FOUR-MOVE INVARIANCE OF GLUCK TWISTS AND STANDARD TWISTED CORK DOUBLES

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Abstract. Let $\mathcal{G}_{m,n}(J)$ be the Gluck twist of the m-twist n-roll spin of a classical knot J. For fixed n, we prove that its oriented diffeomorphism type is independent of m and invariant under four-moves on J. The key local result is a framed compression disk for each crossing-circle product torus, whose push-off is the preferred longitude; even surgery in this direction is trivial relative to the complement. As an application, every longitudinally twisted double of Gompf's cork $C(r, s; h)$, $r, s > 0 > h$, is diffeomorphic to S^4 for every iterate, proving the conjecture attributed to Gompf by Tange. The same conclusion holds for simultaneous meridional and longitudinal twists and finite boundary sums. For each fixed cork, we classify the resulting two-end decompositions of S^4 by the absolute difference of their exponents, allowing orientation reversal and exchange of sides. We also obtain relative extension results for crossing-torus exteriors.

Contents

1. Introduction

Let $S \subset S^4$ be an oriented smooth two-knot. Fix $\nu S \cong S^2 \times D^2$, and let R_v be rotation of S^2 through angle v about a fixed axis. Put

$$\Sigma_S = (S^4 \setminus \operatorname{int} \nu S) \cup_\gamma (S^2 \times D^2), \qquad \gamma(q, e^{iv}) = (R_v q, e^{iv}). \tag{1.1}$$

The loop R_v is the nonzero element of $\pi_1(SO(3)) = \mathbb{Z}/2$, so γ and γ^{-1} have the same boundary mapping class. The manifold Σ_S is a homotopy four-sphere by Gluck's theorem [7]. We call it standard when it is diffeomorphic to S^4.

Let $J \subset S^3$ be an oriented knot. Denote by $\tau^m \rho^n(J)$ its m-twist n-roll spin and by

$$\mathcal{G}_{m,n}(J) = \Sigma_{\tau^m \rho^n(J)} \qquad (m, n \in \mathbb{Z}) \tag{1.2}$$

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the corresponding Gluck manifold. The spinning convention is the marked rim-surgery convention of Hughes–Kim–Miller [13, Proposition 2.1]. A four-move replaces two strands in a ball by the same strands with two additional full twists; both signs and both strand-orientation patterns are allowed. Write $J \sim_4 J'$ for the equivalence relation generated by four-moves and oriented ambient isotopies.

Theorem 1.1. *For every $n \in \mathbb{Z}$, the oriented diffeomorphism type of $\mathcal{G}_{m,n}(J)$ depends only on the four-move class of J. More precisely,*

$$J \sim_4 J' \quad \Longrightarrow \quad \mathcal{G}_{m,n}(J) \cong \mathcal{G}_{m',n}(J') \qquad (m, m' \in \mathbb{Z}). \tag{1.3}$$

In particular,

$$\mathcal{G}_{m,n}(J) \cong \mathcal{G}_{m',n}(J), \qquad J \sim_4 U \implies \mathcal{G}_{m,n}(J) \cong S^4. \tag{1.4}$$

Here U is the unknot.

The roll parameter n is fixed in (1.3). The conclusion is a diffeomorphism of closed four-manifolds. It does not assert an isotopy of the spun knots, or a prescribed identification of their exteriors or surgery-dual spheres.

1.1. **Twisted cork doubles.** Let $C(r, s; h)$ be Gompf's contractible four-manifold associated with the double-twist knot $\kappa(r, -s)$ and an h-framed meridional two-handle. Let f be its longitudinal boundary twist [9]. For $k \in \mathbb{Z}$ set

$$\mathbb{S}_{r,s,h,k} = C(r, s; h) \cup_{f^k} \big(-C(r, s; h)\big). \tag{1.5}$$

The copy of C on the left has its given orientation; the copy on the right has the opposite orientation.

Theorem 1.2. *For all integers r, s, h, k with $r, s > 0 > h$,*

$$\mathbb{S}_{r,s,h,k} \cong S^4. \tag{1.6}$$

For $k \neq 0$, this is the conjecture attributed to Gompf in Tange [23, Conjecture 1.6]. The case $k = 0$ and the even-framing case are already known; the assertion here includes every odd negative framing and every iterate. The deduction uses the two-bridge presentation of $\kappa(r, -s)$ [9, Remark 2.1(c)] and the double-to-Gluck identification of Naylor–Schwartz [18, discussion preceding Corollary 2.6]. The latter identifies an even-framed double with the trivial sphere filling and an odd-framed double with a Gluck filling. It holds for arbitrary J; the unknotting-number-one assumption enters only their subsequent standardness corollary.

Corollary 1.3. *If J is a Conway 2-algebraic knot or a knot represented by a closed three-strand braid, then*

$$\mathcal{G}_{m,n}(J) \cong S^4 \qquad (m, n \in \mathbb{Z}).$$

In particular, this holds for every two-bridge knot.

The classical reductions used here are Przytycki's Proposition 3.2(ii) and Theorem 3.6(a) [20]. No enumeration of knots enters the proof of Theorem 1.1.

1.2. **The relative surgery theorem.** Let $T \subset \operatorname{int} M^4$ be a torus with trivial normal bundle. Suppose that an essential circle $\alpha \subset T$ bounds an embedded disk D, that $D \cap T = \partial D = \alpha$ is clean, and that the normal vector to α in T extends to a nowhere-zero section of ND. Let α^0 be the push-off determined by the inward tangent of D. We prove

$$M_T(\mu_T + 2q\alpha^0) \cong M \qquad (q \in \mathbb{Z}), \tag{1.7}$$

relative to the complement of $\nu(T \cup D)$.

The framing hypothesis makes the two-handle attachment to νT a product attachment. Its outer boundary is $\#_2(S^1 \times S^2)$. The coefficient-k shear induces a diffeomorphism ψ_k with

$$\psi_k = \psi_1^k, \qquad (\psi_k)_* = \mathrm{id} \text{ on } \pi_1\big(\#_2(S^1 \times S^2)\big).$$

Laudenbach's kernel theorem, in the formulation of Brendle–Broaddus–Putman [3, Theorem 2.1 and Corollary 5.2], identifies the kernel of the action on the outer automorphism group with $(\mathbb{Z}/2)^2$. Hence $[\psi_{2q}] = 1$. Extending a boundary isotopy through a collar gives (1.7) with its stated support.

The disk required for the product torus $C \times S^1 \subset \mathcal{G}_{m,n}(J)$ is constructed in Section 4. The two meridional boundary components of the punctured crossing disk are capped by disjoint disks in the standard torus exterior. The product direction projects to normal sections C_+e^{it} and C_-e^{it}, where $C_\pm \neq 0$. The normal orientation on each cap is computed from its induced surface orientation and the ambient orientation. With these signs included, the cancelling phases have boundary degrees $(0, 1, -1)$ in the induced orientation of the pair of pants, for either strand-orientation pattern. A circle-valued extension therefore makes both cap sections constant while preserving the prescribed outer section. This gives a framed disk whose push-off is exactly the preferred longitude λ_C.

For $\ell = \mathrm{lk}(C, J)$ and $(-1/d)$-surgery on C, the preferred longitude and product filling satisfy

$$\lambda_{J^{(d)}} = \lambda_J - d\ell^2\mu_J, \tag{1.8}$$

$$\mathcal{G}_{m,n}(J)_{C\times S^1}(\mu_C - d\lambda_C) \cong \mathcal{G}_{m+nd\ell^2,n}(J^{(d)}). \tag{1.9}$$

The comparison retains the product-circle coordinate. Taking $d = \pm 2$ and then changing the twist parameter proves Theorem 1.1.

1.3. **Further relative conclusions.** The same construction extends every even longitudinal shear over the torus exterior; see Theorem 8.1. A four-move supported away from prescribed disjoint circle tubes admits a comparison fixed on their products with S^1; see Theorem 8.4. Both assertions retain the boundary parametrizations; neither is inferred from an unmarked closed-manifold comparison.

For the cork application one can retain the embedded decomposition of the standard sphere. Let f_0, f_1 be the longitudinal twists in the two end knot exteriors, defined using their inward collars. With

$$\mathscr{D}_{p,q} = C_+ \cup_{f_0^p f_1^q} C_-, \qquad C_+ = C(r, s; h), \quad C_- = -C(r, s; h),$$

their common boundary will be denoted $Y_{p,q}$.

Theorem 1.4. *Fix $r, s > 0 > h$. Then*

$$(\mathscr{D}_{p,q}, C_+) \cong_+ (\mathscr{D}_{p',q'}, C_+) \quad \Longleftrightarrow \quad |p-q| = |p'-q'|. \tag{1.10}$$

The same criterion classifies the pairs $(\mathscr{D}_{p,q}, Y_{p,q})$ under diffeomorphisms without an orientation or side-preservation requirement. Every ambient manifold $\mathscr{D}_{p,q}$ is S^4. The corresponding embedded copies of C are equivalent by orientation-preserving ambient homeomorphisms, with their abstract parametrizations retained.

The proof in Section 9 uses the extending product $f_0 f_1$ and the quotient of the normal longitudinal-twist subgroup by its extending subgroup. Both complementary reparametrizations are accounted for by double cosets. The topological conclusion asserts homeomorphism, not a separately proved ambient topological isotopy.

1.4. **Previous results and scope.** Gluck proved standardness for spun knots [7]; Gordon and Pao proved it for twist-spun knots [11, 19]. Naylor–Schwartz prove standardness for every twist and roll parameter when the classical knot has unknotting number one [18, Theorem 2.4]. Their hypothesis and the four-move hypothesis in Theorem 1.1 are different; no implication between them is used here. For torus knots, Litherland identifies twist–roll spins with twist spins; see [18, Remark 1.11]. Their Gluck manifolds are therefore also covered by the earlier twist-spin results. Hughes–Kim–Miller supply the marked rim-surgery presentation used in Section 2 [13, Proposition 2.1]. Their four-twist periodicity concerns branched double covers [13, Theorem 1.1], not the effect of a classical four-move on the Gluck manifold. The twist-parameter comparison below combines their presentation with Montesinos' extension theorem, in the form stated by Larson [16, proof of Theorem 5.2].

The neighborhood $\nu(T \cup D)$ is the pochette $(S^2 \times D^2) \natural (S^1 \times D^3)$ of Iwase–Matsumoto [14]. Its gluing data are described by a slope and a mod-two framing; see Suzuki [21, Theorem 2.2]. Suzuki–Tange establish a surgery comparison under a boundary-parallel-cord hypothesis [22, Theorem 1.1]. No such hypothesis occurs in (1.7); the conclusion there is relative to the complement of the compression neighborhood. Gompf's torus-twist extension theorem instead assumes disk framing ±1 [10, Theorem 4.1]. Our disk framing is zero, and our surgery coefficient is even. The essential geometric input for Theorem 1.1 is the construction of this framed disk with the specified longitude for every crossing-circle product torus.

Tange proves the handle-framing parity reduction for Gompf's doubles [23, Theorem 2]. Akbulut proves standardness for the iterates of a fixed loose-cork construction [1, Theorem 1.1]. These results, and the standardness theorem of Naylor–Schwartz, give antecedents to Theorem 1.2; the quantifiers in (1.6) concern the entire original family $r, s > 0 > h$. Dai–Mallick–Zemke give a Floer-theoretic criterion producing further corks [5]; their criterion addresses nonextension, rather than standardness of the doubles considered here.

Auckly–Ruberman construct smoothly inequivalent embeddings in S^4 with diffeomorphic complementary regions [2, Theorem 1.1]. For the subfamily $C(2, 1; -2N)$, $N > 0$, their argument already obtains the necessary relation $p = \pm q$ between one-end twisting exponents, by combining nonextension with the action of boundary diffeomorphisms on the longitudinal twist [2, Proposition 2.1 and proof of Theorem 1.1]. Thus neither the embedding phenomenon nor this absolute-exponent obstruction is new here. In Theorem 1.4 we treat every $r, s > 0 > h$, allow twists at both ends, and prove both directions of (1.10). The proof uses the normal subgroup generated by the two end twists and its quotient by the extending subgroup; it does not require the two end exteriors to be individually preserved. Our topological conclusion is equivalence by an ambient homeomorphism retaining the cork parametrization. We do not assert a separate ambient topological isotopy theorem.

Nakanishi's conjecture, that every knot is four-move equivalent to the unknot, would imply standardness of $\mathcal{G}_{m,n}(J)$ for every J, m, n. No four-move reduction for an arbitrary knot is proved here. Nor do we assert that an odd longitudinal shear always extends over the original crossing-torus exterior. The odd-regluing calculation in Section A concerns the compression neighborhood and is independent of the proof of Theorem 1.1.

Conventions. All manifolds and maps are smooth, except where homeomorphisms are explicitly stated. A boundary is oriented by placing its outward normal first. An isomorphism $\cong$ between oriented four-manifolds is orientation-preserving unless otherwise stated. The subscript $+$ emphasizes that condition. All circle coordinates have period 2π. Matrices act on column vectors; columns record images of the ordered source curves. Homology has integral coefficients unless a coefficient group is displayed.

Tubular neighborhoods are closed disk bundles. Corners are rounded in their normal two-dimensional quadrants, constantly along each corner stratum. Attachments use signed collars and are product-type in their normal coordinate. A map relative to a specified exterior is the identity on that exterior. For $J \subset S^3$, write $E(J) = S^3 \setminus \operatorname{int} \nu J$. The meridian μ_J has linking number $+1$ with J, and the preferred longitude λ_J has degree one over J and is null-homologous in $E(J)$. The coefficient $-1/d$ uses the unoriented meridian slope $\mu_C - d\lambda_C$ and inserts d full twists under the fixed local sign convention. Reversing that convention replaces d by $-d$ throughout.

2. The torus presentation and the twist parameter

2.1. **Framed coordinates on the sphere and torus.** Give $S^4 = \partial B^5$ its boundary orientation, using

$$B^5 \subset \mathbb{C}^2 \times \mathbb{R}, \qquad (\Re z_1, \Im z_1, \Re z_2, \Im z_2, z)$$

as positively oriented ambient coordinates. Set

$$S = \{z_1 = 0\} \cap S^4, \qquad T_0 = \{z = 0,\ |z_1| = |z_2| = 2^{-1/2}\}. \tag{2.1}$$

Write $a = \arg z_1$ and $b = \arg z_2$ on T_0. Near T_0, define u by

$$z_1 = \sqrt{\frac{1 - z^2}{2} + u}\, e^{ia}, \qquad z_2 = \sqrt{\frac{1 - z^2}{2} - u}\, e^{ib}. \tag{2.2}$$

Fix $0 < \delta < 1/10$, take $\nu T_0 = \{u^2 + z^2 \le \delta^2\}$ within this coordinate neighborhood, and put

$$X = S^4 \setminus \operatorname{int} \nu T_0, \qquad u = \delta \cos c, \quad z = \delta \sin c \quad \text{on } \partial X. \tag{2.3}$$

The square roots in (2.2) are positive: on $u^2 + z^2 \le \delta^2$,

$$\frac{1 - z^2}{2} \pm u \ge \frac{1 - \delta^2}{2} - \delta > 0.$$

Thus (2.2) is a tubular parametrization with inverse given by the two arguments, $u = (|z_1|^2 - |z_2|^2)/2$, and z.

For the orientation of this chart put $r_i = |z_i|$. Complex rotations reduce the calculation to $a = b = 0$ without changing real orientation. With the position vector first, followed by the coordinate tangents $(\partial_a, \partial_b, \partial_u, \partial_z)$, the determinant is

$$\det \begin{pmatrix} r_1 & 0 & 0 & (2r_1)^{-1} & -z/(2r_1) \\ 0 & r_1 & 0 & 0 & 0 \\ r_2 & 0 & 0 & -(2r_2)^{-1} & -z/(2r_2) \\ 0 & 0 & r_2 & 0 & 0 \\ z & 0 & 0 & 0 & 1 \end{pmatrix} = \frac{r_1^2 + r_2^2 + z^2}{2} = \frac{1}{2}. \tag{2.4}$$

Thus (a, b, u, z) is a positive coordinate system on the sphere. The zero-linking representatives of a, b are fixed as follows. In the equatorial $S^3 = \{z = 0\}$, the maps

$$\begin{aligned} d_a(v) &= (v, \sqrt{1 - |v|^2}, 0), \\ d_b(v) &= (\sqrt{1 - |v|^2}, v, 0), \\ &\quad |v| \le 2^{-1/2}, \end{aligned} \tag{2.5}$$

are embedded disks with boundaries a at $b = 0$ and b at $a = 0$. Their interiors are disjoint from T_0. The fields ∂_b on d_a and ∂_a on d_b are nonzero normal sections agreeing with the corresponding torus-tangent normals at the boundary. The disk d_a intersects S once and transversely at $v = 0$; the disk d_b is disjoint from S because $|z_1| \ge 2^{-1/2}$ there. The restrictions of d_a, d_b to $|v| \le \sqrt{1/2 - \delta}$ have boundaries on ∂X, at $c = \pi$ and $c = 0$, respectively. These restricted disks lie in X. If $c_0, c_1 \in \mathbb{R}$ are lifts of two normal angles, the maps

$$(t, r) \longmapsto (t, 0, (1 - r)c_0 + rc_1), \qquad (t, r) \longmapsto (0, t, (1 - r)c_0 + rc_1)$$

from $S^1 \times [0, 1]$ to ∂X are homotopies between the corresponding parallels. Thus every constant-c parallel of either coordinate circle represents zero in $H_1(X; \mathbb{Z})$. Alexander duality gives

$$H_1(X; \mathbb{Z}) \cong H^2(T_0; \mathbb{Z}) \cong \mathbb{Z};$$

its positive generator is the meridian c. These are the zero-linking coordinates used in the extension theorem below. For $0 < q \le 2^{-1/2}$, the tori

$$T_q = \{z = 0,\ |z_1| = q,\ |z_2| = \sqrt{1 - q^2}\}$$

are parametrized by (a, b). Choose $0 < q_0 < 2^{-1/2}$ and a smooth nondecreasing function $\chi_0 : [0, 1] \to [0, 1]$ equal to 0 near 0 and to 1 near 1. Put $q(t) = (1 - \chi_0(t))2^{-1/2} + \chi_0(t)q_0$ and $u_t = q(t)^2 - 1/2$. Then $q(0) = 2^{-1/2}$, $q(1) = q_0$, and $q(t) > 0$. The compact curve $(u_t, 0)$ lies in the interior of the orbit-coordinate region

$$|u| < (1 - z^2)/2.$$

Choose $\epsilon_0 > 0$ so that all disks of radius $3\epsilon_0$ about this curve lie in that region and away from $\{z_1 = 0\}$. Let $\chi : \mathbb{R}^2 \to [0, 1]$ be smooth, equal to 1 on $B_{\epsilon_0}(0)$ and supported in $B_{2\epsilon_0}(0)$. In the tubular chart define

$$W_t = \dot{u}_t \chi(u - u_t, z)\partial_u, \tag{2.6}$$

and extend it by zero. Its flow is defined on the compact manifold S^4 for the full parameter interval. If $|u-u_0|^2+z^2<\epsilon_0^2$, the solution is

$$(a,b,u,z)\longmapsto(a,b,u+u_t-u_0,z).$$

The two angular coordinates and the ordered transverse frame (∂_u,∂_z) are therefore preserved on the moving torus. The flow is the identity near S. For small q_0, T_{q_0} is the normal-circle bundle over the equator of S. The disks (2.5) identify a, b with the framed curves C_1, C_2 of [13, Proposition 2.1]. If necessary, first decrease the tubular radius so that the entire chosen tube lies in the region on which the displayed flow is translation. Two positive tubular radii give canonically diffeomorphic exteriors by a radial collar isotopy, fixed away from the intervening annulus. This isotopy retains both angular coordinates and the normal-angle coordinate. Thus the framed identification, not only the embedded core torus, is transported by the flow. The marked rim-surgery theorem [13, Proposition 2.1] gives the pair $(S^4,\tau^m\rho^n(J))$ by replacing νT_0 by $E(J)\times S^1$, keeping S, and using

$$\mu_J\longmapsto a,\qquad \lambda_J\longmapsto c,\qquad s\longmapsto ma+b+nc. \tag{2.7}$$

In the notation of that theorem, $C_1=a$, $C_2=b$ and $\mu(T)=c$. The ordered parametrizations, and not only an unmarked group presentation, are part of (2.7).

Lemma 2.1. *Let M be a compact manifold with boundary Y, and let $g_t : Y\to Y$ be an isotopy with $g_0=\mathrm{id}$. For a collar $Y\times[0,\epsilon)\subset M$, the map g_1 extends to a diffeomorphism of M equal to the identity outside that collar. Consequently, isotopic attaching diffeomorphisms give diffeomorphic glued manifolds.*

Proof. Choose $\kappa : [0,\epsilon)\to[0,1]$ equal to 1 near 0 and to 0 for $r\geq\epsilon/2$. Define

$$G(y,r)=(g_{\kappa(r)}(y),r),\qquad G^{-1}(y,r)=(g_{\kappa(r)}^{-1}(y),r).$$

The inverse family is smooth by the inverse function theorem applied to $(t,y)\mapsto(t,g_t(y))$. Both maps equal the identity for $r\geq\epsilon/2$. They extend by the identity on the rest of M. Their restrictions at $r=0$ are g_1 and g_1^{-1}. If $f_0,f_1 : Y_1\to Y_2$ are isotopic attachments, apply the construction to $g_t=f_tf_0^{-1}$. The map equal to G on the second manifold and to the identity on the first respects the equivalence relations because

$$G(f_0(y))=f_1(y)\qquad(y\in Y_1).$$

Its inverse is the union of G^{-1} and the identity. □

2.2. **Regluing along the sphere.**

Proposition 2.2. *There is a presentation*

$$\mathcal{G}_{m,n}(J)\cong X\cup_{B_{m,n}}(E(J)\times S^1), \tag{2.8}$$

where, with $A=a+b$,

$$\mu_J\longmapsto A,\qquad \lambda_J\longmapsto c,\qquad s\longmapsto mA+b+nc. \tag{2.9}$$

The matrix with columns (μ_J,λ_J,s) and rows (a,b,c) is

$$B_{m,n}=\begin{pmatrix}1&0&m\\1&0&m+1\\0&1&n\end{pmatrix}. \tag{2.10}$$

An affine representative can be used, so that the equations also determine its differential on the boundary.

Proof. Choose $\nu S=\{|z_1|\leq\epsilon\}$, where $0<\epsilon<\sqrt{(1-\delta^2)/2-\delta}$. The lower bound following (2.3) then gives $\nu S\cap\nu T_0=\emptyset$. On the exterior of S, define

$$F(z_1,z_2,z)=\left(z_1,\frac{z_1}{|z_1|}z_2,z\right). \tag{2.11}$$

On this exterior $|z_1| \geq \epsilon > 0$. The map is smooth, preserves $|z_1|^2 + |z_2|^2 + z^2$, and has the smooth inverse

$$F^{-1}(z_1, z_2, z) = \left(z_1, \frac{\overline{z}_1}{|z_1|} z_2, z \right).$$

For $\epsilon \leq r < 1$, write $z_1 = re^{iv}$ and $(z_2, z) = \sqrt{1 - r^2}\, q$. On this open dense set, $F(r, v, q) = (r, v, R_v q)$. Its derivative is triangular over the identity on (r, v) and the orientation-preserving tangent map of R_v on S^2; hence its orientation sign is positive. The Cartesian formulas for F and F^{-1} are smooth also where $r = 1$. Their derivatives are inverse there. The orientation sign is locally constant, and so is positive everywhere. Write a point of $\partial \nu S$ as $(e^{iv}, q) \in S^1 \times S^2$. There $F(e^{iv}, q) = (e^{iv}, R_v q)$. If the Gluck attachment is represented by $(e^{iv}, q) \mapsto (e^{iv}, R_{-v} q)$, its composition with F is the identity. Choose radial collars on both sides of this boundary. In these collars F and the attaching map are independent of the radial parameter. Their composition is the identity on the entire boundary collar. Therefore F and the identity on the refilled $S^2 \times D^2$ define a smooth diffeomorphism to S^4. The map F preserves T_0, u and z. In the tubular coordinates it is

$$(a, b, u, z) \longmapsto (a, a + b, u, z).$$

It follows that its boundary action is

$$a \longmapsto a + b, \qquad b \longmapsto b, \qquad c \longmapsto c. \tag{2.12}$$

Put $Z = S^4 \setminus (\operatorname{int} \nu S \cup \operatorname{int} \nu T_0)$. Both orders of replacement are the quotient of

$$Z \sqcup (S^2 \times D^2) \sqcup (E(J) \times S^1)$$

by the same two boundary equivalence relations. The preceding diffeomorphism on $Z \cup (S^2 \times D^2)$ acts on the remaining boundary by (2.12). Its composition with (2.7) is (2.9). The three columns in (2.10) are linearly independent and $\det B_{m,n} = -1$. The orientation signs of the peripheral bases can be fixed directly. The coordinates (a, b, u, z) are positive on S^4 near T_0, and ∂X has outward normal $-\partial_{\sqrt{u^2+z^2}}$; consequently (a, b, c) is negative relative to its boundary orientation. In an oriented knot tube the coordinates (r, μ, λ) are positive. The exterior has outward normal $-\partial_r$, so (μ, λ, s) is negative on $\partial(E(J) \times S^1)$. The determinant -1 therefore gives an orientation-reversing attachment between the oriented boundaries. By [13, Proposition 2.1], the three peripheral classes determine that class up to isotopy. Apply Lemma 2.1 to replace the attaching map by its affine representative. In angular coordinates it is

$$a = \mu + ms, \qquad b = \mu + (m + 1)s, \qquad c = \lambda + ns. \tag{2.13}$$

The inverse affine map is

$$s = b - a, \qquad \mu = (m + 1)a - mb, \qquad \lambda = c - n(b - a). \tag{2.14}$$

All its coefficients are integral. Since $u \circ F = u$ and $z \circ F = z$, no meridional winding is introduced by F. $\square$

2.3. **Boundary maps extending over the torus exterior.** We use the following form of Montesinos' theorem, stated in [16, proof of Theorem 5.2]. In the zero-linking basis (a, b, c), a matrix

$$\begin{pmatrix} p & q & v_1 \\ r & t & v_2 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{with} \quad pt - qr = 1, \quad p + q + r + t \equiv 0 \pmod 2 \tag{2.15}$$

is induced, up to boundary isotopy, by an orientation-preserving diffeomorphism of X. If $f|_{\partial X}$ and a selected representative g induce the same mapping class, then $g \circ (f|_{\partial X})^{-1}$ is isotopic to the identity. Extend this isotopy by Lemma 2.1 and postcompose f. The resulting diffeomorphism of X restricts to g.

Proposition 2.3. *For every $j, m, n \in \mathbb{Z}$ and every knot J, one has $\mathcal{G}_{m+j,n}(J) \cong \mathcal{G}_{m,n}(J)$.*

Proof. The matrix

$$R_j = \begin{pmatrix} 1-j & j & 0 \\ -j & 1+j & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.16}$$

has upper-left determinant $(1-j)(1+j)+j^2 = 1$ and upper-left entry sum 2. It extends over X by (2.15). Moreover,

$$R_j(A) = A, \qquad R_j(c) = c, \qquad R_j(b) = b + jA.$$

In addition,

$$R_j R_k = R_{j+k}, \qquad R_0 = I_3, \qquad R_j^{-1} = R_{-j}.$$

Multiplication on the three columns gives

$$R_j B_{m,n} = \big(A,\ c,\ (m+j)A + b + nc\big) = B_{m+j,n}.$$

Let $f_j : X \to X$ extend the selected affine representative of R_j. The map equal to f_j on X and to the identity on $E(J) \times S^1$ respects the attaching relation and gives the required diffeomorphism. □

Lemma 2.4. *For every $q \in \mathbb{Z}$, the boundary map*

$$E_q : \quad a \longmapsto a, \quad b \longmapsto b, \quad c \longmapsto c + q(a+b) \tag{2.17}$$

extends over X.

Proof. The matrix and its inverse are

$$E_q = \begin{pmatrix} 1 & 0 & q \\ 0 & 1 & q \\ 0 & 0 & 1 \end{pmatrix}, \qquad E_q^{-1} = E_{-q}.$$

The upper-left block has determinant 1 and entry sum 2. Its bottom row is $(0,0,1)$. Thus all hypotheses of (2.15) hold. The selected affine representative extends by Lemma 2.1. □

Lemma 2.5. *For all $m, n \in \mathbb{Z}$, one has $\mathcal{G}_{m,n}(U) \cong S^4$.*

Proof. For the unknot,

$$E(U) \times S^1 = S^1_\mu \times S^1_s \times D^2_w, \qquad \arg w = \lambda_U.$$

Use the unit normal disk coordinate $w' = (u+iz)/\delta$ on νT_0; thus $\arg w' = c$ on ∂X. The map

$$(\mu, s, w) \longmapsto \big(a = \mu + ms,\ b = \mu + (m+1)s,\ w' = e^{ins} w\big) \tag{2.18}$$

has inverse

$$s = b - a, \qquad \mu = (m+1)a - mb, \qquad w = e^{-in(b-a)} w'. \tag{2.19}$$

All angular coefficients are integers, so both maps descend to the torus coordinates. Multiplication of a complex disk coordinate by e^{ins} is smooth at $w = 0$. The torus determinant is $(m+1) - m = 1$, and multiplication in the disk preserves its orientation. Hence (2.18) is orientation preserving. On $|w| = 1$ it is (2.13); it is product-type in the disk radius near that boundary. Its union with the identity on X is therefore a smooth diffeomorphism to $X \cup \nu T_0 = S^4$. □

3. Framed compression and even torus surgery

Definition 3.1. Let M^4 be oriented and let $T \subset \operatorname{int} M$ be an oriented embedded torus with trivial normal bundle. A *framed compression disk* consists of an embedded disk $D \subset \operatorname{int} M$ and an essential curve $\alpha = \partial D \subset T$ such that $D \cap T = \alpha$, the intersection is clean, and a nonzero normal vector to α in T extends to a nowhere-zero section of the rank-two normal bundle ND. Normal bundles may equivalently be defined as quotient bundles. A metric identifies the quotient with an orthogonal complement and does not change this condition.

Choose an oriented parametrization $T = S^1_\alpha \times S^1_\beta$ extending the parametrization of α. Such a parametrization exists because an embedded essential circle on a torus represents a primitive homology class: cutting along it gives an annulus, whose product parametrization gives the complementary circle β. The inward tangent to D, projected to NT along α, is nonzero by the clean-intersection hypothesis. Relative to an initial oriented trivialization of NT, its unit normalization is a map $g : S^1_\alpha \to S^1$. The map $(\alpha, \beta) \mapsto g(\alpha)$ extends it over T. Rotating the initial frame by this map makes the prescribed section the first normal vector along α. The inward tangent consequently determines a push-off $\alpha^0 \subset \partial\nu T$ at a constant normal angle. Choose the resulting product coordinates

$$V = \nu T = S^1_\alpha \times S^1_\beta \times D^2_\theta \tag{3.1}$$

so that $\alpha^0 = S^1_\alpha \times \{\beta_0\} \times \{e^{i\theta_0}\}$. The meridian is $\mu_T = \{\alpha_0, \beta_0\} \times \partial D^2$. A torus surgery is a gluing of $T^2 \times D^2$ to $M \setminus \operatorname{int} V$. Its meridian is the image of $\{\mathrm{pt}\} \times \partial D^2$. The following fact fixes the meaning of a surgery specified by an unoriented primitive meridian class.

Lemma 3.2. *An orientation-preserving diffeomorphism of $\partial(T^2 \times D^2)$ which preserves the unoriented meridian class extends over $T^2 \times D^2$.*

Proof. Its matrix, in the angular basis (x_1, x_2, θ), is

$$P = \begin{pmatrix} L & 0 \\ v^\top & \epsilon \end{pmatrix}, \quad L \in GL(2, \mathbb{Z}), \quad v \in \mathbb{Z}^2, \quad \epsilon \in \{1, -1\}, \quad \epsilon \det L = 1.$$

Define $c_1(w) = w$ and $c_{-1}(w) = \overline{w}$. The formula

$$(x, w) \longmapsto \left(Lx, e^{iv^\top x} c_\epsilon(w)\right) \tag{3.2}$$

is well-defined on $T^2 \times D^2$, because L, v are integral. Its inverse is

$$x = L^{-1}x', \qquad w = c_\epsilon\left(e^{-iv^\top L^{-1}x'} w'\right).$$

The orientation sign is $\det L \cdot \epsilon = 1$. Its boundary matrix is P. The action on $H_1(T^3; \mathbb{Z})$ identifies $\pi_0 \operatorname{Diff}^+(T^3)$ with $SL(3, \mathbb{Z})$; this is also the boundary-isotopy fact used in [13, paragraph following Proposition 2.1]. Hence the given boundary diffeomorphism differs from this representative by a map isotopic to the identity. Extend that difference by Lemma 2.1. □

Let $f_0, f_1 : \partial(T^2 \times D^2) \to \partial(M \setminus \operatorname{int} V)$ have the same unoriented meridian image. Then $f_1^{-1} f_0$ preserves the meridian and is orientation preserving. If H is the extension supplied by Lemma 3.2, the union of H on the inserted piece with the identity on the exterior descends to a diffeomorphism of the two surgeries. The condition $f_1 \circ H = f_0$ on the boundary verifies the quotient relation.

Theorem 3.3. *If D is a framed compression disk for $T \subset M$, surgery on T with meridian*

$$\mu_T + 2q\alpha^0 \qquad (q \in \mathbb{Z}) \tag{3.3}$$

is diffeomorphic to M, relative to the complement of a regular neighborhood of $T \cup D$.

Proof. Let $\Gamma = \nu(T \cup D)$. Remove a small collar of ∂D inside νT. A neighborhood of the remaining disk is a 2-handle attached to V along α^0. Normalize the given nonzero section of ND to unit length in a bundle metric. Together with its oriented perpendicular it trivializes ND. Choose a representative of the second normal vector transverse to T along α; its quotient class is independent of the first normal vector, so this transversality follows from $TD \cap TT = T\alpha$. In a collar of α in D, let $x \geq 0$ be the inward coordinate. Extend the first normal vector to an ambient vector field whose restriction along T is tangent to T, and extend the second vector to a transverse ambient field. Applying their local flows, with parameters β, y, to the points (α, x) of the disk collar defines a map from a neighborhood of $S^1 \times \{0\} \times [0, \epsilon) \times \{0\}$. At $x = \beta = y = 0$, its derivative has as columns the tangent of α, the nonzero tangent-to-T normal, the inward tangent of D, and the second normal vector. These columns are linearly independent. The inverse function theorem gives a local diffeomorphism near every point of

α. There is a common sufficiently small parameter interval on which the map is injective. Otherwise, choose pairs of distinct points with transverse parameters tending to zero and with equal images. Compactness gives limits on α; the embedding of α makes these limits equal. Both points then lie in one of the preceding local inverse neighborhoods, a contradiction. The resulting map is therefore a diffeomorphism onto its image after the intervals are decreased. At $x = y = 0$, the first flow stays in T. At $\beta = y = 0$, the map is the disk-collar parametrization. Thus near α the coordinates (α, β, x, y) satisfy

$$T = \{x = y = 0\}, \quad D = \{\beta = y = 0,\ x \geq 0\}.$$

At the attaching circle $x = \epsilon, y = 0$, the normal quotient frame is $(\partial_\beta, \partial_y)$, whereas $\partial_\theta = \epsilon\partial_y$. The matrix comparing $(\partial_\beta, \partial_y)$ with $(\partial_\beta, \partial_\theta)$ is $\mathrm{diag}(1, \epsilon)$; it is constant along the attaching circle. Its loop in $GL(2, \mathbb{R})$ is therefore constant. The framing supplied by ND has relative winding number zero in the (β, θ) disk. The two-handle attachment is the product attachment

$$j : S^1 \times D^2 \longrightarrow \partial V, \qquad j(e^{i\alpha}, q) = (e^{i\alpha}, b(q)), \tag{3.4}$$

where $b : D^2 \to T^2_{\beta,\theta}$ parametrizes a small disk centered at (β_0, θ_0). Let B be a small disk around (β_0, θ_0) in $T^2_{\beta,\theta}$ and let

$$P = T^2_{\beta,\theta} \setminus \operatorname{int} B.$$

The part of ∂V removed by the handle is $S^1_\alpha \times B$. The other face of the handle is $D^2_\alpha \times \partial B$. Hence

$$\partial\Gamma = (S^1_\alpha \times P) \cup (D^2_\alpha \times \partial P) = \partial(D^2 \times P) \cong \#_2(S^1 \times S^2). \tag{3.5}$$

For the last identification, a disk with two 1-handles is a handle decomposition of P. Multiplication by D^2 gives a 4-dimensional 1-handlebody with two 1-handles; its boundary is the indicated connected sum. For later use the two fillings are distinguished by

$$H_1(\Gamma; \mathbb{Z}) \cong \mathbb{Z}\langle\beta\rangle, \qquad H_2(\Gamma; \mathbb{Z}) \cong \mathbb{Z}\langle[T]\rangle, \tag{3.6}$$

whereas $H_1(D^2 \times P; \mathbb{Z}) \cong \mathbb{Z}^2$ and $H_2(D^2 \times P; \mathbb{Z}) = 0$. The relative handle chain complex has a single generator in degree 2. The relevant exact sequence is

$$0 \longrightarrow H_2(V) \longrightarrow H_2(\Gamma) \longrightarrow \mathbb{Z} \xrightarrow{1 \mapsto (1,0)} \mathbb{Z}\langle\alpha\rangle \oplus \mathbb{Z}\langle\beta\rangle \longrightarrow H_1(\Gamma) \longrightarrow 0.$$

The middle map is injective. Hence $H_2(V) \to H_2(\Gamma)$ is an isomorphism and $H_1(\Gamma) = \mathbb{Z}\langle\beta\rangle$. The other relative groups vanish, so $H_k(\Gamma) = 0$ for $k \geq 3$. Choose $h : T^2_{\beta,\theta} \to S^1$ representing the cohomology class of the θ-coordinate and satisfying $h = 1$ near B. To construct it, choose a slightly larger coordinate disk, a real lift $\widetilde{\theta}$ of θ there, and a cutoff χ equal to one near B. Set $h = e^{i\theta}e^{-i\chi\widetilde{\theta}}$ on the disk and $h = e^{i\theta}$ outside. The real function $\chi\widetilde{\theta}$ is zero near the boundary of the larger coordinate disk and extends by zero to the torus. The maps $e^{-it\chi\widetilde{\theta}}$, $0 \leq t \leq 1$, form a homotopy to 1. Thus $[h] = [e^{i\theta}] \in H^1(T^2; \mathbb{Z})$. For $k \in \mathbb{Z}$ define

$$\phi_k(e^{i\alpha}, \beta, \theta) = \big(e^{i\alpha}h(\beta, \theta)^k, \beta, \theta\big). \tag{3.7}$$

Its inverse is ϕ_{-k}, and it is the identity near $S^1_\alpha \times B$. On peripheral classes it fixes α, β and sends θ to $\theta + k\alpha$. It represents the surgery slope $\mu_T + k\alpha^0$. Let $H = D^2_\alpha \times D^2$ denote the attached handle and let $W = \mathrm{cl}(M \setminus \Gamma)$. The attaching map j in (3.4) satisfies $\phi_k \circ j = j$. In the reglued manifold, the union of the inserted copy V_k with H is therefore the quotient

$$\Gamma_k = V_k \sqcup H \big/ \big(j(e^{i\alpha}, q) \sim (e^{i\alpha}, q)\big),$$

with exactly the same relation as $\Gamma = V \cup_j H$. The identity maps on the two abstract factors descend to a diffeomorphism $I_k : \Gamma_k \to \Gamma$. On $S^1_\alpha \times P$, the map to W is ϕ_k; on $D^2_\alpha \times \partial P$, it is the original handle-face map. Thus, with the original attachment to W understood, the new outer attachment is represented by

$$\psi_k = \phi_k \quad \text{on } S^1_\alpha \times P, \qquad \psi_k = \mathrm{id} \quad \text{on } D^2_\alpha \times \partial P. \tag{3.8}$$

The convention just fixed writes the result as $W \cup_{\psi_k} \Gamma$. The formulas imply

$$\psi_j \psi_k = \psi_{j+k}, \qquad \psi_0 = \mathrm{id}, \qquad \psi_k^{-1} = \psi_{-k}.$$

In particular $\psi_k = \psi_1^k$. Take a basepoint where $h = 1$. Let d_0 generate the fundamental group of $D^2_\alpha \times \partial P$, and orient ∂P so that its image in $\pi_1(P)$ is $[\beta, \theta]$. Van Kampen gives the amalgamated presentation

$$\pi_1(\partial\Gamma) = \frac{(\mathbb{Z}\langle\alpha\rangle \times F(\beta,\theta)) * \langle d_0\rangle}{\langle\!\langle \alpha,\ d_0[\beta,\theta]^{-1}\rangle\!\rangle}.$$

Eliminating d_0 gives

$$\pi_1(\partial\Gamma) = \big(\mathbb{Z}\langle\alpha\rangle \times \pi_1(P)\big)/\langle\!\langle\alpha\rangle\!\rangle = F(\beta,\theta). \tag{3.9}$$

Choose based generators β, θ in P and the basepoint in the region $h = 1$. A based loop $\gamma \subset P$ maps under (3.7) to

$$\alpha^{k\langle [h],[\gamma]\rangle}\gamma \quad \text{in } \mathbb{Z}\langle\alpha\rangle \times \pi_1(P).$$

Thus the images of β, θ before passage to (3.9) are β and $\alpha^k\theta$. Since $\alpha = 1$ in the quotient, the induced automorphism is

$$(\psi_k)_*(\beta) = \beta, \qquad (\psi_k)_*(\theta) = \theta.$$

These are equalities in the free group $F(\beta,\theta)$. Laudenbach's theorem [17], in the formulation of [3, Theorem 2.1 and Corollary 5.2], gives

$$\ker\big[\mathrm{Mod}^+(\#_2(S^1\times S^2)) \longrightarrow \mathrm{Out}(F_2)\big] \cong (\mathbb{Z}/2)^2. \tag{3.10}$$

The class $[\psi_1]$ belongs to the kernel in (3.10), so $[\psi_1]^2 = 1$. The identities of representatives $\psi_k = \psi_1^k$ then imply $[\psi_{2q}] = 1$ for every $q \in \mathbb{Z}$. Let $J_q : \Gamma \to \Gamma$ extend ψ_{2q} by Lemma 2.1. The map

$$W \cup_{\psi_{2q}} \Gamma \longrightarrow W \cup_{\mathrm{id}} \Gamma, \qquad \mathrm{id}_W \cup J_q,$$

is well-defined: the source relation $y \sim \psi_{2q}(y)$ is carried to the target relation $J_q(y) \sim \psi_{2q}(y)$. Both J_q and its inverse have product representatives on boundary collars. The resulting map is a smooth diffeomorphism and restricts to id_W. □

Remark 3.4. The equality $[\psi_{2q}] = 1$ is an assertion in the boundary mapping-class group. The proof constructs the diffeomorphism relative to $M \setminus \operatorname{int}\Gamma$ by extending a boundary isotopy through a collar. It does not deduce this relative conclusion from equality of the diffeomorphism types of the closed regluings. The slope and mod-two framing description of pochette surgery [21, Theorem 2.2] describes the gluing data of this neighborhood. The ±1 disk-framing hypotheses of [8, Lemma 2.2] and [10, Theorem 4.1] differ from the zero relative framing used above.

4. Compression disks for crossing-circle tori

Let $C \subset E(J)$ bound an oriented embedded disk $\Delta \subset S^3$ transverse to J in two points. Their signs may be equal or opposite. Set

$$\ell = \mathrm{lk}(C,J) \in \{-2,0,2\}, \qquad \mathcal{T} = C \times S^1 \subset E(J)\times S^1 \subset \mathcal{G}_{m,n}(J). \tag{4.1}$$

Lemma 4.1. *Let P be an oriented pair of pants. Let $u_i : \partial_i P \to S^1$ be smooth maps, with degrees computed using the induced boundary orientations. They extend to a smooth $u : P \to S^1$ if and only if*

$$\sum_{i=0}^{2} \deg u_i = 0. \tag{4.2}$$

The extension can be made independent of the collar parameter on each boundary collar.

Proof. Let $\omega_{S^1} = z^{-1}dz/(2\pi i)$. If an extension exists, then

$$\sum_i \deg u_i = \int_{\partial P} u^*\omega_{S^1} = \int_P d(u^*\omega_{S^1}) = 0.$$

Conversely, represent P as an oriented planar disk with two disjoint open disks removed. Let their centers be p_1, p_2, and let d_1, d_2 be the required degrees on the inner boundary circles. Define

$$v(\xi) = \left(\frac{\xi - p_1}{|\xi - p_1|}\right)^{-d_1} \left(\frac{\xi - p_2}{|\xi - p_2|}\right)^{-d_2}.$$

An inner boundary has clockwise orientation. The winding number of $\xi - p_j$ on the jth inner circle is -1 and its winding number on the other inner circle is 0. On the outer boundary both winding numbers are 1. Thus the degrees of v are $(-d_1 - d_2, d_1, d_2)$. For each boundary circle let $q_i = u_i/v|_{\partial_i P}$. Its degree is zero. A lift $a_i : \mathbb{R} \to \mathbb{R}$ satisfying $e^{ia_i(t)} = q_i(e^{it})$ obeys

$$a_i(t + 2\pi) - a_i(t) = 2\pi \deg q_i = 0.$$

Hence a_i descends to a smooth real function on the circle. Choose pairwise disjoint collars $c_i : \partial_i P \times [0, \epsilon) \to P$ and a smooth cutoff κ, equal to 1 near 0 and 0 for $r \geq \epsilon/2$. Extend a_i to the collar by $a_i(x)$ and define

$$u(c_i(x, r)) = v(c_i(x, r))e^{i\kappa(r)a_i(x)}, \qquad u = v \text{ off the collars.}$$

The formulas agree on open neighborhoods of the collar ends; they therefore define a smooth map with $u|_{\partial_i P} = u_i$. Finally choose $\varphi : [0, \epsilon) \to [0, \epsilon)$ smooth, equal to 0 near 0 and to r for $r \geq \epsilon/2$. Replacing $u(c_i(x, r))$ by $u(c_i(x, \varphi(r)))$ gives a smooth extension independent of r near every boundary component. The map $(x, r) \mapsto (x, \varphi(r))$ is smooth and agrees with the identity for $r \geq \epsilon/2$; hence the new function agrees with the old function on a neighborhood of every collar end. □

Lemma 4.2. *Let $E \to S$ be a smooth real vector bundle over a compact surface, and let $L \subset S$ be a two-sided embedded circle. Suppose that smooth sections v_+ and v_- on the two sides of L have the same nonzero boundary value v_0. They may be changed in an arbitrarily small collar of L, without introducing a zero, so that their union is smooth. The sections remain unchanged outside that collar.*

Proof. Choose a bundle metric and a connection. Parallel transport normal to L identifies E on a collar with the pullback of $E|_L$. Write both sections in this identification and put $b = \min_{x \in L} |v_0(x)| > 0$. By compactness, decrease the collar width so that

$$|v_\pm(x, r) - v_0(x)| < b/4. \tag{4.3}$$

Let $\kappa(r)$ be a smooth cutoff equal to 1 for $|r|$ sufficiently small and to 0 near the collar ends. On each side set

$$\widetilde{v}_\pm(x, r) = v_0(x) + (1 - \kappa(r))(v_\pm(x, r) - v_0(x)). \tag{4.4}$$

The triangle inequality gives

$$|\widetilde{v}_\pm(x, r)| \geq |v_0(x)| - (1 - \kappa(r))|v_\pm(x, r) - v_0(x)| > 3b/4.$$

Both formulas are the section $v_0(x)$ on an open collar of L. Thus all their derivatives agree there. Near the collar ends they equal the original sections, so they extend by those sections to the asserted smooth section. □

Theorem 4.3. *The circle $\alpha = C \times \{0\}$ bounds a framed compression disk for $\mathcal{T}$. Its disk push-off represents the preferred longitude λ_C in the product framing of $\nu C \times S^1$. The assertion holds for every $m, n \in \mathbb{Z}$ and both orientation patterns of $\Delta \cap J$.*

Proof of Theorem 4.3. The punctured crossing disk. Let $p_+, p_- \in \Delta \cap J$. The subscripts distinguish the two points; they do not denote their intersection signs. Transversality gives disjoint coordinate balls $U_\pm$ in which

$$J \cap U_\pm = \{x_1 = x_2 = 0\}, \qquad \Delta \cap U_\pm = \{x_3 = 0\}.$$

Choose a knot-tube parametrization whose fibers at $p_\pm$ are small disks in the planes $x_3 = 0$. The oriented normal bundle of J in S^3 is trivial. A preliminary frame can be modified away from $U_+ \cup U_-$ by a map $J \to SO(2)$ of any prescribed integer degree: choose a lift on the complementary subarc whose total change is $2\pi k$, constant near its ends. Since the linking number of a framed parallel changes by k, there is a unique degree correction that makes that linking number zero. This produces the preferred framing without changing either of the selected fibers. Shrink the tube so that its intersection with Δ consists only of those two disks; compactness of $\Delta \setminus (U_+ \cup U_-)$ and its disjointness from J permit this choice. Put

$$P_\Delta = \Delta \setminus \operatorname{int}(\Delta \cap \nu J).$$

This is an oriented pair of pants with outer boundary C. Its inner boundaries are geometric meridians of J. Choose the longitude parametrization so that their coordinates are $\lambda = \pi/2$ and $\lambda = 3\pi/2$. Choose an orientation-preserving circle diffeomorphism taking the two selected longitude values to these two values. For an increasing lift $h : \mathbb{R} \to \mathbb{R}$ with $h(x + 2\pi) = h(x) + 2\pi$, the maps $h_t(x) = (1 - t)x + th(x)$ satisfy $h_t'(x) = (1 - t) + th'(x) > 0$ and the same periodicity. They descend to an isotopy from the identity. Extending that isotopy through a boundary collar changes neither the peripheral classes nor the marked gluing class. For phase origins choose a smooth real-valued function $v(\lambda)$ with the two prescribed values and use $(\mu, \lambda) \mapsto (\mu + v(\lambda), \lambda)$. Its isotopy is $(\mu, \lambda) \mapsto (\mu + tv(\lambda), \lambda)$, so it changes neither peripheral homology class. Extend these two isotopies through a boundary collar disjoint from C by Lemma 2.1, and take their products with id_{S^1}. Use the resulting boundary parametrization when selecting the affine representative in (2.13). The product coordinate s is unchanged in this choice, so its transported vector is exactly $m\partial_a + (m + 1)\partial_b + n\partial_c$. Place P_Δ in $E(J) \times \{0\}$. Equation (2.13) sends its inner circles to

$$a = b = t, \qquad c = \pi/2 \quad \text{and} \quad a = b = t, \qquad c = 3\pi/2. \tag{4.5}$$

The parameter t is the positively oriented meridian coordinate; the surface boundary orientations need not agree with it.

The embedded disk. Let

$$D_\pm = \left\{ \left(\frac{w}{\sqrt{2}}, \frac{w}{\sqrt{2}}, \pm\sqrt{1 - |w|^2} \right) \ : \ |w| \le \sqrt{1 - \delta^2} \right\}. \tag{4.6}$$

The coordinate w parametrizes a smooth disk: the square root is smooth on a neighborhood of its closed parameter disk, since $1 - |w|^2 \ge \delta^2 > 0$. At the boundary, $u = 0$, $z = \pm\delta$, and $a = b = \arg w$. The boundary parametrizations are $(a, b, c) = (t, t, \pi/2)$ and $(t, t, 3\pi/2)$, respectively, so they agree with (4.5) as parametrized circles. The interiors lie in $z > \delta$ and $z < -\delta$, respectively. Thus they are disjoint and contained in X. Write $v = \sqrt{u^2 + z^2} - \delta$ on an exterior collar in X. For a sufficiently small $\epsilon > 0$, collar parametrizations of the disks D_+, D_- are

$$c_\pm(t, v) = \left(\sqrt{\frac{1 - (\delta + v)^2}{2}} e^{it}, \sqrt{\frac{1 - (\delta + v)^2}{2}} e^{it}, \pm(\delta + v) \right), \quad 0 \le v < \epsilon. \tag{4.7}$$

In tubular coordinates these have $(a, b, c) = (t, t, \pi/2)$ and $(t, t, 3\pi/2)$, with v as the fourth coordinate. A collar of each inner boundary of P_Δ has coordinates $(\mu, \lambda, s, v) = (t, \lambda_\pm, 0, v)$ for $-\epsilon < v \le 0$. Attach the ambient collars using the affine boundary map and the common signed coordinate v. In the resulting ambient chart the union has the equation

$$(a, b, c, v) = (t, t, \lambda_\pm, v), \qquad -\epsilon < v < \epsilon.$$

The coordinate derivatives in t, v are linearly independent. Thus the union is a smooth embedded surface across both attaching circles. It is

$$D = (P_\Delta \times \{0\}) \cup D_+ \cup D_-. \tag{4.8}$$

For the glued surface, TD is spanned in these charts by $\partial_a + \partial_b$ and ∂_v. The quotient map $TM|_D \to ND = TM|_D/TD$ therefore kills precisely the span of $\partial_a + \partial_b$ and ∂_v on either attaching collar. The differential of (2.13) induces the quotient identification

$$[\partial_s] \longmapsto [m\partial_a + (m+1)\partial_b + n\partial_c] \quad \text{in } TM/\langle \partial_a + \partial_b, \partial_v \rangle. \tag{4.9}$$

The interiors of D_+ and D_- lie in $\operatorname{int} X$ and are disjoint from each other. The pair-of-pants interior lies in $\operatorname{int}(E(J) \times S^1)$, except for its outer boundary, and hence is disjoint from the disks D_+, D_-. Its intersection with $C \times S^1$ is $C \times \{0\}$. Together with the collar parametrizations, these facts prove embeddedness and

$$D \cap \mathcal{T} = \partial D = C \times \{0\}.$$

The two disk orientations are chosen so that the induced orientations on each identified circle are opposite to those of P_Δ. This orients the glued connected surface. Euler characteristic is additive here because the two gluing circles have Euler characteristic zero:

$$\chi(D) = \chi(P_\Delta) + \chi(D_+) + \chi(D_-) - 2\chi(S^1) = -1 + 1 + 1 = 1.$$

The boundary is the single circle C. The classification formula $\chi(D) = 2 - 2g - 1$ gives $g = 0$, so D is a disk. At its outer boundary, its tangent plane is spanned by the tangent of C and the inward tangent to Δ. The tangent plane of $\mathcal{T}$ is spanned by the tangent of C and ∂_s. Their intersection is TC, so the intersection is clean.

The boundary normal section. Use the normal quotient class of ∂_s along $P_\Delta \times \{0\}$. It never vanishes there. Near $C \times \{0\}$ a product metric makes it an orthogonal normal vector to D, and it is the tangent vector to $\mathcal{T}$ transverse to α. Put

$$\rho = \sqrt{\frac{1-\delta^2}{2}}, \qquad \zeta = \frac{z_1 - z_2}{\sqrt{2}}. \tag{4.10}$$

At every point of either disk $D_\pm$, its orthogonal normal plane in S^4 is

$$N = \{(\xi, -\xi, 0) \,:\, \xi \in \mathbb{C}\}. \tag{4.11}$$

For $P(w) = (w/\sqrt{2}, w/\sqrt{2}, z(w))$ and $\xi \in \mathbb{C}$,

$$\langle (\xi, -\xi, 0), P(w) \rangle = 0, \qquad \langle (\xi, -\xi, 0), dP_w(v) \rangle = 0 \quad (v \in \mathbb{C}).$$

These follow by cancellation of the two complex-coordinate real inner products. Thus $N \subset T_{P(w)}S^4 \cap (T_{P(w)}D_\pm)^\perp$. Both spaces have dimension two, so they are equal. The affine boundary map gives

$$\partial_s \longmapsto m\partial_a + (m+1)\partial_b + n\partial_c.$$

At $a = b = t$ its angular normal projection has ζ-coordinate

$$\operatorname{pr}_N\big(m\partial_a + (m+1)\partial_b\big) = \frac{i\rho}{\sqrt{2}}\big(m - (m+1)\big)e^{it} = -\frac{i\rho}{\sqrt{2}}e^{it}. \tag{4.12}$$

At $u = 0$, differentiating (2.2) in u gives $\partial_u\zeta = e^{it}/(\sqrt{2}\rho)$. Since $\partial_c u = -\delta \sin c$ and $\partial_c z = \delta \cos c$, one obtains

$$\operatorname{pr}_N \partial_c = \begin{cases} -\dfrac{\delta}{\sqrt{2}\rho}e^{it}, & c = \pi/2, \\ +\dfrac{\delta}{\sqrt{2}\rho}e^{it}, & c = 3\pi/2. \end{cases} \tag{4.13}$$

Thus the boundary normal sections on the disks D_+, D_- are

$$v_\pm(t) = C_\pm e^{it}, \qquad C_\pm = \frac{1}{\sqrt{2}}\left(-i\rho \mp \frac{n\delta}{\rho}\right). \tag{4.14}$$

For both signs,

$$|C_\pm|^2 = \frac{1}{2}\left(\rho^2 + \frac{n^2\delta^2}{\rho^2}\right) \geq \frac{\rho^2}{2} > 0. \tag{4.15}$$

Thus each boundary section is nonzero, including when $n = 0$. The identity $m - (m+1) = -1$ makes these sections independent of m.

The orientations. Let $w = (z_1 + z_2)/\sqrt{2}$ on the diagonal plane. The change from (z_1, z_2) to (w, ζ) is complex linear and has positive real determinant. At the north pole $z = 1$, the ordered basis

$$(\partial_{\Re w}, \partial_{\Im w}, \partial_{\Re\zeta}, \partial_{\Im\zeta})$$

is positive for TS^4. At the south pole $z = -1$, it is negative. The signs can also be computed away from the poles. In coordinates $(\Re w, \Im w, \Re\zeta, \Im\zeta, z)$, put $w = x + iy$. On either disk $D_\pm$, $x^2 + y^2 + z^2 = 1$ and $z \neq 0$. The position vector and a tangent-normal frame have determinant

$$\det\begin{pmatrix} x & 1 & 0 & 0 & 0 \\ y & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ z & -x/z & -y/z & 0 & 0 \end{pmatrix} = \frac{x^2 + y^2 + z^2}{z} = \frac{1}{z}. \tag{4.16}$$

It is positive on D_+ and negative on D_-. Orient D by the orientation of P_Δ. For each disk $D_\pm$ let $\varepsilon_\pm \in \{1, -1\}$ record agreement of this orientation with its complex w-orientation. Choose a metric on ND equal to the Euclidean metric on the disks D_+, D_-, and let J_N be rotation by $+\pi/2$ in the oriented normal plane. In the coordinate (4.11),

$$J_N = \varepsilon_+ i \quad \text{on } D_+, \qquad J_N = -\varepsilon_- i \quad \text{on } D_-. \tag{4.17}$$

Reversal of the orientation of either cap reverses both its induced boundary orientation and the orientation of its normal plane. Thus (4.17) holds for each of the four values of $(\varepsilon_+, \varepsilon_-)$.

The normal section on the disk. Prescribe the boundary phases

$$u|_C = 1, \qquad u_+(t) = e^{-i\varepsilon_+ t}, \qquad u_-(t) = e^{i\varepsilon_- t}. \tag{4.18}$$

On an inner boundary component, the orientation of P_Δ is opposite to that of the corresponding disk. Relative to the increasing-t parametrization it is therefore $-\varepsilon_\pm$. The degrees in (4.18) are

boundary component	C	∂D_+	∂D_-
degree in the P_Δ orientation	0	1	-1.

(4.19)

The four sign substitutions are:

$(\varepsilon_+, \varepsilon_-)$	boundary signs on P_Δ	phase exponents	deg u_+	deg u_-
$(+1, +1)$	$(-1, -1)$	$(-1, +1)$	$+1$	-1
$(+1, -1)$	$(-1, +1)$	$(-1, -1)$	$+1$	-1
$(-1, +1)$	$(+1, -1)$	$(+1, +1)$	$+1$	-1
$(-1, -1)$	$(+1, +1)$	$(+1, -1)$	$+1$	-1

(4.20)

Thus the degree condition is $0 + 1 - 1 = 0$ in every case. Apply Lemma 4.1 to obtain a smooth $u : P_\Delta \to S^1$, equal to 1 on the outer boundary and constant in each collar coordinate. Let v be the normal quotient class of ∂_s on P_Δ. Define

$$v^u = (\Re u)\, v + (\Im u)\, J_N v. \tag{4.21}$$

Since J_N is an isometry, $\langle v, J_N v\rangle = 0$, and $|u| = 1$, one has $|v^u|^2 = |v|^2 > 0$. At the boundary of D_+, (4.17) makes the ordinary complex multiplier

$$\exp\big(i\varepsilon_+(-\varepsilon_+ t)\big) = e^{-it}.$$

At the boundary of D_- it is

$$\exp\bigl(-i\varepsilon_-(\varepsilon_- t)\bigr) = e^{-it}.$$

Consequently (4.14) becomes the constant vector C_+ on one circle and the constant vector C_- on the other. Extend them as constant sections of the plane (4.11) over the disks D_+, D_-. Apply Lemma 4.2 to the two attaching circles, with the common nonzero values C_+ and C_- in the anti-diagonal coordinates. Choose its disjoint collars to miss C. The resulting section is smooth and nowhere zero on D and restricts to $[\partial_s]$ on a collar of C. In particular, with the relative Euler-number convention determined by that boundary section,

$$e(ND, [\partial_s]|_C) = 0. \tag{4.22}$$

The peripheral class. Near C, the disk (4.8) equals $\Delta \times \{0\}$. Let C_ϵ be its inward parallel, oriented by the projection $C_\epsilon \to C$. A two-sided product neighborhood of Δ in S^3 contains $\Delta \times [-h, h]$. For $h > 0$, the disk $\Delta_h = \Delta \times \{h\}$ is disjoint from C_ϵ, and its boundary C_h is isotopic to C in the complement of C_ϵ. Hence

$$\mathrm{lk}(C_\epsilon, C) = \mathrm{lk}(C_\epsilon, C_h) = C_\epsilon \cdot \Delta_h = 0.$$

Choose the parallels sufficiently close to C that they are disjoint from J. In the basis (μ_C, λ_C, s), the coefficient of λ_C is the projection degree 1, the coefficient of μ_C is the linking number 0, and the coefficient of s is 0 because the curve lies in the slice $s = 0$. Thus

$$[\alpha^0] = (0, 1, 0) = [\lambda_C] \quad \text{in } H_1(\partial(\nu C \times S^1); \mathbb{Z}). \tag{4.23}$$

The constructed disk and normal section satisfy Definition 3.1, with the asserted push-off. ☐

5. Crossing-circle fillings and four-moves

Let $J^{(d)}$ be obtained from J by $-1/d$ surgery on a crossing circle C, where $d \neq 0$. Since C bounds a disk in S^3, its exterior is a solid torus $S^1_{\mu_C} \times D^2_w$, with $\arg w = \lambda_C$ on the boundary. The map

$$(v, w) \longmapsto (v, e^{-idv} w)$$

extends over that solid torus and sends μ_C to $\mu_C - d\lambda_C$, while fixing λ_C. Its inverse sends the slope $\mu_C - d\lambda_C$ to μ_C. Extending this boundary identification across the two attached solid tori identifies the filling along $\mu_C - d\lambda_C$ with the meridional filling, hence with S^3. In the disk-twist convention fixed in Section 1, the two strands crossing the disk acquire d positive full twists. The identification of the filled link exterior is made in dimension three and then multiplied by the identity of S^1; it does not change the product coordinate s.

Lemma 5.1. *In the peripheral coordinates on the unfilled J-boundary,*

$$\mu_{J^{(d)}} = \mu_J, \qquad \lambda_{J^{(d)}} = \lambda_J - d\ell^2 \mu_J, \qquad \ell = \mathrm{lk}(C, J). \tag{5.1}$$

Proof. The first homology of the two-component link exterior is freely generated by μ_J, μ_C. The two preferred longitudes satisfy

$$[\lambda_J] = \ell[\mu_C], \qquad [\lambda_C] = \ell[\mu_J].$$

The map

$$H_1(E(J \cup C); \mathbb{Z}) \longrightarrow \mathbb{Z}^2, \quad [\gamma] \longmapsto (\mathrm{lk}(\gamma, J), \mathrm{lk}(\gamma, C))$$

sends μ_J, μ_C to $(1, 0), (0, 1)$. It is an isomorphism by Alexander duality. The images of λ_J, λ_C are $(0, \ell), (\ell, 0)$, respectively, which gives the displayed relations. Mayer–Vietoris for the filling gives

$$H_1(E(J^{(d)}); \mathbb{Z}) = \frac{\mathbb{Z}\langle \mu_J \rangle \oplus \mathbb{Z}\langle \mu_C \rangle}{\langle -\mu_C + d\ell \mu_J \rangle} \cong \mathbb{Z}\langle \mu_J \rangle. \tag{5.2}$$

The coefficient of μ_C in the relation is -1, so elimination gives $[\mu_C] = d\ell[\mu_J]$ and $[\lambda_J] = d\ell^2[\mu_J]$ without torsion. The surgery is supported in νC, which is disjoint from νJ; the inclusion of each meridional disk of νJ is unchanged. Thus the oriented meridian remains μ_J. A preferred longitude projects with degree one to J and is null-homologous in its exterior. In the quotient homology, the class of $\lambda_J + t\mu_J$ is $(d\ell^2 + t)[\mu_J]$. Since $[\mu_J]$ generates an infinite cyclic group, this class vanishes exactly when $t = -d\ell^2$. The pair of coefficients $(-d\ell^2, 1)$ is primitive. Thus the calculation identifies the preferred slope on the unfilled boundary torus, not only its image in the exterior. □

Proposition 5.2. *Product $-1/d$ surgery on $\mathcal{T} = C \times S^1 \subset \mathcal{G}_{m,n}(J)$ produces*

$$\mathcal{G}_{m+nd\ell^2,n}(J^{(d)}). \tag{5.3}$$

If d is even, this manifold is diffeomorphic to $\mathcal{G}_{m,n}(J)$.

Proof. Taking the product of the Dehn filling with S^1 replaces $E(J) \times S^1$ by $E(J^{(d)}) \times S^1$. By Lemma 5.1, the new boundary map into X is

$$\mu' \longmapsto A, \qquad \lambda' \longmapsto c - d\ell^2 A, \qquad s \longmapsto mA + b + nc. \tag{5.4}$$

Postcompose with the diffeomorphism of X from Lemma 2.4, for $q = d\ell^2$. It fixes the meridian image, sends the longitude image to c, and sends the product circle to

$$mA + b + n(c + d\ell^2 A) = (m + nd\ell^2)A + b + nc.$$

Equivalently, put $q = d\ell^2$ and

$$L_q = \begin{pmatrix} 1 & -q & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The preferred-coordinate change is $B_{m,n}L_q$, and

$$E_q B_{m,n} L_q = \begin{pmatrix} 1 & 0 & m + nq \\ 1 & 0 & m + nq + 1 \\ 0 & 1 & n \end{pmatrix} = B_{m+nq,n}. \tag{5.5}$$

Let $f_q : X \to X$ extend E_q. The new attachment is $B_{m,n}L_q$. Since $f_q \circ (B_{m,n}L_q) = B_{m+nq,n}$ on the boundary after the collar adjustment, $f_q \cup \mathrm{id}_{E(J^{(d)})\times S^1}$ descends to the asserted diffeomorphism. The meridian of the product surgery is $\mu_C - d\lambda_C$. By Theorem 4.3, λ_C is the disk push-off of a framed compression direction on $\mathcal{T}$. If $d = 2q$, Theorem 3.3 applies with coefficient $-2q$. It identifies the surgery result with the original manifold, relative to the complement of the compression neighborhood. □

Proof of Theorem 1.1. The twist-parameter equality in (1.4) is Proposition 2.3. A four-move has coefficient $d = 2$ or $d = -2$ in Proposition 5.2. Both possible strand orientations are included in Theorem 4.3. Applying these two propositions gives

$$\mathcal{G}_{m,n}(J^{(d)}) \cong \mathcal{G}_{m,n}(J) \qquad (d = \pm 2).$$

Let $j_t : S^1 \to S^3$ parametrize an oriented knot isotopy. In a tubular neighborhood of its trace in $[0,1] \times S^3$, extend the section $\partial_t j_t$ of the pulled-back tangent bundle and multiply by a cutoff equal to one on the trace. This gives a smooth time-dependent vector field W_t satisfying $W_t(j_t(x)) = \partial_t j_t(x)$. Its flow F_t exists for $0 \le t \le 1$ because S^3 is compact. Uniqueness for the flow equation gives $F_t \circ j_0 = j_t$. The maps F_t preserve orientation because $F_0 = \mathrm{id}$. The meridian is carried to a meridian of linking number $+1$, and the degree-one null-homologous peripheral curve is carried to the preferred longitude. The product of the exterior diffeomorphism with id_{S^1} respects the three attaching classes. Apply Lemma 2.1 to match the selected attaching representatives. For a sequence $J = J_0, J_1, \ldots, J_N = J'$, compose the resulting N diffeomorphisms. Apply Proposition 2.3 at the endpoint to change m to m'. This gives (1.3). The last assertion follows from Lemma 2.5. Each comparison uses its own compression disk; no simultaneous system of disjoint disks is required. □

6. Classical reductions and standard Gluck manifolds

6.1. **Classical four-move reductions.** A Conway 2-algebraic tangle is obtained from two-strand tangles with at most one crossing by horizontal composition and quarter-turn rotations. A 2-algebraic knot is a closure of such a tangle without introducing additional crossings. Przytycki proves that every 2-algebraic knot and every closed three-braid knot is four-move equivalent to the unknot [20, Proposition 3.2 and Theorem 3.6]. Rational tangles are contained in this class, so their knot closures include all two-bridge knots. Transport the orientation of the initial knot through each local replacement and isotopy. A four-move has the same oriented endpoints before and after the replacement, and Theorem 4.3 includes both strand-orientation patterns. The two orientations of the unknot are equivalent by an orientation-preserving isotopy of S^3: on the coordinate unknot $\{(e^{it},0)\} \subset S^3 \subset \mathbb{C}^2$, the map $(z_1,z_2) \mapsto (\bar{z}_1,\bar{z}_2)$ reverses that circle, has real determinant $+1$, and is joined to the identity in $SO(4)$. Thus the cited reductions give $J \sim_4 U$ with the oriented conventions of this paper. Apply Theorem 1.1 to prove Corollary 1.3. The same deduction applies to every knot for which a four-move reduction is given. In particular, the reduction through twelve crossings proved by Dąbkowski–Jablan–Khan–Sahi [4] and recorded in [20, Introduction and discussion following Theorem 3.7] gives standardness for that list. No enumeration is used in the proof of Theorem 1.1.

6.2. **The one-roll spin of** 7_3. Use the Artin braid group B_3 with relation $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$. Put

$$\beta_j = \sigma_1^j\sigma_2\sigma_1^{-1}\sigma_2. \tag{6.1}$$

The Knot Atlas gives $7_3 = \widehat{\beta_5}$ with this choice of handedness [15]. The braid relation gives

$$\sigma_1\sigma_2\sigma_1^{-1} = \sigma_2^{-1}\sigma_1\sigma_2, \qquad \beta_1 = \sigma_2^{-1}(\sigma_1\sigma_2)\sigma_2.$$

Thus $\widehat{\beta_1}$ is the closure of $\sigma_1\sigma_2$. Deleting the last strand with its single crossing gives the closure of σ_1 in B_2; deleting its last strand gives the one-strand unknot. Each step is the Markov destabilization of a braid of the form $\gamma\sigma_{n-1}^{\pm 1}$ with $\gamma \in B_{n-1}$. For this operation, close γ outside its braid cylinder and replace a short subarc of the closing arc by the additional strand and a Reidemeister-I curl. The added arc lies in a ball disjoint from the other closing arcs. The Reidemeister-I isotopy in this ball identifies the closure of $\gamma\sigma_{n-1}^{\pm 1}$ with that of γ. Here the two instances are $\gamma = \sigma_1, n = 3$ and $\gamma = 1, n = 2$. A four-move removes the initial factor σ_1^4 from β_5. Hence

$$\mathcal{G}_{m,n}(7_3) \cong S^4 \qquad (m,n \in \mathbb{Z}). \tag{6.2}$$

For the one-roll sphere the full marked calculation is as follows. Let $U = \widehat{\beta_1}$ and let C encircle the first two strands immediately before the initial σ_1. Orient the strands downward and orient the crossing disk so that $\ell = 2$. Surgery of coefficient $-1/2$ inserts σ_1^4, giving 7_3; Lemma 5.1 gives $\lambda_{7_3} = \lambda_U - 8\mu_U$. Start with $\mathcal{G}_{-8,1}(U) \cong S^4$. Its even product surgery is trivial by Theorems 3.3 and 4.3. The boundary matrix after filling, and the extending exterior correction, satisfy

$$\begin{pmatrix} 1 & 0 & 8 \\ 0 & 1 & 8 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -8 & -8 \\ 1 & -8 & -7 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} = B_{0,1}. \tag{6.3}$$

Consequently $\Sigma_{\rho^1(7_3)} = \mathcal{G}_{0,1}(7_3) \cong S^4$. Mirroring the braid gives the corresponding computation for the other handedness, with the crossing-surgery signs reversed. More generally, for $q \in \mathbb{Z}$ set

$$J_q = \widehat{\sigma_1^{4q+1}\sigma_2\sigma_1^{-1}}\sigma_2. \tag{6.4}$$

Write $\sigma_1^{4q+1} = (\sigma_1^4)^q\sigma_1$. For $q > 0$, delete the q positive blocks; for $q < 0$, delete the $-q$ negative blocks in $(\sigma_1^{-4})^{-q}\sigma_1$; for $q = 0$ there is no block. Every deletion is a signed four-move. Under $B_3 \to S_3$, $\sigma_1 \mapsto (12)$ and $\sigma_2 \mapsto (23)$, the image is

$$(12)(23)(12)(23) = (132)$$

with rightmost-first composition. Since the number of components of a braid closure is the number of cycles of its strand permutation, every J_q is a knot. Hence $J_q \sim_4 U$ and $\mathcal{G}_{m,n}(J_q) \cong S^4$ for every $q, m, n \in \mathbb{Z}$.

6.3. **Two further examples.**

Corollary 6.1. *For the two explicit eight-plat knots K_A, K_B in* (B.2)*, every twist–roll Gluck manifold is standard:*

$$\mathcal{G}_{m,n}(K_A) \cong S^4, \qquad \mathcal{G}_{m,n}(K_B) \cong S^4 \qquad (m, n \in \mathbb{Z}).$$

Proof. The complete finite reductions in Theorem B.1 give $K_A \sim_4 U$ and $K_B \sim_4 U$. Apply Theorem 1.1. □

7. Twisted doubles

Let $D_J \subset B^4$ be the canonical ribbon disk for $J \# (-J)$, obtained by doubling a knotted arc. Here $-J$ denotes the reverse of the mirror of J. Write

$$C_h(J) = \big(B^4 \setminus \operatorname{int} \nu D_J\big) \cup H_h, \tag{7.1}$$

where H_h is attached along a meridian with integer framing h. We use the markings in Naylor–Schwartz [18, Definition 1.10 and the discussion preceding Corollary 2.6]. The two sides of the ribbon exterior give the swallow-follow torus. Choose the meridian-handle attaching region disjoint from a collar of that torus. The meridional and longitudinal translations define boundary diffeomorphisms g and f, supported in the collar and equal to the identity near the handle.

In angular coordinates $(\mu, \lambda, t) \in T^2 \times [0, 1]$, choose χ equal to zero near 0 and one near 1, and use the representatives

$$g(\mu, \lambda, t) = (\mu + 2\pi\chi(t), \lambda, t), \qquad\qquad f(\mu, \lambda, t) = (\mu, \lambda + 2\pi\chi(t), t). \tag{7.2}$$

They glue to the identity at both collar ends because their terminal translations are integral turns. Their inverses are obtained by negating the translation. Their triangular derivatives have determinant one, and they commute as actual maps. For $p, q \in \mathbb{Z}$ set

$$\mathcal{D}_{h;p,q}(J) = C_h(J) \cup_{g^p f^q} \big(-C_h(J)\big). \tag{7.3}$$

The sign of each generator is chosen to agree with the twist–roll convention in (1.2). Replacing either generator by its inverse only reverses the corresponding integer parameter.

Proposition 7.1. *With these markings,*

$$\mathcal{D}_{h;p,q}(J) \cong \begin{cases} S^4, & h \text{ even}, \\ \mathcal{G}_{p,q}(J), & h \text{ odd}. \end{cases} \tag{7.4}$$

In particular, the diffeomorphism type is independent of p:

$$\mathcal{D}_{h;p,q}(J) \cong \mathcal{D}_{h;0,q}(J). \tag{7.5}$$

Proof. We use two marked constructions from [18], and then check their compatibility with both boundary twists. Definition 1.10 identifies the double of the ribbon-disk pair, using the meridional and longitudinal torus twists, with the twist–roll spun pair $(S^4, \tau^p \rho^q(J))$. For the longitudinal double, the handle identification preceding Corollary 2.6 replaces the meridian handle and its dual in the doubled ribbon exterior by a copy of $S^2 \times D^2$, attached by the hth power of the Gluck boundary map. This identification is local to the handle region and does not use the subsequent unknotting-number-one standardness theorem. To include a meridional twist, it suffices to check that its representative leaves this framed region fixed, as follows.

Let A_h be the attaching region of the meridian handle, equipped with its framed tubular coordinates. The chosen representatives satisfy

$$(g^p f^q)|_{A_h} = \mathrm{id}_{A_h}, \qquad D(g^p f^q)|_{TA_h} = \mathrm{id}_{TA_h}\,. \tag{7.6}$$

Consequently the two handle pieces have exactly the attaching maps and normal framings in the cited local identification; its exponent remains h. After their removal, the remaining marked manifold is exactly the exterior of $\tau^p \rho^q(J)$ supplied by Definition 1.10. We obtain

$$\mathcal{D}_{h;p,q}(J) \cong E\big(\tau^p \rho^q(J)\big) \cup_{\gamma^h} (S^2 \times D^2). \tag{7.7}$$

In the local identification the filling map is represented by

$$\gamma^h(x, e^{i\theta}) = (R_{h\theta} x, e^{i\theta}). \tag{7.8}$$

Its isotopy class depends only on h mod 2, since the rotation loop generates $\pi_1(SO(3)) \cong \mathbb{Z}/2$. For even h, Lemma 2.1 replaces γ^h by the identity; the resulting filling restores the original S^4. For odd h, the attachment is isotopic to γ, giving $\mathcal{G}_{p,q}(J)$. This proves (7.4). For even h, both sides of (7.5) are S^4; for odd h, apply Proposition 2.3. □

For the longitudinal subfamily, write $\mathcal{D}_{h,q}(J) = \mathcal{D}_{h;0,q}(J)$. Thus the usual double formula is

$$\mathcal{D}_{h,q}(J) \cong \begin{cases} S^4, & h \equiv 0 \pmod 2, \\ \mathcal{G}_{0,q}(J), & h \equiv 1 \pmod 2. \end{cases} \tag{7.9}$$

Corollary 7.2. *If $J \sim_4 U$, then $\mathcal{D}_{h;p,q}(J) \cong S^4$ for every $h, p, q \in \mathbb{Z}$.*

Proof. For even h, use the first line of (7.4). For odd h, use its second line and Theorem 1.1. The theorem includes every pair of twist and roll parameters, hence includes (p, q). □

Proof of Theorem 1.2. Gompf identifies $C(r, s; h)$ with $C_h(\kappa(r, -s))$ in the stated meridional handle convention [9, Section 1]. The knot $\kappa(r, -s)$ is two-bridge [9, Remark 2.1(c)], so Corollary 1.3 applies. Its longitudinal double in (1.5) is $\mathcal{D}_{h;0,k}(\kappa(r, -s))$. Apply Corollary 7.2. All $r, s > 0$ give two-bridge knots of this form; Corollary 7.2 holds for every integer iterate and every handle framing. Restricting to $h < 0$ therefore retains every parameter in the asserted original cork family. Reversing the longitude convention replaces k by $-k$, which does not change the set of asserted integers. □

Corollary 7.3. *For $r, s > 0 > h$ and $p, q \in \mathbb{Z}$, every double of $C(r, s; h)$ obtained by the simultaneous meridional and longitudinal twist $g^p f^q$ is S^4.*

Proof. Substitute $J = \kappa(r, -s)$ in Corollary 7.2. □

Proposition 7.4. *Let $r \geq 1$, let $J_1, \ldots, J_r$ be knots with $J_i \sim_4 U$, and let*

$$C = C_{h_1}(J_1) \natural \cdots \natural C_{h_r}(J_r).$$

Choose the boundary-connected-sum balls disjoint from the torus-twist supports. Extend each $g_i^{p_i} f_i^{q_i}$ by the identity to ∂C, and put $\Phi = \prod_i g_i^{p_i} f_i^{q_i}$. Then

$$C \cup_\Phi (-C) \cong S^4. \tag{7.10}$$

Proof. For two summands, the boundary-connected-sum handle is $[-1, 1] \times D^3$. The gluing map is the identity near its feet. Doubling its lateral boundary therefore gives

$$[-1, 1] \times (D^3 \cup_{S^2} D^3) = [-1, 1] \times S^3.$$

The two endpoint S^3's bound the balls removed from the two individual doubles. Thus this product is precisely a connected-sum neck, with product collars on its attachments. The complement of its interior consists of those punctured doubles. Iterating over the tree of boundary-sum handles gives

$$C \cup_\Phi (-C) \cong \mathop{\#}_{i=1}^{r} \mathcal{D}_{h_i;p_i,q_i}(J_i). \tag{7.11}$$

Each summand is S^4 by Corollary 7.2. The connected sums in (7.11) use coordinate balls. After choosing diffeomorphisms of the summands to S^4, choose these balls in coordinate charts; their complements in S^4 are standard balls. The connected sum of finitely many copies of S^4 is therefore S^4. The case of one summand is the same corollary. □

8. Relative exterior maps

The local proof gives an extension on the exterior side as well as the relative closed-surgery comparison.

Theorem 8.1. *Let W be a compact oriented four-manifold with a boundary component parametrized by $(\alpha, \beta, \theta) \in T^3$. Suppose that $S^1_\alpha \times \{(\beta_0, \theta_0)\}$ bounds a properly embedded disk whose normal framing is the product framing $(\partial_\beta, \partial_\theta)$. For every $q \in \mathbb{Z}$, there is an orientation-preserving diffeomorphism of W, equal to the identity near its other boundary components, whose restriction to the specified boundary collar is*

$$(\alpha, \beta, \theta, r) \longmapsto (\alpha + 2q\theta, \beta, \theta, r). \tag{8.1}$$

Its support lies in a collar of that boundary component together with a neighborhood of the disk.

Proof. The collar and the disk neighborhood form

$$N = (T^3 \times [0, 1]) \cup h^2_0. \tag{8.2}$$

The zero framing makes the attachment $S^1_\alpha \times B$, for a disk $B \subset T^2_{\beta,\theta}$. Its outgoing boundary is the manifold in (3.5). The map (3.7), applied on every collar level, is the identity near this entire attaching region. Extend it by the identity on the handle. For coefficient $2q$, the resulting diffeomorphism of N restricts to ϕ_{2q} incoming and to ψ_{2q} outgoing, with the localized representatives of (3.7) and (3.8). By (3.10) and the full fundamental-group calculation in (3.9), ψ_{-2q} is isotopic to the identity. Extend such an isotopy into an outgoing collar by Lemma 2.1, and postcompose. On a smaller outgoing collar the composite is exactly $\psi_{-2q}\psi_{2q} = \mathrm{id}$. It therefore extends by the identity over the rest of W.

It remains to select the affine incoming representative. In the notation used to construct h in Theorem 3.3, the maps

$$(e^{i\alpha}, \beta, \theta) \longmapsto \left(e^{i\alpha} e^{2qi\theta} e^{-2qi(1-t)\chi\widetilde{\theta}}, \beta, \theta\right), \qquad 0 \le t \le 1, \tag{8.3}$$

are an isotopy from the localized shear to the affine shear. The phase has modulus one; its reciprocal gives the inverse at every t. The cutoff logarithm is a globally smooth real function, so the isotopy is well-defined across the boundary of its coordinate disk. Apply Lemma 2.1 to the ratio of this isotopy with its initial map, on the incoming collar only. The outgoing identity is unchanged. All factors preserve orientation and have smooth inverses; their composition proves the assertion. □

Corollary 8.2. *For the crossing torus $\mathcal{T} = C \times S^1$ in $\mathcal{G}_{m,n}(J)$, let*

$$Z_{m,n}(J, C) = \mathrm{cl}\big(\mathcal{G}_{m,n}(J) \setminus (\nu C \times S^1)\big).$$

For every $q \in \mathbb{Z}$, the map

$$(\mu_C, \lambda_C, s) \longmapsto (\mu_C, \lambda_C + 2q\mu_C, s) \tag{8.4}$$

extends over this exterior, with the indicated product formula on a boundary collar.

Proof. Trim the disk of Theorem 4.3 at the boundary of the selected tube. Its proper boundary is the preferred longitude, and its normal framing is the product framing by the local coordinate calculation in Theorem 3.3. Apply Theorem 8.1 with $(\alpha, \beta, \theta) = (\lambda_C, s, \mu_C)$. This reordering is an even permutation and retains the orientation convention. □

Corollary 8.3. *If $V = T^2 \times D^2$ is attached to the exterior in Corollary 8.2 by the shear of coefficient k, changing k to $k + 2q$ gives a diffeomorphism which is the identity on the abstract inserted V.*

Proof. Let F_q be the exterior map supplied by that corollary. On the common boundary,

$$F_q\phi_k = \phi_{2q}\phi_k = \phi_{k+2q}. \tag{8.5}$$

Hence $F_q \cup \mathrm{id}_V$ respects the two attaching equivalence relations. Its inverse is $F_q^{-1} \cup \mathrm{id}_V$. Their product formulas on the collars give smoothness of both glued maps. □

Theorem 8.4. *Let $C_1, \dots, C_r$ be disjoint oriented circles in $E(J)$, with fixed disjoint tubular parametrizations. Suppose J is changed to J' by a finite sequence of oriented isotopies and four-moves whose crossing disks lie in balls disjoint from these tubes. Assume the isotopies are the identity near the tubes. Then, for every $m, n \in \mathbb{Z}$, there is a diffeomorphism*

$$\left(\mathcal{G}_{m,n}(J), \coprod_{i=1}^{r} \nu C_i \times S^1\right) \longrightarrow \left(\mathcal{G}_{m,n}(J'), \coprod_{i=1}^{r} \nu C_i \times S^1\right) \tag{8.6}$$

which is the identity on all the indicated abstract tubes and on smaller surrounding collars.

Proof. For one move, let A be its crossing circle and let Q be the ball containing its crossing disk Δ_A. The disk in Theorem 4.3 consists of

$$(\Delta_A \setminus \mathrm{int}\, \nu J) \times \{0\} \quad \text{and two caps in } X.$$

The first piece misses every auxiliary tube because Q does; the caps miss them because those tubes lie in the product piece. The coordinate changes on the J-boundary can be supported in a collar disjoint from the auxiliary tubes. Compactness then permits a regular neighborhood of the crossing torus and its disk disjoint from all those tubes. The diffeomorphism in Theorem 3.3 is the identity off this neighborhood.

The three-dimensional filling comparison can be supported in the crossing ball and is the identity near every C_i. For example, choose a product neighborhood of Δ_A in Q; the disk-twist replacement is the identity outside that neighborhood, and the standard filled ball is identified with Q relative to its boundary. The preferred longitude of each auxiliary circle is unchanged by this ambient identification, since meridional linking number and projection degree are preserved. Equivalently, the possible surgery correction involves $\mathrm{lk}(A, C_i)$, which is zero because $\Delta_A \cap C_i = \emptyset$. Taking the product with id_{S^1} preserves the product coordinates. The exterior corrections in Lemma 2.4 and Proposition 2.3 act on X and are the identity on the entire product piece. Thus the composite comparison for one four-move is the identity near all the auxiliary tubes.

For an isotopy of J fixed near the tubes, extend its velocity field by zero on their neighborhoods and integrate as in the proof of Theorem 1.1. The resulting ambient isotopy retains the parametrizations, and its product with id_{S^1} has the same property. Compose the finitely many comparisons in their prescribed order. Every inverse retains the same tubes, proving (8.6). □

Corollary 8.5. *Under the hypotheses of Theorem 8.4, restriction of* (8.6) *gives a diffeomorphism of the auxiliary-torus exteriors equal to the identity on their marked boundary collars. For any prescribed boundary diffeomorphism, smooth extendability is the same for the two exteriors.*

Proof. Let Ψ denote the restricted map, and suppose g' extends the boundary map on the target exterior. Then $\Psi^{-1}g'\Psi$ extends the identical boundary representative on the source exterior. The inverse implication uses $\Psi g \Psi^{-1}$. The formulas are smooth and have smooth inverses, and the collar identity for Ψ makes both boundary restrictions exact. □

9. The longitudinal decomposition of a cork double

Fix $r, s > 0 > h$ throughout this section, and put

$$K = \kappa(r, -s), \qquad P = E(K), \qquad C = C(r, s; h), \qquad Y = \partial C. \tag{9.1}$$

The objects classified below are the pairs consisting of a double and one of its cork regions, and the pairs consisting of a double and the common boundary of its regions. The absolute-exponent obstruction for $C(2,1;-2N)$ is due to Auckly–Ruberman [2, Section 2]; the two-end argument below uses the extension quotient in Proposition 9.3.

9.1. **The end exteriors and their twists.** We use the following boundary data from Gompf's construction [9, Section 1, the paragraphs preceding Theorem 1.5]. The manifold C is $I \times P$ with an h-framed meridional two-handle. Its boundary contains two oppositely oriented end copies P_0, P_1 of P. Their unordered pair is determined, up to isotopy, by the JSJ decomposition of Y. For $|h| > 1$, the two end boundaries are distinct canonical tori. For $|h| = 1$, the intervening product is suppressed: P_0 and P_1 are the two sides of the remaining canonical torus. This is the boundary description also used in the opening of [10, Section 3]. Only the preservation of the unordered end pieces, not preservation of either chosen end separately, is used below.

We also use the following two facts:

$$f^k \text{ does not extend over } C \quad (k \neq 0), \tag{9.2}$$

by [9, Theorem 1.5]; and K admits a strong inversion, since it is two-bridge [6, Section 2.3]. A strong inversion is an orientation-preserving involution of S^3 preserving K and reversing its orientation, with two fixed points on K.

We first fix the end parametrizations. Move the meridian-handle attaching circle from $\{1\} \times P$ to the middle of $I \times \partial P$. This motion follows a meridional annulus in a collar of ∂P and the rounded interval–collar quadrant. It can be chosen independent of the meridian angle. The comparison of the ordered normal frames is then constant in that angle, so the transported integer framing acquires no additional winding. Extend the velocity of this circle motion to a vector field in its tubular neighborhood and multiply by a cutoff; the resulting flow gives an ambient isotopy of the boundary. Its collar extension identifies the two handle attachments. Choose the original longitudinal twist on the end side of this attaching motion. Its supporting torus can be isotoped to the collar of P_0 through the product region disjoint from the attachment. This identifies its class with the first end twist below, up to one common choice of longitude sign.

Use coordinates $x \in (\mathbb{R}/2\pi\mathbb{Z})^2$ on ∂P, and let $v \in \mathbb{Z}^2$ be the longitude vector. For an inward collar $\partial P \times [0, \varepsilon]$, let χ be zero near 0 and one near ε. Define

$$q_v(x, a) = (x + 2\pi\chi(a)v, a) \quad \text{on the collar}, \qquad q_v = \mathrm{id} \quad \text{elsewhere in } P. \tag{9.3}$$

The formulas agree on a neighborhood of the inner collar end because translation by $2\pi v$ is the identity. They are the identity near the actual boundary. The inverse is q_{-v}, the derivative is triangular with diagonal $(I_2, 1)$, and

$$q_v q_w = q_{v+w}. \tag{9.4}$$

Let f_i be this map on P_i, extended by the identity on the remainder of Y. The supports are disjoint; hence $f_0 f_1 = f_1 f_0$ as actual maps. Choose the common longitude sign so that $[f_0] = [f]$.

Lemma 9.1. *The product $z = f_0 f_1$ extends to an orientation-preserving diffeomorphism of C.*

Proof. Use $\mathrm{id}_I \times q_v$ on $I \times P$. Choose the attaching parametrization $j_h : S^1 \times D^2 \to \partial(I \times P)$ with image in the region where $q_v = \mathrm{id}$. Then

$$(\mathrm{id}_I \times q_v) \circ j_h = j_h, \qquad D(\mathrm{id}_I \times q_v) \circ Dj_h = Dj_h.$$

These identities retain the entire framed attaching map. It extends by the identity on the handle, for every integer h. The inverse uses q_{-v} on the product and the identity on the handle. The formulas agree on open attaching collars and therefore also on the rounded corners. The boundary restriction is f_0 on one end, f_1 on the other, and the identity on the rest of Y. □

9.2. **The extension subgroup.** Define

$$\mathcal{M} = \pi_0 \operatorname{Diff}^+(Y), \qquad \mathcal{H} = \operatorname{im}\big(\pi_0 \operatorname{Diff}^+(C) \longrightarrow \mathcal{M}\big), \qquad \mathcal{A} = \langle [f_0], [f_1] \rangle. \tag{9.5}$$

Restriction respects products, inverses and isotopies, so $\mathcal{H}$ is a subgroup. It is also the extension subgroup for the underlying copy $-C$: reversing the orientations of both source and target does not change the sign of a self-map. By Lemma 2.1, if one representative of a boundary mapping class extends, then every representative extends.

Lemma 9.2. *The subgroup $\mathcal{A}$ is abelian and normal in $\mathcal{M}$.*

Proof. Commutativity follows from the disjoint-support representatives. For either end exterior,

$$\ker\big(H_1(\partial P_i; \mathbb{Z}) \longrightarrow H_1(P_i; \mathbb{Z})\big) = \mathbb{Z}\lambda_i. \tag{9.6}$$

Indeed, the meridian maps to a generator of $H_1(P_i) = \mathbb{Z}$, and the preferred longitude maps to zero. Thus a peripheral class $a\mu_i + b\lambda_i$ maps to a times the generator. Naturality shows that any diffeomorphism between the end pieces sends λ_i to $\pm\lambda_j$.

Let $g \in \operatorname{Diff}^+(Y)$. By the stated canonicity of the end decomposition, an ambient isotopy makes g permute P_0, P_1. The isotopy extends by integrating a time-dependent extension of its velocity field; compactness supplies the full time interval. Tubular-neighborhood uniqueness [12, Chapter 4] makes g product-type on smaller collars of the end tori. A homotopy of the cutoff in (9.3) moves the twist support into those collars without changing its mapping class.

The translation inclusion induces an isomorphism

$$\pi_1(T^2) \longrightarrow \pi_1(\operatorname{Diff}_0(T^2)), \tag{9.7}$$

as used for torus twists in [10, Section 2]. Evaluation at a fixed point is its inverse on fundamental groups: evaluation composed with translation is the identity on T^2. Conjugation of a translation loop by the torus restriction of g has evaluated class $g_*(\lambda_i) = \pm\lambda_j$. Changing the evaluation basepoint only conjugates a loop in the abelian group $\pi_1(T^2)$ and therefore has no effect. By (9.7), the conjugated loop is homotopic, with its endpoints fixed, to the corresponding signed longitude translation. A smooth such homotopy gives an isotopy of the supported collar maps by applying its loop at every collar level; the maps are the identity near the collar ends throughout. This proves

$$[g][f_i][g]^{-1} = [f_j]^{\pm 1}. \tag{9.8}$$

Hence $g\mathcal{A}g^{-1} \subseteq \mathcal{A}$. Applying the same argument to g^{-1} proves equality. ☐

Proposition 9.3. *One has*

$$\mathcal{A} \cap \mathcal{H} = \langle [z] \rangle, \qquad \mathcal{A}/(\mathcal{A} \cap \mathcal{H}) \cong \mathbb{Z}, \qquad [f_0^p f_1^q] \longmapsto p - q. \tag{9.9}$$

Proof. Since f_0 and f_1 commute,

$$f_0^p f_1^q = f_0^{p-q} z^q. \tag{9.10}$$

Every power of z extends by Lemma 9.1. If $[f_0^p f_1^q] \in \mathcal{H}$, multiplication by $[z]^{-q} \in \mathcal{H}$ gives $[f_0]^{p-q} \in \mathcal{H}$. By (9.2) and the representative-correction lemma, this implies $p = q$. Conversely, $p = q$ makes the word z^q, which extends. Thus the intersection is exactly $\langle [z] \rangle$.

To check the coordinate without assuming $\mathcal{A} \cong \mathbb{Z}^2$, suppose two exponent pairs represent the same element. Their quotient is the identity, hence extends; the criterion just proved forces equality of their differences. Thus $p - q$ is well-defined, is additive by commutativity, and is onto since $[f_0]$ maps to 1. Its kernel is the displayed intersection, proving the quotient assertion. ☐

9.3. **Diffeomorphisms of decomposed doubles.** For $u \in \operatorname{Diff}^+(Y)$ write

$$\mathscr{D}_u = C_+ \cup_u C_-, \qquad x_+ \sim u(x)_-, \quad C_+ = C, \quad C_- = -C. \tag{9.11}$$

The common boundary is denoted Y_u.

Lemma 9.4. *For $u, v \in \mathrm{Diff}^+(Y)$,*

$$(\mathscr{D}_u, C_+) \cong_+ (\mathscr{D}_v, C_+) \iff \mathcal{H}[u]\mathcal{H} = \mathcal{H}[v]\mathcal{H}. \tag{9.12}$$

Proof. A diffeomorphism of the indicated pairs restricts to a diffeomorphism of each half, since the closure of the complement of the positive half is the negative half. Let a, b be its two boundary restrictions. The attaching relations agree precisely when

$$b \circ u = v \circ a. \tag{9.13}$$

Hence $[v] = [b][u][a]^{-1}$ with $[a], [b] \in \mathcal{H}$. This proves necessity.

Conversely, a double-coset equality supplies extending representatives a, b for which this equality holds in $\mathcal{M}$. Put $b' = vau^{-1}$. Then b' is isotopic to b, so it extends by Lemma 2.1. Choose its extension and that of a product-type near the boundary, using collar uniqueness relative to their boundary values. The equality $b'u = va$ holds as an equality of actual maps, so the two extensions glue smoothly. Their inverse restrictions satisfy the inverse equality and glue to the inverse diffeomorphism. □

Lemma 9.5. *If $\mathcal{H}[f_0]^m\mathcal{H} = \mathcal{H}[f_0]^n\mathcal{H}$, then $|m| = |n|$.*

Proof. Write $[f_0]^n = a[f_0]^m b$ with $a, b \in \mathcal{H}$. Normality of $\mathcal{A}$ gives

$$ab = \left(a[f_0]^m a^{-1}\right)^{-1}[f_0]^n \in \mathcal{A} \cap \mathcal{H}. \tag{9.14}$$

Normality gives $a\mathcal{A}a^{-1} = \mathcal{A}$; since $a \in \mathcal{H}$ and $\mathcal{H}$ is a subgroup, $a\mathcal{H}a^{-1} = \mathcal{H}$. Intersecting yields $a(\mathcal{A} \cap \mathcal{H})a^{-1} = \mathcal{A} \cap \mathcal{H}$. It induces an automorphism η of the quotient in (9.9). That quotient is $\mathbb{Z}$, whose automorphisms send 1 to 1 or -1. Projecting $[f_0]^n = (a[f_0]^m a^{-1})(ab)$ therefore gives

$$n = \eta(m) = m \quad \text{or} \quad n = -m. \tag{9.15}$$

Both alternatives imply $|m| = |n|$. □

9.4. **A diffeomorphism realizing the sign change.**

Lemma 9.6. *There exists $R \in \mathrm{Diff}^+(C)$ such that*

$$[R|_Y][f_0][R|_Y]^{-1} = [f_0]^{-1}. \tag{9.16}$$

Proof. Let ρ be a strong inversion of K. Average a Riemannian metric under its order-two action. Near a fixed point of K, choose an invariant arclength parameter λ and a complex normal coordinate z so that

$$\rho(\lambda, z) = (-\lambda, \bar{z}). \tag{9.17}$$

Indeed, its derivative on the knot tangent is -1. The normal derivative is an orthogonal involution of determinant -1, hence a reflection. Choose an eigenframe at the fixed point and transport it by the invariant normal connection. Equivariance of the exponential map gives (9.17) on a neighborhood, not merely at the derivative level.

Choose the meridional two-handle attachment at that fixed longitude and at the middle interval coordinate. The map $\mathrm{id}_I \times \rho$ reverses its meridian angle μ and acts as a fixed reflection J_0 on its normal two-plane. If $R_{h\mu}$ describes the integer framing, then

$$J_0 R_{h\mu} = R_{-h\mu} J_0. \tag{9.18}$$

In complex notation this is $\overline{e^{ih\mu}w} = e^{-ih\mu}\overline{w}$. Thus the framed action on the attaching region is extended over the two-handle by

$$(z, w) \longmapsto (\bar{z}, \overline{w}). \tag{9.19}$$

On its attaching face this reads $(e^{i\mu}, w) \mapsto (e^{-i\mu}, \overline{w})$, precisely the preceding framed map. The real derivative of complex conjugation on each disk is $\mathrm{diag}(1, -1)$. The product derivative has determinant $(-1)(-1) = 1$, so the extension preserves the four-dimensional orientation. The formulas agree on invariant product collars and have the same involutive inverses there. They define $R \in \mathrm{Diff}^+(C)$.

The map retains each end and its inward collar direction. It sends the preferred longitude to its negative: it preserves the null-homologous peripheral subgroup and reverses the projection degree onto the oriented knot. The translation-loop calculation in (9.7) then sends the longitudinal twist to its inverse. No meridional correction is possible because the kernel in (9.6) has rank one. This proves (9.16) for every integer framing h. □

9.5. **Classification and embedded regions.** Put $u_{p,q} = f_0^p f_1^q$ and $\mathscr{D}_{p,q} = \mathscr{D}_{u_{p,q}}$. By (9.10) and Lemma 9.4,

$$(\mathscr{D}_{p,q}, C_+) \cong_+ (\mathscr{D}_{p-q,0}, C_+). \tag{9.20}$$

Indeed, the omitted factor is the extending power z^q. The same diffeomorphism sends common boundary to common boundary.

Proof of Theorem 1.4. After (9.20), a diffeomorphism of oriented pairs implies equality of the corresponding double cosets by Lemma 9.4. The necessary absolute-exponent equality follows from Lemma 9.5. Conversely, equal absolute exponents are either equal or opposite. Equal exponents give identical attachments. Opposite exponents give equal double cosets by the extending conjugating map of Lemma 9.6. Thus Lemma 9.4 proves (1.10) in both directions.

We next remove the orientation and side markings. Gompf proves that $C(r, s; h)$ is orientation-reversingly diffeomorphic to $C(r, s; -h)$, while these manifolds are not orientation-preservingly homeomorphic for $h \neq 0$ [9, Section 1]. An orientation-reversing self-homeomorphism of $C(r, s; h)$, composed with the former map, would give the excluded orientation-preserving homeomorphism. Hence C has no orientation-reversing self-homeomorphism.

For $\mathscr{D}_k = \mathscr{D}_{k,0}$, the identity maps between the opposite abstract halves define an orientation-reversing diffeomorphism

$$S_k : \mathscr{D}_k \longrightarrow \mathscr{D}_{-k}, \qquad S_k(x_+) = x_-, \quad S_k(x_-) = x_+. \tag{9.21}$$

The source relation $x_+ \sim f_0^k(x)_-$ becomes $x_- \sim f_0^k(x)_+$, exactly the target relation for exponent $-k$. The inverse is S_{-k}, and signed product collars verify smoothness.

A diffeomorphism carrying one common-boundary hypersurface to another permutes their two complementary halves. If it preserves ambient orientation, it cannot exchange the halves, since its restriction would give an orientation-reversing self-map of the underlying C. If it reverses ambient orientation, it cannot retain each half for the same reason. Thus it either preserves orientation and sides, or reverses orientation and exchanges sides. In the second case compose with the target half-swap; the resulting map preserves orientation and sides and has target exponent with reversed sign. In both cases the already proved classification forces equality of absolute exponents. The converse follows from the oriented pair maps, which also carry the common boundary to the common boundary.

By Theorem 1.2, every $\mathscr{D}_{k,0}$ is the standard S^4; (9.20) gives the same conclusion for $\mathscr{D}_{p,q}$. Choose orientation-preserving standardizations $\Theta_u : \mathscr{D}_u \to S^4$ and set

$$e_u = \Theta_u|_{C_+}. \tag{9.22}$$

Its complementary closure is $\Theta_u(C_-)$ and therefore has oriented diffeomorphism type $-C$. Changing Θ_u postcomposes e_u with an ambient diffeomorphism, so the classification of its image is independent of this choice.

Every orientation-preserving boundary diffeomorphism of a contractible smooth four-manifold extends to a homeomorphism, by Freedman's theorem as stated in [9, Definition 1.1 and its discussion]. Choose an extension $\overline{u} : C \to C$ of u. It preserves orientation: contractibility and the long exact sequence give an isomorphism $H_4(C, Y; \mathbb{Z}) \to H_3(Y; \mathbb{Z})$, and naturality transfers the boundary degree $+1$ to the relative orientation class. The map

$$Q_u = \mathrm{id}_{C_+} \cup \overline{u}_{C_-} : \mathscr{D}_{\mathrm{id}} \longrightarrow \mathscr{D}_u \tag{9.23}$$

is a homeomorphism. Indeed, it takes $x_+ \sim x_-$ to $x_+ \sim u(x)_-$; the identity and $\overline{u}^{-1}$ define its inverse on the two factors. For two attachments u, v,

$$F_{u,v} = \Theta_v Q_v Q_u^{-1} \Theta_u^{-1}, \qquad F_{u,v} e_u = e_v. \tag{9.24}$$

This is an orientation-preserving ambient homeomorphism retaining the positive parametrization. No ambient topological isotopy is required for, or asserted by, this conclusion. □

Corollary 9.7. *For each $r, s > 0 > h$, there are embeddings $e_j : C(r, s; h) \hookrightarrow S^4$, $j \geq 0$, with pairwise inequivalent images under arbitrary ambient diffeomorphisms. Their complementary closures are all diffeomorphic to $-C(r, s; h)$, and the embeddings are related by orientation-preserving ambient homeomorphisms retaining their parametrizations.*

Proof. Take $u = f_0^j$ in (9.22). An ambient diffeomorphism carrying one image to another also carries their boundaries to each other. The unoriented hypersurface classification forces $|i| = |j|$, hence $i = j$ for nonnegative indices. The assertions about the complement and the homeomorphism are (9.22) and (9.24). □

Use of AI tools

AI tools were used during manuscript preparation for editorial revision and for assistance in checking arguments and citations. The author is responsible for all mathematical content and references.

Appendix A. Odd regluing and the compressed sphere

This appendix is independent of the proof of Theorem 1.1. Let (T, D) and Γ be as in Section 3. Compressing T along D gives a sphere S_D: remove an annular neighborhood of α in T and attach two disjoint parallel copies of D supplied by its framing. Use the framed disk neighborhood $D^2 \times [-\epsilon, \epsilon]_\beta \times [-\epsilon, \epsilon]_y$ from Theorem 3.3. The two copies of the disk are $D^2 \times \{\pm\epsilon\} \times \{0\}$. Together with the closure of the complementary annulus in T, they form S_D. They can be oriented so that

$$[S_D] = [T] \quad \text{in } H_2(\Gamma; \mathbb{Z}) \ : \tag{A.1}$$

the difference is the boundary of the oriented product $D^2 \times [-\epsilon, \epsilon] \times \{0\}$, with the collar pieces included. The normal bundle of T is trivial, so $[T]^2 = 0$. It follows from (A.1) that $[S_D]^2 = 0$. The Euler number of the oriented normal bundle of S_D is this self-intersection, and an oriented rank-two bundle over S^2 is classified by that Euler number. Thus $\nu S_D \cong S^2 \times D^2$. Let γ be the arc $\{0\} \times [-\epsilon, \epsilon] \times \{0\}$ in the disk neighborhood, shortened so that its endpoints lie on the two copies of D^2. In the same product coordinates, a neighborhood of $T \cup D$ is a neighborhood of $S_D \cup \gamma$. The product $D^2 \times [-\epsilon, \epsilon] \times \{0\}$ contains the two disks on its boundary and the annulus that is replaced in the compression. A neighborhood of the interior of γ is $[-1, 1] \times D^3$ and is attached to νS_D along its two endpoint 3-balls. The attaching embeddings of the two endpoint 3-balls can be made product embeddings in the displayed disk coordinates. A 1-handle attached to two disjoint 3-balls in the connected boundary of $S^2 \times D^2$ gives its boundary connected sum with $S^1 \times D^3$. The coordinate choices restrict to the identity on the sphere core. Consequently,

$$(\Gamma, S_D) \cong \big((S^2 \times D^2) \natural (S^1 \times D^3),\ S^2 \times \{0\}\big). \tag{A.2}$$

The circle β generates $H_1(\Gamma; \mathbb{Z})$, and θ is the meridian of the sphere summand. The first assertion is (3.6). For the second, choose a point on the part of S_D equal to the complementary annulus of T. Its normal circle agrees with the original disk meridian θ. This identifies θ with the normal meridian of S_D in (A.2).

Proposition A.1. *For the boundary maps in* (3.8), ψ_{2q} *is isotopic to the identity, and* ψ_{2q+1} *represents the Gluck boundary twist on the sphere summand in* (A.2). *In particular,*

$$M_T(\mu_T + k\alpha^0) \cong \begin{cases} M, & k \text{ even}, \\ M_{S_D}^{\mathrm{Gluck}}, & k \text{ odd}. \end{cases} \tag{A.3}$$

For odd k, the map ψ_k does not extend over Γ.

Proof. Let $Y_\Gamma = \partial\Gamma$. Its sphere-twist kernel in (3.10) carries the derivative invariant

$$\mathfrak{t} : \ker(\mathrm{Mod}^+(Y_\Gamma) \to \mathrm{Out}(F_2)) \longrightarrow H^1(Y_\Gamma; \mathbb{Z}/2).$$

This is an isomorphism, and the class of a twist about an embedded sphere is its mod-two Poincaré dual [3, Lemma 5.1 and Corollary 5.2]. Spin structures on an oriented manifold, when they exist, form an affine space over its first mod-two cohomology. To obtain this description, compare lifts of the oriented transition functions to the spin group. Two lifts differ by signs on double overlaps; the compatibility on triple overlaps is the cocycle relation. Changing local lifts changes the sign cocycle by a coboundary. Conversely, multiplication by a sign cocycle gives a spin structure. This construction commutes with restriction and pullback. If g acts trivially on $H^1(Y_\Gamma; \mathbb{Z}/2)$, then

$$(g^*(\sigma + v) - (\sigma + v)) - (g^*\sigma - \sigma) = g^*v - v = 0. \tag{A.4}$$

Thus $g^*\sigma - \sigma$ is independent of σ. If g_1, g_2 lie in the kernel of (3.10), they act trivially on $H^1(Y_\Gamma; \mathbb{F}_2)$, and

$$(g_1 g_2)^*\sigma - \sigma = g_2^*(g_1^*\sigma - \sigma) + (g_2^*\sigma - \sigma) = (g_1^*\sigma - \sigma) + (g_2^*\sigma - \sigma). \tag{A.5}$$

For a generating sphere twist, choose its dual circle through a fixed point of the rotation axis. Along that circle its derivative rotates the tangent two-plane of the sphere once. The spin lift of a 2π rotation ends at -1; the other basis circles miss the support. The spin translation is therefore the mod-two class dual to the twisting sphere. It agrees with $\mathfrak{t}$ on the generating sphere twists, and hence on the entire kernel. This identifies the two homomorphisms without assuming an equality of their extensions outside the kernel. Use the auxiliary filling $H = D^2_w \times P$, whose boundary is Y_Γ. This is not the filling Γ. The map

$$\Phi_k(w, x) = (h(x)^k w, x) \tag{A.6}$$

is a diffeomorphism of H whose boundary is ψ_k. The surface $P = T^2 \setminus \operatorname{int} B$ is a once-punctured torus. It deformation retracts onto a graph. An oriented real vector bundle over a graph is trivial: trivialize it over a spanning tree and extend the trivialization over each remaining edge using connectedness of the structure group. Applying this to TP gives a trivialization of TP. Together with the coordinate frame on D^2_w, it gives a product trivialization of TH. For $\xi \in T_w D^2$ and $v \in T_x P$, differentiation gives

$$D\Phi_k|_{(w,x)}(\xi, v) = \big(h(x)^k \xi + w\, d(h^k)_x(v), v\big). \tag{A.7}$$

The homotopy $(w, x) \mapsto (tw, x)$ retracts H to its zero section. On that section the mixed term is zero, and the derivative is rotation by $h(x)^k$ on the w-plane and the identity on TP. The loop that rotates one oriented two-plane through 2π represents the nonzero element of $\pi_1(SO(4)) \cong \mathbb{Z}/2$: its lift to Spin(4) starts at 1 and ends at -1. A 4π rotation lifts to a closed loop. This computes the reduction modulo two of its winding number. To compare this computation with the boundary derivative, use an outward-normal splitting $TH|_{Y_\Gamma} = TY_\Gamma \oplus \mathbb{R}$. The derivative of a boundary-preserving orientation-preserving diffeomorphism is, in this splitting, triangular with positive normal diagonal entry. Multiplying its tangential-normal off-diagonal term by $1 - t$ and replacing the positive normal entry a by $(1 - t)a + t$ gives a homotopy through invertible maps to the tangent derivative direct sum the identity. The inclusion Spin(3) $\to$ Spin(4) preserves the element -1. Hence the spin-lift sign computed from $D\Phi_k$ is the same sign for the boundary tangent derivative. Since h has degree zero on β and degree one on θ, the change of spin structure on H, restricted to its boundary, is

$$\mathfrak{t}(\psi_k) = k\,\bar\theta, \qquad \bar\theta(\beta) = 0, \quad \bar\theta(\theta) = 1. \tag{A.8}$$

Restriction identifies the H^1 generators here: H retracts to P, and the inclusion $Y_\Gamma \to H$ induces the isomorphism of the free groups on β, θ. In (A.2), the Gluck map on the $S^2 \times \partial D^2$ boundary, made the identity near the connected-sum ball, is a sphere twist dual to the meridian θ. It lies in the kernel of (3.10) and has derivative class $\bar\theta$. The injectivity of $\mathfrak{t}$ on that kernel and (A.8) identify ψ_k with the stated parity class. Applying the same collar adjustment as in Theorem 3.3 proves (A.3) relative to the complement of Γ. The factors in (A.2) are spin: $T(S^1 \times D^3)$ is trivial, and $TS^2 \oplus \varepsilon^2 \cong \varepsilon^4$ by the tangent–normal splitting of $S^2 \subset \mathbb{R}^3$. Their spin structures join over the connecting one-handle, whose attaching balls are contractible. Fix one such structure σ_Γ. The boundary structures extending over Γ form $\sigma_\Gamma|_{Y_\Gamma} + \operatorname{im} H^1(\Gamma; \mathbb{F}_2)$, with difference space

$$\operatorname{im}\big(H^1(\Gamma; \mathbb{Z}/2) \to H^1(Y_\Gamma; \mathbb{Z}/2)\big) = \langle \bar\beta \rangle. \tag{A.9}$$

An odd ψ_k translates this subset by $\bar\theta$, by (A.8). The resulting affine subset is disjoint from the original one. An extension over Γ would preserve the set of extending spin structures, which is impossible. □

Remark A.2. The last assertion is relative to Γ. It does not assert that an odd surgery changes the closed ambient manifold. A diffeomorphism of $M \setminus \operatorname{int} \Gamma$ with the required boundary restriction could still identify the two closed gluings. No such diffeomorphism is asserted in Theorem 3.3.

Appendix B. Two finite four-move reductions

This appendix specifies two knots and complete finite reductions to the unknot. The local rules below, together with the registers and the explicit inverse-II insertions, determine every intermediate diagram. The initial crossings are labelled consecutively from zero in the displayed braid word. The label of every surviving crossing is unchanged by a transition. Labels absent from the current state may be reused for the new crossings of an inverse-II insertion; all such insertions are specified in Table 3.

Words are read from top to bottom, in written order. At the top and bottom of an eight-strand plat, join consecutive endpoints. Put

$$u = (\sigma_2\sigma_1\sigma_3\sigma_2)^2, \qquad v = (\sigma_4\sigma_3\sigma_5\sigma_4)^2, \qquad d = uvu^{-1}v^{-1}, \tag{B.1}$$

$$t = \sigma_2\sigma_4\sigma_6, \qquad p_A = \sigma_6\sigma_7\sigma_2\sigma_3,$$

$$p_B = \sigma_2\sigma_3\sigma_4\sigma_3\sigma_5\sigma_6\sigma_5\sigma_4\sigma_3\sigma_2, \qquad K_A = \operatorname{Plat}(tp_A^{-1}dp_A), \qquad K_B = \operatorname{Plat}(tp_B^{-1}dp_B). \tag{B.2}$$

The two words have respectively 43 and 55 letters. Each of u, v is a pure braid: its four-letter factor exchanges adjacent two-strand blocks and its square restores their positions. Thus d and its conjugates are pure, and both displayed plat words have the permutation of $t = (23)(45)(67)$. Superimposing the top matching with the pulled-back bottom matching gives the alternating cycle

$$1, 2, 5, 6, 8, 7, 4, 3, 1. \tag{B.3}$$

It contains all eight endpoints, so each plat is a knot. No doubled-delta orientation identification is needed in this appendix.

Theorem B.1. *The knots in* (B.2) *satisfy*

$$K_A \sim_4 U \text{ by at most } 17 \text{ four-moves}, \qquad K_B \sim_4 U \text{ by at most } 28 \text{ four-moves}. \tag{B.4}$$

The bounds are not asserted to be minimal.

B.1. **Diagrams and elementary replacements.** A crossing with label c has clockwise ports $4c, 4c+1, 4c+2, 4c+3$. Even ports pass under, and opposite ports belong to the same strand. Let e be the fixed-point-free involution pairing ports into edges, and put

$$r(4c+i) = 4c + (i + 1 \bmod 4). \tag{B.5}$$

A positive braid crossing has its northwest, northeast, southeast and southwest ports in the order $(3, 0, 1, 2)$; a negative crossing has order $(0, 1, 2, 3)$, each offset by $4c$. Joining consecutive braid arcs and the specified plat caps determines the initial involution e. The face successor and directed strand successor are

$$q \longmapsto e(r(q)), \qquad q \longmapsto e(r^2(q)). \tag{B.6}$$

The graph with vertex set the ports and edges $\{q, e(q)\}$ and $\{q, r^2(q)\}$ has degree two, with the two edge types alternating. Each of its components has two directed traversals; the permutation $e \circ r^2$ has the corresponding two cycles. Thus the number of non-crossingless link components is half the number of those permutation cycles. Crossingless components are recorded separately.

The cyclic port order thickens the graph to an oriented ribbon surface. Capping each face cycle gives its closed oriented surface. For a connected projection with v vertices, there are $2v$ edges; thus

$$v - 2v + f = 2 \quad \Longleftrightarrow \quad f - v = 2. \tag{B.7}$$

For genus g, the Euler characteristic is $2 - 2g$; hence the displayed equality gives $g = 0$. Conversely, the ribbon surface and its capping disks construct a cellular embedding in that sphere. Assign opposite crossing ports to one strand, and separate the two local strands in a small normal ball according to the declared over–under parity. This produces the link diagram represented by the state. Two realizations of the same oriented rotation system give orientation-preservingly homeomorphic cellular decompositions: map the vertex disks and edge strips by their parametrizations and extend across each face disk. Smoothing on collars gives equivalent spherical diagrams. No unbounded face is specified or needed. For several projection components, impose the Euler equality on each component. Every nonempty intermediate diagram below has one knot component and hence one projection component. Its monogon, digon and triangle faces therefore bound empty disks in the reconstructed spherical diagram.

For an RI deletion at crossing c, require $e(q) = r(q)$ at some port of c. This is precisely an empty monogon. For an RII deletion at two distinct crossings, require a digon with ports a, b satisfying

$$e(r(a)) = b, \qquad e(r(b)) = a, \qquad r(a) \equiv b \pmod 2. \tag{B.8}$$

The last equality says that one local strand is over at both crossings and the other is under at both. The indicated push across the empty disk is therefore a Reidemeister-II isotopy. Deletion reconnects each surviving port by following the strand through opposite ports until another surviving port is reached:

$$q_0 = e(q), \qquad q_{j+1} = e(r^2(q_j)) \text{ while the crossing containing } q_j \text{ is deleted.} \tag{B.9}$$

A closed strand contained entirely in the deleted vertices contributes one crossingless component instead of being discarded. An inverse-II insertion is specified by all six new edges in Table 3; deleting its inserted empty digon must recover the exact old involution. Thus every insertion is verified as the inverse of the stated local isotopy.

For an RIII move, the recorded ports (a, b, c) belong to distinct crossings and satisfy

$$e(r(a)) = b, \quad e(r(b)) = c, \quad e(r(c)) = a, \qquad b \equiv 0, \quad c \equiv 1 \pmod 2. \tag{B.10}$$

The edge from $r(b)$ to c is over at both ends and is the top strand. If a is even, the edge from $r(a)$ to b is the middle strand and the edge from $r(c)$ to a is the bottom strand. If a is odd, these two roles are exchanged. In either case there is a consistent top–middle–bottom ordering, not a cyclic crossing pattern. Replace the exterior-port list O by N in the stated order, retaining every attached exterior edge, and insert the three indicated internal edges:

$$\begin{aligned} O &= (r^{-1}c, r^{-2}c, r^{-1}a, r^{-2}a, r^{-1}b, r^{-2}b), \\ N &= (a, r(b), b, r(c), c, r(a)), \\ \text{internal edges} &= (r^{-1}a, r^2 b),\ (r^{-1}b, r^2 c),\ (r^{-1}c, r^2 a). \end{aligned} \tag{B.11}$$

Following opposite ports before and after this replacement retains the same three pairs of exterior strand ends and their height ordering. It is the Reidemeister-III isotopy across the empty triangular face.

Finally, an empty digon with the opposite parity from (B.8) is a two-strand twist of two crossings with equal sign. Rotate all its vertex attachments by one port at each of its two crossings. The over- and under-crossings are both reversed, giving $s^2 \leftrightarrow s^{-2}$. This replacement is one classical four-move followed by two RII deletions, since

$$s^2 \longmapsto s^{-4}s^2 \longmapsto s^{-2}. \tag{B.12}$$

The inverse uses the reversed sequence. No assumption about the orientations of these two strands along the whole knot is required.

B.2. **Finite verification.**

Proof of Theorem B.1. Construct the initial diagrams from (B.2) and the prescribed port orders. The registers below specify the next local replacement at every step. For each row, verify its monogon, digon or triangle conditions, or the six edges of its inverse-II insertion. Apply (B.9) or (B.11), or rotate the two twist vertices as specified above. Then check the edge involution, the four ports at every surviving vertex, the spherical face condition, the number of through-strand components, and the indicated output crossing count. These are finite operations on integer-labelled ports. The accompanying verifier reconstructs the initial diagram from the words, performs exactly these operations, and records the full edge pairings after every row. Its accepted counts are

	RI$^-$	RII$^-$	RIII	RII$^+$	IV	total
K_A	17	22	273	9	17	338
K_B	19	19	113	1	28	180

(B.13)

Here IV denotes the twist reversal justified by (B.12). The input-to-output crossing balances are

$$43 + 2\cdot 9 - 17 - 2\cdot 22 = 0, \qquad 55 + 2\cdot 1 - 19 - 2\cdot 19 = 0. \tag{B.14}$$

At the final row, the edge pairing has empty domain and the separately recorded crossingless-component count is one. The two separately implemented transition calculations and the complete output states are included in the verification package. In the second calculation, removal of vertices is performed by connected components of the deleted-edge and opposite-port graph; its agreement with (B.9) is checked at every surviving port. Each undirected output edge is checked separately for the reciprocal identity $e(e(q)) = q$, and every crossing is checked for the presence of its four ports. Thus each endpoint is a crossingless unknot. Every other row is an isotopy, and each IV row costs one four-move. Concatenating the rows proves the two bounds in (B.4). The local geometric justification is the preceding analysis of the moves, not a state hash. □

B.3. **The complete registers.** In the tables, the entries for I, II and IV are crossing labels; the entries for III are ports. The II$^+$ entries refer to the explicitly named new vertices in Table 3. The last column records the crossing count before and after the operation. Read each left block downwards and then each right block downwards.

The register for K_A.

Step	Move	Labels	Count	Step	Move	Labels	Count
1	I	(40)	$43 \to 42$	170	III	$(60, 128, 139)$	$18 \to 18$
2	I	(42)	$42 \to 41$	171	IV	$(17, 21)$	$18 \to 18$
3	II	$(4, 7)$	$41 \to 39$	172	III	$(98, 88, 95)$	$18 \to 18$
4	II	$(0, 8)$	$39 \to 37$	173	III	$(146, 54, 143)$	$18 \to 18$
5	II	$(3, 9)$	$37 \to 35$	174	III	$(71, 104, 87)$	$18 \to 18$
6	I	(10)	$35 \to 34$	175	I	(26)	$18 \to 17$
7	I	(11)	$34 \to 33$	176	III	$(60, 124, 135)$	$17 \to 17$
8	III	$(158, 146, 153)$	$33 \to 33$	177	III	$(100, 78, 71)$	$17 \to 17$
9	III	$(5, 52, 63)$	$33 \to 33$	178	III	$(154, 54, 151)$	$17 \to 17$
10	III	$(9, 24, 23)$	$33 \to 33$	179	III	$(62, 126, 133)$	$17 \to 17$

Step	Move	Labels	Count	Step	Move	Labels	Count
11	I	(6)	$33 \to 32$	180	III	(54, 134, 125)	$17 \to 17$
12	III	(118, 166, 123)	$32 \to 32$	181	III	(102, 76, 69)	$17 \to 17$
13	III	(158, 142, 149)	$32 \to 32$	182	III	(76, 140, 151)	$17 \to 17$
14	III	(134, 158, 139)	$32 \to 32$	183	III	(96, 90, 93)	$17 \to 17$
15	III	(110, 166, 115)	$32 \to 32$	184	III	(76, 144, 155)	$17 \to 17$
16	III	(139, 118, 143)	$32 \to 32$	185	III	(62, 130, 137)	$17 \to 17$
17	III	(106, 112, 165)	$32 \to 32$	186	III	(54, 138, 129)	$17 \to 17$
18	III	(126, 158, 131)	$32 \to 32$	187	III	(98, 88, 95)	$17 \to 17$
19	III	(23, 52, 71)	$32 \to 32$	188	III	(94, 62, 99)	$17 \to 17$
20	III	(137, 116, 141)	$32 \to 32$	189	III	(78, 146, 153)	$17 \to 17$
21	III	(11, 68, 55)	$32 \to 32$	190	IV	(15, 22)	$17 \to 17$
22	III	(88, 158, 85)	$32 \to 32$	191	III	(78, 142, 149)	$17 \to 17$
23	III	(9, 70, 53)	$32 \to 32$	192	III	(52, 136, 131)	$17 \to 17$
24	III	(9, 154, 145)	$32 \to 32$	193	III	(100, 78, 71)	$17 \to 17$
25	III	(80, 158, 77)	$32 \to 32$	194	III	(52, 132, 127)	$17 \to 17$
26	III	(139, 118, 143)	$32 \to 32$	195	III	(152, 52, 149)	$17 \to 17$
27	III	(135, 118, 151)	$32 \to 32$	196	III	(102, 76, 69)	$17 \to 17$
28	I	(29)	$32 \to 31$	197	III	(97, 92, 61)	$17 \to 17$
29	III	(11, 152, 147)	$31 \to 31$	198	III	(61, 128, 139)	$17 \to 17$
30	III	(158, 74, 65)	$31 \to 31$	199	III	(100, 78, 71)	$17 \to 17$
31	III	(9, 154, 145)	$31 \to 31$	200	III	(61, 124, 135)	$17 \to 17$
32	III	(135, 138, 9)	$31 \to 31$	201	III	(144, 52, 141)	$17 \to 17$
33	III	(156, 72, 67)	$31 \to 31$	202	III	(63, 126, 133)	$17 \to 17$
34	III	(102, 120, 165)	$31 \to 31$	203	III	(63, 130, 137)	$17 \to 17$
35	III	(131, 110, 143)	$31 \to 31$	204	III	(78, 86, 89)	$17 \to 17$
36	III	(149, 10, 137)	$31 \to 31$	205	III	(131, 62, 95)	$17 \to 17$
37	III	(100, 122, 167)	$31 \to 31$	206	III	(76, 84, 91)	$17 \to 17$
38	III	(104, 114, 167)	$31 \to 31$	207	III	(102, 76, 69)	$17 \to 17$
39	III	(166, 98, 105)	$31 \to 31$	208	III	(146, 54, 143)	$17 \to 17$
40	III	(135, 122, 155)	$31 \to 31$	209	III	(154, 54, 151)	$17 \to 17$
41	III	(149, 152, 121)	$31 \to 31$	210	III	(100, 78, 71)	$17 \to 17$
42	III	(158, 74, 65)	$31 \to 31$	211	III	(54, 134, 125)	$17 \to 17$
43	III	(151, 154, 123)	$31 \to 31$	212	III	(78, 86, 89)	$17 \to 17$
44	III	(111, 130, 137)	$31 \to 31$	213	III	(129, 60, 93)	$17 \to 17$
45	III	(133, 120, 153)	$31 \to 31$	214	III	(54, 138, 129)	$17 \to 17$
46	III	(166, 94, 101)	$31 \to 31$	215	II	(13, 15)	$17 \to 15$
47	III	(127, 130, 9)	$31 \to 31$	216	III	(76, 84, 91)	$15 \to 15$
48	III	(127, 110, 151)	$31 \to 31$	217	III	(102, 76, 69)	$15 \to 15$
49	III	(4, 158, 23)	$31 \to 31$	218	III	(93, 96, 91)	$15 \to 15$
50	III	(111, 126, 133)	$31 \to 31$	219	III	(91, 136, 131)	$15 \to 15$
51	III	(164, 92, 103)	$31 \to 31$	220	III	(91, 132, 127)	$15 \to 15$
52	III	(158, 70, 61)	$31 \to 31$	221	III	(149, 152, 91)	$15 \to 15$
53	III	(164, 96, 107)	$31 \to 31$	222	IV	(24, 32)	$15 \to 15$
54	III	(151, 8, 139)	$31 \to 31$	223	III	(151, 154, 89)	$15 \to 15$
55	III	(133, 136, 11)	$31 \to 31$	224	III	(100, 78, 71)	$15 \to 15$
56	III	(23, 56, 75)	$31 \to 31$	225	III	(89, 134, 125)	$15 \to 15$
57	III	(166, 98, 105)	$31 \to 31$	226	III	(102, 76, 69)	$15 \to 15$
58	III	(156, 68, 63)	$31 \to 31$	227	III	(76, 140, 151)	$15 \to 15$
59	III	(135, 138, 9)	$31 \to 31$	228	III	(138, 130, 89)	$15 \to 15$
60	III	(9, 150, 141)	$31 \to 31$	229	IV	(17, 21)	$15 \to 15$
61	III	(158, 70, 61)	$31 \to 31$	230	III	(136, 128, 91)	$15 \to 15$
62	III	(7, 54, 61)	$31 \to 31$	231	III	(91, 132, 127)	$15 \to 15$
63	III	(92, 78, 23)	$31 \to 31$	232	III	(76, 144, 155)	$15 \to 15$
64	III	(70, 48, 75)	$31 \to 31$	233	III	(141, 144, 91)	$15 \to 15$
65	III	(89, 84, 9)	$31 \to 31$	234	III	(149, 152, 91)	$15 \to 15$
66	III	(164, 96, 107)	$31 \to 31$	235	III	(78, 142, 149)	$15 \to 15$
67	III	(111, 114, 121)	$31 \to 31$	236	III	(91, 102, 93)	$15 \to 15$
68	III	(54, 48, 59)	$31 \to 31$	237	III	(76, 140, 151)	$15 \to 15$
69	III	(166, 98, 105)	$31 \to 31$	238	III	(132, 76, 137)	$15 \to 15$

Step	Move	Labels	Count	Step	Move	Labels	Count
70	III	$(52,50,57)$	$31 \to 31$	239	III	$(89,100,95)$	$15 \to 15$
71	III	$(91,86,11)$	$31 \to 31$	240	III	$(101,88,155)$	$15 \to 15$
72	III	$(164,96,107)$	$31 \to 31$	241	III	$(68,88,147)$	$15 \to 15$
73	III	$(54,48,59)$	$31 \to 31$	242	III	$(134,78,139)$	$15 \to 15$
74	III	$(127,114,155)$	$31 \to 31$	243	III	$(88,126,85)$	$15 \to 15$
75	III	$(96,86,23)$	$31 \to 31$	244	III	$(132,76,137)$	$15 \to 15$
76	III	$(107,102,111)$	$31 \to 31$	245	III	$(97,88,131)$	$15 \to 15$
77	III	$(62,48,67)$	$31 \to 31$	246	I	(22)	$15 \to 14$
78	III	$(48,80,91)$	$31 \to 31$	247	II^+	$(39,40)$	$14 \to 16$
79	III	$(98,84,21)$	$31 \to 31$	248	II^+	$(41,42)$	$16 \to 18$
80	III	$(5,56,67)$	$31 \to 31$	249	III	$(134,78,139)$	$18 \to 18$
81	III	$(125,112,153)$	$31 \to 31$	250	III	$(78,142,149)$	$18 \to 18$
82	III	$(96,86,23)$	$31 \to 31$	251	II	$(24,41)$	$18 \to 16$
83	III	$(111,86,77)$	$31 \to 31$	252	II	$(25,39)$	$16 \to 14$
84	III	$(7,58,65)$	$31 \to 31$	253	III	$(76,140,151)$	$14 \to 14$
85	III	$(11,148,143)$	$31 \to 31$	254	III	$(132,76,137)$	$14 \to 14$
86	III	$(149,10,137)$	$31 \to 31$	255	IV	$(37,40)$	$14 \to 14$
87	III	$(96,48,93)$	$31 \to 31$	256	IV	$(31,36)$	$14 \to 14$
88	II	$(12,27)$	$31 \to 29$	257	III	$(84,76,171)$	$14 \to 14$
89	III	$(5,56,67)$	$29 \to 29$	258	III	$(76,130,125)$	$14 \to 14$
90	III	$(127,114,155)$	$29 \to 29$	259	IV	$(21,31)$	$14 \to 14$
91	III	$(7,58,65)$	$29 \to 29$	260	IV	$(34,35)$	$14 \to 14$
92	III	$(131,114,147)$	$29 \to 29$	261	II^+	$(43,44)$	$14 \to 16$
93	I	(28)	$29 \to 28$	262	III	$(127,172,87)$	$16 \to 16$
94	III	$(135,122,155)$	$28 \to 28$	263	III	$(124,86,179)$	$16 \to 16$
95	III	$(94,76,21)$	$28 \to 28$	264	II^+	$(45,46)$	$16 \to 18$
96	III	$(166,94,101)$	$28 \to 28$	265	I	(43)	$18 \to 17$
97	III	$(151,8,139)$	$28 \to 28$	266	I	(44)	$17 \to 16$
98	III	$(139,122,147)$	$28 \to 28$	267	II	$(19,46)$	$16 \to 14$
99	III	$(9,150,141)$	$28 \to 28$	268	IV	$(34,35)$	$14 \to 14$
100	III	$(141,144,121)$	$28 \to 28$	269	IV	$(23,38)$	$14 \to 14$
101	III	$(149,152,121)$	$28 \to 28$	270	IV	$(21,31)$	$14 \to 14$
102	III	$(11,152,147)$	$28 \to 28$	271	II^+	$(46,47)$	$14 \to 16$
103	III	$(98,84,21)$	$28 \to 28$	272	III	$(141,190,149)$	$16 \to 16$
104	III	$(164,92,103)$	$28 \to 28$	273	III	$(142,148,187)$	$16 \to 16$
105	II	$(30,41)$	$28 \to 26$	274	III	$(182,128,125)$	$16 \to 16$
106	III	$(5,56,67)$	$26 \to 26$	275	III	$(84,182,169)$	$16 \to 16$
107	III	$(9,154,145)$	$26 \to 26$	276	III	$(138,190,133)$	$16 \to 16$
108	III	$(21,54,69)$	$26 \to 26$	277	III	$(143,186,133)$	$16 \to 16$
109	III	$(7,58,65)$	$26 \to 26$	278	II^+	$(48,49)$	$16 \to 18$
110	III	$(54,104,99)$	$26 \to 26$	279	III	$(141,184,135)$	$18 \to 18$
111	III	$(81,76,9)$	$26 \to 26$	280	III	$(139,132,187)$	$18 \to 18$
112	III	$(5,56,67)$	$26 \to 26$	281	III	$(86,180,171)$	$18 \to 18$
113	III	$(156,72,67)$	$26 \to 26$	282	III	$(170,190,87)$	$18 \to 18$
114	I	(16)	$26 \to 25$	283	II^+	$(50,51)$	$18 \to 20$
115	III	$(83,78,11)$	$25 \to 25$	284	III	$(168,188,85)$	$20 \to 20$
116	III	$(54,100,95)$	$25 \to 25$	285	III	$(180,130,127)$	$20 \to 20$
117	III	$(154,54,151)$	$25 \to 25$	286	III	$(183,200,131)$	$20 \to 20$
118	III	$(6,156,21)$	$25 \to 25$	287	III	$(180,130,207)$	$20 \to 20$
119	III	$(146,54,143)$	$25 \to 25$	288	II^+	$(52,53)$	$20 \to 22$
120	III	$(11,148,143)$	$25 \to 25$	289	III	$(137,134,185)$	$22 \to 22$
121	III	$(21,58,73)$	$25 \to 25$	290	III	$(170,190,209)$	$22 \to 22$
122	II	$(1,14)$	$25 \to 23$	291	III	$(140,150,185)$	$22 \to 22$
123	III	$(54,138,129)$	$23 \to 23$	292	III	$(185,68,163)$	$22 \to 22$
124	III	$(9,150,141)$	$23 \to 23$	293	III	$(168,188,211)$	$22 \to 22$
125	III	$(52,136,131)$	$23 \to 23$	294	III	$(147,154,185)$	$22 \to 22$
126	III	$(81,76,9)$	$23 \to 23$	295	II	$(17,35)$	$22 \to 20$
127	III	$(99,20,127)$	$23 \to 23$	296	II	$(21,53)$	$20 \to 18$
128	III	$(156,68,63)$	$23 \to 23$	297	II	$(31,50)$	$18 \to 16$

Step	Move	Labels	Count	Step	Move	Labels	Count
129	III	$(89, 84, 9)$	$23 \to 23$	298	II	$(33, 37)$	$16 \to 14$
130	II	$(2, 39)$	$23 \to 21$	299	II	$(48, 49)$	$14 \to 12$
131	III	$(97, 22, 125)$	$21 \to 21$	300	III	$(182, 128, 205)$	$12 \to 12$
132	III	$(131, 126, 21)$	$21 \to 21$	301	III	$(205, 186, 153)$	$12 \to 12$
133	III	$(139, 134, 21)$	$21 \to 21$	302	IV	$(36, 51)$	$12 \to 12$
134	III	$(54, 134, 125)$	$21 \to 21$	303	III	$(170, 190, 209)$	$12 \to 12$
135	III	$(21, 146, 153)$	$21 \to 21$	304	III	$(204, 184, 155)$	$12 \to 12$
136	III	$(74, 60, 71)$	$21 \to 21$	305	IV	$(40, 52)$	$12 \to 12$
137	III	$(54, 138, 129)$	$21 \to 21$	306	III	$(206, 186, 153)$	$12 \to 12$
138	III	$(60, 84, 79)$	$21 \to 21$	307	III	$(138, 144, 95)$	$12 \to 12$
139	III	$(98, 54, 95)$	$21 \to 21$	308	III	$(94, 206, 139)$	$12 \to 12$
140	III	$(21, 142, 149)$	$21 \to 21$	309	III	$(139, 162, 187)$	$12 \to 12$
141	III	$(62, 86, 77)$	$21 \to 21$	310	III	$(95, 162, 155)$	$12 \to 12$
142	III	$(77, 80, 21)$	$21 \to 21$	311	I	(40)	$12 \to 11$
143	III	$(86, 74, 91)$	$21 \to 21$	312	III	$(204, 184, 155)$	$11 \to 11$
144	II	$(5, 18)$	$21 \to 19$	313	III	$(92, 204, 137)$	$11 \to 11$
145	III	$(106, 54, 103)$	$19 \to 19$	314	II^{+}	$(53, 54)$	$11 \to 13$
146	III	$(54, 78, 85)$	$19 \to 19$	315	III	$(146, 152, 187)$	$13 \to 13$
147	III	$(60, 88, 83)$	$19 \to 19$	316	III	$(139, 94, 213)$	$13 \to 13$
148	I	(20)	$19 \to 18$	317	II	$(36, 53)$	$13 \to 11$
149	III	$(52, 76, 87)$	$18 \to 18$	318	III	$(138, 216, 95)$	$11 \to 11$
150	III	$(60, 84, 79)$	$18 \to 18$	319	III	$(94, 206, 139)$	$11 \to 11$
151	III	$(104, 52, 101)$	$18 \to 18$	320	IV	$(51, 54)$	$11 \to 11$
152	III	$(90, 76, 87)$	$18 \to 18$	321	III	$(152, 94, 131)$	$11 \to 11$
153	III	$(76, 140, 151)$	$18 \to 18$	322	III	$(218, 204, 131)$	$11 \to 11$
154	III	$(96, 52, 93)$	$18 \to 18$	323	I	(32)	$11 \to 10$
155	III	$(52, 136, 131)$	$18 \to 18$	324	III	$(185, 204, 93)$	$10 \to 10$
156	III	$(76, 144, 155)$	$18 \to 18$	325	III	$(153, 218, 137)$	$10 \to 10$
157	III	$(100, 60, 105)$	$18 \to 18$	326	IV	$(45, 47)$	$10 \to 10$
158	III	$(52, 132, 127)$	$18 \to 18$	327	II	$(45, 54)$	$10 \to 8$
159	III	$(104, 90, 101)$	$18 \to 18$	328	III	$(187, 206, 95)$	$8 \to 8$
160	III	$(92, 60, 97)$	$18 \to 18$	329	III	$(137, 92, 205)$	$8 \to 8$
161	III	$(132, 76, 137)$	$18 \to 18$	330	III	$(190, 152, 211)$	$8 \to 8$
162	III	$(96, 90, 93)$	$18 \to 18$	331	III	$(211, 206, 95)$	$8 \to 8$
163	III	$(124, 76, 129)$	$18 \to 18$	332	II	$(47, 51)$	$8 \to 6$
164	III	$(144, 52, 141)$	$18 \to 18$	333	IV	$(38, 42)$	$6 \to 6$
165	III	$(126, 78, 131)$	$18 \to 18$	334	II	$(23, 38)$	$6 \to 4$
166	III	$(134, 78, 139)$	$18 \to 18$	335	I	(42)	$4 \to 3$
167	III	$(152, 52, 149)$	$18 \to 18$	336	IV	$(46, 52)$	$3 \to 3$
168	III	$(78, 142, 149)$	$18 \to 18$	337	II	$(34, 46)$	$3 \to 1$
169	III	$(78, 146, 153)$	$18 \to 18$	338	I	(52)	$1 \to 0$

The register for K_B.

Step	Move	Labels	Count	Step	Move	Labels	Count
1	II	$(0, 3)$	$55 \to 53$	91	III	$(139, 134, 95)$	$14 \to 14$
2	II	$(12, 13)$	$53 \to 51$	92	IV	$(35, 28)$	$14 \to 14$
3	I	(14)	$51 \to 50$	93	III	$(107, 128, 123)$	$14 \to 14$
4	II	$(11, 15)$	$50 \to 48$	94	III	$(137, 132, 93)$	$14 \to 14$
5	II	$(51, 52)$	$48 \to 46$	95	III	$(105, 130, 121)$	$14 \to 14$
6	II	$(53, 54)$	$46 \to 44$	96	IV	$(24, 25)$	$14 \to 14$
7	III	$(178, 184, 191)$	$44 \to 44$	97	III	$(107, 128, 123)$	$14 \to 14$
8	II	$(44, 48)$	$44 \to 42$	98	III	$(139, 134, 95)$	$14 \to 14$
9	II	$(45, 47)$	$42 \to 40$	99	III	$(150, 116, 101)$	$14 \to 14$
10	I	(43)	$40 \to 39$	100	III	$(105, 130, 121)$	$14 \to 14$
11	II	$(46, 49)$	$39 \to 37$	101	III	$(148, 118, 103)$	$14 \to 14$
12	I	(42)	$37 \to 36$	102	III	$(95, 122, 129)$	$14 \to 14$

Step	Move	Labels	Count	Step	Move	Labels	Count
13	III	(26, 32, 31)	36 → 36	103	IV	(30, 34)	14 → 14
14	II	(2, 8)	36 → 34	104	IV	(24, 25)	14 → 14
15	II	(6, 50)	34 → 32	105	IV	(23, 26)	14 → 14
16	III	(28, 4, 21)	32 → 32	106	III	(131, 94, 121)	14 → 14
17	II	(4, 7)	32 → 30	107	III	(95, 130, 137)	14 → 14
18	I	(5)	30 → 29	108	IV	(35, 28)	14 → 14
19	II	(1, 9)	29 → 27	109	III	(93, 128, 139)	14 → 14
20	I	(10)	27 → 26	110	III	(129, 92, 123)	14 → 14
21	II	(19, 21)	26 → 24	111	IV	(30, 34)	14 → 14
22	II	(20, 22)	24 → 22	112	III	(94, 122, 131)	14 → 14
23	III	(70, 66, 75)	22 → 22	113	IV	(35, 28)	14 → 14
24	I	(18)	22 → 21	114	III	(92, 120, 129)	14 → 14
25	III	(165, 154, 161)	21 → 21	115	IV	(24, 25)	14 → 14
26	I	(38)	21 → 20	116	III	(94, 122, 131)	14 → 14
27	III	(94, 66, 99)	20 → 20	117	III	(150, 116, 103)	14 → 14
28	III	(66, 100, 111)	20 → 20	118	III	(92, 120, 129)	14 → 14
29	III	(66, 104, 115)	20 → 20	119	IV	(25, 36)	14 → 14
30	III	(120, 66, 117)	20 → 20	120	III	(94, 122, 131)	14 → 14
31	III	(128, 66, 125)	20 → 20	121	III	(132, 94, 139)	14 → 14
32	III	(165, 150, 157)	20 → 20	122	III	(94, 144, 143)	14 → 14
33	III	(163, 158, 149)	20 → 20	123	III	(142, 100, 95)	14 → 14
34	I	(39)	20 → 19	124	III	(114, 142, 119)	14 → 14
35	III	(98, 68, 95)	19 → 19	125	IV	(25, 36)	14 → 14
36	III	(132, 126, 65)	19 → 19	126	III	(146, 102, 119)	14 → 14
37	III	(147, 142, 95)	19 → 19	127	I	(29)	14 → 13
38	III	(134, 124, 67)	19 → 19	128	III	(115, 146, 93)	13 → 13
39	III	(130, 64, 127)	19 → 19	129	III	(104, 128, 121)	13 → 13
40	III	(122, 64, 119)	19 → 19	130	III	(120, 132, 105)	13 → 13
41	III	(145, 140, 93)	19 → 19	131	III	(93, 140, 103)	13 → 13
42	III	(165, 138, 145)	19 → 19	132	IV	(32, 33)	13 → 13
43	III	(68, 108, 103)	19 → 19	133	III	(107, 122, 135)	13 → 13
44	III	(68, 112, 107)	19 → 19	134	IV	(36, 37)	13 → 13
45	III	(120, 66, 117)	19 → 19	135	III	(108, 136, 93)	13 → 13
46	III	(128, 66, 125)	19 → 19	136	III	(93, 130, 107)	13 → 13
47	III	(165, 134, 141)	19 → 19	137	III	(130, 134, 109)	13 → 13
48	II	(16, 41)	19 → 17	138	III	(141, 130, 115)	13 → 13
49	III	(109, 112, 149)	17 → 17	139	I	(32)	13 → 12
50	III	(116, 68, 121)	17 → 17	140	III	(92, 120, 109)	12 → 12
51	III	(111, 114, 151)	17 → 17	141	I	(27)	12 → 11
52	III	(139, 134, 95)	17 → 17	142	IV	(35, 28)	11 → 11
53	III	(118, 70, 123)	17 → 17	143	III	(105, 120, 133)	11 → 11
54	III	(109, 112, 149)	17 → 17	144	IV	(34, 30)	11 → 11
55	III	(106, 120, 69)	17 → 17	145	III	(102, 98, 137)	11 → 11
56	III	(113, 108, 161)	17 → 17	146	III	(146, 150, 137)	11 → 11
57	III	(104, 122, 71)	17 → 17	147	I	(34)	11 → 10
58	III	(116, 68, 121)	17 → 17	148	IV	(35, 28)	10 → 10
59	III	(137, 132, 93)	17 → 17	149	IV	(36, 37)	10 → 10
60	III	(124, 68, 129)	17 → 17	150	II^+	(38, 39)	10 → 12
61	III	(68, 132, 143)	17 → 17	151	III	(96, 148, 157)	12 → 12
62	III	(68, 136, 147)	17 → 17	152	III	(114, 142, 157)	12 → 12
63	III	(115, 110, 163)	17 → 17	153	III	(153, 96, 145)	12 → 12
64	II	(17, 40)	17 → 15	154	III	(114, 92, 153)	12 → 12
65	III	(101, 104, 149)	15 → 15	155	I	(39)	12 → 11
66	III	(97, 150, 107)	15 → 15	156	III	(140, 154, 95)	11 → 11
67	III	(147, 142, 95)	15 → 15	157	IV	(35, 28)	11 → 11
68	III	(107, 124, 119)	15 → 15	158	IV	(36, 37)	11 → 11
69	III	(145, 140, 93)	15 → 15	159	III	(151, 154, 145)	11 → 11
70	III	(107, 128, 123)	15 → 15	160	I	(38)	11 → 10
71	III	(93, 150, 115)	15 → 15	161	IV	(23, 26)	10 → 10

Step	Move	Labels	Count	Step	Move	Labels	Count
72	III	$(105, 130, 121)$	$15 \to 15$	162	III	$(135, 104, 123)$	$10 \to 10$
73	IV	$(36, 37)$	$15 \to 15$	163	IV	$(24, 25)$	$10 \to 10$
74	IV	$(24, 25)$	$15 \to 15$	164	II	$(25, 26)$	$10 \to 8$
75	III	$(99, 124, 103)$	$15 \to 15$	165	III	$(120, 98, 115)$	$8 \to 8$
76	I	(31)	$15 \to 14$	166	IV	$(28, 33)$	$8 \to 8$
77	III	$(107, 128, 123)$	$14 \to 14$	167	III	$(140, 120, 151)$	$8 \to 8$
78	III	$(128, 110, 99)$	$14 \to 14$	168	III	$(145, 132, 93)$	$8 \to 8$
79	III	$(99, 106, 137)$	$14 \to 14$	169	III	$(93, 140, 115)$	$8 \to 8$
80	III	$(100, 128, 93)$	$14 \to 14$	170	III	$(150, 96, 93)$	$8 \to 8$
81	III	$(97, 104, 139)$	$14 \to 14$	171	I	(23)	$8 \to 7$
82	III	$(133, 136, 107)$	$14 \to 14$	172	III	$(141, 150, 145)$	$7 \to 7$
83	III	$(102, 130, 95)$	$14 \to 14$	173	I	(37)	$7 \to 6$
84	III	$(130, 108, 97)$	$14 \to 14$	174	IV	$(35, 36)$	$6 \to 6$
85	IV	$(37, 36)$	$14 \to 14$	175	III	$(145, 98, 143)$	$6 \to 6$
86	III	$(135, 138, 105)$	$14 \to 14$	176	I	(24)	$6 \to 5$
87	IV	$(25, 24)$	$14 \to 14$	177	IV	$(35, 36)$	$5 \to 5$
88	III	$(95, 150, 113)$	$14 \to 14$	178	II	$(35, 28)$	$5 \to 3$
89	III	$(145, 142, 95)$	$14 \to 14$	179	II	$(36, 33)$	$3 \to 1$
90	III	$(105, 130, 121)$	$14 \to 14$	180	I	(30)	$1 \to 0$

B.4. **All inverse-II insertions.** Each row specifies four edges from surviving ports to new ports and the two internal edges between the new crossings. The previous two edges among those four surviving ports are removed. All other edges are unchanged. The labels in each pair specify the new edge involution in both directions. The finite computation is supplied as `verify_revision.py`,

TABLE 3. Explicit inverse-II insertions.

Knot	Step	New edges
K_A	247	$(101, 159)$, $(148, 163)$, $(100, 156)$, $(69, 162)$, $(158, 160)$, $(161, 157)$
K_A	248	$(96, 166)$, $(129, 170)$, $(99, 167)$, $(130, 169)$, $(165, 171)$, $(168, 164)$
K_A	261	$(124, 172)$, $(129, 178)$, $(87, 173)$, $(170, 177)$, $(175, 179)$, $(176, 174)$
K_A	264	$(92, 181)$, $(77, 187)$, $(125, 182)$, $(78, 186)$, $(180, 184)$, $(185, 183)$
K_A	271	$(160, 185)$, $(149, 191)$, $(68, 186)$, $(142, 190)$, $(184, 188)$, $(189, 187)$
K_A	278	$(146, 194)$, $(125, 196)$, $(145, 195)$, $(126, 199)$, $(193, 197)$, $(198, 192)$
K_A	283	$(126, 200)$, $(199, 206)$, $(125, 201)$, $(196, 205)$, $(203, 207)$, $(204, 202)$
K_A	288	$(191, 209)$, $(87, 215)$, $(170, 210)$, $(84, 214)$, $(208, 212)$, $(213, 211)$
K_A	314	$(95, 213)$, $(206, 219)$, $(139, 214)$, $(207, 218)$, $(212, 216)$, $(217, 215)$
K_B	150	$(101, 152)$, $(96, 158)$, $(146, 153)$, $(149, 157)$, $(155, 159)$, $(156, 154)$

with the two literal registers, all output edge pairings, a second transition implementation, and a source-indexed list of the evaluated predicates. No Jones or Alexander polynomial calculation, orientation identification with a doubled block, or relative-tangle assertion is used in these reductions. The initial diagrams, local replacement rules, and complete registers suffice.

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