satisfies, for some $\lambda, \Lambda, C > 0$,

$$\lambda g_* \leq B(s,t) \leq \Lambda g_*, \qquad \|B(s,t)\|_{C^8(X,g_*)} \leq C.$$

Choose a compact set $K \subset \mathbb{H}_X$ containing all $p(B(s,t))$ and $\gamma(t)$ on this parameter rectangle. Lemma 2.6, with $U_0 = U_1 = X$ and $\ell = 8$, gives a number $\varepsilon > 0$ and a map $\mathcal{R}$ for these bounds and K. Since $p(B(s,0)) = \gamma(0)$ for every s,

$$\sup_{0 \leq s \leq 1} d_{\mathbb{H}}\big(p(B(s,t)), \gamma(t)\big) \longrightarrow 0 \quad \text{as } t \longrightarrow 0.$$

After decreasing δ, the supremum is less than ε for $|t| < \delta$. Hence

$$G(s,t) = \mathcal{R}\big(B(s,t), \gamma(t)\big)$$

is smooth and satisfies $p(G(s,t)) = \gamma(t)$. Near $s = 0$ one has $B(s,t) = k_t$, and near $s = 1$ one has $B(s,t) = l_t$. The identity $\mathcal{R}(g, p(g)) = g$ therefore gives $G(s,t) = k_t$ and $G(s,t) = l_t$ on the respective endpoint neighborhoods.

Choose $0 < \tau' < \delta$ and a smooth function $\theta : (-\tau', \tau') \to [0,1]$ equal to zero near its left end and to one near its right end. Then $t \mapsto G(\theta(t), t)$ has period $\gamma(t)$ and agrees with the two given lifts near the corresponding ends. □

Proof of Theorem C. Via $u \mapsto \mathbb{R}u$ and (2.5), Theorem A gives a smooth section $s : \mathbb{H}_X \to \operatorname{Met}^\infty(X)$ of p. Put $g_t^* = s(\gamma(t))$. Let $A \subset [0,1]$ consist of $0, 1$ and the parameters of the prescribed germs. For each $a \in A$ carrying a prescribed germ, choose a representative k_t^a common to all the germs there; compatibility ensures that such a representative exists. At an endpoint with no prescribed germ, Proposition 2.3 and Lemma 2.4 give a local lift k_t^a through the prescribed metric. Thus $k_i^i = g_i$ for $i = 0, 1$.

Choose pairwise disjoint relative open intervals I_a about the points $a \in A$, on which k_t^a is defined, and smaller closed relative interval neighborhoods $J_a \Subset I_a$ of a. Choose one point in each component

of $I_a \setminus J_a$. On a small interval about a point to the left of J_a, apply Lemma 12.3, with the parameter translated to that point, to g_t^*, k_t^a; to the right, apply it to k_t^a, g_t^*. Choose these intervals inside their respective components. At $a = 0$ or 1, there is only one such interval. Take the resulting lifts on the smaller intervals supplied by the lemma, and k_t^a between them; at an endpoint take k_t^a between that endpoint and its sole gluing interval. Take g_t^* elsewhere outside the gluing intervals. Agreement near their ends gives a smooth lift g_t on $[0, 1]$ with $g_t = k_t^a$ on each J_a. It has all the prescribed germs and endpoint metrics.

If a smooth family of volume forms ν_t is prescribed, apply N_{ν_t} from (2.12) pointwise in t. By Lemma 2.5, the resulting lift is smooth, has volume form ν_t, and agrees with g_t wherever $dV_{g_t} = \nu_t$. Compatibility therefore preserves every prescribed germ and endpoint metric. □

13. Consequences

13.1. **Chambers.** In this and the next subsection, X is a closed connected oriented smooth four-manifold with $b^+(X) = 1$.

Proof of Corollary 1.1. Let $C \subset \mathbb{H}_X$ be nonempty, connected and open. As a connected smooth manifold, C is smoothly path connected [70, p. 49 and proof of Lemma 1.10]. For $g_0, g_1 \in p^{-1}(C)$, choose a smooth path in C from $p(g_0)$ to $p(g_1)$. Theorem C gives a lift joining g_0 to g_1 in $p^{-1}(C)$. If $dV_{g_0} = dV_{g_1} = \nu$, its volume clause gives the lift in $p^{-1}(C) \cap \mathrm{Met}_\nu^\infty(X)$.

Let $\{W_\xi\}_{\xi\in\Xi}$ be the locally finite family of walls, and put

$$\mathcal{U} = \mathbb{H}_X \setminus \bigcup_{\xi\in\Xi} W_\xi.$$

Local finiteness makes $\mathcal{U}$ open, and its components are open since $\mathbb{H}_X$ is locally path connected. On each component, every function $u \mapsto Q_X(u, \xi)$ has a constant strict sign. Conversely, let $u_0, u_1 \in \mathcal{U}$ have the same sign for every such pairing. Set

$$u_t = \frac{(1-t)u_0 + tu_1}{\sqrt{Q_X((1-t)u_0 + tu_1, (1-t)u_0 + tu_1)}}, \qquad 0 \le t \le 1. \tag{13.1}$$

Since $Q_X(u_0, u_1) \ge 1$,

$$Q_X((1-t)u_0 + tu_1, (1-t)u_0 + tu_1) = 1 + 2t(1-t)\big(Q_X(u_0, u_1) - 1\big) \ge 1.$$

The numerator is a path of positive-square vectors starting in $\mathcal{P}_X$, so $u_t \in \mathbb{H}_X$. For each $\xi \in \Xi$, the numerator of $Q_X(u_t, \xi)$ is $(1-t)Q_X(u_0, \xi) + tQ_X(u_1, \xi)$, which has their common strict sign. Thus $u_t \in \mathcal{U}$, and u_0, u_1 lie in the same chamber.

By continuity of p, every connected subset of $p^{-1}(\mathcal{U})$ has image in one chamber. Each chamber inverse image is nonempty by Theorem B and smoothly path connected by the first paragraph. These inverse images are therefore exactly the connected components of $p^{-1}(\mathcal{U})$. The same argument applies to $p^{-1}(\mathcal{U}) \cap \mathrm{Met}_\nu^\infty(X)$. □

13.2. **Scalar periods.**

Corollary 13.1. *Let $\xi \in V_X$ satisfy $Q_X(\xi, \xi) < 0$, and define*

$$p_\xi(g) = Q_X(p(g), \xi).$$

The map $p_\xi : \mathrm{Met}^\infty(X) \to \mathbb{R}$ is surjective, and every level set is smoothly path connected. Every smooth $f : [0, 1] \to \mathbb{R}$ admits a smooth lift g_t with $p_\xi(g_t) = f(t)$, arbitrary endpoint metrics g_i satisfying $p_\xi(g_i) = f(i)$ for $i = 0, 1$, and finitely many compatible prescribed germs of such lifts. A compatible smooth family of volume forms may also be prescribed.

Proof. Put $c = -Q_X(\xi,\xi) > 0$ and $q = b^-(X) \geq 1$. Choose a Q_X-orthonormal basis $e_0, e_1, \dots, e_q$ with $e_0 \in \mathcal{P}_X$ and $e_1 = \xi/\sqrt{c}$. The hyperboloid has global coordinates

$$u(x) = \sqrt{1+|x|^2}\, e_0 + \sum_{j=1}^{q} x_j e_j, \qquad x \in \mathbb{R}^q,$$

where $|x|$ is the Euclidean norm, and $Q_X(u(x),\xi) = -\sqrt{c}\, x_1$. Surjectivity of p [56, Theorem 1.2] therefore gives surjectivity of p_ξ.

Fix f, the endpoint metrics and the prescribed germs. Write $x = (x_1, y)$ with $y \in \mathbb{R}^{q-1}$; if $q = 1$, this last space is $\{0\}$. Define y_i by

$$p(g_i) = u\big(-f(i)/\sqrt{c}, y_i\big), \qquad i = 0, 1,$$

and put $\bar{y}(t) = (1-t)y_0 + t y_1$. Let $A \subset [0,1]$ be the finite set of parameters of the prescribed germs. At each $a \in A$, choose a representative k_t^a common to the germs there. Choose pairwise disjoint relative open intervals I_a about these points, on which the representatives are defined, with $I_a \cap \{0,1\} \subset \{a\}$. Define $y^a(t)$ on I_a by

$$p(k_t^a) = u\big(-f(t)/\sqrt{c}, y^a(t)\big).$$

Choose smooth functions $\chi_a : [0,1] \to [0,1]$, equal to one near a and with support contained in I_a [70, Corollary to Theorem 1.11, p. 11], and set

$$y(t) = \bar{y}(t) + \sum_{a \in A} \chi_a(t)\big(y^a(t) - \bar{y}(t)\big), \qquad \gamma(t) = u\big(-f(t)/\sqrt{c}, y(t)\big).$$

Each summand is extended by zero outside I_a. Thus γ is smooth and agrees with $p(k_t^a)$ near each a. Compatibility gives $y^a(i) = y_i$ when $a = i \in \{0,1\}$; all other summands vanish at i. Hence $\gamma(i) = p(g_i)$. Theorem C, applied to γ with the given metric germs and endpoints, gives the required lift and any compatible prescribed volume forms. Taking f constant proves smooth path connectedness of every level set, also with a fixed volume form. □

13.3. **Smooth families of middle Hodge inner products.**

Corollary 13.2. *Let M^{2m} be closed, connected and oriented, with $m \geq 2$. Let B be a finite-dimensional smooth manifold, $A : B \to \mathcal{D}(Q_M)$ a smooth map, and ν_b a jointly smooth family of positive volume forms on M. There is a jointly smooth family of metrics g_b such that*

$$\Gamma_m(g_b) = A(b), \qquad dV_{g_b} = \nu_b.$$

Each g_b is regular for $\Gamma_m|_{\mathrm{Met}^\infty_{\nu_b}(M)}$.

Proof. Fix a positive volume form ν_0 and let s_{ν_0} be the section of Theorem A. Put

$$g_b = N_{\nu_b}\big(s_{\nu_0}(A(b))\big) = \left(\frac{\nu_b}{\nu_0}\right)^{1/m} s_{\nu_0}(A(b)).$$

Lemma 2.5 gives joint smoothness and the two identities. For each fixed b, it also gives regularity on the image of the section $N_{\nu_b} \circ s_{\nu_0}$. □

The case of compact B is also proved in [51, Corollary 7.6].

Suppose now that $m \geq 3$ is odd. The lattice L_M with its unimodular alternating pairing Q_M is fixed. A compatible complex structure J on V_M satisfies

$$J^2 = -I, \qquad Q_M(Jv, Jw) = Q_M(v,w), \qquad Q_M(v, Jv) > 0 \quad (v \neq 0).$$

The positive Hermitian form

$$H_J(v,w) = Q_M(v, Jw) - i Q_M(v,w),$$

complex-linear in its first argument, has imaginary part $-Q_M$. It determines the principal polarization on $(V_M, J)/L_M$. The correspondence $J \mapsto Q_M J$ identifies these marked structures with $\mathcal{D}(Q_M)$. For a metric g, the polarized torus $(V_M, J_M(g))/L_M$ is the Lazzeri Jacobian of (M, g), with its lattice marking [54, Definition 1.2 and Notation 1.4]. Given a positive volume form ν, the metric $g = s_\nu(Q_M J)$ from Theorem A satisfies $J_M(g) = J$ by (2.1) and has volume form ν. Corollary 13.2 applies to smooth finite-dimensional families of these marked structures.

13.4. **The other degrees.**

Corollary 13.3. *Let M be a closed connected oriented smooth manifold of dimension $n \geq 3$, and let $1 \leq p \leq n-1$ with $2p \neq n$. The map Γ_p has a global smooth section over the cone of positive definite forms on $H^p(M;\mathbb{R})$. Every metric in its image is regular for Γ_p.*

Proof. If $H^p(M;\mathbb{R}) = 0$, the target is a point and any fixed metric suffices. Suppose $H^p(M;\mathbb{R}) \neq 0$. For $2p < n$, this is Theorem 3.1. Suppose $2p > n$ and set $q = n - p$, so that $1 \leq q < n/2$. Choose bases $(a_i)_{i=1}^r$ of $H^q(M;\mathbb{R})$ and $(b_j)_{j=1}^r$ of $H^p(M;\mathbb{R})$, and put

$$C_{ij} = \langle a_i \smile b_j, [M] \rangle.$$

Poincaré duality makes C invertible [23, Proposition 3.38]. Fix $g \in \mathrm{Met}^\infty(M)$ and let $\alpha_i = h_g^q(a_i)$ and $\beta_j = h_g^p(b_j)$. Write A_g, B_g for the Gram matrices in these bases. The Hodge star is an L^2-isometry from harmonic q-forms onto harmonic p-forms [70, Chapter 2, Exercise 13; §6.1(4)–(5)]. Define the matrix S_g by

$$*_g\, \alpha_j = \sum_{i=1}^{r} (S_g)_{ij} \beta_i.$$

The definitions and the isometry give

$$A_g = C S_g, \qquad A_g = S_g^{\mathsf{T}} B_g S_g.$$

Substituting $S_g = C^{-1} A_g$ yields

$$B_g = C^{\mathsf{T}} A_g^{-1} C.$$

Let σ_q be the section of Theorem 3.1. For $B = B^{\mathsf{T}} > 0$, set

$$\sigma_p(B) = \sigma_q(C B^{-1} C^{\mathsf{T}}).$$

Since $C B^{-1} C^{\mathsf{T}} > 0$, this defines a smooth map. The displayed identity gives $\Gamma_p(\sigma_p(B)) = B$; its differential is

$$D\Gamma_p\big|_{\sigma_p(B)} \circ D\sigma_p\big|_B = \mathrm{id}_{\mathrm{Sym}(r)}.$$

Thus every metric in the image of σ_p is regular. □

13.5. **Retractions over connected open sets.** Let X be a closed connected oriented smooth four-manifold with $b^+(X) = 1$, and fix a positive smooth volume form ν. Identify $\mathbb{H}_X$ with $\mathcal{D}_X$ by $u \mapsto \mathbb{R}u$. Under (2.5), Theorem A gives a section $s_\nu : \mathbb{H}_X \to \mathrm{Met}_\nu^\infty(X)$ of p.

Proposition 13.4. *Let $U \subset \mathbb{H}_X$ be nonempty, open, and connected, and put $M_U = p^{-1}(U) \cap \mathrm{Met}_\nu^\infty(X)$. Then M_U is smoothly path connected and $s_\nu(U)$ is a retract of M_U. For every $g \in M_U$ and every $k \geq 1$, the homomorphism*

$$p_* : \pi_k(M_U, g) \longrightarrow \pi_k(U, p(g))$$

has a right inverse. For every abelian coefficient group and in every degree, p_* on singular homology is split surjective and p^* on singular cohomology is split injective.

Proof. Smooth path connectedness is Corollary 1.1. The map $r = s_\nu \circ p : M_U \to s_\nu(U)$ satisfies $r|_{s_\nu(U)} = \mathrm{id}$. Fix $g \in M_U$ and $k \geq 1$, and put $u = p(g)$. By Theorem B, there is a path δ from g to $s_\nu(u)$ in $p^{-1}(u) \cap \mathrm{Met}^\infty_\nu(X)$. Let

$$\beta_\delta : \pi_k(M_U, s_\nu(u)) \longrightarrow \pi_k(M_U, g)$$

be the change-of-basepoint isomorphism [23, Proposition 1.5; §4.1, pp. 341–342]. Since $p \circ \delta$ is constant, naturality gives

$$p_* \beta_\delta (s_\nu)_* = \beta_{p\circ\delta}(p \circ s_\nu)_* = \mathrm{id}\,.$$

Thus $\beta_\delta(s_\nu)_*$ is the required right inverse. The identities $p_*(s_\nu)_* = \mathrm{id}$ on singular homology and $(s_\nu)^* p^* = \mathrm{id}$ on singular cohomology follow from functoriality, with any abelian coefficient group [23, pp. 111, 153, 201]. □

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