FUNDAMENTAL-GROUP EPIMORPHISMS IN NONCOLLAPSED RICCI LIMITS

ANASS NIFA

Abstract. Let $\varphi : G \twoheadrightarrow Q$ be an epimorphism of finitely presented groups. We characterize when φ is induced, under fixed markings, by continuous Gromov–Hausdorff approximation maps for a sequence of metrics on a fixed closed manifold, with a uniform lower Ricci bound, bounded diameter and a positive lower volume bound. A realization with a compact semilocally simply connected limit exists in some dimension if and only if $\ker \varphi$ is finitely normally generated by finite-order elements of G. Every such epimorphism has a realization on a fixed closed oriented four-manifold with constant positive volume, two-sided Ricci bounds and uniformly bounded L^2 curvature. The limit is an orbifold with one isolated flat singularity for each nonidentity element in a chosen finite list of finite-order normal generators of the kernel. We construct two resolutions of one orbifold whose fundamental groups are Thompson's group V and the trivial group, and whose scalar eigenvalues have the same limit at each fixed index. We also give a five-dimensional nonnegative-Ricci sequence with infinite dihedral fundamental group and simply connected limit. Finally, in every fixed dimension, under uniform lower Ricci, diameter and volume bounds, a heat integral comparing a manifold with its universal cover diverges exactly when the homotopy systole tends to zero.

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2020 *Mathematics Subject Classification.* 53C21, 53C23, 57M10, 57R65, 20E06.

Key words and phrases. Ricci curvature, Gromov–Hausdorff convergence, fundamental group, ALE metric, rational homology ball, normal closure.

1. Introduction

We determine which epimorphisms of finitely presented groups arise from continuous Gromov–Hausdorff approximation maps under a uniform lower Ricci bound, bounded diameter and noncollapse. The groups and the homomorphism are prescribed. The approximating metrics are required to lie on one fixed closed manifold. The criterion is algebraic: the kernel must be finitely normally generated by elements of finite order. Every epimorphism satisfying this condition can already be realized in dimension four, with two-sided Ricci bounds and bounded curvature energy.

Marked realizations. For a map $f : (Y, d_Y) \to (X, d_X)$, put

$$\operatorname{dis} f = \sup_{y,z\in Y} |d_X(f(y), f(z)) - d_Y(y, z)|.$$

An η-approximation has $\operatorname{dis} f \leq \eta$ and η-dense image. A sequence $f_i : Y_i \to X$ is a sequence of approximation maps if f_i is an η_i-approximation for some $\eta_i > 0$ with $\eta_i \to 0$. For $S \subset G$, let $\langle\!\langle S\rangle\!\rangle_G$ denote its normal closure in G, and put $C_p = \mathbb{Z}/p\mathbb{Z}$ for $p \geq 2$. The fundamental group of an orbifold will always mean that of its underlying topological space.

A *marked realization* of a homomorphism $\varphi : G \to Q$ consists of a fixed manifold M, metrics g_i on M, a compact metric space X, basepoints $p \in M$ and $q \in X$, continuous Gromov–Hausdorff approximation maps $f_i : (M, g_i) \to X$, and isomorphisms

$$\alpha : \pi_1(M, p) \xrightarrow{\sim} G, \qquad \beta : \pi_1(X, q) \xrightarrow{\sim} Q, \qquad \beta(f_i)_*\alpha^{-1} = \varphi.$$

For each i, a path c_i from q to $f_i(p)$ is included in the data; thus

$$(f_i)_*[\gamma] = [c_i * (f_i \circ \gamma) * c_i^{-1}].$$

Replacing c_i by c_i' conjugates this homomorphism by $[c_i' * c_i^{-1}] \in \pi_1(X, q)$. The markings α, β are independent of i. Continuity of f_i is part of the definition; it is not a consequence of the metric approximation inequalities.

Theorem 1.1 (Realization criterion). *Let G, Q be finitely presented groups and let $\varphi : G \twoheadrightarrow Q$ be an epimorphism. The following conditions are equivalent.*

(i) *There are finitely many finite-order elements $t_1, \dots, t_s \in G$ such that*

$$\ker \varphi = \langle\!\langle t_1, \dots, t_s\rangle\!\rangle_G.$$

(ii) *For some integer $n \geq 2$, the epimorphism has a marked realization by metrics g_i on one fixed closed smooth n-manifold M and a compact semilocally simply connected geodesic space X, with constants $D, v > 0$ such that*

$$\operatorname{Ric}_{g_i} \geq -(n-1)g_i, \qquad \operatorname{diam}(M, g_i) \leq D, \qquad \operatorname{Vol}(M, g_i) \geq v. \tag{1.1}$$

(iii) *The epimorphism has a marked realization by metrics g_i on one fixed closed oriented smooth four-manifold M and a compact connected four-dimensional Riemannian orbifold X, with constants $D, V_0, E > 0$ such that*

$$-3g_i \leq \operatorname{Ric}_{g_i} \leq 3g_i, \qquad \operatorname{Vol}(M, g_i) = V_0, \qquad \operatorname{diam}(M, g_i) \leq D, \tag{1.2}$$

$$\int_M |\operatorname{Rm}_{g_i}|^2 \, dv_{g_i} \leq E, \qquad (M, g_i) \xrightarrow{\mathrm{GH}} X. \tag{1.3}$$

The orbifold is smooth outside finitely many isolated points and is flat near each of them.

After identity elements have been omitted from the list in (i)*, let $p_j \geq 2$ be the exact order of t_j. The realization in* (iii) *can be chosen with exactly s singular points, whose round links are $L(p_j^2, p_j - 1)$. The constants may depend on the presentation and the chosen normal generators. The empty list is permitted when φ is an isomorphism.*

The dimension in (ii) is allowed to depend on φ. The equivalence with (iii) says that allowing other dimensions does not enlarge the class of realizable epimorphisms. It makes no assertion about realization in each prescribed dimension, or about constants uniform over presentations or dimensions. The bounds in (iii) allow negative Ricci curvature.

Short loops and the necessity of the criterion. Anderson [1] studied short geodesics under lower Ricci and noncollapse hypotheses. Sormani–Wei [10] developed covering-space methods for Gromov–Hausdorff convergence. Pan–Wei [8, Main Theorem 1.1] proved semilocal simple connectivity of noncollapsed Ricci limits, and Wang [13] removed the noncollapse assumption. We retain semilocal simple connectivity as an explicit hypothesis and study the homomorphisms of the given continuous approximation maps.

Let $N_r(M)$ be the normal subgroup generated by conjugates of rectifiable loops of length less than r, with no length restriction on the paths used to transport their basepoints. Suppose $f_i : M_i \to X$ are continuous approximation maps, where X is compact, geodesic and semilocally simply connected. Under the bounds (1.1), Proposition 3.2 proves that, for every sufficiently small fixed $r > 0$,

$$(f_i)_* : \pi_1(M_i) \twoheadrightarrow \pi_1(X), \qquad \ker(f_i)_* = N_{9r}(M_i) \quad \text{for all sufficiently large } i.$$

The identification is with the kernel of the specified map f_i. It uses the open-Rips quotient correspondence of Virk [12, Proposition 5.6 and Theorem 5.7]; Lemma 3.1 proves the version, with its natural discretization homomorphism, used here.

Lemma 2.2 bounds the order of the subgroup generated by short loops at one basepoint. In particular, the normal generators of $N_{9r}(M_i)$ have uniformly bounded finite order. This does not make their normal closure finite, or even a torsion group. Pan–Wei [8, Remark 7.2, p. 4059] observed this distinction, corrected the finite-kernel assertion in [10, Theorem 1.4], and gave an infinite dihedral kernel. Finite presentation of the target permits the extraction of finitely many torsion normal generators; the required group-theoretic argument is Lemma 3.3.

Prescribing the homomorphism. Panov–Petrunin [9, Theorem 1.3] constructed a finitely presented hyperbolic group Γ such that every finitely presented group Q can be written

$$Q \cong \Gamma' / \langle\!\langle \mathrm{Tor}(\Gamma') \rangle\!\rangle_{\Gamma'}$$

for some finite-index subgroup $\Gamma' \leq \Gamma$, where $\mathrm{Tor}(\Gamma')$ is the set of its finite-order elements. Their Theorem 1.6 also realizes every finitely presented group as the fundamental group of the underlying space of a compact oriented hyperbolic four-orbifold. In these statements the source subgroup Γ' may vary with Q. Theorem 1.1 instead starts with a given epimorphism $G \twoheadrightarrow Q$. Its kernel need not contain all torsion elements of G, and the quotient map must be induced by continuous approximation maps from smooth Ricci-bounded manifolds on one fixed underlying manifold.

For the implication (i)⇒(iii), we use the rational balls of Fintushel–Stern [4, Lemma 2.1] and the Ricci-flat asymptotically locally Euclidean (ALE) metrics of Şuvaina [11, Proposition 3.3]. For an element t_j of order p_j, the relevant filling has boundary $L(p_j^2, p_j - 1)$ and boundary homomorphism $C_{p_j^2} \twoheadrightarrow C_{p_j}$. Relative surgery constructs a compact four-manifold Z with $\pi_1(Z) = G$ for which this boundary generator maps to t_j. The inclusion of Z into the filled manifold induces an isomorphism on fundamental groups. The corresponding cone attachment induces the quotient by $\langle\!\langle t_j \rangle\!\rangle_G$. For all boundary components, the two fundamental groups are

$$G \qquad \text{and} \qquad G/\langle\!\langle t_1, \ldots, t_s \rangle\!\rangle_G.$$

The metric on Z is fixed. On the attached pieces, the differentiated ALE estimates of Kröncke–Szabó [7, Definition 1.4 and Theorem 1.16] permit interpolation between a scaled Ricci-flat metric and its flat cone, with a uniform two-sided Ricci bound. Four-dimensional scaling preserves the L^2 curvature integral. The construction, including the boundary markings and continuous approximation maps, is given in Sections 4–5.

Examples and two resolutions of one orbifold. Put

$$D_\infty = C_2 * C_2 = \langle a, b \mid a^2 = b^2 = 1\rangle. \tag{1.4}$$

The next realization gives explicit metrics and curvature formulas for this kernel, with the stronger bounds in (1.2)–(1.3).

Theorem 1.2. *There exist a fixed closed oriented smooth four-manifold M, a simply connected compact four-dimensional Riemannian orbifold X, constants $D, V_0, E > 0$, and smooth metrics g_i on M such that*

$$\pi_1(M) \cong D_\infty, \qquad \pi_1(X) = 1, \tag{1.5}$$

and (1.2)–(1.3) *hold. The space X has exactly two singular points. Each has a neighbourhood isometric, up to one common constant scaling, to a truncated flat cone over $S^3/\langle \mathbf{j}\rangle$. There are continuous Gromov–Hausdorff approximation maps $f_i : M \to X$. Their induced homomorphisms have infinite kernel. If $\Sigma \subset M$ is the union of the two quotient zero sections, then $M \setminus \Sigma$ is identified with the smooth locus of X; on every compact subset of this locus the metrics converge smoothly.*

Here S^3 is the group of unit quaternions and $\mathbf{j}$ acts by right multiplication. The round quotient is isometric, without a specified orientation, to $L(4, 1)$. Sections 6–8 give the order-two filling, its curvature tensor and the proof.

Brena [3] constructed noncollapsed sequences of closed four-manifolds with nonnegative Ricci curvature, fundamental groups C_2 and 1, and the same simply connected limit. A diagonal involution applied to his C_2 sequence gives the following result.

Theorem 1.3. *There is a sequence of closed smooth five-manifolds P_i and constants $D, v > 0$ such that*

$$\mathrm{Ric}_{P_i} \geq 0, \qquad \mathrm{diam}(P_i) \leq D, \qquad \mathrm{Vol}(P_i) \geq v, \qquad \pi_1(P_i) \cong D_\infty, \tag{1.6}$$

and P_i converges in Gromov–Hausdorff distance to a simply connected compact length space. No orientability assertion is made for this sequence.

This construction is proved in Section 9. It does not use the cutoff metrics of Theorem 1.2, whose Ricci tensor has a negative eigenvalue in the transition annulus.

Let V be Thompson's infinite finitely presented simple group; we use the prefix-replacement description in Bleak–Quick [2]. Every nonidentity involution normally generates V. Theorem 1.1 therefore applies to $\mathsf{V} \twoheadrightarrow 1$. Replacing the same orbifold point by a second ALE filling gives another resolution.

Corollary 1.4. *Let V be Thompson's infinite finitely presented simple group. There are fixed closed oriented smooth four-manifolds M, N, a simply connected compact Riemannian orbifold X with one isolated singular point, and sequences of metrics g_i, h_i with common fixed positive volume, bounded diameter, two-sided Ricci bounds and bounded curvature energy, such that*

$$\pi_1(M) \cong \mathsf{V}, \qquad \pi_1(N) = 1, \qquad (M, g_i) \xrightarrow{\mathrm{GH}} X \xleftarrow{\mathrm{GH}} (N, h_i). \tag{1.7}$$

Every finite-dimensional complex representation of $\pi_1(M)$ is trivial. There are no nontrivial finite-sheeted connected coverings of either M or N. For every fixed $j \geq 0$, the scalar eigenvalues of the two sequences converge to the same jth eigenvalue of X.

The topology and representation assertions are proved in Section 10. Spectral convergence follows from the measured convergence of these resolutions and Gigli–Mondino–Savaré [5, Theorem 7.8]. It is convergence at each fixed eigenvalue index, not exact isospectrality of the smooth metrics and not a uniform comparison over all indices.

A heat integral and the homotopy systole. Let (M^n, g) be closed, with $n \geq 2$, let $\widetilde{M}$ be its universal Riemannian covering, and let $p_M, p_{\widetilde{M}}$ be the heat kernels for $\Delta = -\operatorname{div}\nabla$. For $\mu > 0$, put

$$\mathcal{U}_\mu(M, g) = \int_0^\infty e^{-\mu t} \left(\int_M \left[p_M(t, x, x) - p_{\widetilde{M}}(t, \widetilde{x}, \widetilde{x}) \right] dv_g(x) \right) \frac{dt}{t},$$

where $\widetilde{x}$ is any lift of x. The covering diagonal is independent of this choice. Lemma 11.1 proves that $\mathcal{U}_\mu$ is finite and nonnegative for every fixed closed metric. The proof uses the Gaussian heat-kernel estimate of Jiang–Li–Zhang [6, Theorem 1.2].

Write $\operatorname{sys}_1(M, g)$ for the infimum of the lengths of noncontractible rectifiable loops, with value $+\infty$ when M is simply connected. For every fixed $n \geq 2$ and $D, v, \mu > 0$, Theorem 11.2 gives, on the class (1.1),

$$\mathcal{U}_\mu(M_i, g_i) \longrightarrow +\infty \qquad \Longleftrightarrow \qquad \operatorname{sys}_1(M_i, g_i) \longrightarrow 0.$$

No convergence of the metric spaces is assumed in this criterion. In a marked realization, a nontrivial kernel forces divergence of $\mathcal{U}_\mu$, whereas a trivial kernel gives a bounded sequence (Corollary 11.3). For the resolutions in Corollary 1.4, the integral diverges on M and is zero on N. The integral uses the heat kernel of the universal cover; it is not asserted to be determined by the scalar spectrum of M.

Conventions. Manifolds are connected unless stated otherwise. A closed manifold is compact and without boundary. Balls are open. We normalize Hausdorff measure to agree with Riemannian volume. Path concatenation is read from left to right. The notation $-W$ means W with the opposite orientation. If (L, γ) is closed, $\mathrm{C}_{\leq a}(L)$ is the completion of $((0, a] \times L, dr^2 + r^2\gamma)$ at $r = 0$; the added point is its vertex, and $\mathrm{C}_{<a}(L)$ denotes $r < a$. Throughout $\Delta = -\operatorname{div}\nabla$, $|\operatorname{Ric}_g|_{\mathrm{op}} = \sup_{|u|_g=1} |\operatorname{Ric}_g(u, u)|$, and $|\operatorname{Rm}|$ is the full four-tensor norm. Ricci inequalities are inequalities of covariant quadratic forms. Constants $C, c > 0$ in uniform estimates are independent of i and of $\epsilon \downarrow 0$, unless a dependence is stated; their values may change between estimates.

2. Finite subgroups generated by short loops

Let ω_n be the Euclidean unit-ball volume and put $\sigma_{n-1} = n\omega_n$. Write

$$\mathsf{V}_n(r) = \sigma_{n-1} \int_0^r \sinh^{n-1} t \, dt \tag{2.1}$$

for the volume of a ball in the simply connected space of constant sectional curvature -1. Let (M, g) be complete and connected, and let $\pi : (\widetilde{M}, \widetilde{g}) \to (M, g)$ be its universal Riemannian covering, with deck group Γ. The lifted metric is complete. For $x \in M$, let $\mathscr{L}_x$ be the set of rectifiable loops based at x whose class in $\pi_1(M, x)$ is nontrivial. Fix $\widetilde{x} \in \pi^{-1}(x)$. A loop representing $\gamma \in \Gamma$ lifts to a path from $\widetilde{x}$ to $\gamma\widetilde{x}$, while the projection of a minimizing segment between these two points represents the same deck transformation. Hence

$$s_g(x) := \inf_{\ell \in \mathscr{L}_x} L_g(\ell) = \inf_{\gamma \in \Gamma \setminus \{1\}} d(\widetilde{x}, \gamma\widetilde{x}). \tag{2.2}$$

The convention is $\inf \varnothing = +\infty$. The orbit $\Gamma\widetilde{x} = \pi^{-1}(x)$ is closed and discrete. By completeness and the Hopf–Rinow theorem, its intersection with any closed bounded ball is finite. If $\gamma_0 \neq 1$ and $R = d(\widetilde{x}, \gamma_0\widetilde{x})$, the nonidentity orbit points in $\overline{B}_R(\widetilde{x})$ form a nonempty finite set. Their positive distances have a minimum, which is the infimum in (2.2). For a minimizing path c from x to y and $\ell \in \mathscr{L}_y$, the loop $c * \ell * c^{-1}$ belongs to $\mathscr{L}_x$ and has length $2d(x, y) + L_g(\ell)$. Taking infima, and then interchanging x, y, gives

$$|s_g(x) - s_g(y)| \leq 2d(x, y) \qquad (\Gamma \neq 1).$$

Lemma 2.1. *Let $n \geq 2$ and let (M^n, g, p) be complete and connected, with $\operatorname{Ric}_g \geq -(n-1)g$ and $\operatorname{Vol}(B_1(p)) \geq v > 0$. Let Γ be its universal deck group. Then*

$$\#\{\gamma \in \Gamma : d(\widetilde{p}, \gamma\widetilde{p}) \leq R\} \leq \frac{\mathsf{V}_n(R+1)}{v} \qquad (R \geq 0). \tag{2.3}$$

Proof. Let $U = B_1(p) \setminus \mathrm{Cut}(p)$. For $x \in U$, let $s(x)$ be the endpoint of the lift, starting at $\widetilde{p}$, of the unique minimizing segment from p to x. The inverse exponential map is smooth on the complement of the cut locus; thus s is measurable. Set $E = s(U) \subset B_1(\widetilde{p})$. Since the cut locus has measure zero and $\pi \circ s = \mathrm{Id}_U$, the local-isometry change-of-variables formula gives

$$\mathrm{Vol}(E) = \mathrm{Vol}(U) = \mathrm{Vol}(B_1(p)) \geq v.$$

If $\gamma E \cap \gamma' E \neq \emptyset$, then $\gamma s(x) = \gamma' s(x')$ for some $x, x' \in U$. Projection gives $x = x'$, so $(\gamma')^{-1}\gamma$ fixes $s(x)$. A deck transformation with a fixed point is the identity. Therefore $\gamma = \gamma'$.

If $d(\widetilde{p}, \gamma\widetilde{p}) \leq R$, then $\gamma E \subset B_{R+1}(\widetilde{p})$. For every finite collection F of these elements, Bishop comparison on the complete covering gives

$$|F| v \leq \sum_{\gamma \in F} \mathrm{Vol}(\gamma E) \leq \mathrm{Vol}(B_{R+1}(\widetilde{p})) \leq \mathsf{V}_n(R+1).$$

An infinite set has finite subsets of arbitrarily large cardinality. The displayed estimate therefore bounds the cardinality of the whole set by $\mathsf{V}_n(R+1)/v$. □

Lemma 2.2. *Fix $n \geq 2$, $D, v > 0$. There are an integer $A \geq 1$ and $\eta > 0$ such that, on every closed n-manifold with*

$$\mathrm{Ric}_g \geq -(n-1)g, \qquad \mathrm{diam}(M,g) \leq D, \qquad \mathrm{Vol}(M,g) \geq v, \tag{2.4}$$

the subgroup of $\pi_1(M,x)$ generated by loops based at x of length less than η has order at most A, for every $x \in M$.

Proof. Since $B_{D+1}(x) = M$, Bishop–Gromov comparison gives

$$\mathrm{Vol}(B_r(x)) \geq v \frac{\mathsf{V}_n(r)}{\mathsf{V}_n(D+1)} \geq v_* r^n \quad (0 < r \leq 1), \qquad v_* := \frac{v\omega_n}{\mathsf{V}_n(D+1)}. \tag{2.5}$$

Choose an integer $A \geq \max\{1, \mathsf{V}_n(2)/v_*\}$ and put $\eta = (2A)^{-1}$. Lemma 2.1 implies

$$\#\{\gamma \in \Gamma : d(\widetilde{x}, \gamma\widetilde{x}) \leq 1\} \leq A.$$

Put

$$S = \{\gamma \in \Gamma : d(\widetilde{x}, \gamma\widetilde{x}) < \eta\}, \qquad H = \langle S \rangle, \qquad B_k = S^k \quad (k \geq 0),$$

where $B_0 = \{1\}$. Since $S = S^{-1}$ and $1 \in S$, the sets B_k are increasing. If $B_{k+1} = B_k$, then $B_k S = B_k$ and induction gives $B_{k+j} = B_k$ for every $j \geq 0$. Since $H = \bigcup_{j \geq 0} B_j$, this implies $B_k = H$. If $|H| > A$ and $|B_A| \leq A$, none of the A inclusions $B_0 \subset \cdots \subset B_A$ can be an equality. They are therefore strict, and $|B_A| \geq |B_0| + A = A + 1$, a contradiction. Thus $|H| > A$ implies $|B_A| \geq A + 1$. On the other hand, for $1 \leq k \leq A$ and $\gamma = s_1 \cdots s_k$ with $s_j \in S$, isometry of the deck action gives

$$d(\widetilde{x}, \gamma\widetilde{x}) \leq \sum_{j=1}^{k} d(s_1 \cdots s_{j-1}\widetilde{x}, s_1 \cdots s_j \widetilde{x}) = \sum_{j=1}^{k} d(\widetilde{x}, s_j \widetilde{x}) < A\eta = \tfrac{1}{2}.$$

The identity has displacement zero. Thus B_A is contained in the set counted above, whose cardinality is at most A, a contradiction. It follows that $|H| \leq A$. Every loop based at x of length less than η represents an element of S, so its generated subgroup is contained in H. □

For every $x \in M$, a loop based at x of length less than η represents an element of order at most A. This assertion does not bound the normal subgroup generated by these elements as x varies.

3. Kernels of continuous approximation maps

For a connected geodesic space A and $r > 0$, write $\mathcal{R}_r(A)$ for the abstract simplicial complex whose simplices are the finite subsets of A of diameter strictly less than r. We write $\pi_1(\mathcal{R}_r(A))$ for the fundamental group of its geometric realization, endowed with the weak topology. Its basepoint is the vertex corresponding to the chosen basepoint of A. Cellular approximation shows that its two-skeleton determines this group. Fix $a_0 \in A$ and define

$$N_r(A) = \langle\!\langle [\alpha * \beta * \alpha^{-1}] : \ \alpha(0) = a_0, \ \alpha(1) = \beta(0) = \beta(1), \ \beta \text{ rectifiable}, \ L(\beta) < r \rangle\!\rangle_{\pi_1(A,a_0)}.$$

Here α is any continuous path with the specified endpoints. For a semilocally simply connected geodesic space A, $N_{3r}(A)$ is Virk's length subgroup $\mathcal{L}(A, 3r, \pi_1)$, and every continuous loop has a rectifiable representative. The following lemma is therefore the open-Rips quotient correspondence [12, Proposition 5.6 and Theorem 5.7], with its discretization map specified for use in Proposition 3.2.

Lemma 3.1. *If A is connected, geodesic and semilocally simply connected, discretizing loops gives an epimorphism compatible with changes of scale*

$$q_r : \pi_1(A) \twoheadrightarrow \pi_1(\mathcal{R}_r(A)), \qquad \ker q_r = N_{3r}(A). \tag{3.1}$$

The inclusion map between Rips complexes induces a surjection on fundamental groups for every increase of scale. If every loop of length less than $3r$ is nullhomotopic in A, then q_r is an isomorphism.

Proof. Let $\gamma : [0,1] \to A$ be a loop at a_0. Uniform continuity gives a partition $0 = t_0 < \cdots < t_m = 1$ such that $\operatorname{diam} \gamma([t_{j-1}, t_j]) < r$ for every j. Associate the edge loop

$$D_r(\gamma; t_0, \dots, t_m) = (\gamma(t_0), \dots, \gamma(t_m)).$$

Inserting a partition point replaces an edge by two edges in a single simplex. Two partitions have a common refinement, so they give the same based edge-homotopy class. For a based homotopy $H : [0,1]^2 \to A$, choose a rectangular subdivision whose closed rectangles have images of diameter less than r, and divide each rectangle into two triangles. Their vertex images define a simplicial homotopy between the boundary edge loops. The constant side edges map to a_0. Hence D_r defines a homomorphism q_r on $\pi_1(A, a_0)$; concatenation of partitions gives the homomorphism identity.

For each unoriented Rips edge $\{x, y\}$, choose a minimizing geodesic in one orientation and denote the reverse path by $g_{yx} = g_{xy}^{-1}$. Put $g_{xx}(t) = x$. Consecutive repeated vertices in an edge path are deleted. The length of g_{xy} is less than r, so every finite set of its points has diameter less than r. Its discretization is edge-homotopic to (x, y). Replacing the edges of an edge loop by these paths proves surjectivity of q_r.

Put $\Pi = \pi_1(A, a_0)/N_{3r}(A)$. The elementary edge-path relations are deletion of (x, y, x) and replacement of (x, y, z) by (x, z) when $\{x, y, z\}$ is a simplex. Geodesic replacement sends the first relation to a path followed by its inverse. In the second relation, the two paths differ by $g_{xy} * g_{yz} * g_{zx}$, whose length is

$$d(x, y) + d(y, z) + d(z, x) < 3r.$$

After transport to a_0, its class belongs to $N_{3r}(A)$. Thus geodesic replacement induces $e_r : \pi_1(\mathcal{R}_r(A)) \to \Pi$.

Choose a finite cover of $\gamma([0,1])$ by balls $B_{\rho_j}(z_j)$ such that $2\rho_j < r$ and every loop in $B_{3\rho_j}(z_j)$ is nullhomotopic in A. Semilocal simple connectivity and local path connectivity give these balls. A Lebesgue number for their inverse images under γ gives a partition for which each subarc lies in one ball. Write $(z, \rho) = (z_j, \rho_j)$ for that subarc. If $x, y \in B_\rho(z)$, a minimizing geodesic from x to y lies in $B_{3\rho}(z)$, because

$$d(z, w) \le d(z, x) + d(x, w) < \rho + 2\rho$$

for every point w of that geodesic. The subarc followed by the reverse geodesic is therefore nullhomotopic in A. The chosen path g_{xy} is also minimizing and has this property. Consequently $e_r q_r$ is the quotient homomorphism $\pi_1(A, a_0) \to \Pi$, and $\ker q_r \subset N_{3r}(A)$.

Let β be rectifiable with $L(\beta) < 3r$. Divide its parameter interval into three consecutive subarcs, each of length less than r. The sampled vertices of each subarc belong to a simplex, so its edge path is homotopic to its endpoint edge. The three endpoint edges bound a simplex. Hence $q_r[\beta] = 1$ at its basepoint. For every joining path α, $q_r[\alpha * \beta * \alpha^{-1}] = 1$, which gives $N_{3r}(A) \subset \ker q_r$.

For $r < R$, refine a partition until every subarc has image diameter less than r. The same vertices compute q_r and q_R, so $(\iota_{r,R})_* q_r = q_R$. Since q_R is onto, $(\iota_{r,R})_*$ is onto. The final assertion follows from $\ker q_r = N_{3r}(A)$. □

Under the geometric hypotheses of Proposition 3.2, the limit is semilocally simply connected by Pan–Wei [8, Main Theorem 1.1]. Wang [13] proves this conclusion without the noncollapse assumption. The proposition retains semilocal simple connectivity as an explicit hypothesis; it also assumes that the approximation maps are continuous.

Proposition 3.2. *Let $n \geq 2$. Suppose closed smooth n-manifolds (M_i, g_i) satisfy* (2.4) *and converge to a compact semilocally simply connected geodesic space X. Suppose continuous Gromov–Hausdorff approximation maps $f_i : M_i \to X$ have distortion and density error tending to zero. There are $r_* > 0$ and $A = A(n, D, v) \in \mathbb{N}$ with the following property: for every fixed $0 < r < r_*$ there is $i(r)$ such that*

$$(f_i)_* \text{ is onto}, \qquad \ker(f_i)_* = N_{9r}(M_i) \qquad (i \geq i(r)). \tag{3.2}$$

The displayed kernel is normally generated by elements of orders at most A. The constant r_* may depend on X.

Proof. Compactness and semilocal simple connectivity give $\ell_X > 0$ such that every loop in X with image diameter less than ℓ_X contracts in X. For each $x \in X$, choose a path-connected open neighbourhood U_x on which the inclusion homomorphism to $\pi_1(X, x)$ is trivial. Such neighbourhoods exist because a geodesic space is locally path connected. Any loop based at another point of U_x is conjugate, inside U_x, to a loop based at x. Choose a finite subcover and let ℓ_X be a positive Lebesgue number for it. In particular every loop of length less than ℓ_X is nullhomotopic.

Let η and A be the constants in Lemma 2.2 and set $r_* = \min\{\ell_X/12, \eta/9\}$. Fix $0 < r < r_*$. Let f_i be an α_i-approximation, where $\alpha_i \to 0$, and fix $p_i \in M_i$, $q_i = f_i(p_i)$. Choose $h_i : X \to M_i$ with $h_i(q_i) = p_i$ and $d_X(f_i h_i(y), y) \leq \alpha_i$. For $y, z \in X$ and $x \in M_i$,

$$|d_{M_i}(h_i(y), h_i(z)) - d_X(y, z)| \leq \alpha_i + d_X(f_i h_i(y), y) + d_X(f_i h_i(z), z) \leq 3\alpha_i,$$
$$d_{M_i}(h_i f_i(x), x) \leq d_X(f_i h_i f_i(x), f_i(x)) + \alpha_i \leq 2\alpha_i.$$

Choose $i(r)$ such that $\alpha_i < r/10$ for $i \geq i(r)$. For a finite subset S the diameter bounds

$$\operatorname{diam} f_i(S) \leq \operatorname{diam} S + \alpha_i, \qquad \operatorname{diam} h_i(S) \leq \operatorname{diam} S + 3\alpha_i$$

then give the based simplicial maps

$$\mathcal{R}_r(M_i) \xrightarrow{a_i} \mathcal{R}_{2r}(X) \xrightarrow{b_i} \mathcal{R}_{3r}(M_i) \xrightarrow{c_i} \mathcal{R}_{4r}(X), \tag{3.3}$$

where a_i, c_i use f_i and b_i uses h_i. If x, x' belong to a simplex of $\mathcal{R}_r(M_i)$, then

$$d_{M_i}(h_i f_i(x), h_i f_i(x')) \leq d_{M_i}(x, x') + 4\alpha_i,$$
$$d_{M_i}(x, h_i f_i(x')) \leq d_{M_i}(x, x') + 2\alpha_i.$$

Thus the union of its vertices and their $h_i f_i$-images has diameter less than $r + 4\alpha_i < 3r$. Hence $b_i a_i$ is contiguous to the inclusion from scale r to $3r$. For vertices y, y' of a simplex of $\mathcal{R}_{2r}(X)$,

$$d_X(f_i h_i(y), f_i h_i(y')) \leq d_X(y, y') + 2\alpha_i,$$
$$d_X(y, f_i h_i(y')) \leq d_X(y, y') + \alpha_i.$$

The union of the original and image vertices has diameter less than $2r + 2\alpha_i < 4r$. Hence $c_i b_i$ is contiguous to the inclusion from scale $2r$ to $4r$. Both contiguities fix the respective basepoints.

By Lemma 3.1, $(b_i)_*(a_i)_*$ is surjective. The map $(c_i)_*(b_i)_*$ is an isomorphism: the discretization maps identify both its source and target with $\pi_1(X)$, since $12r < \ell_X$. Therefore

$$\operatorname{im}(b_i)_* = \pi_1(\mathcal{R}_{3r}(M_i)), \qquad \ker(b_i)_* \subset \ker\big((c_i)_*(b_i)_*\big) = 1.$$

Thus $(b_i)_*$ is an isomorphism, and so is $(c_i)_* = \big((c_i)_*(b_i)_*\big)(b_i)_*^{-1}$. It follows that

$$\pi_1(M_i) \xrightarrow{q_{3r}} \pi_1(\mathcal{R}_{3r}(M_i)) \xrightarrow{(c_i)_*} \pi_1(\mathcal{R}_{4r}(X)) \cong \pi_1(X) \tag{3.4}$$

is onto with kernel $N_{9r}(M_i)$.

For a loop γ based at p_i, choose a subdivision whose subarcs have image diameter less than $3r$ in M_i. Each image subarc under f_i has diameter less than $3r + \alpha_i < 4r$; it is a continuous subarc because f_i is continuous. The vertices used by c_i are exactly those of the discretized loop $f_i\gamma$. Consequently

$$(c_i)_* q_{3r}[\gamma] = q_{4r}(f_i)_*[\gamma].$$

Since q_{4r} on X is an isomorphism, this identity gives

$$(f_i)_* = q_{4r}^{-1}(c_i)_* q_{3r}, \qquad \ker(f_i)_* = N_{9r}(M_i).$$

In particular, the kernel is that of the given continuous map.

If $[\alpha * \beta * \alpha^{-1}]$ is a generator in the definition of $N_{9r}(M_i)$, then $L(\beta) < 9r < \eta$. Lemma 2.2 applied at $\beta(0)$ gives $\operatorname{ord}[\beta] \le A$. The map

$$T_\alpha : \pi_1(M_i, \beta(0)) \longrightarrow \pi_1(M_i, p_i), \qquad [\lambda] \longmapsto [\alpha * \lambda * \alpha^{-1}]$$

is an isomorphism with inverse $T_{\alpha^{-1}}$. It preserves orders, so $\operatorname{ord} T_\alpha[\beta] \le A$. This proves the assertion for every $i \ge i(r)$. ☐

Lemma 3.3 (Finite normal generation). *Let G be finitely generated, let Q be finitely presented, and let $\varphi : G \twoheadrightarrow Q$ be an epimorphism. If* $\ker\varphi = \langle\!\langle S \rangle\!\rangle_G$ *for a subset $S \subset G$, then* $\ker\varphi = \langle\!\langle T \rangle\!\rangle_G$ *for some finite subset $T \subset S$.*

Proof. Put $K = \ker\varphi$. Let $x_1, \dots, x_k$ generate G, and take a finite presentation $Q = \langle y_1, \dots, y_l \mid S_1, \dots, S_m \rangle$. Choose words $u_a(y)$ representing $\varphi(x_a)$ and words $v_b(x)$ representing y_b in the images of the x_a. Then

$$Q = \big\langle x_1, \dots, x_k \;\big|\; S_j(v(x)),\ x_a u_a(v(x))^{-1} \big\rangle.$$

Let Q' denote the group defined by the displayed presentation. Define $A : Q' \to Q$ by $A(x_a) = u_a(y)$ and $B : Q \to Q'$ by $B(y_b) = v_b(x)$. The relations $S_j(v(x)) = 1$ make B well-defined. In Q, the equality $v_b(u(y)) = y_b$ holds by the choice of v_b, so every defining relation of Q' maps to the identity under A. Moreover,

$$AB(y_b) = v_b(u(y)) = y_b, \qquad BA(x_a) = u_a(v(x)) = x_a.$$

Thus A and B are inverse. If $F(x)$ is the free group on the x_a, its kernel under $F(x) \to Q$ is normally generated by the finitely many displayed relators. The inclusion $\ker(F(x) \to G) \subset \ker(F(x) \to Q)$ and passage to the quotient G show that their images normally generate K.

Let $r_1, \dots, r_N \in K$ be these images. For each ν, choose an expression

$$r_\nu = \prod_{j=1}^{m_\nu} g_{\nu j} s_{\nu j}^{\varepsilon_{\nu j}} g_{\nu j}^{-1}, \qquad s_{\nu j} \in S, \quad \varepsilon_{\nu j} \in \{1, -1\}.$$

The set $T = \{s_{\nu j} : 1 \le \nu \le N,\ 1 \le j \le m_\nu\}$ is finite, is contained in S, and satisfies

$$K = \langle\!\langle r_1, \dots, r_N \rangle\!\rangle_G \subset \langle\!\langle T \rangle\!\rangle_G \subset K.$$

Hence $K = \langle\!\langle T \rangle\!\rangle_G$. ☐

Corollary 3.4. *Under the hypotheses of Proposition 3.2, if $\pi_1(M_i)$ is torsion-free, then $(f_i)_*$ is an isomorphism for all sufficiently large i.*

Proof. For all sufficiently large i, Proposition 3.2 gives an epimorphism $(f_i)_*$ and a set S_i of finite-order elements with $\ker(f_i)_* = \langle\!\langle S_i \rangle\!\rangle$. Torsion-freeness implies $S_i \subset \{1\}$. Hence $\ker(f_i)_* = 1$, and $(f_i)_*$ is an isomorphism. ☐

No conclusion that the whole kernel is a torsion group follows from Proposition 3.2. The infinite-order word ab in (1.4) belongs to the kernel of Theorem 1.2.

4. Metric replacement and Ricci-flat ALE fillings

Lemma 4.1. *Let Y_i and Y be compact length spaces obtained by attaching, respectively, compact connected length spaces $A_{i,1}, \dots, A_{i,s}$ and $B_{i,1}, \dots, B_{i,s}$ to a common compact length space Z_i. The attachment identifies closed subsets of corresponding pieces with the same nonempty closed sets $S_{i,j} \subset Z_i$, which are pairwise disjoint. In both unions, assume that the distance is the infimum of the lengths of finite concatenations of paths in the constituent spaces. The length metric on Z_i is the same in the two unions. Suppose*

$$\operatorname{diam} A_{i,j} \le \eta_i, \qquad \operatorname{diam} B_{i,j} \le \eta_i \quad (1 \le j \le s), \qquad \eta_i \longrightarrow 0,$$

where all diameters are intrinsic. Regard each piece as its image in the corresponding union and put

$$C_i = \{(z,z) : z \in Z_i\} \cup \bigcup_{j=1}^{s} (A_{i,j} \times B_{i,j}).$$

For a relation $C \subset Y_i \times Y$*, define*

$$\operatorname{dis} C = \sup_{(u,v),(u',v') \in C} |d_{Y_i}(u,u') - d_Y(v,v')|.$$

Then $\operatorname{dis} C_i \le (s+4)\eta_i$*. In particular* $d_{\mathrm{GH}}(Y_i, Y) \le (s+4)\eta_i/2$.

Proof. For a finite concatenation $c = (c_1, \dots, c_m)$, write $\mathscr{L}(c) = \sum_{\nu=1}^{m} L(c_\nu)$, with each summand measured in its assigned constituent space. The hypothesis identifies the union distance with the infimum of $\mathscr{L}$. Fix $x, x' \in Z_i$ and $\alpha > 0$. Choose a finite concatenation c_0 of paths in the pieces of Y_i joining x to x' with $\mathscr{L}(c_0) < d_{Y_i}(x,x') + \alpha$. Assign each constituent path to one piece. For $0 \le j \le s$, consider the union consisting of Z_i, the pieces $B_{i,k}$ with $k \le j$, and the pieces $A_{i,k}$ with $k > j$, with the specified identifications. If no constituent of c_{j-1} is assigned to $A_{i,j}$, put $c_j = c_{j-1}$. Otherwise, let u_j be the initial point of the first such constituent and v_j the terminal point of the last. Both points belong to $S_{i,j}$: the endpoints x, x' are in Z_i, and a transition to or from $A_{i,j}$ occurs only through $S_{i,j}$. Choose a path b_j in $B_{i,j}$ from u_j to v_j with $L(b_j) < \eta_i + \alpha$. Write $c_{j-1} = c^- * a_j * c^+$, where a_j is the concatenation from the specified occurrence of u_j to that of v_j, and put $c_j = c^- * b_j * c^+$. Then

$$\mathscr{L}(c_j) = \mathscr{L}(c_{j-1}) - \mathscr{L}(a_j) + L(b_j) \le \mathscr{L}(c_{j-1}) + \eta_i + \alpha,$$

and c_j has no constituent assigned to $A_{i,k}$ for $k \le j$. Induction gives a concatenation c_s in Y and

$$d_Y(x,x') \le \mathscr{L}(c_s) < d_{Y_i}(x,x') + s\eta_i + (s+1)\alpha.$$

Letting $\alpha \downarrow 0$ and interchanging the two unions yields

$$|d_Y(x,x') - d_{Y_i}(x,x')| \le s\eta_i.$$

For $(u,v) \in C_i$, choose $z = u = v$ if the pair belongs to the diagonal of Z_i, and choose $z \in S_{i,j}$ if it belongs to $A_{i,j} \times B_{i,j}$. In either case $d_{Y_i}(u,z), d_Y(v,z) \le \eta_i$. For two such pairs, with choices z, z', the triangle inequality gives

$$\begin{aligned} |d_{Y_i}(u,u') - d_Y(v,v')| &\le d_{Y_i}(u,z) + d_{Y_i}(u',z') + |d_{Y_i}(z,z') - d_Y(z,z')| \\ &\quad + d_Y(z,v) + d_Y(z',v') \le (s+4)\eta_i. \end{aligned}$$

Both projections of C_i are onto. The correspondence characterization of Gromov–Hausdorff distance proves the last assertion. □

Lemma 4.2 (Distances across the gluing hypersurfaces). *Let Y be a compact smooth Riemannian manifold without boundary, or a compact Riemannian orbifold without boundary with finitely many isolated singular points, each having a flat conical neighbourhood. Cut Y along finitely many disjoint closed smooth hypersurfaces in its regular part. The distance on Y is the infimum of the sums of intrinsic lengths of finite concatenations of paths in the resulting pieces.*

Proof. Let $D(x,y)$ be the infimum in the statement. Every admissible finite concatenation is a path from x to y, so $d_Y(x,y) \le D(x,y)$. Fix a rectifiable path $c : [0,1] \to Y$ from x to y and $\epsilon > 0$. Write H for the union of the cutting hypersurfaces.

In the orbifold case, choose pairwise disjoint sets $K_j = \mathrm{C}_{\le a}(L_j)$ about all s singular points, with $K_j \cap H = \emptyset$ and $2sa < \epsilon/3$. For each j, let I_j be the inverse image of K_j under the current path. If $I_j \ne \emptyset$, replace the restriction between $\min I_j$ and $\max I_j$ by the radial path from its first endpoint to the vertex followed by the radial path to its second endpoint. The replacement has length at most $2a$ and lies in K_j. Since $K_j \cap K_k = \emptyset$ for $j \ne k$, the replacement introduces no intersection with another K_k. After these finitely many replacements the path c' satisfies

$$L(c') \le L(c) + 2sa < L(c) + \epsilon/3.$$

Its radial restrictions lie in single cut pieces; its other restrictions form a finite family of paths in the regular part. In the smooth case put $c' = c$ and use its single restriction.

Approximation of rectifiable Riemannian paths by piecewise smooth paths, relative to their endpoints, gives a path c'' with $L(c'') < L(c') + \epsilon/3$ on these regular restrictions. At an endpoint $z \in H$, prescribe a segment $t \mapsto \exp_z(tv_z)$ or its reverse, where v_z is a nonzero normal vector to H; the segment lengths can be chosen arbitrarily small and included in this error bound. Choose the remaining subdivision vertices outside H. Relative transversality, applied to the finitely many smooth restrictions and fixing the prescribed endpoint segments, gives c''' with

$$L(c''') < L(c'') + \epsilon/3 < L(c) + \epsilon, \qquad c''' \pitchfork H$$

on their interiors. The length inequality follows from continuity of length in the C^1 topology on each compact parameter interval. The inverse image of H in each such interval is closed and discrete; the transverse endpoint segments exclude accumulation at its endpoints. It is therefore finite. Subdivision at these inverse images, together with the radial restrictions, gives an admissible finite concatenation. Its sum of intrinsic lengths is $L(c''')$. Consequently

$$D(x, y) \le L(c) + \epsilon.$$

Taking the infimum over c and then letting $\epsilon \downarrow 0$ gives $D(x, y) \le d_Y(x, y)$. □

Fix a nonnegative function $b \in C_c^\infty((1, 2))$ with $\int_{\mathbb{R}} b = 1$, and put

$$\chi(s) = 1 - \int_{-\infty}^{s} b(u)\, du \qquad (s \ge 0). \tag{4.1}$$

Then $0 \le \chi \le 1$, $\chi' \le 0$, $\chi = 1$ near $[0, 1]$, and $\chi = 0$ near $[2, \infty)$.

For $p \ge 2$, let

$$L_p = S^3/\Gamma_p, \qquad \Gamma_p = C_{p^2}, \qquad (z_1, z_2) \mapsto (e^{2\pi i/p^2} z_1, e^{2\pi i(p-1)/p^2} z_2). \tag{4.2}$$

Since $p^2 - (p-1)(p+1) = 1$, one has $\gcd(p-1, p^2) = 1$. A power of the displayed generator that fixes a point of S^3 is the identity: at least one coordinate is nonzero, and either weight then forces the exponent to be divisible by p^2. Thus the action is free. Let γ_p be the round quotient metric. The rational homology and boundary-group data occur in [4, Lemma 2.1]; the existence of the Ricci-flat ALE metrics is [11, Proposition 3.3].

Lemma 4.3. *For every $p \ge 2$ there is a complete connected oriented Ricci-flat ALE four-manifold (B_p, h_p) with compact core W_p and one end $(R_0, \infty) \times L_p$ such that*

$$H_*(W_p; \mathbb{Q}) \cong H_*(D^4; \mathbb{Q}), \qquad \pi_1(W_p) = C_p, \qquad \pi_1(L_p) = C_{p^2} \twoheadrightarrow C_p. \tag{4.3}$$

In suitable end coordinates,

$$h_p = g_C + e, \qquad g_C = dR^2 + R^2\gamma_p, \qquad |\nabla^j e|_{g_C} \le C_j R^{-4-j} \quad (j = 0, 1, 2). \tag{4.4}$$

In (4.4), $\nabla = \nabla^{g_C}$. *Also* $\int_{B_p} |\mathrm{Rm}_{h_p}|^2\, dv_{h_p} < \infty$.

Proof. With $n = p$ and $m = 1$, [11, Proposition 3.3] gives a Ricci-flat ALE rational homology ball (B_p, h_p) with $\pi_1(B_p) = C_p$ and end link $L(p^2, p-1)$. Choose R_0 strictly inside the radial domain of an end chart. The chart then extends across the level $R = R_0$, and the complement of $\{R > R_0\}$ is a compact smooth core with boundary L_p. The universal cover of B_p is the simply connected A_{p-1} ALE manifold in the construction of [11, Sections 3–5]. This cover has one end. Let $\widetilde{B}_p \to B_p$ denote this finite covering. The inverse image of a compact core is compact. Each component of the inverse image of the product end is a product of (R_0, ∞) with a connected finite cover of L_p; it is noncompact. If there were two components, deleting the inverse image of the core would separate two ends of $\widetilde{B}_p$, contrary to its ALE description. The restricted covering over the end is therefore connected. Its monodromy has a transitive action on the fibre C_p by translations, so its image is C_p. Under a choice of generators, the boundary homomorphism is $\mathbb{Z}/p^2\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$, $[a]_{p^2} \mapsto [a]_p$. On the end, the homotopy $(R,\theta)\mapsto((1-t)R+tR_0,\theta)$ extends by the identity on the truncated core. It is a deformation retraction of the complete space onto that core.

The differentiated order-four coordinates follow from Kröncke–Szabó [7, Theorem 1.16 and Definition 1.4] in dimension four. These coordinates retain the round quotient link. Choose R_0 inside this coordinate domain and truncate at that level; the radial retraction above applies in these coordinates as well. For sufficiently large R, (4.4) implies $\frac{1}{2}g_C \le h_p \le \frac{3}{2}g_C$. With $\nabla=\nabla^{g_C}$ and $A=\nabla^{h_p}-\nabla$, the coefficients satisfy

$$A^k_{ij}=\tfrac{1}{2}h_p^{k\ell}(\nabla_i e_{j\ell}+\nabla_j e_{i\ell}-\nabla_\ell e_{ij}),\qquad \nabla h_p^{-1}=-h_p^{-1}(\nabla e)h_p^{-1}.$$

Thus $|A|_{g_C}\le CR^{-5}$ and $|\nabla A|_{g_C}\le C(R^{-6}+R^{-10})$. Since $\mathrm{Rm}_{g_C}=0$, the identity

$$R(h_p)^k{}_{\ell ij}=\nabla_i A^k_{j\ell}-\nabla_j A^k_{i\ell}+A^k_{im}A^m_{j\ell}-A^k_{jm}A^m_{i\ell}$$

gives $|\mathrm{Rm}_{h_p}|_{h_p}\le CR^{-6}$. Also $dv_{h_p}\le CR^3\,dR\,dv_{\gamma_p}$. For a sufficiently large R_2,

$$\int_{\{R\ge R_2\}}|\mathrm{Rm}_{h_p}|^2\,dv_{h_p}\le C\,\mathrm{Vol}(L_p,\gamma_p)\int_{R_2}^{\infty}R^{-9}\,dR=\frac{C\,\mathrm{Vol}(L_p,\gamma_p)}{8R_2^8}<\infty.$$

The complementary region is compact and the metric is smooth, so its curvature energy is finite. □

Lemma 4.4. *Let Z be a fixed compact connected oriented smooth four-manifold with boundary components $L_{p_1},\dots,L_{p_s}$, carrying a fixed smooth metric which equals $dr^2+r^2\gamma_{p_j}$ on an outer collar $1\le r<1+\eta$ at each component. Fill these boundaries by the cores W_{p_j} of Lemma 4.3, with opposite boundary orientations. On the resulting fixed closed manifold M there are metrics $\widehat{g}_\epsilon$ which converge to*

$$X=Z\cup_{\partial Z}\coprod_{j=1}^{s}\mathrm{C}_{\le 1}(L_{p_j}),\tag{4.5}$$

with uniformly bounded diameter, two-sided Ricci tensor and curvature energy, and with volume tending to $\mathcal{H}^4(X)>0$. The convergence is realized by continuous maps equal to the identity on the common complement. After volume normalization and one fixed scaling, the metrics satisfy (1.2)–(1.3). *The same conclusions hold for any Ricci-flat ALE fillings with the same round ends and the differentiated decay in* (4.4), *without a restriction on their fundamental groups.*

Proof. For one filling, scale h_p by ϵ^2 and set $r=\epsilon R$. In the coordinate r, (4.4) becomes

$$h_{p,\epsilon}=dr^2+r^2\gamma_p+e_\epsilon,\qquad |\nabla^j e_\epsilon|_{g_C}\le C_j\epsilon^4 r^{-4-j}\quad (j=0,1,2).\tag{4.6}$$

Let χ be as in (4.1). Set $\delta=\sqrt{\epsilon}$ and decrease ϵ so that $\epsilon R_0<\delta$ and $2\delta<1$. On the compact core retain $\epsilon^2 h_p$, and on the end define

$$k_{p,\epsilon}=\begin{cases} h_{p,\epsilon}, & \epsilon R_0\le r\le\delta,\\ g_C+\chi(r/\delta)e_\epsilon, & \delta\le r\le 2\delta,\\ g_C, & 2\delta\le r\le 1.\end{cases}\tag{4.7}$$

The cutoff is one near its inner boundary and zero near its outer boundary. On this annulus, $|\chi(r/\delta)e_\epsilon|_{g_C}\le C\epsilon^2$. Choose ϵ_0 so that $C\epsilon_0^2\le 1/2$. Then

$$\tfrac{1}{2}g_C\le g_C+\chi(r/\delta)e_\epsilon\le\tfrac{3}{2}g_C\qquad(0<\epsilon<\epsilon_0),$$

which proves positive definiteness.

Put $q_\epsilon=\chi(r/\delta)e_\epsilon$ and $g=g_C+q_\epsilon$. For $\nabla=\nabla^{g_C}$, the identities

$$\nabla^2 r=\frac{g_C-dr^2}{r},\qquad d\chi(r/\delta)=\delta^{-1}\chi'(r/\delta)\,dr,$$

$$\nabla^2\chi(r/\delta)=\delta^{-2}\chi''(r/\delta)\,dr\otimes dr+\delta^{-1}\chi'(r/\delta)\nabla^2 r$$

give $|\nabla^j\chi(r/\delta)|\le C_j\delta^{-j}$ on the annulus. The product rule and (4.6) therefore give

$$|\nabla^j q_\epsilon|_{g_C}\le C_j\sum_{a=0}^{j}\delta^{-a}\epsilon^4\delta^{-4-(j-a)}\le C_j'\epsilon^4\delta^{-4-j}\qquad(0\le j\le 2).$$

For $\nabla = \nabla^{g_C}$ the connection difference is

$$A^k_{ij} = \frac{1}{2} g^{k\ell}(\nabla_i q_{\epsilon,j\ell} + \nabla_j q_{\epsilon,i\ell} - \nabla_\ell q_{\epsilon,ij}), \qquad \nabla g^{-1} = -g^{-1}(\nabla q_\epsilon) g^{-1}.$$

Since g_C is flat,

$$R(g)^k{}_{\ell ij} = \nabla_i A^k_{j\ell} - \nabla_j A^k_{i\ell} + A^k_{im} A^m_{j\ell} - A^k_{jm} A^m_{i\ell}.$$

The inequality $\frac{1}{2} g_C \le g \le \frac{3}{2} g_C$ gives $|g^{-1}|_{g_C} \le C$. Consequently

$$|A|_{g_C} \le C\epsilon^4 \delta^{-5}, \qquad |\nabla A|_{g_C} \le C(\epsilon^4 \delta^{-6} + \epsilon^8 \delta^{-10}).$$

The curvature identity therefore gives

$$|\operatorname{Rm}| \le C\big(\epsilon^4\delta^{-6} + \epsilon^8\delta^{-10}\big) = C(\epsilon + \epsilon^3) \le 2C\epsilon \quad (\epsilon \le 1). \tag{4.8}$$

On $r \le \delta$ the metric is Ricci-flat, and on $r \ge 2\delta$ it is flat. On the transition annulus, $|\operatorname{Ric}|_{\mathrm{op}} \le C|\operatorname{Rm}| \le C\epsilon$ by (4.8). The inner curvature energy is at most the fixed energy of h_p, by four-dimensional scaling invariance. On the transition annulus $T_\epsilon = \{\delta \le r \le 2\delta\}$,

$$\operatorname{Vol}(T_\epsilon) \le C \int_\delta^{2\delta} r^3\, dr \le C\delta^4, \qquad \int_{T_\epsilon} |\operatorname{Rm}|^2\, dv \le C\epsilon^2\delta^4 = C\epsilon^4.$$

Choose R_0 large enough that h_p and g_C are uniformly bilipschitz on $R \ge R_0$, and fix a point o in the compact core. Every point of the end with $\epsilon R_0 \le r \le 2\delta$ is joined to the level $r = \epsilon R_0$ by a radial path of length at most $C\delta$. Let D_p be the intrinsic diameter of the compact core in h_p. Each of its points is joined to o by a path in the core of length at most $(D_p + 1)\epsilon$ for the scaled metric. Put $C_0 = D_p + 1$. If K_ϵ denotes the union of the core and the region $r \le 2\delta$, then

$$\operatorname{diam}_{\mathrm{intr}}(K_\epsilon) \le 2C\delta + 2C_0\epsilon, \qquad \operatorname{Vol}(K_\epsilon) \le C_1\epsilon^4 + C_2 \int_{\epsilon R_0}^{2\delta} r^3\, dr \le C_3\delta^4.$$

The metric is exactly conical near $r = 1$ and hence matches the given metric on Z on a full collar. Fix $R_1 > 2R_0$ and use the core truncated at $R = R_1$ as W_p, with boundary collar coordinate $r = R/R_1$. Put $a = R_0$, $b = R_1$ and $d = \epsilon^{-1} - R_0$. For sufficiently small ϵ, $d > 0$. Choose disjoint endpoint intervals $I_a, I_b \subset [a, b]$ and smooth functions $0 \le \theta_a, \theta_b \le 1$, equal to 1 near the corresponding endpoint and supported in those intervals, such that

$$|I_a| + \frac{|I_b|}{\epsilon R_1} < d/2.$$

Choose $0 < c < d/(2(b-a))$ and set

$$q_0 = \theta_a + \frac{\theta_b}{\epsilon R_1} + c(1 - \theta_a - \theta_b).$$

Then $q_0 > 0$ and $\int_a^b q_0 < d$. For a nonnegative $\psi \in C_c^\infty((a,b))$ with integral one, put

$$q = q_0 + \left(d - \int_a^b q_0\right)\psi, \qquad \Psi_\epsilon(R) = R_0 + \int_{R_0}^R q(u)\, du.$$

Thus $\Psi_\epsilon' > 0$, $\Psi_\epsilon(R) = R$ near R_0, and $\Psi_\epsilon(R) = R/(\epsilon R_1)$ near R_1; the last equality follows from the derivative there and $\Psi_\epsilon(R_1) = \epsilon^{-1}$. Extend the resulting radial map by the identity on the compact core. It is a diffeomorphism from W_p to the truncation at $R = \epsilon^{-1}$ and satisfies $\epsilon\Psi_\epsilon(R) = R/R_1$ on the outer collar. Pullback therefore fixes both the smooth manifold and its gluing collar.

Let v_j be the vertex of the jth cone. On the corresponding filling, define

$$f_\epsilon(x) = \begin{cases} v_j, & x \text{ lies in the core or } r(x) \le \delta, \\ (2(r-\delta), \theta), & \delta \le r \le 2\delta, \\ (r, \theta), & r \ge 2\delta. \end{cases}$$

Set $f_\epsilon = \mathrm{Id}$ on Z. At $r = 2\delta$ both formulas give $(2\delta, \theta)$, and on $\delta \le r \le 2\delta$,

$$d_X(f_\epsilon(r,\theta), v_j) \le 2(r - \delta).$$

Thus the formulas define a continuous map. Every cone point (t,θ) with $0 < t \le 2\delta$ is the image of $(\delta + t/2, \theta)$; the remaining points are fixed. Hence f_ϵ is onto. Apply Lemma 4.2 to the hypersurfaces $r = 2\delta$. It verifies the distance hypothesis of Lemma 4.1 for both M and X. The graph of f_ϵ is contained in the correspondence of Lemma 4.1, with all inserted and removed pieces of diameter at most $C\delta$. Thus

$$\operatorname{dis} f_\epsilon \le C\delta, \qquad \operatorname{diam}(M, \widehat{g}_\epsilon) \le \operatorname{diam}(X) + C\delta.$$

On the common complement the two metrics agree. Since the total volumes of both the inserted pieces and the removed cone pieces are at most $C\delta^4$, one also has

$$\left|\operatorname{Vol}(M, \widehat{g}_\epsilon) - \mathcal{H}^4(X)\right| \le C\delta^4.$$

The fixed metric on Z contributes a bounded Ricci tensor and finite curvature energy.

Multiply each metric by c_ϵ^2, where $c_\epsilon = (\mathcal{H}^4(X)/\operatorname{Vol}(M, \widehat{g}_\epsilon))^{1/4} \to 1$. Choose $c_* > 0$ with $c_\epsilon \ge c_*$ and a common bound $|\operatorname{Ric}_{\widehat{g}_\epsilon}|_{\mathrm{op}} \le C_0$. Fix $\Lambda > 0$ such that $C_0/(\Lambda^2 c_*^2) \le 3$, and put $g_\epsilon = \Lambda^2 c_\epsilon^2 \widehat{g}_\epsilon$. Then

$$|\operatorname{Ric}_{g_\epsilon}|_{\mathrm{op}} \le 3, \qquad \operatorname{Vol}(M, g_\epsilon) = \Lambda^4 \mathcal{H}^4(X).$$

The diameter remains bounded because $c_\epsilon \to 1$, and the curvature energy is unchanged by constant rescaling in dimension four:

$$|\operatorname{Rm}_{a^2 g}|^2_{a^2 g} = a^{-4} |\operatorname{Rm}_g|^2_g, \qquad dv_{a^2 g} = a^4 dv_g.$$

For the maps with target distance Λd_X,

$$\operatorname{dis}\big(f_\epsilon : (M, g_\epsilon) \to (X, \Lambda d_X)\big) \le \Lambda c_\epsilon \operatorname{dis}\big(f_\epsilon : (M, \widehat{g}_\epsilon) \to (X, d_X)\big) + \Lambda |c_\epsilon - 1| \operatorname{diam}(X, d_X) \longrightarrow 0.$$

Thus the rescaled metrics converge to $(X, \Lambda d_X)$. Equation (2.5) gives uniform noncollapse. □

5. Relative surgery and realization

Lemma 5.1. *Let G be finitely presented, and let $t_1, \dots, t_s \in G$ have exact orders $p_1, \dots, p_s \ge 2$. There is a compact connected oriented smooth four-manifold Z with boundary $\coprod_j (-L_{p_j})$ and an identification $\pi_1(Z) \cong G$ such that the jth boundary homomorphism sends a generator of $C_{p_j^2}$ to t_j.*

Proof. Choose a presentation $G = \langle x_1, \dots, x_k \mid R_1, \dots, R_m \rangle$ and words $w_j(x)$ representing t_j. Form the interior connected sum Z_0 of $-W_{p_1}, \dots, -W_{p_s}$ and k copies of $S^1 \times S^3$; set $Z_0 = S^4$ if $s + k = 0$. Fix a basepoint and paths to all boundary components. Van Kampen's theorem gives

$$\pi_1(Z_0) = \langle x_1, \dots, x_k, a_1, \dots, a_s \mid a_j^{p_j} = 1 \ (1 \le j \le s) \rangle,$$

and the jth boundary generator has image a_j.

Choose free homotopy classes in $\operatorname{Int} Z_0$ represented by the words

$$R_1, \dots, R_m, \qquad a_j w_j(x)^{-1} \quad (1 \le j \le s). \tag{5.1}$$

Let S be the disjoint union of $m + s$ circles and let $c : S \to \operatorname{Int} Z_0$ be a smooth representative of these classes. Put

$$\mathcal{Z} = \{(x, z, A) \in J^1(S, \operatorname{Int} Z_0) : A = 0\}, \qquad \Delta_Z = \{(z, z) : z \in \operatorname{Int} Z_0\},$$

and let $S^{(2)} = \{(x, y) \in S \times S : x \ne y\}$. One-jet and two-point multijet transversality give an arbitrarily small smooth perturbation, still denoted by c, such that

$$j^1 c \pitchfork \mathcal{Z}, \qquad (c \times c)|_{S^{(2)}} \pitchfork \Delta_Z.$$

The two target strata have codimension four, whereas their source dimensions are one and two. Both inverse images are empty. Hence c is an injective immersion; compactness of S makes it an embedding. The perturbation can be chosen in a normal neighbourhood of the original map, so pointwise minimizing segments give a homotopy between the two maps. Its restrictions $c_\nu : S^1 \hookrightarrow \operatorname{Int} Z_0$ therefore represent the prescribed free homotopy classes and have pairwise disjoint images.

Orient each circle. Its normal bundle is an oriented rank-three bundle over S^1. Choose a fibre metric and a compatible connection. Parallel transport an oriented orthonormal frame along $[0, 1]$; write $e(1) = e(0)T$ at the identified endpoints, with $T \in \mathrm{SO}(3)$. Choose a smooth path $A : [0, 1] \to \mathrm{SO}(3)$

equal to I near 0 and to T^{-1} near 1. Such a path exists since SO(3) is path connected. The frame $e(t)A(t)$ descends smoothly to S^1: on an interval about the identified endpoint, its two restrictions are parallel frames with the same value, and hence agree by uniqueness of parallel transport. Thus the normal bundle is smoothly trivial. Choose pairwise disjoint framed tubular neighbourhoods $T_v \cong S^1 \times D^3$ and put

$$Z^\circ = Z_0 \setminus \bigcup_v \operatorname{Int} T_v.$$

Write $C = \bigcup_v c_v(S^1)$. Relative transversality gives

$$\pi_1(Z_0 \setminus C) \xrightarrow{\sim} \pi_1(Z_0).$$

Relative transversality moves paths with endpoints outside C off C, since $1 + 1 < 4$; in particular, the complement is path connected and the displayed homomorphism is onto. A nullhomotopy of a loop in the complement can be perturbed relative to its boundary to avoid C, since $2 + 1 < 4$. The homomorphism is therefore injective. In each punctured normal fibre of radius one, the homotopy

$$v \longmapsto \left((1-t) + \frac{t}{|v|}\right) v \qquad (0 < |v| \le 1)$$

fixes the unit sphere and retracts the fibre onto it. Extended by the identity outside the tubular neighbourhoods, it retracts $Z_0 \setminus C$ onto Z°. Hence $\pi_1(Z^\circ) \to \pi_1(Z_0)$ is an isomorphism.

Attach $D^2 \times S^2$ to every $\partial T_v = S^1 \times S^2$. Choose the orientation of each attached piece so that the framed boundary identification reverses the boundary orientations. Collars give open covers to which van Kampen's theorem applies. Since $D^2 \times S^2$ is simply connected and $\pi_1(S^1 \times S^2)$ is generated by its circle factor, the resulting manifold Z satisfies

$$\pi_1(Z) = \pi_1(Z_0)/\langle\langle [c_1], \dots, [c_{m+s}] \rangle\rangle.$$

Eliminating the generators a_j yields

$$\pi_1(Z) = \langle x, a \mid a_j^{p_j}, R_\ell, a_j w_j(x)^{-1} \rangle = \langle x \mid R_\ell, w_j(x)^{p_j} \rangle = G,$$

because $t_j^{p_j} = 1$ in G. The surgeries preserve the given orientations and do not meet ∂Z_0. The joining paths may be perturbed relative to their endpoints to avoid the circles and then retracted into Z°. Under the quotient map just computed, the jth boundary generator has image $a_j = w_j(x) = t_j$. Thus $\partial Z = \coprod_j (-L_{p_j})$ with the required boundary maps. □

Proof of Theorem 1.1. (i)⟹(iii). Discard any identity elements from the list in (i). Suppose the remaining list is nonempty and write $p_j = \operatorname{ord}(t_j)$. Let Z be the manifold in Lemma 5.1. Choose disjoint boundary collars and prescribe the metrics $dr^2 + r^2\gamma_{p_j}$ on them. Take any smooth Riemannian metric g_0 on Z and a smooth function $0 \le \theta \le 1$, supported in the boundary collars and equal to one near the boundary. On the collars set $g_Z = \theta(dr^2 + r^2\gamma_{p_j}) + (1-\theta)g_0$ and set $g_Z = g_0$ off them. This is smooth and positive definite. Apply Lemma 4.4 with the fillings W_{p_j}.

For one attachment the group pushout is $G *_{C_{p_j^2}} C_{p_j}$. It has the presentation obtained by adding a generator a with $a^{p_j} = 1$ and $a = t_j$. The first relation already holds in G; thus the pushout is G. Each attachment preserves the original identification of G. All fillings together give $\pi_1(M) = G$. Attaching the corresponding cone imposes the relation $t_j = 1$. Therefore

$$\pi_1(X) = G/\langle\langle t_1, \dots, t_s \rangle\rangle_G \cong Q. \tag{5.2}$$

Choose $p \in \operatorname{Int} Z$ and let $j_M : Z \to M$ and $j_X : Z \to X$ be the inclusions. The preceding pushout calculation shows that $(j_M)_*$ is an isomorphism. Let $\theta : \pi_1(Z, p) \to G$ be the marking from Lemma 5.1, and put $\alpha = \theta(j_M)_*^{-1}$. The quotient identification in (5.2) is the unique isomorphism $\beta : \pi_1(X, p) \to Q$ such that $\beta(j_X)_* = \varphi\theta$. The maps from Lemma 4.4 satisfy $f_i \circ j_M = j_X$ and $f_i(p) = p$. Hence

$$\beta(f_i)_*\alpha^{-1} = \beta(f_i)_*(j_M)_*\theta^{-1} = \beta(j_X)_*\theta^{-1} = \varphi.$$

Lemma 4.4 gives $|\operatorname{Ric}_{g_i}|_{\mathrm{op}} \le 3$, $\operatorname{Vol}(M, g_i) = V_0 > 0$, $\sup_i \operatorname{diam}(M, g_i) < \infty$ and $\sup_i \int_M |\operatorname{Rm}_{g_i}|^2 \, dv_{g_i} < \infty$.

At each conical point, radial retraction identifies the fundamental group of every punctured cone neighbourhood with $C_{p_j^2} \neq 1$. If a four-ball chart existed there, choose a smaller cone neighbourhood inside the chart and a larger cone neighbourhood containing the chart. The inclusion between the two punctured cones induces an isomorphism on π_1, but factors through the simply connected punctured four-ball. This is impossible. Thus these are exactly the s singular points.

If the list is empty, φ is an isomorphism. Realize G by a closed oriented four-manifold, starting with a connected sum of $S^1 \times S^3$ and performing the relator surgeries of Lemma 5.1; use S^4 when the group is trivial. Use a fixed smooth metric, a constant sequence and the identity maps, with the marking on the target composed with φ. Constant scaling gives the stated normalization.

(iii)⇒(ii). A compact connected Riemannian orbifold is a geodesic space. At a smooth point it has a contractible coordinate ball; at each singular point in (iii) it has a radially contractible cone neighbourhood. It is therefore locally contractible and semilocally simply connected. Take $n = 4$ and $v = V_0$.

(ii)⇒(i). Fix an index i_0 for which Proposition 3.2 applies. Let $S \subset \pi_1(M, p)$ consist of its finite-order normal generators of $\ker(f_{i_0})_*$. The marking identity gives

$$\ker \varphi = \alpha(\ker(f_{i_0})_*) = \langle\!\langle \alpha(S) \rangle\!\rangle_G.$$

Every element of $\alpha(S)$ has finite order, since α is an isomorphism. Lemma 3.3 gives a finite subset with the same normal closure. Its elements still have finite order. □

6. The order-two rational ball

Let $\mathbf{i}, \mathbf{j}, \mathbf{k}$ denote the quaternion units. Identify S^3 with SU(2), and consider

$$E = \mathrm{SU}(2) \times_{\mathrm{U}(1)} \mathbb{C}, \qquad (u, z) \sim (ue^{\mathrm{i}\theta}, e^{-2\mathrm{i}\theta} z). \tag{6.1}$$

The base is $S^2 = \mathrm{SU}(2)/\mathrm{U}(1)$. The associated-bundle norm is $\|[u, z]\| = |z|$; it is well-defined because the fibre action has modulus one. For $b > 0$, put $E_{\leq b} = \{[u, z] : |z| \leq b\}$. Fibre contraction retracts each $E_{\leq b}$ onto the simply connected zero section.

Lemma 6.1. *The map*

$$\iota[u, z] = [u\mathbf{j}, \overline{z}] \tag{6.2}$$

is a free, orientation-preserving involution of E preserving each $E_{\leq b}$. Let $W = E_{\leq 1/2}/\langle \iota \rangle$. Then W is a compact connected oriented smooth four-manifold with

$$W \simeq \mathbb{RP}^2, \qquad \pi_1(W) = C_2, \qquad \partial W = L := S^3/\langle \mathbf{j} \rangle, \qquad \pi_1(L) = C_4. \tag{6.3}$$

The boundary homomorphism is the surjection $C_4 \to C_2$.

Proof. For every $\theta \in \mathbb{R}$,

$$[ue^{\mathrm{i}\theta}\mathbf{j}, \overline{e^{-2\mathrm{i}\theta} z}] = [u\mathbf{j}e^{-\mathrm{i}\theta}, e^{2\mathrm{i}\theta}\bar{z}] = [u\mathbf{j}, \bar{z}].$$

Thus (6.2) is well-defined. Since $|\overline{z}| = |z|$, it preserves $E_{\leq b}$ for every $b > 0$. Its square is $[u, z] \mapsto [-u, z] = [u, z]$, since the equivalence with $\theta = \pi$ does not change z.

The projection $\pi_E([u, z]) = u\mathbf{i}u^{-1}$ satisfies

$$\pi_E(\iota[u, z]) = u\mathbf{j}\mathbf{i}\mathbf{j}^{-1}u^{-1} = -u\mathbf{i}u^{-1} = -\pi_E([u, z]).$$

Thus ι has no fixed point. In a local bundle trivialization it reverses the orientations of the base and of the complex fibre. Let $\mathcal{V} = \ker d\pi_E$. The derivative preserves the exact sequence

$$0 \longrightarrow \mathcal{V} \longrightarrow TE \xrightarrow{d\pi_E} \pi_E^* TS^2 \longrightarrow 0.$$

Its induced maps on $\mathcal{V}$ and $\pi_E^* TS^2$ reverse orientation. The determinant identity for this exact sequence gives $\operatorname{sign}\det d\iota = (-1)(-1) = 1$. The free finite quotient is therefore smooth and oriented. The homotopy $[u, z] \mapsto [u, (1 - s)z]$, $0 \leq s \leq 1$, commutes with ι and is a deformation retraction onto the zero section. Its quotient is a deformation retraction of W onto $S^2/(x \sim -x) = \mathbb{RP}^2$. The rational cellular chain complex of $\mathbb{RP}^2$ is $0 \to \mathbb{Q} \xrightarrow{2} \mathbb{Q} \xrightarrow{0} \mathbb{Q} \to 0$. Thus $H_0(W; \mathbb{Q}) = \mathbb{Q}$ and $H_j(W; \mathbb{Q}) = 0$ for $j > 0$.

For $b > 0$, the map $u \mapsto [u, b]$ identifies the fibre level $|z| = b$ with $S^3/\{\pm 1\}$: every class has such a representative, and $[u, b] = [v, b]$ holds exactly when $v = \pm u$. There ι is right multiplication by $\mathbf{j}$; quotienting gives L. Right multiplication J by $\mathbf{j}$ is orthogonal on $\mathbb{R}^4$ and satisfies $J^2 = -\mathrm{Id}$. Choose a unit vector v and a unit vector w perpendicular to $\mathrm{span}\{v, Jv\}$. Then (v, Jv, w, Jw) is an orthonormal basis. In the corresponding identification with $\mathbb{C}^2$, $J(z_1, z_2) = (\mathrm{i}z_1, \mathrm{i}z_2)$. Hence the round quotient is isometric, without a chosen orientation, to $L(4, 1)$, and its fundamental group is C_4. The restricted two-sheeted boundary covering $S^3/\{\pm 1\} \to L$ is connected. Its monodromy onto C_2 is surjective. It is the restriction of the universal covering of W, whose total space is the disc bundle over S^2. Hence it is the composition $\pi_1(L) \to \pi_1(W) \cong C_2$, proving the final assertion. □

Give L the round quotient metric γ. Then

$$\mathrm{Vol}(L, \gamma) = \frac{\sigma_3}{4} = \frac{\pi^2}{2}. \tag{6.4}$$

This rational ball is the order-two case of the construction in [11, Proposition 3.3]; the involution also occurs in Brena's construction [3, discussion preceding Theorem 1]. Section 7 gives the Ricci tensor and the curvature-energy bound for the metrics used here.

7. Curvature estimates and Ricci-bounded truncations

Choose the left-invariant coframe $\omega_1, \omega_2, \omega_3$ dual to $\mathbf{j}, \mathbf{k}, \mathbf{i}$. Thus $\sum \omega_i^2$ is the curvature-one round metric and the brackets of the dual fields are $[X_1, X_2] = 2X_3$ and its cyclic permutations. The individual forms are used on the covering space. Right multiplication by $\mathbf{j}$ acts on the dual basis by $\mathrm{Ad}_{\mathbf{j}^{-1}}(\mathbf{j}, \mathbf{k}, \mathbf{i}) = (\mathbf{j}, -\mathbf{k}, -\mathbf{i})$. Hence $R_{\mathbf{j}}^*(\omega_1, \omega_2, \omega_3) = (\omega_1, -\omega_2, -\omega_3)$. The tensors ω_3^2 and $\omega_1^2 + \omega_2^2$ descend to L.

Lemma 7.1. *For a positive smooth function $F(r)$, set*

$$h_F = F^{-1} dr^2 + r^2(\omega_1^2 + \omega_2^2) + r^2 F \omega_3^2. \tag{7.1}$$

In the orthonormal frame $e_0 = \sqrt{F}\,\partial_r$, $e_1 = r^{-1}X_1$, $e_2 = r^{-1}X_2$, $e_3 = (r\sqrt{F})^{-1}X_3$, its Ricci tensor is diagonal:

$$\mathrm{Ric}_{00} = \mathrm{Ric}_{33} = -\frac{1}{2}F'' - \frac{5}{2r}F', \tag{7.2}$$

$$\mathrm{Ric}_{11} = \mathrm{Ric}_{22} = \frac{4(1-F)}{r^2} - \frac{F'}{r}. \tag{7.3}$$

If $F = 1 - m(r)r^{-4}$, these identities become

$$\mathrm{Ric}_{00} = \mathrm{Ric}_{33} = \frac{rm'' - 3m'}{2r^5}, \qquad \mathrm{Ric}_{11} = \mathrm{Ric}_{22} = \frac{m'}{r^5}. \tag{7.4}$$

The volume density is $r^3\, dr\, dv_\gamma$ on the quotient. For $F = 1 - \epsilon^4 r^{-4}$,

$$\mathrm{Ric}_{h_F} = 0, \qquad |\mathrm{Rm}_{h_F}|^2 = 384\epsilon^8 r^{-12}. \tag{7.5}$$

Proof. Write $\alpha = \sqrt{F}/r$, $\beta = \alpha + F'/(2\sqrt{F})$ and $\tau = 2/(r\sqrt{F})$. The nonzero brackets, together with their negatives, are

$$\begin{aligned} &[e_0, e_1] = -\alpha e_1, \quad [e_0, e_2] = -\alpha e_2, \quad [e_0, e_3] = -\beta e_3, \\ &[e_1, e_2] = 2\alpha e_3, \quad [e_2, e_3] = \tau e_1, \quad [e_3, e_1] = \tau e_2. \end{aligned} \tag{7.6}$$

Koszul's formula in an orthonormal frame is

$$2\langle \nabla_{e_a} e_b, e_c \rangle = \langle [e_a, e_b], e_c \rangle - \langle [e_b, e_c], e_a \rangle + \langle [e_c, e_a], e_b \rangle. \tag{7.7}$$

It gives $\nabla_{e_0} e_a = 0$ for all a, $\nabla_{e_i} e_0 = \alpha_i e_i$ and $\nabla_{e_i} e_i = -\alpha_i e_0$, with $(\alpha_1, \alpha_2, \alpha_3) = (\alpha, \alpha, \beta)$, and

$$\begin{aligned} &\nabla_{e_1} e_2 = \alpha e_3, \quad \nabla_{e_2} e_1 = -\alpha e_3, \qquad \nabla_{e_1} e_3 = -\alpha e_2, \\ &\nabla_{e_2} e_3 = \alpha e_1, \quad \nabla_{e_3} e_1 = (\tau - \alpha) e_2, \quad \nabla_{e_3} e_2 = -(\tau - \alpha) e_1. \end{aligned} \tag{7.8}$$

All other coefficients vanish. If C^q_{ab} and Γ^q_{ab} denote the bracket and connection coefficients, respectively, then

$$\langle R(e_a, e_b)e_d, e_c\rangle = e_a\Gamma^c_{bd} - e_b\Gamma^c_{ad} + \sum_q \left(\Gamma^q_{bd}\Gamma^c_{aq} - \Gamma^q_{ad}\Gamma^c_{bq} - C^q_{ab}\Gamma^c_{qd}\right).$$

Use $R(u, v)w = \nabla_u\nabla_v w - \nabla_v\nabla_u w - \nabla_{[u,v]}w$. For every smooth function $u = u(r)$, $e_0u = \sqrt{F}\,u'$ and $e_iu = 0$ for $i \in \{1, 2, 3\}$. Define

$$A = -\frac{F'}{2r}, \qquad B = -\frac{F''}{2} - \frac{3F'}{2r}, \qquad C = \frac{4(1-F)}{r^2}. \tag{7.9}$$

In the ordered basis $(01, 02, 03, 23, 31, 12)$ of two-forms, the matrix with entries $\langle R(e_a, e_b)e_d, e_c\rangle$ is

$$\begin{pmatrix} A & 0 & 0 & A & 0 & 0 \\ 0 & A & 0 & 0 & A & 0 \\ 0 & 0 & B & 0 & 0 & -2A \\ A & 0 & 0 & A & 0 & 0 \\ 0 & A & 0 & 0 & A & 0 \\ 0 & 0 & -2A & 0 & 0 & C \end{pmatrix}. \tag{7.10}$$

The coefficients used in this matrix satisfy

$$-e_0\alpha - \alpha^2 = \alpha(\alpha - \beta) = A, \qquad -e_0\beta - \beta^2 = B, \qquad 2\alpha\tau - 4\alpha^2 = C,$$
$$-e_0(\tau - \alpha) - \beta(\tau - \alpha) = -2A.$$

With $R_{ab,cd} = \langle R(e_a, e_b)e_d, e_c\rangle$, the independent nonzero entries are

$$R_{01,01} = R_{02,02} = R_{23,23} = R_{31,31} = A,$$
$$R_{03,03} = B, \qquad R_{12,12} = C,$$
$$R_{01,23} = R_{02,31} = A, \qquad R_{03,12} = -2A.$$

The identities $R_{ab,cd} = -R_{ba,cd} = -R_{ab,dc} = R_{cd,ab}$ determine the other entries; substitution in the coefficient formula gives zero for entries not listed. Hence

$$\mathrm{Ric}_{bd} = \sum_{a=0}^{3} R_{ab,ad}, \qquad (\mathrm{Ric}_{00}, \mathrm{Ric}_{11}, \mathrm{Ric}_{22}, \mathrm{Ric}_{33}) = (2A + B, 2A + C, 2A + C, 2A + B),$$

and $\mathrm{Ric}_{bd} = 0$ for $b \neq d$. Since

$$F' = -m'r^{-4} + 4mr^{-5}, \qquad F'' = -m''r^{-4} + 8m'r^{-5} - 20mr^{-6},$$

substitution gives (7.4).

The determinant of (7.1) in the coframe $(dr, \omega_1, \omega_2, \omega_3)$ is r^6. If $m = \epsilon^4$, then $(A, B, C) = (-2, 4, 4)\epsilon^4 r^{-6}$. Antisymmetry in each index pair gives

$$|\,\mathrm{Rm}\,|^2 = \sum_{a,b,c,d=0}^{3} R^2_{ab,cd} = 4\sum_{a<b,\,c<d} R^2_{ab,cd} = 4(16A^2 + B^2 + C^2) = 384\epsilon^8 r^{-12}.$$

□

Proposition 7.2. *There are smooth metrics k_ϵ on W, $0 < \epsilon < \epsilon_0$, with an end coordinate $\epsilon < r \leq 1$, such that k_ϵ equals $dr^2 + r^2\gamma$ for $r \geq 2\sqrt{\epsilon}$, and*

$$|\,\mathrm{Ric}_{k_\epsilon}|_{\mathrm{op}} \leq C\epsilon, \qquad \mathrm{Vol}(W, k_\epsilon) = \frac{\mathrm{Vol}(L)}{4}(1 - \epsilon^4), \tag{7.11}$$

$$\mathrm{diam}_{\mathrm{intr}}\{r \leq 2\sqrt{\epsilon}\} \leq C\sqrt{\epsilon}, \qquad \int_W |\,\mathrm{Rm}_{k_\epsilon}|^2\,dv \leq 24\pi^2 + C\epsilon^4. \tag{7.12}$$

The set in the diameter estimate includes the zero section. On $r \leq \sqrt{\epsilon}$, the pullback of k_ϵ to the canonical disc-bundle double cover is the scaled Eguchi–Hanson metric. On that region, k_ϵ is the quotient of this restriction by the free involution (6.2).

Proof. Fix $0 < \epsilon_0 < 1/16$ and use the cutoff χ in (4.1). Put

$$\delta = \sqrt{\epsilon}, \qquad m_\epsilon(r) = \epsilon^4 \chi(r/\delta), \qquad F_\epsilon(r) = 1 - \frac{m_\epsilon(r)}{r^4}. \tag{7.13}$$

For $\epsilon < r < \delta$, $F_\epsilon = 1 - \epsilon^4 r^{-4} > 0$. For $r \geq \delta$, $F_\epsilon \geq 1 - \epsilon^2 \geq 1/2$ after decreasing ϵ_0. Use (7.1) with this F_ϵ.

Set $\rho = \frac{1}{2}\sqrt{r^4 - \epsilon^4}$. Then $r^4 = \epsilon^4 + 4\rho^2$ and $dr = 2\rho r^{-3}\, d\rho$. On the disc-bundle double cover, the metric for $r \leq \delta$ is

$$(\epsilon^4 + 4\rho^2)^{-1/2}(d\rho^2 + 4\rho^2\omega_3^2) + (\epsilon^4 + 4\rho^2)^{1/2}(\omega_1^2 + \omega_2^2). \tag{7.14}$$

On $S^3/\{\pm 1\}$ the connection form $2\omega_3$ has fibre period 2π. In a local trivialization write $2\omega_3 = d\psi + \eta$. With $x + \mathrm{i}y = \rho e^{\mathrm{i}\psi}$,

$$d\rho^2 + \rho^2 d\psi^2 = dx^2 + dy^2, \qquad \rho^2 d\psi = x\, dy - y\, dx.$$

In these coordinates,

$$d\rho^2 + \rho^2(d\psi + \eta)^2 = dx^2 + dy^2 + 2(x\, dy - y\, dx)\eta + (x^2 + y^2)\eta^2.$$

Here $\alpha\beta = (\alpha \otimes \beta + \beta \otimes \alpha)/2$ for one-forms α, β. Both scalar coefficients in (7.14) are smooth functions of $x^2 + y^2$. At $x = y = 0$ the tensor is $\epsilon^{-2}(dx^2 + dy^2) + \epsilon^2(\omega_1^2 + \omega_2^2)$; the second term is the pullback of a positive metric on the base. The cross terms vanish there. Thus the extension is smooth and positive definite at the zero section. The metric is invariant under (6.2) and descends to a smooth metric on its free quotient.

The cutoff makes the metric exactly conical for $r \geq 2\delta$. On $\delta \leq r \leq 2\delta$,

$$|m_\epsilon'| \leq C\epsilon^4\delta^{-1}, \qquad |m_\epsilon''| \leq C\epsilon^4\delta^{-2}.$$

Equation (7.4) gives $|\operatorname{Ric}|_{\mathrm{op}} \leq C\epsilon^4\delta^{-6} = C\epsilon$. For the full curvature tensor, (7.9) becomes

$$A = \frac{m_\epsilon'}{2r^5} - \frac{2m_\epsilon}{r^6}, \qquad B = \frac{m_\epsilon''}{2r^4} - \frac{5m_\epsilon'}{2r^5} + \frac{4m_\epsilon}{r^6}, \qquad C = \frac{4m_\epsilon}{r^6}.$$

Since $|m_\epsilon| \leq \epsilon^4$ and $\delta \leq r \leq 2\delta$,

$$|A| + |B| + |C| \leq C_0\epsilon^4\delta^{-6} = C_0\epsilon, \qquad |\operatorname{Rm}|^2 = 4(16A^2 + B^2 + C^2) \leq C_1\epsilon^2.$$

The Ricci tensor vanishes on $r \leq \delta$ and on $r \geq 2\delta$. The volume formula follows by integrating $r^3\, dr\, dv_\gamma$ from ϵ to one; the zero section has volume zero.

The radial length from ϵ to δ is

$$\epsilon \int_1^{\delta/\epsilon} (1 - u^{-4})^{-1/2}\, du \leq \delta + C\epsilon. \tag{7.15}$$

For $1 < u \leq 2$,

$$1 - u^{-4} = \frac{(u-1)(u^3 + u^2 + u + 1)}{u^4} \geq \frac{u-1}{4}.$$

For $u \geq 2$,

$$(1 - u^{-4})^{-1/2} - 1 = \frac{u^{-4}}{\sqrt{1 - u^{-4}}(1 + \sqrt{1 - u^{-4}})} \leq 2u^{-4}.$$

Thus the nonnegative difference has integral at most $\int_1^2 2(u-1)^{-1/2}\, du + \int_2^\infty 2u^{-4}\, du = 49/12$. Consequently the left side of (7.15) is at most $\delta - \epsilon + (49/12)\epsilon$. Let h_0 be the fixed quotient metric induced by $\omega_1^2 + \omega_2^2$ on the zero section. Its induced metric is $\epsilon^2 h_0$, and hence its diameter is $\epsilon \operatorname{diam}(\mathbb{RP}^2, h_0)$. Every point with $r \leq 2\delta$ has a radial path to that section of length at most $\delta + C\epsilon + \sqrt{2}\delta$. It follows that

$$\operatorname{diam}_{\mathrm{intr}}\{r \leq 2\delta\} \leq 2(1 + \sqrt{2})\delta + 2C\epsilon + \epsilon \operatorname{diam}(\mathbb{RP}^2, h_0) \leq C'\delta.$$

By (7.5), the curvature energy of the inner part is at most

$$384\epsilon^8 \operatorname{Vol}(L) \int_\epsilon^\infty r^{-9}\, dr = 48 \operatorname{Vol}(L) = 24\pi^2.$$

On $T_\epsilon = \{\delta \le r \le 2\delta\}$,

$$\mathrm{Vol}(T_\epsilon) = \mathrm{Vol}(L)\int_\delta^{2\delta} r^3\,dr = \frac{15}{4}\mathrm{Vol}(L)\delta^4, \qquad \int_{T_\epsilon} |\mathrm{Rm}|^2\,dv \le C\epsilon^2\delta^4 = C\epsilon^4.$$

On $r \ge 2\delta$, the curvature tensor vanishes.

Use the fixed disc bundle $0 \le \rho_0 \le 1/2$, with outer collar coordinate $r = \sqrt{2\rho_0}$. Choose an increasing smooth map

$$f_\epsilon : [0, 1/2] \longrightarrow [0, \tfrac{1}{2}\sqrt{1-\epsilon^4}]$$

which equals ρ_0 on a neighbourhood of $[0, \epsilon^4]$ and $\sqrt{\rho_0^2 - \epsilon^4/4}$ for $2\epsilon \le \rho_0 \le 1/2$. Put $a = \epsilon^4$, $b = 2\epsilon$ and $g(\rho_0) = \sqrt{\rho_0^2 - \epsilon^4/4}$ near $[b, 1/2]$. Then $0 < a < b < 1/2$ and $d := g(b) - a > 0$. Choose disjoint endpoint intervals $I_a, I_b \subset [a, b]$ such that

$$|I_a| + \int_{I_b} g'(u)\,du < d/2.$$

Let $0 \le \theta_a, \theta_b \le 1$ be smooth, supported in I_a, I_b, and equal to 1 near the respective endpoints. For $0 < c < \min\{1, d/(2(b-a))\}$, set

$$q_0 = \theta_a + \theta_b g' + c(1 - \theta_a - \theta_b).$$

The product $\theta_b g'$ is extended by zero away from I_b. Disjointness of the supports gives $q_0 > 0$, with $q_0 = 1$ near a and $q_0 = g'$ near b. Moreover,

$$\int_a^b q_0 \le |I_a| + \int_{I_b} g' + c(b-a) < d.$$

Take $\psi \in C_c^\infty((a,b))$ with $\psi \ge 0$ and $\int_a^b \psi = 1$, and put

$$q = q_0 + \left(g(b) - a - \int_a^b q_0\right)\psi, \qquad f_\epsilon(\rho_0) = a + \int_a^{\rho_0} q \quad (a \le \rho_0 \le b).$$

Then $q > 0$, $f_\epsilon = \rho_0$ near a, and $f_\epsilon = g$ near b. Extend by these two formulas to $[0, 1/2]$. This gives the required smooth increasing bijection with strictly positive derivative. The induced fibre map is smooth at zero because $f_\epsilon(\rho_0) = \rho_0$ there. It commutes with fibre conjugation and descends to W. Near the boundary, $4f_\epsilon(\rho_0)^2 + \epsilon^4 = 4\rho_0^2 = r^4$, so the end coordinate agrees with the fixed coordinate $r = \sqrt{2\rho_0}$ whenever $\rho_0 \ge 2\epsilon$. Every compact subset of the punctured fixed disc bundle is eventually in this range. Pullback by these diffeomorphisms preserves all preceding estimates and fixes the conical tensor on each such compact subset. □

Since $\chi(1) = 1$ and $\chi(2) = 0$, there is $s \in (1,2)$ with $\chi'(s) < 0$. At $r = \delta s$, formula (7.4) gives $\mathrm{Ric}_{11} = \epsilon^4\delta^{-1}\chi'(s)r^{-5} < 0$. For every $0 < \epsilon < \epsilon_0$, the Ricci tensor of k_ϵ therefore has a negative eigenvalue at some point of the annulus. Formula (7.5) also gives $|\mathrm{Rm}| = \sqrt{384}\,\epsilon^{-2}$ on the zero section by continuity; there is no uniform pointwise curvature bound there.

8. The infinite dihedral realization

Proof of Theorem 1.2. Remove an interior four-disc from each of two copies of $-W$ and identify the resulting three-sphere boundaries by an orientation-reversing diffeomorphism. Denote the interior connected sum by Z. Removing a disc does not change π_1, and the gluing three-sphere is simply connected. Van Kampen therefore gives

$$\pi_1(Z) = \langle a, b \mid a^2 = b^2 = 1\rangle. \tag{8.1}$$

Its two boundary components are copies of L. Choose a basepoint in the connected-sum region and joining paths to the two boundary components. Under these paths, their chosen generators map to a, b, respectively, by Lemma 6.1.

Glue a copy of W to each component and put

$$M = Z \cup_{L \sqcup L} (W \sqcup W), \qquad X = Z \cup_{L \sqcup L} (\mathrm{C}_{\le 1}(L) \sqcup \mathrm{C}_{\le 1}(L)). \tag{8.2}$$

All boundary identifications are the fixed identity maps of the underlying L; the orientations of Z and the fillings are opposite at the boundary. The first attachment to M adds the relations $c^2 = 1$ and $c = a$, and the second adds $d^2 = 1$ and $d = b$. Thus $\pi_1(M) = D_\infty$. The cone attachments instead add $a = b = 1$, so $\pi_1(X) = 1$. Collars make these applications of van Kampen applications to open path-connected covers. The manifold is diffeomorphic to $D(W)\#D(W)$, where $D(W) = W \cup_L (-W)$.

The assignments $a(t) = -t$ and $b(t) = 2 - t$ define a homomorphism from the group in (8.1) to $\mathrm{Isom}(\mathbb{R})$. With composition of maps, $(ab)^k(t) = t - 2k$ for every $k \in \mathbb{Z}$. Hence ab has infinite order, and (1.5) follows.

Fix disjoint boundary collars of Z, with coordinate $1 \le r < 1 + \eta$, and let g_0 be any smooth Riemannian metric on Z. Let θ be a smooth function supported in these collars, with $0 \le \theta \le 1$ and $\theta = 1$ on $1 \le r < 1 + \eta/3$. Set

$$g_Z = \theta(dr^2 + r^2\gamma) + (1 - \theta)g_0$$

on the collars and $g_Z = g_0$ off them. The two definitions agree near the boundary of the collar support. For every nonzero tangent vector v, $g_Z(v, v) > 0$; hence g_Z is smooth and positive definite. Its curvature is bounded by compactness of Z. Adjoin the exact cone metrics for $0 < r \le 1$ to obtain the metric on X. Each vertex is a flat $\mathbb{R}^4/C_4$ orbifold point, with the specified free sphere action. Suppose a vertex v has a manifold chart. After restricting the chart, there are $0 < r < R$ and a chart neighbourhood $U \cong B^4$ such that

$$v \in \mathrm{C}_{<r}(L) \subset U \subset \mathrm{C}_{<R}(L),$$

with v corresponding to the centre of B^4. Choose a basepoint in $\mathrm{C}_{<r}(L) \setminus \{v\}$. The inclusion of the punctured cones induces an isomorphism $C_4 \to C_4$, since both retract radially onto L. It also factors through $\pi_1(U \setminus \{v\}) = \pi_1(B^4 \setminus \{0\}) = 1$, a contradiction. All other points have smooth neighbourhoods.

Use Proposition 7.2 in the two fillings in M, and denote the resulting smooth metric by $\widehat{g}_\epsilon$. The metrics agree as tensors on a full collar of $r = 1$, so they define one smooth metric. Lemma 4.2, applied at $r = 2\sqrt{\epsilon}$, identifies both distances with the infima required in Lemma 4.1. The underlying manifold and all identifications are fixed. With

$$V_\infty = \mathrm{Vol}(Z) + \frac{\pi^2}{4}, \qquad V_\epsilon = \mathrm{Vol}(M, \widehat{g}_\epsilon) = V_\infty - \frac{\pi^2}{4}\epsilon^4, \tag{8.3}$$

we have $V_\epsilon \to V_\infty > 0$.

By (7.12), the sets $r \le 2\sqrt{\epsilon}$ have intrinsic diameter at most $C\sqrt{\epsilon}$; the corresponding cone pieces have diameter at most $4\sqrt{\epsilon}$. Their complements carry the same metric. Each point in a filling can be joined to its boundary by a path of length at most $1 + C\sqrt{\epsilon}$. Paths in Z therefore give

$$\mathrm{diam}(M, \widehat{g}_\epsilon) \le \mathrm{diam}_{\mathrm{intr}}(Z) + 2 + 2C\sqrt{\epsilon_0}.$$

Moreover, with $E_Z = \int_Z |\mathrm{Rm}_{g_Z}|^2\, dv_{g_Z}$,

$$|\mathrm{Ric}_{\widehat{g}_\epsilon}|_{\mathrm{op}} \le \max\{\|\mathrm{Ric}_{g_Z}\|_{L^\infty,\mathrm{op}}, C\epsilon_0\}, \qquad \int_M |\mathrm{Rm}_{\widehat{g}_\epsilon}|^2\, dv \le E_Z + 48\pi^2 + 2C\epsilon_0^4.$$

Thus there are constants D_0, C_0, E_0 such that

$$\mathrm{diam}(M, \widehat{g}_\epsilon) \le D_0, \qquad |\mathrm{Ric}_{\widehat{g}_\epsilon}|_{\mathrm{op}} \le C_0, \qquad \int_M |\mathrm{Rm}_{\widehat{g}_\epsilon}|^2\, dv \le E_0. \tag{8.4}$$

The last inequality uses two copies of (7.12) and the fixed curvature energy of Z.

Put $\delta = \sqrt{\epsilon}$. Define f_ϵ to be the identity on Z and on the regions $r \ge 2\delta$. In either filling, with vertex v in the corresponding cone, set

$$f_\epsilon(x) = v \quad (r(x) \le \delta), \qquad f_\epsilon(r, \theta) = (2(r - \delta), \theta) \quad (\delta \le r \le 2\delta).$$

The latter formula tends to v as $r \downarrow \delta$, uniformly in θ, and equals (r, θ) at $r = 2\delta$. Thus f_ϵ is continuous and onto. Its graph is contained in the correspondence of Lemma 4.1, with $s = 2$ and $\eta_\epsilon \le C\delta$. Therefore $\mathrm{dis}\, f_\epsilon \le 6C\delta \to 0$, which proves the convergence.

Put $c_\epsilon = (V_\infty/V_\epsilon)^{1/4}$. Choose a constant $\Lambda > 0$ with $C_0/\Lambda^2 \le 3$, and set

$$g_\epsilon = \Lambda^2 c_\epsilon^2 \widehat{g}_\epsilon. \tag{8.5}$$

Here $c_\epsilon \ge 1$ and $c_\epsilon \to 1$. Constant scaling leaves the covariant Ricci tensor unchanged and divides its metric eigenvalues by the squared distance factor. In dimension four, $|\mathrm{Rm}_{a^2 g}|^2_{a^2 g} = a^{-4}|\mathrm{Rm}_g|^2_g$ and $dv_{a^2 g} = a^4 dv_g$, so $\int |\mathrm{Rm}|^2\, dv$ is unchanged. Equations (8.3) and (8.4) prove all the bounds in the theorem, with $V_0 = \Lambda^4 V_\infty$. The limit metric is multiplied by Λ^2. For the target distance Λd_X,

$$\operatorname{dis} f_\epsilon \le \Lambda c_\epsilon \operatorname{dis}(f_\epsilon : (M, \widehat{g}_\epsilon) \to X) + \Lambda |c_\epsilon - 1| \operatorname{diam} X \longrightarrow 0.$$

Equation (2.5) gives a common positive unit-ball volume lower bound. Since $f_\epsilon|_Z = \mathrm{Id}_Z$, its homomorphism is the quotient by $\langle\!\langle a, b\rangle\!\rangle_{D_\infty} = D_\infty$. Let Σ be the two quotient zero sections. The radial identifications in Proposition 7.2 give a fixed diffeomorphism $M \setminus \Sigma \to X_{\mathrm{reg}}$. For every compact $K \subset X_{\mathrm{reg}}$, $\widehat{g}_\epsilon|_K = g_X|_K$ for all sufficiently small ϵ. Hence $g_\epsilon|_K = \Lambda^2 c_\epsilon^2 g_X|_K$ converges in C^∞ to $\Lambda^2 g_X|_K$. Choose any sequence $\epsilon_i \downarrow 0$. □

Proposition 8.1. *For the sequence in Theorem 1.2, choose a basepoint p in the interior of Z. For every $r > 0$,*

$$N_r(M, g_i) = D_\infty$$

for all sufficiently large i. At every basepoint the subgroup generated by loops of length less than the constant η of Lemma 2.2 has uniformly bounded finite order.

Proof. Let γ generate $\pi_1(\mathbb{RP}^2)$ in either quotient zero section, and let ℓ_0 be its length for the fixed metric h_0 of Proposition 7.2. Equations (7.14) and (8.5) give

$$L_{g_i}(\gamma) = \Lambda c_{\epsilon_i} \epsilon_i \ell_0 \longrightarrow 0.$$

Denote these local loops by γ_1, γ_2. Let α_j join p to the basepoint of γ_j, using the fixed exterior paths and radial paths in the fillings. Then

$$[\alpha_1 * \gamma_1 * \alpha_1^{-1}] = a, \qquad [\alpha_2 * \gamma_2 * \alpha_2^{-1}] = b.$$

For all sufficiently large i, $L_{g_i}(\gamma_j) < r$ for $j = 1, 2$. The definition of N_r imposes no length bound on α_j. It follows that

$$D_\infty = \langle\!\langle a, b\rangle\!\rangle_{D_\infty} \subset N_r(M, g_i) \subset D_\infty.$$

The last assertion is Lemma 2.2, applied to (1.2). □

The product ab has infinite order. Thus a kernel normally generated by finite-order elements need not be a torsion group.

9. Diagonal involutions and nonnegative Ricci curvature

Proposition 9.1. *Let A_i be closed connected smooth m-manifolds such that*

$$\mathrm{Ric}_{A_i} \ge 0, \qquad \operatorname{diam}(A_i) \le D, \qquad \operatorname{Vol}(A_i) \ge v > 0, \qquad \pi_1(A_i) = C_2.$$

Suppose $A_i \to A$ in Gromov–Hausdorff distance and A is simply connected. Then, after passage to a subsequence, there are closed smooth $(m+1)$-manifolds P_i with

$$\mathrm{Ric}_{P_i} \ge 0, \quad \operatorname{diam}(P_i) \le 3D + 1, \quad \operatorname{Vol}(P_i) \ge 2v, \quad \pi_1(P_i) = D_\infty, \tag{9.1}$$

which converge to a simply connected compact length space.

Proof. Let $q_i : \widetilde{A}_i \to A_i$ be the universal double covering, with involution τ_i. Fix $u \in \widetilde{A}_i$ and let w be a midpoint of a minimizing segment from u to $\tau_i u$, of length l. Isometry of τ_i gives $d(u, w) = d(u, \tau_i w) = l/2$. Lifting and projecting minimizing paths gives the quotient-distance formula, and hence

$$\frac{l}{2} = \min\{d(u, w), d(u, \tau_i w)\} = d_{A_i}(q_i(u), q_i(w)) \le D.$$

For arbitrary $u, v \in \widetilde{A}_i$, a minimizing path between their projections lifts from u to either v or $\tau_i v$ and has length at most D. In the second case, $d(u, v) \le d(u, \tau_i v) + d(\tau_i v, v) \le D + 2D$. Consequently

$$\operatorname{diam}(\widetilde{A}_i) \le 3D. \tag{9.2}$$

The covers have volume $2\operatorname{Vol}(A_i)$ and Ricci curvature at least zero. For $0 < r \le 1$, a disjoint family of balls of radius $r/2$ contains at most $(2(3D+1)/r)^m$ members. Indeed, Bishop–Gromov comparison bounds the volume of each ball below by $(r/(2(3D+1)))^m \operatorname{Vol}(\widetilde{A}_i)$. A maximal r-separated set therefore has uniformly bounded cardinality and its radius-r balls cover $\widetilde{A}_i$. Gromov compactness gives a subsequence converging to a compact metric space Y. Choose maps $a_i : \widetilde{A}_i \to Y$ of distortion and density error $\zeta_i \to 0$. Let $\{y_j\}_{j\ge 1}$ be dense in Y, and choose $x_{i,j}$ with $a_i x_{i,j} \to y_j$. A diagonal subsequence satisfies $a_i \tau_i x_{i,j} \to z_j$ for every j. Isometry of τ_i gives

$$d_Y(z_j, z_k) = \lim_i d_{\widetilde{A}_i}(\tau_i x_{i,j}, \tau_i x_{i,k}) = d_Y(y_j, y_k).$$

Hence $y_j \mapsto z_j$ extends to an isometric map $\tau : Y \to Y$. Given $b > 0$, choose finitely many y_j forming a b-net in Y. For $x \in \widetilde{A}_i$ choose one of them with $d_Y(a_i x, y_j) < b$. The triangle inequality gives

$$\begin{aligned} d_Y(a_i\tau_i x, \tau a_i x) &\le d_{\widetilde{A}_i}(x, x_{i,j}) + \zeta_i + d_Y(a_i\tau_i x_{i,j}, z_j) + d_Y(y_j, a_i x) \\ &\le 2b + 2\zeta_i + d_Y(a_i x_{i,j}, y_j) + d_Y(a_i \tau_i x_{i,j}, z_j). \end{aligned}$$

The last two errors tend to zero uniformly over the finite net. Taking the supremum over x, then $\limsup_i$, and then $b \downarrow 0$ proves

$$E_i := \sup_x d_Y(a_i\tau_i x, \tau a_i x) \longrightarrow 0.$$

Since $\tau_i^2 = \mathrm{Id}$, $d_Y(a_i x, \tau^2 a_i x) \le 2E_i$. For $y \in Y$, choose x_i with $d_Y(a_i x_i, y) \le \zeta_i$. Then

$$d_Y(y, \tau^2 y) \le 2\zeta_i + 2E_i \longrightarrow 0.$$

Thus $\tau^2 = \mathrm{Id}$. In particular τ is onto. Let $q : Y \to Y/\langle\tau\rangle$ be the quotient map. The relation

$$\mathcal{D}_i = \{(q_i(x), q(a_i(x))) : x \in \widetilde{A}_i\}$$

contains a pair for every representative $x \in \widetilde{A}_i$. The identities

$$\begin{aligned} d_{A_i}(q_i(x), q_i(x')) &= \min\{d(x,x'), d(x, \tau_i x')\}, \\ d_{Y/\langle\tau\rangle}(q(a_i(x)), q(a_i(x'))) &= \min\{d_Y(a_i(x), a_i(x')), d_Y(a_i(x), \tau a_i(x'))\} \end{aligned}$$

give $\operatorname{dis}\mathcal{D}_i \le \zeta_i + E_i$. Its first projection is onto and its second is ζ_i-dense. Define

$$\widehat{\mathcal{D}}_i = \{(u,v) : \text{there is } (u,w) \in \mathcal{D}_i \text{ with } d(v,w) \le \zeta_i\}.$$

Both projections of $\widehat{\mathcal{D}}_i$ are onto, and the triangle inequality gives $\operatorname{dis}\widehat{\mathcal{D}}_i \le \operatorname{dis}\mathcal{D}_i + 2\zeta_i \le 3\zeta_i + E_i$. Thus $A_i \to Y/\langle\tau\rangle$. Uniqueness of the compact Gromov–Hausdorff limit gives $A \cong Y/\langle\tau\rangle$ isometrically. No freeness or effectiveness of τ is used. For $y, z \in Y$, choose approximating points y_i, z_i, and let m_i be a midpoint of a minimizing segment joining them. A subsequence of $a_i(m_i)$ converges to $m \in Y$, and the distortion estimate gives $d_Y(y,m) = d_Y(m,z) = d_Y(y,z)/2$. Thus every pair has a midpoint. Put $l = d_Y(y,z)$. If $l = 0$, use the constant path. Otherwise successive midpoint subdivisions define γ at dyadic parameters in $[0,1]$. For dyadic $s < t$, subdivision and the triangle inequality give

$$d_Y(\gamma(s), \gamma(t)) \le l(t-s), \qquad l \le ls + d_Y(\gamma(s), \gamma(t)) + l(1-t).$$

Thus $d_Y(\gamma(s), \gamma(t)) = l|t-s|$. Completeness extends γ to a minimizing segment on $[0,1]$. In particular, Y is path connected.

Let $S^1_2 = \mathbb{R}/2\mathbb{Z}$ be the circle of length two and set

$$P_i = (\widetilde{A}_i \times S^1_2)/\langle (u,t) \mapsto (\tau_i u, -t)\rangle. \tag{9.3}$$

This is a free isometric involution, even at the fixed points of circle reflection, because τ_i has no fixed point. The quotient is a closed smooth manifold. For the product metric,

$$\operatorname{Ric}_{\widetilde{A}_i \times S^1_2}((v,a),(v,a)) = \operatorname{Ric}_{\widetilde{A}_i}(v,v) \ge 0.$$

The quotient map is a local isometry of degree two. Consequently,

$$\operatorname{Ric}_{P_i} \ge 0, \qquad \operatorname{diam}(P_i) \le \sqrt{(3D)^2 + 1} \le 3D + 1, \qquad \operatorname{Vol}(P_i) = \frac{2\operatorname{Vol}(\widetilde{A}_i)}{2} = 2\operatorname{Vol}(A_i).$$

The simply connected manifold $\widetilde{A_i} \times \mathbb{R}$ covers P_i. Its full deck group is generated by

$$T(u,t) = (u, t+2), \qquad R(u,t) = (\tau_i u, -t).$$

It has the relations $R^2 = 1$ and $RTR = T^{-1}$. Every element has a unique form T^k or $T^k R$; equality of two such maps is excluded by their actions on the real coordinate, and equality within a form forces the same integer. Each T^k with $k \neq 0$ acts freely, and $T^k R$ acts freely because of its τ_i component. If a compact set K has real-coordinate projection in $[-B, B]$, then either $T^k K \cap K \neq \emptyset$ or $T^k RK \cap K \neq \emptyset$ implies $|2k| \leq 2B$. Only finitely many group elements meet this condition, so the action is properly discontinuous. These maps give the full equivalence relation defining (9.3). Thus their group is $\pi_1(P_i) = D_\infty$.

Choose approximation maps $u_i : \widetilde{A_i} \to Y$ with metric, density and equivariance errors at most $\eta_i \to 0$; thus $d_Y(u_i \tau_i x, \tau u_i x) \leq \eta_i$. The quotient distance is

$$\min\left\{ \sqrt{d_{\widetilde{A_i}}(x,x')^2 + d_{S^1_2}(t,t')^2}, \sqrt{d_{\widetilde{A_i}}(x,\tau_i x')^2 + d_{S^1_2}(t,-t')^2} \right\}.$$

Replacing x, x' by $u_i x, u_i x'$ changes the first term by at most η_i and the second by at most $2\eta_i$. Here $|\sqrt{a^2+c^2} - \sqrt{b^2+c^2}| \leq |a-b|$ for $a,b,c \geq 0$. The relation pairing $[x,t]$ with $[u_i x, t]$ therefore has distortion at most $2\eta_i$ and η_i-dense target projection. Extending that projection to all target points increases its distortion by at most $2\eta_i$. Hence

$$P_i \longrightarrow P := (Y \times S^1_2)/\langle (y,t) \mapsto (\tau y, -t) \rangle.$$

The product $Y \times S^1_2$ is geodesic. Indeed, constant-speed minimizing segments in the two factors, parametrized on $[0,1]$, give a minimizing segment for the product distance. For two orbits of the finite isometric action, choose representatives realizing the minimum in the quotient-distance formula. Projection of a minimizing segment between them has length at most that minimum and at least the distance of its endpoints. The two lengths are equal. Thus P is a compact geodesic space.

The compact space P is homeomorphic to

$$(Y \times [0,1])/\big((y,0) \sim (\tau y, 0),\ (y,1) \sim (\tau y, 1)\big). \tag{9.4}$$

The map $Y \times [0,1] \to P$ given by $(y,t) \mapsto [y,t]$ is continuous and onto. If $0 < t < 1$, equality $[y,t] = [y',t']$ with $0 \leq t' \leq 1$ forces $(y',t') = (y,t)$. At $t = 0$ or $t = 1$, it identifies precisely y and τy. It therefore factors through a continuous bijection from the quotient in (9.4) to P. The domain is compact and P is Hausdorff, so this bijection is a homeomorphism.

The coordinate t descends continuously to (9.4). Its inverse images $U_0 = \{t < 2/3\}$ and $U_1 = \{t > 1/3\}$ are open. The maps $[y,t] \mapsto [y, (1-s)t]$ on U_0 and $[y,t] \mapsto [y, 1-(1-s)(1-t)]$ on U_1 are well-defined deformation retractions onto the endpoint copies of $Y/\langle \tau \rangle = A$. Moreover $U_0 \cap U_1 = Y \times (1/3, 2/3)$ is path connected. Van Kampen gives

$$\pi_1(P) = \pi_1(U_0) *_{\pi_1(U_0 \cap U_1)} \pi_1(U_1) = 1 *_{\pi_1(Y)} 1 = 1.$$

□

Proof of Theorem 1.3. Use Brena's sequence with fundamental group C_2 [3, unnumbered main theorem and its proof]. It consists of closed four-manifolds with nonnegative Ricci curvature, uniformly bounded diameters and uniformly positive volumes. The proof identifies its limit as the spherical suspension of a connected scaled round lens space. Let L denote that connected lens space and U_+, U_- the open cone subsets of its suspension. Radial contraction gives $\pi_1(U_+) = \pi_1(U_-) = 1$, while $U_+ \cap U_- \cong L \times (-1,1)$ is path connected. Van Kampen gives

$$\pi_1(\operatorname{Susp} L) = 1 *_{\pi_1(L)} 1 = 1.$$

Proposition 9.1 applies with $m = 4$. □

The limit in (9.4) is computed using only the path connectivity of Y and the simple connectivity of $Y/\langle \tau \rangle$. No simple-connectivity assumption on Y is used.

10. Two resolutions of one orbifold

Let V act on the binary Cantor space by finite prefix replacements. We use its finite presentability and simplicity as recalled in [2]. Infinitude also follows from the subgroups $(C_2)^m$ constructed below.

Lemma 10.1. *The group V contains an involution $\vartheta \neq 1$, and $\langle\!\langle \vartheta \rangle\!\rangle_{\mathsf{V}} = \mathsf{V}$. Every finite-dimensional complex representation of V is trivial. The group has no proper subgroup of finite index.*

Proof. For a finite binary word a, let $[a]$ be its cylinder in $\{0,1\}^{\mathbb{N}}$. If $[a] \cap [b] = \emptyset$, define

$$\tau_{a,b}(a\xi) = b\xi, \qquad \tau_{a,b}(b\xi) = a\xi, \qquad \tau_{a,b}(x) = x \quad (x \notin [a] \cup [b]).$$

Set $N = \max\{|a|, |b|\}$. The complement of $[a] \cup [b]$ is the disjoint union of those length-N cylinders which are disjoint from $[a]$ and $[b]$. Together with $[a]$ and $[b]$, they form a finite prefix partition. On this partition $\tau_{a,b}$ is a prefix substitution; hence $\tau_{a,b} \in \mathsf{V}$. Its square is the identity and it interchanges the two nonempty cylinders, so it has order two. In particular, $\vartheta = \tau_{0,1}$ is a nonidentity involution. Given $m \geq 1$, choose N with $2^N \geq 2m$ and distinct words $a_1, b_1, \dots, a_m, b_m$ of length N. The involutions τ_{a_j,b_j} have disjoint supports. They commute. If $J \subset \{1,\dots,m\}$ is nonempty and $j \in J$, then $\prod_{k\in J} \tau_{a_k,b_k}$ sends $[a_j]$ to $[b_j]$, so this product is not the identity. Thus the homomorphism

$$(C_2)^m \longrightarrow \mathsf{V}, \qquad (e_1,\dots,e_m) \longmapsto \prod_{j=1}^{m} \tau_{a_j,b_j}^{e_j}$$

is injective.

Simplicity gives $\langle\!\langle \vartheta \rangle\!\rangle_{\mathsf{V}} = \mathsf{V}$ for every $\vartheta \neq 1$. Suppose $\rho : \mathsf{V} \to \mathrm{GL}_r(\mathbb{C})$ is nontrivial. Its kernel is a normal subgroup of V and is not V. Simplicity therefore gives $\ker\rho = 1$. Restriction to $(C_2)^{r+1}$ gives an injection into $\mathrm{GL}_r(\mathbb{C})$. Each image involution is diagonalizable, since its minimal polynomial divides $(T-1)(T+1)$. Let $T_1,\dots,T_{r+1}$ be the commuting image involutions. For $\epsilon \in \{1,-1\}^{r+1}$, set

$$P_\epsilon = \prod_{j=1}^{r+1} \frac{I + \epsilon_j T_j}{2}.$$

The identities $T_j^2 = I$ give $P_\epsilon^2 = P_\epsilon$, $P_\epsilon P_{\epsilon'} = 0$ for $\epsilon \neq \epsilon'$, and $\sum_\epsilon P_\epsilon = I$. On $\operatorname{im} P_\epsilon$, each T_j is multiplication by ϵ_j. The direct sum of these images is $\mathbb{C}^r$; choosing bases in the nonzero summands gives simultaneous diagonalizability. In a simultaneous eigenbasis, the image is contained in $\{\mathrm{diag}(\epsilon_1,\dots,\epsilon_r) : \epsilon_j \in \{1,-1\}\}$, which has order 2^r. An injective homomorphism from $(C_2)^{r+1}$ would give $2^{r+1} \leq 2^r$, a contradiction. Thus every such ρ is trivial.

If $H \leq \mathsf{V}$ has finite index d, the action on V/H defines a homomorphism $\mathsf{V} \to S_d \hookrightarrow \mathrm{GL}_d(\mathbb{C})$. It is trivial. Since the action is transitive, $d = 1$. □

Proposition 10.2. *There exist fixed closed oriented smooth four-manifolds M, N, a compact simply connected Riemannian orbifold X, and metrics g_i, h_i satisfying the common geometric bounds in Corollary 1.4, such that $\pi_1(M) = \mathsf{V}$, $\pi_1(N) = 1$ and both sequences converge to X. The space X has one singular point p, with round link isometric to $L(4,1)$. There are continuous Gromov–Hausdorff approximation maps $f_i : (M, g_i) \to X$ and loops γ_i based at x_i such that*

$$[\gamma_i] \neq 1, \qquad L_{g_i}(\gamma_i) \to 0, \qquad f_i(x_i) \to p.$$

Conversely, every sequence of noncontractible based loops of lengths tending to zero has basepoints whose images under f_i tend to p.

Proof. Fix an involution $\vartheta \in \mathsf{V} \setminus \{1\}$. Lemma 5.1, using the explicit W of Lemma 6.1, gives a compact oriented four-manifold Z with boundary $-L$, fundamental group V, and boundary homomorphism sending a generator of C_4 to ϑ. Give Z a fixed metric with an exact flat-cone collar. Put

$$M = Z \cup_L W, \qquad X = Z \cup_L \mathrm{C}_{\leq 1}(L). \tag{10.1}$$

The pushout calculation in (5.2) gives $\pi_1(M) = \mathsf{V}$ and $\pi_1(X) = \mathsf{V}/\langle\!\langle \vartheta \rangle\!\rangle = 1$. Use the metrics of Proposition 7.2 on the attached W. With one copy of (W, k_ϵ), equations (7.11)–(7.12) and Lemma 4.1

give the Ricci, diameter and energy bounds, volume convergence, and continuous approximation maps before normalization.

There is also a simply connected Ricci-flat ALE four-manifold of type A_3, obtained from the four-centre hyperkähler construction recalled in [11, Section 3]. Its end is the round quotient of S^3 by the cyclic action

$$(z_1, z_2) \longmapsto (\mathrm{i}z_1, -\mathrm{i}z_2).$$

The real orthogonal map $(z_1, z_2) \mapsto (z_1, \overline{z}_2)$ conjugates this action to $(z_1, z_2) \mapsto (\mathrm{i}z_1, \mathrm{i}z_2)$. The resulting round quotient is isometric to L. Choose a compact core A of this ALE manifold and reverse its orientation if necessary, so that the chosen boundary isometry is orientation reversing for the gluing to Z. Put

$$N = Z \cup_L A. \tag{10.2}$$

The end retraction gives $\pi_1(A) = 1$. Van Kampen therefore gives $\pi_1(N) = \mathsf{V}/\langle\!\langle \vartheta \rangle\!\rangle = 1$.

The last assertion of Lemma 4.4 applies to A: its metric is Ricci-flat, its asymptotic cone is the flat cone over L, and [7, Theorem 1.16] gives the differentiated decay (4.4). The curvature-energy estimate in Lemma 4.3 is finite. Thus N has metrics converging to the same metric on X. The two unnormalized volumes tend to $V_\infty = \mathcal{H}^4(X) > 0$. Normalize each sequence to V_∞ and choose one constant Λ large enough for both Ricci bounds. Their final common volume is $\Lambda^4 V_\infty$, and both limits have distance Λ times the original distance of X. Take the maxima of the two diameter and energy bounds.

Fix a generator loop in the zero section of W. Its image in $\pi_1(M)$ is $\vartheta \neq 1$, and its length is at most $C\epsilon_i$ after normalization. The map f_i sends its basepoint to p. This proves the existence assertion.

Let $K \subset X \setminus \{p\}$ be nonempty and compact. Choose a finite family of coordinate balls $U_a \Subset X \setminus \{p\}$ and nonempty open sets $V_a \Subset U_a$, $1 \le a \le m$, such that $K \subset \bigcup_{a=1}^m V_a$. For all sufficiently large i, their closures are identified with subsets of M on which $g_i = c_i^2 k_X$, where $c_i \to 1$ and k_X is the final limit metric. The replaced region has image contained in $B_{3\Lambda\sqrt{\epsilon_i}}(p)$. For sufficiently large i, $3\Lambda\sqrt{\epsilon_i} < d_X(p, K)$; therefore $f_i^{-1}(K)$ is the identified copy of K. Put

$$d_a = d_X(\overline{V_a}, X \setminus U_a) > 0, \qquad \ell_K = \tfrac{1}{2} \min_a d_a > 0.$$

Positivity follows from compactness of $\overline{V_a}$ and the inclusion $\overline{V_a} \subset U_a$. Take i large enough that $c_i \ge 1/2$. Suppose a rectifiable loop γ based in V_a leaves U_a, and let t_0 be its first exit time. Its initial segment lies in $\overline{U_a}$ and satisfies

$$L_{g_i}(\gamma) \ge L_{g_i}(\gamma|_{[0,t_0]}) = c_i L_{k_X}(\gamma|_{[0,t_0]}) \ge c_i d_X(\gamma(0), \gamma(t_0)) \ge c_i d_a \ge \ell_K.$$

Every loop based in V_a of length less than ℓ_K therefore lies in the contractible set U_a. Hence

$$\inf\{s_{g_i}(x) \,:\, f_i(x) \in K\} \ge \ell_K \qquad (i \text{ sufficiently large}).$$

If $f_i(x_i)$ failed to converge to p, some subsequence would belong to $K = X \setminus B_\rho(p)$ for a fixed $\rho > 0$. The last inequality would contradict $L_{g_i}(\gamma_i) \to 0$ for noncontractible γ_i based at x_i. □

Proposition 10.3. *For the metrics in Proposition 10.2, every fixed scalar eigenvalue, counted with multiplicity starting at $\lambda_0 = 0$, satisfies*

$$\lim_{i\to\infty} \lambda_j(M, g_i) = \lambda_j(X) = \lim_{i\to\infty} \lambda_j(N, h_i), \qquad j \ge 0. \tag{10.3}$$

The operator on X is associated with the closed quadratic form $\mathcal{E}_X = 2\,\mathrm{Ch}_X$ on $L^2(X, \mathcal{H}^4)$. Here the Cheeger energy is

$$\mathrm{Ch}_X(f) = \frac{1}{2} \inf_{f_k} \liminf_{k\to\infty} \int_X (\mathrm{lip}\, f_k)^2 \, d\mathcal{H}^4,$$

where f_k ranges over real-valued Lipschitz functions converging to f in L^2, and

$$\mathrm{lip}\, f_k(x) = \limsup_{y\to x,\, y\neq x} \frac{|f_k(y) - f_k(x)|}{d(x,y)}.$$

The complex form is its complexification. Every rank-r Hermitian vector bundle on either resolving manifold, equipped with a flat unitary connection ∇, has connection Laplacian $\nabla^*\nabla$ unitarily equivalent to r copies of the scalar Laplacian.

Proof. Denote either resolving sequence by (Y_i, k_i) and its continuous approximation map by f_i. Let $A_i \subset Y_i$ and $B_i \subset X$ be the replaced regions. Their volumes tend to zero. On their common complement, $k_i = c_i^2 k_X$ with $c_i \to 1$, where k_X includes the common final factor Λ^2. Hence, for every $\phi \in C(X)$,

$$\left|\int_{Y_i} \phi \circ f_i \, dv_{k_i} - \int_X \phi \, d\mathcal{H}^4\right| \le \|\phi\|_\infty \big(\mathrm{Vol}_{k_i}(A_i) + \mathcal{H}^4(B_i) + |c_i^4 - 1|\mathcal{H}^4(X)\big) \longrightarrow 0.$$

Together with the vanishing distortions this gives measured Gromov–Hausdorff convergence. Fix $z_0 \in \operatorname{Int} Z$ and use its identified copy as basepoint in both resolving manifolds and in X. The approximation maps satisfy $f_i(z_0) = z_0$; the measured convergence is therefore pointed for these basepoints. Divide all measures by their common total mass $V_0 > 0$.

The smooth Ricci lower bound gives $\mathrm{RCD}(-3, \infty)$ for each normalized metric-measure space. By [5, Theorem 7.2], the limit has the same condition. The smooth measures have full support. Every nonempty open subset of X meets its smooth locus in an open set of positive volume, so the limiting measure has full support as well. Each L^2 space is infinite dimensional: a smooth open subset contains arbitrarily many disjoint balls of positive measure. Let m_i be the normalized reference measure and let $x_i = z_0$ be the chosen basepoint, also for the limiting space. A common diameter bound D gives, for every $u \in L^2(m_i)$,

$$\left(\int d_i(x, x_i)^2 |u(x)|^2 \, dm_i(x)\right)^{1/2} \le D\|u\|_{L^2(m_i)}.$$

This is exactly condition (6.16) of [5, Definition 6.4] with constants $(D, 0)$. The compact embedding and spectral convergence in [5, Proposition 6.7 and Theorem 7.8] therefore apply. Their form is $2\,\mathrm{Ch}$, and on a smooth manifold

$$2\,\mathrm{Ch}(f) = \int |\nabla f|^2 \, dm.$$

Its associated operator has convention $-\operatorname{div}\nabla$. The first min–max value in that theorem is our $\lambda_0 = 0$. Multiplication of m by V_0 multiplies both terms of the Rayleigh quotient by V_0 and does not change an eigenvalue. This proves (10.3).

Let E be a Hermitian vector bundle equipped with a flat unitary connection ∇. Its holonomy on M is a homomorphism $\mathsf{V} \to \mathrm{U}(r)$, and is trivial by Lemma 10.1. On N it is trivial because N is simply connected. Fix an orthonormal basis at y_0. For any point y and path $c : y_0 \to y$, let $e_a(y)$ be its parallel transport. If c' is another such path, trivial holonomy of $c * (c')^{-1}$ gives the same vector. On a coordinate ball these vectors are obtained by solving the parallel-transport equations along a smooth family of paths; they are smooth. For any path d from y to z, path independence applied to $c * d$ gives $P_d e_a(y) = e_a(z)$. Thus $\nabla e_a = 0$ and $\langle e_a, e_b\rangle = \delta_{ab}$. On either manifold Y, the map

$$U : L^2(Y)^r \longrightarrow L^2(Y, E), \qquad U(f_1, \dots, f_r) = \sum_{a=1}^r f_a e_a$$

is unitary. For smooth f_a,

$$\nabla U(f_1, \dots, f_r) = \sum_{a=1}^r df_a \otimes e_a, \qquad \int_Y |\nabla U(f)|^2 \, dv = \sum_{a=1}^r \int_Y |df_a|^2 \, dv.$$

Since U and U^{-1} preserve the L^2 norm and the displayed energy, they extend to inverse isometries between $W^{1,2}(Y)^r$ and the form domain of $\nabla^*\nabla$. The representation theorem for closed nonnegative forms therefore gives

$$U^{-1}(\nabla^*\nabla)U = \bigoplus_{a=1}^r \Delta_Y, \qquad D(\nabla^*\nabla) = U\big(D(\Delta_Y)^r\big).$$

□

Proof of Corollary 1.4. Proposition 10.2 gives the two manifolds, the common limit, and the geometric bounds. Lemma 10.1 gives triviality of every finite-dimensional complex representation of $\pi_1(M) = \mathsf{V}$. A connected finite-sheeted covering corresponds to a finite-index subgroup of the fundamental group. The only such subgroup of V is V itself, and the trivial group has only itself as a subgroup. Thus every connected finite-sheeted covering of either manifold has degree one. Proposition 10.3 gives the stated eigenvalue convergence. □

Equation (10.3) is coordinatewise convergence. It asserts neither exact isospectrality for a finite i nor a uniform comparison over all eigenvalue indices. It does not compare the orbifold fundamental group of X with the groups of the resolutions.

11. A heat integral and the homotopy systole

Let $n \geq 2$ and let (M^n, g) be closed and connected. Let $\pi : (\widetilde{M}, \widetilde{g}) \to (M, g)$ be its universal Riemannian covering. Write $p_M(t, x, y)$ and $p_{\widetilde{M}}(t, u, w)$ for the heat kernels of $e^{-t\Delta}$, with $\Delta = -\operatorname{div}\nabla$. Deck transformations preserve the heat kernel, so $p_{\widetilde{M}}(t, \widetilde{x}, \widetilde{x})$ is independent of $\widetilde{x} \in \pi^{-1}(x)$. Put

$$Q_M(t, x) = p_M(t, x, x) - p_{\widetilde{M}}(t, \widetilde{x}, \widetilde{x}), \tag{11.1}$$

$$H_M(t) = \int_M Q_M(t, x)\, dv_g(x), \tag{11.2}$$

$$\mathcal{U}_\mu(M, g) = \int_0^\infty e^{-\mu t} H_M(t)\, \frac{dt}{t}, \qquad \mu > 0. \tag{11.3}$$

For an infinite fundamental group, the second diagonal integral is an integral over one fundamental domain, not a trace on the noncompact universal cover. No ordinary zeta determinant on that cover is asserted.

Put

$$\operatorname{sys}_1(M, g) = \inf_{x \in M} s_g(x) = \inf\{L_g(\gamma) : \gamma \text{ is a noncontractible rectifiable loop}\}. \tag{11.4}$$

The infimum of the empty set is $+\infty$. If $\pi_1(M) \neq 1$, then s_g is positive and 2-Lipschitz by the discussion of (2.2); compactness gives a point where its positive minimum is attained. The deck-displacement infimum at that point is also attained. Thus a noncontractible loop of length $\operatorname{sys}_1(M, g)$ exists.

Lemma 11.1. *For a fixed closed (M^n, g), the quantity $\mathcal{U}_\mu(M, g)$ is finite and nonnegative. On the class*

$$\operatorname{Ric}_g \geq -(n-1)g, \qquad \operatorname{diam}(M, g) \leq D, \qquad \operatorname{Vol}(M, g) \geq v > 0, \tag{11.5}$$

there are positive constants depending only on n, D, v such that

$$0 \leq Q_M(t, x) \leq C t^{-n/2} \exp\left(-\frac{s_g(x)^2}{Ct}\right), \qquad 0 < t \leq 1. \tag{11.6}$$

When $s_g(x) = +\infty$, the right side is interpreted as zero. If a nonidentity deck transformation moves a lift of x by at most $0 < \ell < 1$, then

$$H_M(t) \geq c \qquad (\ell^2 \leq t \leq 1). \tag{11.7}$$

Proof. *(a) Gaussian bounds.* Assume first (11.5). The comparison used in (2.5) gives, at every basepoint, $\operatorname{Vol}(B_r(x)) \geq v_* r^n$ for $0 < r \leq 1$. If $u \in \widetilde{M}$ and $x = \pi(u)$, lifting minimizing segments from x gives $\pi(B_r(u)) = B_r(x)$. The area formula for the local isometry, whose multiplicity is at least one on $B_r(x)$, gives $\operatorname{Vol}(B_r(u)) \geq \operatorname{Vol}(B_r(x))$. Bishop comparison gives the upper bound. Thus

$$v_* r^n \leq \operatorname{Vol}(B_r(u)) \leq C r^n \qquad (u \in \widetilde{M},\ 0 < r \leq 1).$$

The complete smooth covering satisfies $\mathrm{RCD}^*(-(n-1), n)$. Theorem 1.2 of [6], with $K = -(n-1)$, $N = n$ and Gaussian parameter 1, gives constants $C_1, C_2 > 0$ depending only on n such that

$$\frac{\exp(-d(u, w)^2/(3t) - C_2 t)}{C_1 \operatorname{Vol}(B_{\sqrt{t}}(w))} \leq p_{\widetilde{M}}(t, u, w) \leq \frac{C_1 \exp(-d(u, w)^2/(5t) + C_2 t)}{\operatorname{Vol}(B_{\sqrt{t}}(w))}.$$

Combining this inequality with the preceding volume bounds gives, for $0 < t \le 1$,

$$p_{\widetilde{M}}(t,u,w) \le Ct^{-n/2}e^{-d(u,w)^2/(Ct)}, \qquad d(u,w) \le 3\sqrt{t} \implies p_{\widetilde{M}}(t,u,w) \ge ct^{-n/2}. \tag{11.8}$$

On $0 < t \le 1$, the factors from the cited theorem satisfy $e^{-C} \le e^{-Ct} \le 1 \le e^{Ct} \le e^{C}$. Consequently the constants in (11.8) depend only on n, D, v.

The mass-preservation identity preceding [6, Theorem 3.1], applied to nonnegative functions in $L^1 \cap L^2$, and symmetry of the kernel give

$$\int_{\widetilde{M}} p_{\widetilde{M}}(t,u,w)\, dv_{\widetilde{g}}(w) = 1 \qquad (t > 0).$$

Write the left side as $S_t(u)$. Mass preservation and symmetry give, for every nonnegative $\phi \in L^1 \cap L^2$,

$$\int_{\widetilde{M}} \phi(u) S_t(u)\, dv(u) = \int_{\widetilde{M}} e^{-t\Delta_{\widetilde{M}}}\phi\, dv = \int_{\widetilde{M}} \phi\, dv.$$

Tonelli's theorem therefore gives $S_t = 1$ almost everywhere. Fix $t > 0$, $o \in \widetilde{M}$ and $b > 0$. The Gaussian bound and Bishop comparison imply, for $R > b$,

$$\sup_{u \in B_b(o)} \int_{\widetilde{M} \setminus B_R(o)} p_{\widetilde{M}}(t,u,w)\, dv(w) \le C_t e^{-(R-b)^2/(2C_t)} \sum_{j=0}^{\infty} \mathsf{V}_n(j+1) e^{-j^2/(2C_t)} \longrightarrow 0.$$

The series converges because $\mathsf{V}_n(j+1) \le C_n e^{(n-1)(j+1)}$. The integral over $B_R(o)$ is continuous in u by continuity of the kernel on compact sets. Hence S_t is continuous. Full support of Riemannian measure gives $S_t(u) = 1$ at every u.

(b) Periodization. Let Γ be the deck group. Isometric invariance of the minimal heat kernel gives

$$p_{\widetilde{M}}(t, \gamma u, \gamma w) = p_{\widetilde{M}}(t,u,w) \quad (\gamma \in \Gamma).$$

Thus its diagonal is independent of the chosen lift. For each x, choose a lift $\widetilde{x}$ and put $L_\gamma(x) = d(\widetilde{x}, \gamma\widetilde{x})$. Replacing $\widetilde{x}$ by $\delta\widetilde{x}$ replaces the index γ by $\delta^{-1}\gamma\delta$; hence the multiset of displacements, its counting function and every sum used below are independent of this choice. These assertions concern the pointwise sums and do not require a continuous section of the covering. Lemma 2.1 and the unit-ball lower bound give

$$\#\{\gamma : L_\gamma(x) \le R\} \le Ae^{B(R+1)} \qquad (R \ge 0). \tag{11.9}$$

We prove the identity

$$p_M(t,x,y) = \sum_{\gamma \in \Gamma} p_{\widetilde{M}}(t, \widetilde{x}, \gamma\widetilde{y}). \tag{11.10}$$

For a fixed $o \in \widetilde{M}$, $\Gamma = \bigcup_{m \ge 1}\{\gamma : d(o, \gamma o) \le m\}$ is countable by (11.9). Changing the lifts to $\delta\widetilde{x}, \eta\widetilde{y}$ reindexes the sum by the bijection $\gamma \mapsto \delta^{-1}\gamma\eta$. Thus any convergent sum in (11.10) is independent of both lifts. Let F_0 be the measurable section obtained by lifting minimizing paths from a fixed basepoint outside its cut locus. The translates γF_0, $\gamma \in \Gamma$, are pairwise disjoint and cover the inverse image of the complement of that cut locus. The omitted set is $\pi^{-1}(\mathrm{Cut}(x_0))$, where x_0 is the chosen basepoint. It has measure zero, since the cut locus has measure zero and a countable atlas of local isometries covers $\widetilde{M}$. Thus F_0 is a fundamental domain up to a null set. For each compact interval $[\tau, T] \subset (0, \infty)$, the Gaussian upper estimate and the positive lower bound for $\mathrm{Vol}(B_{\min\{\sqrt{\tau},1\}}(u))$ give

$$p_{\widetilde{M}}(t,u,w) \le C_{\tau,T} \exp(-d(u,w)^2/C_{\tau,T}) \quad (\tau \le t \le T).$$

This estimate, (11.9), and

$$d(\widetilde{x}, \gamma\widetilde{y}) \ge d(\widetilde{x}, \gamma\widetilde{x}) - d(\widetilde{x}, \widetilde{y})$$

give the following locally uniform majorant. Fix $o \in \widetilde{M}$ and restrict $\widetilde{x}, \widetilde{y}$ to a compact subset of $B_b(o)$. Then

$$d(\widetilde{x}, \gamma\widetilde{y}) \ge d(o, \gamma o) - 2b.$$

On a compact positive-time interval, the sum over $j \le d(o, \gamma o) < j+1$ is bounded, for all sufficiently large j, by $C\exp(B(j+2) - (j-2b)^2/C)$. These bounds are summable andindependent of $\widetilde{x}, \widetilde{y}$. The

Weierstrass test therefore gives absolute locally uniform convergence. For real-valued $f \in C(M)$, Tonelli's theorem for $|f|$ and the change of variable $w = \gamma v$ give

$$\int_{F_0} \sum_{\gamma \in \Gamma} p_{\widetilde{M}}(t, \widetilde{x}, \gamma v) f(\pi v)\, dv = \int_{\widetilde{M}} p_{\widetilde{M}}(t, \widetilde{x}, w) f(\pi w)\, dw.$$

Put $F = f \circ \pi$. This function is bounded and uniformly continuous, since f is uniformly continuous on compact M and π is 1-Lipschitz. Let

$$P_t F(u) = \int_{\widetilde{M}} p_{\widetilde{M}}(t, u, w) F(w)\, dv_{\widetilde{g}}(w), \qquad \omega_F(\delta) = \sup_{d(u,w) \le \delta} |F(u) - F(w)|.$$

Conservativeness gives $|P_t F| \le \|F\|_\infty$ and, for $0 < \delta \le 1$,

$$|P_t F(u) - F(u)| \le \omega_F(\delta) + 2\|F\|_\infty \int_{d(u,w) \ge \delta} p_{\widetilde{M}}(t, u, w)\, dv_{\widetilde{g}}(w).$$

For $0 < t \le 1$, splitting the Gaussian exponent and using Bishop's volume bound on the annuli $j \le d(u, w) < j + 1$ gives

$$\begin{aligned}\int_{d(u,w) \ge \delta} p_{\widetilde{M}}(t, u, w)\, dv_{\widetilde{g}}(w) &\le C t^{-n/2} e^{-\delta^2/(2Ct)} \sum_{j=0}^{\infty} \mathsf{V}_n(j+1) e^{-j^2/(2C)} \\ &\le C' t^{-n/2} e^{-\delta^2/(2Ct)} \longrightarrow 0.\end{aligned}$$

Bishop's model volume satisfies $\mathsf{V}_n(j + 1) \le C_n e^{(n-1)(j+1)}$, so the displayed series is bounded by a convergent Gaussian series. For each fixed $\delta > 0$,

$$\limsup_{t \downarrow 0} \|P_t F - F\|_\infty \le \omega_F(\delta).$$

Since $\omega_F(\delta) \to 0$ as $\delta \downarrow 0$, $\|P_t F - F\|_\infty \to 0$.

Fix $o \in \widetilde{M}$. Choose $\chi_R \in C_c^\infty(\widetilde{M})$ with $0 \le \chi_R \le 1$ and $\chi_R = 1$ on $B_R(o)$, and put $F_R = \chi_R F$. The closed ball $\overline{B}_R(o)$ is compact by Hopf–Rinow, so a smooth compactly supported function equal to one there exists. Since F is bounded, $F_R \in L^1 \cap L^2$, and

$$u_R(t, u) := \int_{\widetilde{M}} p_{\widetilde{M}}(t, u, w) F_R(w)\, dv(w) = e^{-t\Delta_{\widetilde{M}}} F_R(u)$$

satisfies $(\partial_t + \Delta_{\widetilde{M}}) u_R = 0$ for $t > 0$. Fix $0 < \tau < T < \infty$ and a compact set $K \subset B_{R_0}(o)$. Theorem 1.2 of [6], the lower ball-volume estimate, and Bishop comparison imply, with $C_{\tau,T}$ independent of R,

$$\begin{aligned}\sup_{[\tau,T] \times K} |u_R - P_t F| &\le C_{\tau,T} \|F\|_\infty e^{-(R-R_0)^2/(2C_{\tau,T})} \sum_{j=0}^{\infty} \mathsf{V}_n(j+1) e^{-j^2/(2C_{\tau,T})} \\ &\longrightarrow 0 \qquad (R \to \infty).\end{aligned}$$

Here $R > R_0$: on $\operatorname{supp}(1 - \chi_R)$ one has $d(u, w) \ge R - R_0$; splitting the Gaussian exponent gives the displayed bound. The series is finite. For every $\psi \in C_c^\infty((0, \infty) \times \widetilde{M})$,

$$\int_0^\infty \int_{\widetilde{M}} u_R(-\partial_t \psi + \Delta_{\widetilde{M}} \psi)\, dv\, dt = 0.$$

Local uniform convergence permits passage to the limit. Thus $(\partial_t + \Delta_{\widetilde{M}}) P_t F = 0$ distributionally. Interior parabolic regularity on the smooth covering makes $P_t F$smooth for $t > 0$. Isometric invariance of the kernel and $F \circ \gamma = F$ imply $P_t F \circ \gamma = P_t F$ for every $\gamma \in \Gamma$. Let $u(t, \cdot)$ be the descended solution on the compact base. For $0 < \tau < t$, the parabolic maximum principle gives

$$\|u(t, \cdot) - e^{-t\Delta_M} f\|_\infty \le \|u(\tau, \cdot) - e^{-\tau\Delta_M} f\|_\infty.$$

Both terms on the right converge uniformly to f as $\tau \downarrow 0$, so $u(t, \cdot) = e^{-t\Delta_M} f$. Equality against every $f \in C(M)$ and continuity of the kernels give (11.10).

(c) Integrability. Equation (11.10) gives

$$Q_M(t,x)=\sum_{\gamma\neq 1} p_{\widetilde{M}}(t,\widetilde{x},\gamma\widetilde{x})\geq 0.$$

Split the exponent in (11.8) into two equal parts. Every nonidentity displacement is at least $s_g(x)$. Summing the remaining factor over integer annuli and using (11.9) gives

$$\sum_{\gamma\neq 1} e^{-L_\gamma(x)^2/(Ct)}\leq e^{-s_g(x)^2/(2Ct)}\sum_{\gamma\in\Gamma} e^{-L_\gamma(x)^2/(2Ct)}\leq Ce^{-s_g(x)^2/(2Ct)}.$$

For $0<t\leq 1$, (11.9) bounds the last series by

$$A\sum_{m=0}^{\infty}\exp\left(B(m+2)-\frac{m^2}{2C}\right).$$

The inequality $Bm\leq m^2/(4C)+CB^2$ bounds this by

$$Ae^{2B+CB^2}\sum_{m=0}^{\infty} e^{-m^2/(4C)}<\infty.$$

This bound is independent of x and t and proves (11.6), after enlarging C.

For a fixed nonsimply connected M, the positive continuous function s_g has a positive minimum ℓ_g. Put $a=\ell_g^2/C>0$. The substitution $u=a/t$ gives

$$\int_0^1 t^{-1-n/2}e^{-a/t}\,dt=a^{-n/2}\int_a^{\infty} u^{n/2-1}e^{-u}\,du<\infty.$$

By (11.6), this bounds the small-time part of (11.3) up to a fixed factor. For $t\geq 1$, the eigenvalues of Δ_M are nonnegative, so

$$0\leq H_M(t)\leq\sum_{j\geq 0} e^{-t\lambda_j(M,g)}\leq \operatorname{Tr}(e^{-\Delta_M})<\infty.$$

It follows that

$$\int_1^{\infty} e^{-\mu t}H_M(t)\,\frac{dt}{t}\leq \operatorname{Tr}(e^{-\Delta_M})\frac{e^{-\mu}}{\mu}<\infty.$$

For $a>0$, the identities $\Delta_{a^2g}=a^{-2}\Delta_g$ and $dv_{a^2g}=a^n\,dv_g$ imply $p_{a^2g}(t,x,y)=a^{-n}p_g(t/a^2,x,y)$ on the base and its covering. Changing variables $t=a^2s$ in the integral gives

$$Q_{a^2g}(t,x)=a^{-n}Q_g(t/a^2,x),\qquad \mathcal{V}_\mu(M,a^2g)=\mathcal{V}_{\mu a^2}(M,g).$$

For an arbitrary fixed closed metric g, put

$$K=\max\{0,-\min_{|v|_g=1}\operatorname{Ric}_g(v,v)\},\qquad a=\max\{1,\sqrt{K/(n-1)}\}.$$

Then

$$\operatorname{Ric}_{a^2g}\geq -(n-1)a^2g,\qquad \operatorname{diam}(M,a^2g)=a\operatorname{diam}(M,g),\qquad \operatorname{Vol}(M,a^2g)=a^n\operatorname{Vol}(M,g)>0.$$

Finiteness for the rescaled metric with parameter μ/a^2 gives $\mathcal{V}_\mu(M,g)=\mathcal{V}_{\mu/a^2}(M,a^2g)<\infty$. If M is simply connected, $Q_M=0$.

(d) A nontrivial deck transformation. Suppose $d(\widetilde{x},\gamma\widetilde{x})\leq\ell<1$ with $\gamma\neq 1$. For $y\in B_{\sqrt{t}}(x)$, lift a minimizing path from x to y, starting at $\widetilde{x}$, and denote its endpoint by $\widetilde{y}$. If $\ell^2\leq t\leq 1$, then

$$d(\widetilde{y},\gamma\widetilde{y})\leq 2\sqrt{t}+\ell\leq 3\sqrt{t}.$$

Since $\gamma\neq 1$, its term occurs in $Q_M(t,y)$. Equations (11.10) and (11.8) give

$$Q_M(t,y)\geq p_{\widetilde{M}}(t,\widetilde{y},\gamma\widetilde{y})\geq ct^{-n/2}\qquad (y\in B_{\sqrt{t}}(x)).$$

Therefore

$$H_M(t)\geq\int_{B_{\sqrt{t}}(x)} Q_M(t,y)\,dv_g(y)\geq ct^{-n/2}\operatorname{Vol}(B_{\sqrt{t}}(x))\geq cv_*,\qquad \ell^2\leq t\leq 1.$$

□

Theorem 11.2 (Systolic criterion). *Fix $n \geq 2$, $D, v, \mu > 0$. On the class of closed n-manifolds satisfying* (11.5)*, there is $c = c(n, D, v) > 0$ such that*

$$\mathcal{U}_\mu(M, g) \geq 2ce^{-\mu} \log \frac{1}{\mathrm{sys}_1(M, g)} \qquad \big(0 < \mathrm{sys}_1(M, g) < 1\big). \tag{11.11}$$

For every $s_0 > 0$, there is $C = C(n, D, v, \mu, s_0) < \infty$ such that

$$\mathrm{sys}_1(M, g) \geq s_0 \quad \Longrightarrow \quad \mathcal{U}_\mu(M, g) \leq C. \tag{11.12}$$

Consequently, for every sequence in this class,

$$\mathcal{U}_\mu(M_i, g_i) \longrightarrow +\infty \quad \Longleftrightarrow \quad \mathrm{sys}_1(M_i, g_i) \longrightarrow 0. \tag{11.13}$$

Proof. Let $\ell = \mathrm{sys}_1(M, g) < 1$. By (11.4), a nonidentity deck transformation has displacement ℓ at some point. Lemma 11.1 gives

$$\mathcal{U}_\mu(M, g) \geq ce^{-\mu} \int_{\ell^2}^1 \frac{dt}{t} = 2ce^{-\mu} \log \frac{1}{\ell}.$$

This proves (11.11).

Suppose $\mathrm{sys}_1(M, g) \geq s_0$. Bishop comparison and $B_{D+1}(x) = M$ give $\mathrm{Vol}(M, g) \leq \mathsf{V}_n(D + 1)$. The upper bound in Lemma 11.1 therefore implies

$$H_M(t) \leq C_0 t^{-n/2} e^{-s_0^2/(C_0 t)} \qquad (0 < t \leq 1),$$

with C_0 depending only on n, D, v. Set $a = s_0^2/C_0 > 0$. Then

$$\int_0^1 e^{-\mu t} H_M(t) \frac{dt}{t} \leq C_0 a^{-n/2} \int_a^\infty u^{n/2-1} e^{-u}\, du.$$

The same heat-kernel upper estimate applied to the base at $t = 1$ gives $p_M(1, x, x) \leq C_1$, since $\mathrm{Vol}(B_1(x)) \geq v_*$. Hence

$$\mathrm{Tr}(e^{-\Delta_M}) = \int_M p_M(1, x, x)\, dv_g(x) \leq C_1 \mathsf{V}_n(D + 1).$$

The nonnegativity of the scalar Laplacian and of Q_M gives

$$\int_1^\infty e^{-\mu t} H_M(t) \frac{dt}{t} \leq C_1 \mathsf{V}_n(D + 1) \int_1^\infty e^{-\mu t}\, dt = C_1 \mathsf{V}_n(D + 1) \frac{e^{-\mu}}{\mu}.$$

These two estimates prove (11.12). They also apply when the manifold is simply connected, in which case $H_M = 0$.

If $\mathrm{sys}_1(M_i, g_i) \to 0$, equation (11.11) gives divergence of $\mathcal{U}_\mu$. Conversely, if the systoles do not tend to zero, there are $s_0 > 0$ and a subsequence on which they are at least s_0. Equation (11.12) bounds $\mathcal{U}_\mu$ on that subsequence, excluding convergence to $+\infty$. □

No Gromov–Hausdorff convergence is assumed in Theorem 11.2. Its assertion holds for every fixed $n \geq 2$; the constants depend on n.

Corollary 11.3 (Kernels and heat divergence). *Under the hypotheses of Proposition 3.2, fix $\mu > 0$. Then*

$$\mathcal{U}_\mu(M_i, g_i) \longrightarrow +\infty \quad \Longleftrightarrow \quad (f_i)_* \text{ is noninjective for all sufficiently large } i.$$

If $(f_i)_*$ is injective for all sufficiently large i, then $\sup_i \mathcal{U}_\mu(M_i, g_i) < \infty$. In particular, in a marked realization satisfying condition (ii) *of Theorem 1.1,*

$$\ker \varphi \neq 1 \implies \mathcal{U}_\mu(M, g_i) \longrightarrow +\infty, \qquad \ker \varphi = 1 \implies \sup_i \mathcal{U}_\mu(M, g_i) < \infty.$$

Proof. Let $r_* > 0$ be the constant in Proposition 3.2. For every fixed $0 < r < r_*$, equation (3.2) gives

$$\ker(f_i)_* = N_{9r}(M_i) \qquad (i \geq i(r)). \tag{11.14}$$

Suppose $(f_i)_*$ is eventually noninjective. Let I satisfy $\ker(f_i)_* \neq 1$ for $i \geq I$. Given $a > 0$, choose $0 < r < \min\{r_*, a/9\}$. For $i \geq \max\{I, i(r)\}$, $N_{9r}(M_i) \neq 1$. If every rectifiable loop β with $L(\beta) < 9r$ were nullhomotopic, every conjugate in the defining family of $N_{9r}(M_i)$ would be the identity. This would give $N_{9r}(M_i) = 1$. Hence such a noncontractible β exists, and $\mathrm{sys}_1(M_i, g_i) \le L(\beta) < 9r < a$. Thus $\mathrm{sys}_1(M_i, g_i) \to 0$, and Theorem 11.2 gives divergence.

Fix one admissible $r_0 > 0$. If $(f_i)_*$ is injective and $i \ge i(r_0)$, then (11.14) gives $N_{9r_0}(M_i) = 1$. For a rectifiable loop β of length less than $9r_0$, choose a path α from the fixed basepoint to $\beta(0)$. Then $[\alpha * \beta * \alpha^{-1}] = 1$; the basepoint-change isomorphism gives $[\beta] = 1$. Thus $\mathrm{sys}_1(M_i, g_i) \ge 9r_0$. Equation (11.12) gives one upper bound for $\mathcal{U}_\mu$ on all such indices. If injectivity holds at infinitely many indices, select an increasing sequence $i_k \ge i(r_0)$ of such indices. The same upper bound applies to every $\mathcal{U}_\mu(M_{i_k}, g_{i_k})$, so the full sequence cannot tend to $+\infty$. If injectivity holds for $i \ge I$, put $J = \max\{I, i(r_0)\}$. With this uniform bound denoted by C,

$$\sup_i \mathcal{U}_\mu(M_i, g_i) \le \max\{C, \mathcal{U}_\mu(M_1, g_1), \dots, \mathcal{U}_\mu(M_{J-1}, g_{J-1})\} < \infty,$$

where the finite maximum is omitted when $J = 1$. Finiteness of each displayed term is Lemma 11.1. The marked-realization assertions follow from $\beta(f_i)_*\alpha^{-1} = \varphi$. □

Corollary 11.4. *For the sequences in Proposition 10.2 and every fixed $\mu > 0$,*

$$\mathcal{U}_\mu(M, g_i) \longrightarrow +\infty, \qquad \mathcal{U}_\mu(N, h_i) = 0. \tag{11.15}$$

Every flat unitary connection on a finite-rank Hermitian vector bundle over either manifold has trivial holonomy and the eigenvalue multiplicities described in Proposition 10.3.

Proof. Let γ_i be the image of a fixed generator loop in the quotient zero section of (W, k_{ϵ_i}). Its class in $\pi_1(M) = \mathsf{V}$ is $\vartheta \ne 1$, and $L_{g_i}(\gamma_i) \le C\epsilon_i$. For a lift $\widetilde{x}_i$ of its basepoint, the lifted loop joins $\widetilde{x}_i$ to $\sigma_i\widetilde{x}_i$ for a deck transformation $\sigma_i \ne 1$, with

$$d(\widetilde{x}_i, \sigma_i\widetilde{x}_i) \le L_{g_i}(\gamma_i) \le C\epsilon_i.$$

Lemma 11.1 gives $H_M(t) \ge c$ for $C^2\epsilon_i^2 \le t \le 1$, for large i. Hence

$$\mathcal{U}_\mu(M, g_i) \ge ce^{-\mu} \int_{C^2\epsilon_i^2}^1 \frac{dt}{t} \ge c_1 \log\frac{1}{\epsilon_i} - C_1 \longrightarrow +\infty.$$

The manifold N is simply connected, so its universal covering is itself and $Q_N(t, x) = 0$ for every $t > 0$ and $x \in N$. The last assertion is Proposition 10.3. □

The integral (11.3) depends on the heat kernel of the universal covering. This corollary makes no assertion that the scalar spectrum of a single metric determines its fundamental group.

Statements and declarations

Funding. The author received no funding for this work.
Competing interests. The author declares no financial or non-financial competing interests relevant to this work.

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Toulouse, France
Email address: nifa.anass@gmail.com