FINITE SPECTRAL DETERMINATION OF COMPACT HOMOGENEOUS MANIFOLDS

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Abstract. A finite initial segment of a prescribed scalar Laplace spectrum determines the full spectrum among closed homogeneous manifolds of the same dimension. The realizations are represented by finitely many compact semialgebraic sets of positive invariant cometrics. They are therefore smoothly compact modulo diffeomorphisms and have finitely many diffeomorphism types. Among globally naturally reductive metrics, each spectrum has only finitely many isometry classes. For these metrics, a finite set of scalar eigenvalues also gives a local Lipschitz estimate for the metric modulo diffeomorphisms, simultaneously in every differentiability order. An explicit family on $\mathrm{SU}(2) \times S^1$ has fixed volume and uniform relative convergence of all positive eigenvalues, while its scalar curvature tends to $-\infty$.

1. Introduction

All manifolds are smooth, closed, connected and of positive dimension. Homogeneity means global homogeneity. The nonnegative scalar Laplacian and its eigenvalues are denoted by

$$\Delta_g = -\operatorname{div}_g \nabla, \qquad 0 = \lambda_0(g) < \lambda_1(g) \le \lambda_2(g) \le \cdots .$$

Eigenvalues are repeated according to multiplicity. Put $\Lambda_N(g) = (\lambda_1(g), \dots, \lambda_N(g))$. A cometric is the inverse of a Riemannian metric, regarded as a positive symmetric contravariant tensor.

1.1. Finite spectral determination.

Theorem A. *Let $\Sigma = (0 = \sigma_0 < \sigma_1 \le \sigma_2 \le \cdots)$ be the scalar spectrum of a closed connected smooth Riemannian n-manifold, where $n > 0$. There is an integer $N = N(n, \Sigma)$ such that, for every homogeneous n-manifold (M, g),*

$$\Lambda_N(g) = (\sigma_1, \dots, \sigma_N) \quad \Longleftrightarrow \quad \operatorname{Spec}(\Delta_g) = \Sigma.$$

There are finitely many homogeneous presentations $M_i = G_i/H_i$ and compact semialgebraic sets $\mathcal{K}_i$ of positive invariant cometrics on M_i such that

(i) *every homogeneous realization of Σ is isometric to a metric represented in some $\mathcal{K}_i$;*

(ii) *every member of every $\mathcal{K}_i$ has spectrum Σ.*

Consequently, these realizations have finitely many diffeomorphism types and are compact for smooth convergence modulo diffeomorphisms.

The transitive group, the isotropy subgroup, the topology and the toral factors may vary. The reference manifold need not be homogeneous. If Σ has no homogeneous realization, some finite initial segment has no homogeneous realization in dimension n.

Schueth [18] constructed continuous isospectral families of globally homogeneous metrics.

Dai–Wei [7, Theorem 1.2] prove finite spectral determination on compact thin families of unit-volume metrics; their thinness condition is local containment in a finite-dimensional real analytic variety. Their Theorem 1.3 applies to normalized Einstein metrics with a positive lower bound for the injectivity radius and an upper bound for the diameter. Their hypotheses include compact families of unit-volume invariant metrics on a fixed homogeneous space. Here, equality of finitely many

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initial eigenvalues gives the compact positive parameter sets and the finite list of presentations. In dimension three, the recent preprint of Li–Wu–Zhou [16, Theorem 1.2] proves smooth compactness of the full isospectral class, without a homogeneity assumption.

1.2. **Naturally reductive metrics.** A globally naturally reductive metric admits a transitive presentation satisfying (8.1). On a closed manifold the presenting group can be taken compact and connected, by Lemma 8.4. The isotropy subgroup may be disconnected.

Theorem B. *A prescribed scalar Laplace spectrum has only finitely many isometry classes of closed connected globally naturally reductive realizations. In particular, it has only finitely many normal homogeneous realizations up to isometry.*

Theorems A and B imply that, in a fixed dimension, a sufficiently long prescribed initial segment has only finitely many naturally reductive realizations up to isometry; see Corollary 8.12. Yu [22, arXiv v2, Theorems 4.1 and 4.2] proves finiteness of the isotropy conjugacy classes for a fixed compact group with a fixed normal metric, and for a fixed semisimple group with a varying normal metric. The normal homogeneous case of Theorem B is [22, arXiv v2, Conjecture 4.1].

Sutton [20] constructed nonisometric isospectral pairs of normal homogeneous spaces.

There is also a local estimate. If g_0 is globally naturally reductive, then there are $N \geq 1$, a smooth neighborhood $\mathcal{U}$ of $\mathrm{Diff}(M) \cdot g_0$, and constants $C_r > 0$ such that every globally naturally reductive $g \in \mathcal{U}$ has a diffeomorphism ϕ with

$$\|\phi^* g - g_0\|_{C^r(g_0)} \leq C_r \max_{1 \leq j \leq N} |\lambda_j(g) - \lambda_j(g_0)|, \qquad r = 0, 1, \dots . \tag{1.1}$$

The same ϕ and N occur in all differentiability orders. The transitive action and the naturally reductive complement may vary; see Theorem 8.1. Gordon–Sutton [12, Theorem 4.1] prove local spectral isolation for naturally reductive left-invariant metrics on a fixed compact simple Lie group. Gordon–Schueth–Sutton [11, Theorem 2.3] prove local determination of a bi-invariant metric on a compact connected Lie group by finitely many distinct eigenvalues among left-invariant metrics of no greater volume.

For a compact transitive action $B \curvearrowright M$, an invariant symmetric tensor on $\mathfrak{b}$ defines a central quadratic operator. When its principal symbol is positive, this operator is the Laplacian of a naturally reductive metric. Conversely, every metric in the stated class has such a realization; see Lemma 8.4. The inducing tensor may be indefinite or degenerate on $\mathfrak{b}$. For a nondegenerate invariant inducing form, the generalized Casimir formula and scalar action are given in [1, Lemmas 2.6 and 2.9].

In a fixed space of central quadratic operators, finitely many scalar representation functionals form linear coordinates. Eigenvalues near an isolated reference cluster remain in its fixed interval of indices. This gives (1.1) on the fixed operator space. The slice theorem and the finiteness of conjugacy classes of semisimple subgroups with maximal centralizing tori reduce the varying-action case to finitely many such spaces. Their closedness also proves smooth closure of the naturally reductive class. Theorem B then follows from Theorem A and local isolation.

1.3. **Representation multiplicities and compactness.** Let G be compact, connected and semisimple, and let $H \subset G$ be closed. Fix an invariant inner product on $\mathfrak{g}$. An invariant cometric A determines, in each irreducible unitary representation π, a Hermitian matrix $L_\pi^H(A)$ on the fixed space V_π^H. Its entries are linear in A, and

$$\mathrm{mult}_g(\mu) = \sum_{\pi \in \widehat{G}} \dim V_\pi \, \dim \ker\big(L_\pi^H(A) - \mu I\big).$$

A summand contributing to an eigenvalue of multiplicity m_μ has $\dim V_\pi \leq m_\mu$. The Weyl dimension formula leaves only finitely many such representations. Requiring their total contribution to be at least m_μ is a finite union of polynomial rank conditions. This condition is necessary for multiplicity exactly m_μ and sufficient for multiplicity at least m_μ. The Hilbert basis theorem reduces the intersection

over all prescribed eigenvalues to a finite subintersection. On compact positive parameter sets, these identities give finite spectral determination; see Lemma 2.8.

An unbounded semialgebraic family of cometrics on a fixed presentation contains an escaping Nash arc with a nonzero positive semidefinite normalized limit A_∞. A vector with bounded spectral energy is annihilated, in the limit, by every direction in im A_∞. The persistent multiplicities therefore occur on a quotient of smaller dimension. Its Weyl count cannot contain the prescribed spectrum. At a bounded singular limit, adding εI and applying the Weyl law bounds the volume from above, whereas the determinant formula makes it tend to infinity. These arguments involve the Weyl law only for fixed positive metrics.

For varying isotropy, Yu's theorem [21, Theorem 1.1] gives, after conjugation and passage to a subsequence, $H_j \subset H$ and $V_\pi^{H_j} = V_\pi^H$ eventually for each fixed π. Finite-dimensional tensor representations make the quadratic operators descend to G/H. Their spectra contain the prescribed initial segments. The dimension comparison then leaves finitely many isotropy types.

On a semisimple homogeneous space times a torus, the corresponding matrices have the form

$$L_\pi(A) + \sum_{a,b} B_{ab}(m_a I - T_\pi(V_a))(m_b I - T_\pi(V_b)), \qquad T_\pi = -i\, d\pi.$$

Here $B > 0$, the V_a are invariant vector fields, and m ranges over the prescribed Fourier cosets. Representation dimension no longer bounds m. Lattice reduction and multiplicity counts first bound A and B from above. A common Cartan minimization bounds the components of the V_a perpendicular to that Cartan: an unbounded component would annihilate all persistent eigenspaces in one nonzero Lie algebra direction and lower the Weyl exponent.

The remaining unbounded components are toral. Their separated weights give a positive limiting operator. Exact spectral persistence then implies polynomial factorization of the original matrix pencils in the Fourier variables. The variational formula for sums of eigenvalues makes the limiting weight projections reduce the original operators. This gives an actual right-torus symmetry. Integral shears consequently produce isometric representatives with bounded parameters and positive limits.

1.4. **Relative spectral convergence.** For two closed connected metrics put

$$d_{\mathrm{sp}}(g,h) = \sup_{j\ge 1} \left|\log \frac{\lambda_j(g)}{\lambda_j(h)}\right| \in [0,\infty]. \tag{1.2}$$

On $\mathrm{SU}(2)\times S^1$, Proposition 9.1 gives left-invariant metrics g_i of the same volume as the round product Q, and diffeomorphisms F_i, such that

$$d_{\mathrm{sp}}(g_i, Q) \longrightarrow 0, \qquad F_i^* g_i \longrightarrow Q \quad \text{in } C^0, \qquad \mathrm{Scal}(g_i) \longrightarrow -\infty.$$

Thus uniform relative convergence of the entire positive spectrum does not imply smooth precompactness, even with fixed volume and C^0 convergence modulo diffeomorphisms. The parameters are a horizontal anisotropy tending to zero and an integral shear tending to infinity.

2. Invariant Laplacians and polynomial spectral conditions

2.1. **The homogeneous Laplacian.**

Lemma 2.1. *Let D_1, D_2 be real scalar differential operators of order at most two on a smooth manifold. Suppose that both are formally self-adjoint for the same positive smooth density, that $D_1(1) = D_2(1) = 0$, and that their quadratic principal symbols agree. Then $D_1 = D_2$.*

Proof. The difference $D_1 - D_2$ has order at most one and annihilates constants, so it is a real vector field Y. Its formal adjoint is $-Y - \operatorname{div} Y$. Equality of the first-order symbols gives $Y = -Y$, hence $Y = 0$. □

Let G be a compact connected Lie group, let $H \subset G$ be closed, and let Q be an $\mathrm{Ad}(G)$-invariant inner product on $\mathfrak{g}$. Put $\mathfrak{m} = \mathfrak{h}^{\perp_Q}$ and

$$\mathcal{P}(G,H) = \{A \in \mathrm{Sym}(\mathfrak{m}) \,:\, A > 0, A\,\mathrm{Ad}(h) = \mathrm{Ad}(h)A \text{ for every } h \in H\}.$$

The invariant metric corresponding to $A \in \mathcal{P}(G,H)$ is

$$(g_A)_{eH}(X,Y) = Q(A^{-1}X, Y). \tag{2.1}$$

The $\mathrm{Ad}(H)$-invariance of this inner product makes its translates independent of the representative of each coset. Extend A by zero on $\mathfrak{h}$. Unless otherwise indicated, its determinant means $\det_{\mathfrak{m}} A$. Fix a Q-orthonormal basis $X_1, \dots, X_D$ of $\mathfrak{g}$.

For a unitary representation $\pi : G \to U(V_\pi)$ define

$$L_\pi(A) = -\sum_{i,j=1}^{D} A_{ij}\, d\pi(X_i) d\pi(X_j). \tag{2.2}$$

Since $d\pi(X_i)^* = -d\pi(X_i)$ and $A_{ij} = A_{ji}$, one has $L_\pi(A)^* = L_\pi(A)$. For $h \in H$,

$$\pi(h) L_\pi(A) \pi(h)^{-1} = L_\pi(\mathrm{Ad}(h) A\, \mathrm{Ad}(h)^{-1}) = L_\pi(A).$$

Thus V_π^H is invariant under $L_\pi(A)$. Write $L_\pi^H(A) = L_\pi(A)|_{V_\pi^H}$.

Proposition 2.2. *For $\pi \in \widehat{G}$, on the π-summand of $L^2(G/H;\mathbb{C})$ the scalar Laplacian of g_A is $I_{V_\pi^*} \otimes L_\pi^H(A)$. Consequently,*

$$\mathrm{Spec}(\Delta_{g_A}) = \bigsqcup_{\pi \in \widehat{G}} d_\pi \ \mathrm{Spec}\big(L_\pi^H(A)\big), \qquad d_\pi = \dim_{\mathbb{C}} V_\pi. \tag{2.3}$$

The union includes multiplicities; a summand with $V_\pi^H = 0$ is empty. The trivial representation contributes one zero eigenvalue. Every other eigenvalue in the union is positive.

Proof. For $X \in \mathfrak{g}$ put

$$(\mathcal{X} f)(x) = \left.\frac{d}{dt}\right|_{t=0} f(x \exp(tX)).$$

Haar measure is invariant under the flow $x \mapsto x\exp(tX)$, so $\mathcal{X}^* = -\mathcal{X}$. The operator

$$\mathcal{D}_A = -\sum_{i,j} A_{ij} \mathcal{X}_i \mathcal{X}_j$$

is formally self-adjoint. Conjugation by right translation through $h \in H$ replaces its coefficient tensor by $\mathrm{Ad}(h) A\, \mathrm{Ad}(h)^{-1} = A$. Hence $\mathcal{D}_A$ preserves right H-invariant functions and descends to G/H. Its principal symbol at eH is A; at xH it is the image of A under left translation by x. This equals the principal symbol of Δ_{g_A}. Integrating right H-invariant functions on G shows that the descended operator is symmetric for quotient Haar probability.

The Riemannian measure of an invariant metric is a constant multiple of the invariant probability measure. Both $\mathcal{D}_A$ and Δ_{g_A} annihilate constants. Lemma 2.1 therefore gives $\mathcal{D}_A = \Delta_{g_A}$.

For $v \in V_\pi^H$ and $\eta \in V_\pi^*$, the function $f_{\eta,v}(xH) = \eta(\pi(x)v)$ is well defined, and differentiation gives

$$\mathcal{D}_A f_{\eta,v} = f_{\eta, L_\pi(A)v}.$$

The Peter–Weyl decomposition on G [4, Chapter III], followed by the orthogonal projection $f(x) \mapsto \int_H f(xh)\, dh$, gives

$$L^2(G/H;\mathbb{C}) = \widehat{\bigoplus_{\pi \in \widehat{G}}} V_\pi^* \otimes V_\pi^H. \tag{2.4}$$

This proves (2.3). If $\Delta_{g_A} f = 0$, then $\int_{G/H} |df|^2_{g_A} = 0$. Thus $df = 0$, and f is constant because G/H is connected. Constants occur precisely in the trivial summand. □

2.2. **Finite representation lists.** In the conventions of [2, 3], the irreducibles of the simply connected cover of a compact connected semisimple group have highest weights

$$\Lambda = \sum_{i=1}^{r} a_i \omega_i, \qquad a_i \in \mathbb{Z}_{\geq 0}.$$

Only those trivial on the covering kernel occur for the given global group. If Φ^+ is a positive system and ρ its half sum, then

$$d_\Lambda = \prod_{\alpha \in \Phi^+} \frac{\langle \Lambda + \rho, \alpha^\vee \rangle}{\langle \rho, \alpha^\vee \rangle}. \tag{2.5}$$

For the inner product Q, the Casimir eigenvalue is $c_\Lambda = \langle \Lambda, \Lambda + 2\rho \rangle$, where the pairing is induced by the dual of Q.

Lemma 2.3. *For each integer $M \geq 1$, a compact connected semisimple group has only finitely many irreducible representations of dimension at most M.*

Proof. Each positive coroot is a nonnegative integral combination of the simple coroots. Therefore every factor in (2.5) is at least one. The factor corresponding to the simple root α_i is $a_i + 1$. Thus $d_\Lambda \leq M$ implies $0 \leq a_i \leq M - 1$ for every i. There are at most M^r such weights. Representations of the given group form a subset of this finite list. □

Lemma 2.4 (Volume formula). *For $A \in \mathcal{P}(G, H)$,*

$$\mathrm{Vol}(g_A) = \mathrm{Vol}(g_I)(\det_{\mathfrak{m}} A)^{-1/2}. \tag{2.6}$$

Proof. By (2.1), the ratio of the Riemannian densities at eH is $(\det_{\mathfrak{m}} A)^{-1/2}$. Both densities are invariant, so their ratio is constant on G/H. Integration gives the formula. □

2.3. **Analytic conventions and estimates.** For a closed smooth Riemannian manifold, the elliptic estimate is [10, Lemmas 1.3.4(c) and 1.3.5]

$$\|f\|_{H^{s+2}} \leq C_s\big(\|\Delta f\|_{H^s} + \|f\|_{L^2}\big) \qquad (s \in \mathbb{Z}_{\geq 0}). \tag{2.7}$$

The scalar Laplacian admits an orthonormal basis of smooth eigenfunctions [10, Lemma 1.6.3]. On a closed n-manifold, Sobolev embedding gives the continuous inclusion $H^s(M) \hookrightarrow C^r(M)$ whenever $s > r + n/2$ [10, Lemma 1.3.4(b)].

Lemma 2.5. *Finite sums of Laplace eigenfunctions are dense in $C^\infty(M)$ for its usual Fréchet topology.*

Proof. Let (ϕ_j) be an orthonormal eigenbasis and $f \in C^\infty(M)$. Put $a_j = \int_M f\overline{\phi_j}\, d\,\mathrm{Vol}$. For every integer $k \geq 0$, integration by parts and Parseval's identity give

$$\sum_j (1 + \lambda_j)^{2k} |a_j|^2 = \|(1 + \Delta)^k f\|_{L^2}^2 < \infty.$$

For $f_N = \sum_{j \leq N} a_j \phi_j$, the same identity gives $\|(1 + \Delta)^k (f - f_N)\|_{L^2} \to 0$. Iterating (2.7) gives $f_N \to f$ in H^{2k}. For each r, choose $2k > r + n/2$ and apply Sobolev embedding. Then $f_N \to f$ in C^r. □

For $0 \neq f \in C^\infty(M)$, put $R_g(f) = \int_M |df|_g^2\, d\,\mathrm{Vol} \,/\, \int_M |f|^2\, d\,\mathrm{Vol}$. The variational formula is

$$\lambda_k(g) = \inf_{\substack{E \subset C^\infty(M) \\ \dim E = k+1}} \sup_{0 \neq f \in E} R_g(f). \tag{2.8}$$

Indeed, $E = \mathrm{span}(\phi_0, \dots, \phi_k)$ gives the upper bound. Every $(k+1)$-dimensional E contains a nonzero function orthogonal to $\phi_0, \dots, \phi_{k-1}$; its eigenfunction expansion gives $R_g(f) \geq \lambda_k(g)$. The same argument in a finite orthonormal eigenbasis gives the variational formula for Hermitian matrices. In particular, for two Hermitian matrices of the same size,

$$|\lambda_k(B) - \lambda_k(B')| \leq \|B - B'\|_{\mathrm{op}}.$$

2.4. **Weyl counting and Casimir bounds.** For a fixed closed smooth d-manifold Y, let $N_Y(T) = \#\{j : \lambda_j(Y) \le T\}$. The spectral-function asymptotic for $1 + \Delta_Y$, integrated over Y, gives [14, Theorem 1.1]

$$N_Y(T) \sim \frac{\omega_d}{(2\pi)^d} \operatorname{Vol}(Y) T^{d/2}. \tag{2.9}$$

The count for a point is one. For a sequence $\Sigma = (\sigma_0, \sigma_1, \ldots)$, write $N_\Sigma(T) = \#\{j \ge 0 : \sigma_j \le T\}$ and $N_\Sigma^+(T) = \#\{j \ge 1 : \sigma_j \le T\}$. Formula (2.9) concerns each fixed positive metric. The notation $\operatorname{Vol}(\Sigma)$ denotes the volume recovered from its leading coefficient. For $\pi \in \widehat{G}$, put $T_\pi(X) = -i\, d\pi(X)$. The $\mathrm{Ad}(G)$-invariance of Q and Schur's lemma give $\sum_i T_\pi(X_i)^2 = c_\pi I$ with $c_\pi \ge 0$. Consequently,

$$\|T_\pi(X)\| \le \sqrt{c_\pi}\, \|X\|_Q. \tag{2.10}$$

For $\|X\|_Q = 1$, extend X to an orthonormal basis. Its squared Hermitian operator is a positive summand of $c_\pi I$; homogeneity proves the formula. The bi-invariant Laplacian on G has finite-dimensional spectral subspaces below every fixed bound. Since each irreducible π occurs in $L^2(G)$, only finitely many π satisfy $c_\pi \le T$ for a fixed T.

2.5. **Eventual equalities and compact parameter sets.**

Lemma 2.6 (Eventual polynomial identities). *Let (x_j) be a sequence in $\mathbb{R}^d$ and let $\{p_\alpha : \alpha \in A\} \subset \mathbb{R}[X_1, \ldots, X_d]$. Suppose*

$$\text{for every } \alpha \text{ there is } J_\alpha \text{ such that } p_\alpha(x_j) = 0 \quad (j \ge J_\alpha).$$

Then there is J such that $p_\alpha(x_j) = 0$ for every α and every $j \ge J$.

Proof. The ideal $I = (p_\alpha : \alpha \in A)$ is finitely generated. Each member of a finite generating set is a finite polynomial combination of the p_α. The finitely many p_α occurring in these combinations therefore generate I themselves. Write them as $p_{\alpha_1}, \ldots, p_{\alpha_s}$ and put $J = \max_{1 \le i \le s} J_{\alpha_i}$. If $I = 0$, take $J = 1$ instead. For every α, there are polynomials $q_{\alpha,i}$ such that

$$p_\alpha = \sum_{i=1}^{s} q_{\alpha,i} p_{\alpha_i}.$$

Evaluation at x_j, $j \ge J$, makes each summand zero. □

For rational functions, the lemma applies after clearing denominators, provided the denominators are nonzero at the points under consideration.

Lemma 2.7 (Uniform finite spectral support). *Let P be a topological space. For each α in a countable set, let $B_\alpha(x)$ be a continuous Hermitian matrix of fixed finite size, and let w_α be a positive integer. Suppose that*

$$\bigsqcup_\alpha w_\alpha \operatorname{Spec}(B_\alpha(x)) = (\lambda_0(x) \le \lambda_1(x) \le \cdots)$$

has finite multiplicities, tends to infinity, and has continuous indexed eigenvalues. If $K \subset P$ is compact, then for every $T \in \mathbb{R}$ there is a finite set $\mathcal{A}$ such that

$$x \in K, \quad \alpha \notin \mathcal{A} \quad \Longrightarrow \quad \operatorname{Spec}(B_\alpha(x)) \subset (T, \infty).$$

Proof. If $K = \emptyset$, take $\mathcal{A} = \emptyset$. Otherwise fix $x \in K$. Choose $R > \max\{T, \lambda_0(x)\}$ outside the spectrum at x, and let $m \ge 1$ be the number of its eigenvalues below R, counted with multiplicity. The block roots contributing these m occurrences lie in finitely many blocks $\mathcal{A}_x$, since every $w_\alpha \ge 1$. Continuity of their ordered matrix eigenvalues preserves all these occurrences below R on a neighborhood U_x of x. Shrinking U_x by indexed spectral continuity gives

$$\lambda_{m-1}(z) < R < \lambda_m(z) \qquad (z \in U_x).$$

Thus the selected block roots account for every occurrence below R on U_x. No block outside $\mathcal{A}_x$ has a root at most T there. A finite subcover $U_{x_1}, \ldots, U_{x_k}$ of K gives $\mathcal{A} = \bigcup_{i=1}^k \mathcal{A}_{x_i}$. □

Lemma 2.8 (Finite polynomial determination). *Let $P \subset \mathbb{R}^m$ be open and let $K \subset P$ be compact. For each α in a countable set, let $B_\alpha(x)$ be a fixed-size Hermitian matrix whose real and imaginary entries are polynomials in x, and let w_α be a positive integer. Suppose that*

$$\bigsqcup_\alpha w_\alpha \operatorname{Spec}(B_\alpha(x)) = (0 = \lambda_0(x) < \lambda_1(x) \le \lambda_2(x) \le \cdots)$$

is a nonnegative discrete spectrum, with finite multiplicities and $\lambda_j(x) \to \infty$ as $j \to \infty$. Assume that every λ_j is continuous on P. Fix a sequence $\Sigma = (0 = \sigma_0 < \sigma_1 \le \sigma_2 \le \cdots)$ tending to infinity. Then some N satisfies

$$x \in K, \quad \lambda_j(x) = \sigma_j \ (1 \le j \le N) \quad \Longrightarrow \quad \lambda_j(x) = \sigma_j \quad (j \ge 0).$$

If K is semialgebraic, the full isospectral subset of K is compact and semialgebraic. Each block spectrum is constant on every connected component of that subset.

Proof. Let $F = \{x \in K : \lambda_j(x) = \sigma_j \text{ for every } j\}$. Continuity makes F closed in K, hence compact. Fix $x_* \in F$. For every α, expand

$$\det(zI - B_\alpha(x)) - \det(zI - B_\alpha(x_*)) = \sum_\ell f_{\alpha,\ell}(x) z^\ell.$$

By the finite-generation argument in the proof of Lemma 2.6, the ideal generated by all $f_{\alpha,\ell}$ in $\mathbb{R}[x_1, \dots, x_m]$ is generated by finitely many of these polynomials. Let $\mathcal{A}_*$ be the finite set of their block indices.

Let E_* be the finite set of eigenvalues of $B_\alpha(x_*)$, $\alpha \in \mathcal{A}_*$. Enlarge $\mathcal{A}_*$, if necessary, so that E_* is nonempty. Choose N_* with $\sigma_{N_*} > \max E_*$. For each $\mu \in E_*$ choose an open interval I_μ containing μ such that the intervals are disjoint,

$$I_\mu \cap \{\sigma_j : j \ge 0\} = \{\mu\}, \qquad \sup I_\mu < \sigma_{N_*}.$$

Finite-dimensional eigenvalue continuity gives a neighborhood U_* of x_* in which every eigenvalue of each selected block remains in the interval containing its value at x_*. Suppose $x \in U_*$ and the first N_* eigenvalues agree with Σ. A selected block eigenvalue lies below $\lambda_{N_*}(x) = \sigma_{N_*}$, so it belongs to the prescribed finite scalar list. The interval condition makes it equal to its value at x_*. The selected characteristic polynomials are therefore unchanged. Every $f_{\alpha,\ell}$ vanishes, and all block spectra, with their fixed outer multiplicities, agree with those at x_*. Thus x has spectrum Σ.

Take finitely many U_* covering F, and let N_0 be the maximum of the corresponding indices. If $F = \emptyset$, take their union empty and $N_0 = 1$. The complement K_0 of this union in K is compact and contains no realization. For each $x \in K_0$, choose $j(x)$ with $\lambda_{j(x)}(x) \ne \sigma_{j(x)}$. Continuity preserves this inequality on a neighborhood of x. A finite subcover of K_0 has a finite set of associated indices. Their maximum together with N_0 is a determining N.

By Lemma 2.7, only finitely many blocks have an eigenvalue at most $T = \sigma_N + 1$ at some point of K. Let $B(x)$ be the direct sum of these blocks, with w_α copies of $B_\alpha(x)$. Its entries are polynomial. If its size is less than $N + 1$, no parameter can realize the prescribed initial segment, so the realization set is empty. Otherwise, requiring its first $N + 1$ ordered eigenvalues to be $\sigma_0, \dots, \sigma_N$ is semialgebraic: introduce ordered real root variables, equate elementary symmetric functions with its characteristic coefficients, and project the resulting semialgebraic set. The projection theorem for semialgebraic sets applies [6, Theorem 1.1]. Blocks omitted from $B(x)$ have all eigenvalues above T, so the condition is exactly scalar initial segment agreement. The first part identifies its intersection with K with F. Finally each ordered eigenvalue of a fixed block is continuous on F and takes values in the discrete set $\{\sigma_j\}$. It is constant on every connected component. □

Lemma 2.9 (Uniform comparison of positive cometrics). *On a fixed compact homogeneous presentation, let $C_0 > 0$ and $C > 0$ be invariant cometrics and suppose*

$$\|C_0^{-1/2}(C - C_0)C_0^{-1/2}\|_{\mathrm{op}} \le \eta < 1.$$

Then, for every $j \ge 1$,

$$(1 - \eta)\lambda_j(C_0) \le \lambda_j(C) \le (1 + \eta)\lambda_j(C_0).$$

Proof. The matrix norm bound is equivalent to $(1-\eta)C_0 \le C \le (1+\eta)C_0$. All invariant metric measures are constant multiples of the same invariant probability measure ν. The Rayleigh quotient of g_C can therefore be written as

$$R_C(f) = \frac{\int C(df, df)\, d\nu}{\int |f|^2\, d\nu}.$$

The matrix inequalities bound this quotient by $(1 \pm \eta)R_{C_0}(f)$. Taking the supremum over each finite-dimensional trial space and then the infimum in the min–max formula gives the eigenvalue inequalities. □

If $C \to C_0 > 0$, the norm in Lemma 2.9 tends to zero. Hence every indexed eigenvalue is continuous in the positive-cometric cone. Lemma 2.7 applies to each compact subset of that cone.

Lemma 2.10. *Let E be a finite-dimensional real normed space and $T_\alpha : E \to F_\alpha$ linear maps into finite-dimensional normed spaces. Put $K = \bigcap_\alpha \ker T_\alpha$. There are indices $\alpha_1, \dots, \alpha_m$, with $m \le \dim E - \dim K$, for which*

$$K = \bigcap_{i=1}^{m} \ker T_{\alpha_i}.$$

If $K \neq E$, there is $c > 0$ such that

$$c\ \operatorname{dist}(x, K) \le \max_{1 \le i \le m} \|T_{\alpha_i} x\| \qquad (x \in E). \tag{2.11}$$

Proof. Set $E_0 = E$. If $E_j \neq K$, choose $v_j \in E_j \setminus K$. There is an index α_{j+1} with $T_{\alpha_{j+1}} v_j \neq 0$. Then $E_{j+1} = E_j \cap \ker T_{\alpha_{j+1}}$ contains K and has dimension strictly smaller than $\dim E_j$. After at most $\dim E - \dim K$ steps one obtains $E_m = K$. If $K = E$, the empty collection suffices. Otherwise pass to the quotient norm on E/K. On its unit sphere the continuous function $[x] \mapsto \max_i \|T_{\alpha_i} x\|$ is everywhere positive. Its minimum c is positive by compactness. Homogeneity proves (2.11). □

2.6. **A nonsingular limit criterion.**

Lemma 2.11. *On a fixed closed homogeneous n-manifold M, fix a positive invariant reference cometric I and identify symmetric cometrics with self-adjoint endomorphisms by its inverse metric. Let $C_j > 0$ belong to a fixed finite-dimensional invariant-cometric space. Suppose C_j is bounded, $C_j \to C_0 \ge 0$, and $\lambda_k(C_j) \to \sigma_k$ for every k, where Σ is a smooth n-dimensional spectrum. Then $C_0 > 0$.*

Proof. For every $\varepsilon > 0$, one has $C_j \le C_0 + \varepsilon I$ for all sufficiently large j. All invariant measures have the same normalized probability measure, so min–max gives

$$\sigma_k \le \lambda_k(C_0 + \varepsilon I) \quad (k \ge 0), \qquad N_{C_0+\varepsilon I}(T) \le N_\Sigma(T).$$

Apply (2.9) to this one fixed positive cometric and the reference spectrum, then let $T \to \infty$. Thus

$$\operatorname{Vol}(g_I) \det_{T_oM}(C_0 + \varepsilon I)^{-1/2} \le \operatorname{Vol}(\Sigma), \qquad o \in M.$$

Singularity of C_0 would make the left side tend to infinity as $\varepsilon \downarrow 0$. This is impossible. □

3. A fixed homogeneous presentation

Let G be a compact connected semisimple Lie group and let H be a closed subgroup with $M = G/H$ of positive dimension n; H need not be connected and the action need not be effective. Fix an $\mathrm{Ad}(G)$-invariant inner product Q on $\mathfrak{g}$, put $\mathfrak{m} = \mathfrak{h}^{\perp_Q}$, and identify invariant cometrics with the cone

$$\mathcal{P} = \{C \in \operatorname{Sym}(\mathfrak{m})^{\mathrm{Ad}(H)} : C > 0\}.$$

The corresponding metric is $g_C(X, Y) = Q(C^{-1}X, Y)$ at the base point. All Laplacians act on the common Hilbert space defined by invariant probability measure. Let

$$0 = \lambda_0(C) < \lambda_1(C) \le \lambda_2(C) \le \cdots$$

for the scalar eigenvalues, repeated with multiplicity. The normal homogeneous Laplacian is $\Delta_Q = \Delta_I$. In the notation of Eq. (2.2), $A_\pi(C) = L^H_\pi(C)$.

Theorem 3.1. *Let $\Sigma = (0 = \sigma_0 < \sigma_1 \le \sigma_2 \le \cdots)$ be the scalar spectrum of a closed connected smooth Riemannian n-manifold. There exists an integer $N = N(G, H, \Sigma)$ such that*

$$(\lambda_1(C), \dots, \lambda_N(C)) = (\sigma_1, \dots, \sigma_N), \qquad C \in \mathcal{P},$$

implies $\mathrm{Spec}(\Delta_C) = \Sigma$.

The full isospectral locus

$$\mathcal{I}_\Sigma = \{C \in \mathcal{P} \,:\, \mathrm{Spec}(\Delta_C) = \Sigma\}$$

is compact and semialgebraic. It has finitely many connected components, and the spectrum of every representation multiplicity block is constant on each component.

If $\mathcal{I}_\Sigma = \emptyset$, no invariant metric on G/H has the determining initial segment.

3.1. **Polynomial conditions for spectral inclusion.** Peter–Weyl decomposition gives

$$L^2(G/H; \mathbb{C}) = \widehat{\bigoplus}_{\pi \in \widehat{G}} V_\pi^* \otimes W_\pi, \qquad W_\pi = V_\pi^H.$$

Put $d_\pi = \dim V_\pi$. On this summand the invariant Laplacian is $I \otimes A_\pi(C)$, with $A_\pi(C)$ Hermitian and linear in C. For $w \in W_\pi$, one has $d\pi(Y)w = 0$ for every $Y \in \mathfrak{h}$. Adding a Q-orthonormal basis of $\mathfrak{h}$ to one of $\mathfrak{m}$ therefore gives $A_\pi(I) = c_\pi I$. Positivity of the quadratic form in (2.2) yields

$$C \ge aI \quad \Longrightarrow \quad A_\pi(C) \ge a c_\pi I. \tag{3.1}$$

The linear block formula extends to all invariant symmetric C, whether or not positive definite.

For each distinct positive reference eigenvalue μ, denote its multiplicity by m_μ and set

$$Z_\mu = \left\{ C \,:\, \sum_{d_\pi \le m_\mu} d_\pi \dim\ker(A_\pi(C) - \mu I) \ge m_\mu \right\}. \tag{3.2}$$

Zero multiplicity spaces are omitted. The sum in (3.2) is finite by Lemma 2.3.

Each Z_μ is a real algebraic set: take a finite union over tuples of required kernel dimensions whose weighted sum is at least m_μ, and express the rank bounds by vanishing minors. Replace complex equations by their real and imaginary parts. Since $V(I_1) \cup V(I_2) = V(I_1 I_2)$, these finite unions are algebraic. Hilbert's basis theorem gives a finite nonempty list of reference values $\mu_1, \dots, \mu_s$ for which

$$Z := \bigcap_{\mu > 0,\, \mu \in \Sigma} Z_\mu = \bigcap_{i=1}^{s} Z_{\mu_i}. \tag{3.3}$$

Apply the finite-generation argument in the proof of Lemma 2.6 to the ideal generated by all defining polynomials of the sets Z_μ. The selected original equations involve only finitely many reference values. If this list is empty, add any positive reference value.

For $C \in Z \cap \mathcal{P}$, every positive $\mu \in \Sigma$ occurs with multiplicity at least m_μ. Conversely, if its scalar multiplicity is exactly m_μ, then every representation contributing to this eigenspace has $d_\pi \le m_\mu$; hence $C \in Z_\mu$. For each selected μ_i, put $b_i = \max\{k \,:\, \sigma_k = \mu_i\}$. Agreement through the index $1 + \max_i b_i$ gives exactly m_{μ_i} occurrences of every μ_i, and consequently implies $C \in Z$.

3.2. **The kernel of a positive semidefinite direction.**

Lemma 3.2. *Let $0 \ne V \ge 0$ be an $\mathrm{Ad}(H)$-invariant symmetric endomorphism of $\mathfrak{m}$. There is a closed subgroup $K \supset H$ with $\dim(G/K) < n$ such that*

$$\ker A_\pi(V) = V_\pi^K \quad \text{for every } \pi \in \widehat{G}.$$

Consequently,

$$\sum_{c_\pi \le T} d_\pi \dim\ker A_\pi(V) = N_{G/K,Q}(T) = o(T^{n/2}), \tag{3.4}$$

where the count on the right includes zero and refers to the normal homogeneous metric on G/K.

Proof. Write $V = \sum_{a=1}^{r} X_a \otimes X_a$, with $X_a \in \mathfrak{m}$, and let K be the closure of the subgroup generated by H and all one-parameter groups $\exp(\mathbb{R}X_a)$. In a unitary realization ρ_π on V_π, the quadratic form of $A_\pi(V)$ on W_π is

$$\langle A_\pi(V)w, w\rangle = \sum_{a=1}^{r} \|d\rho_\pi(X_a)w\|^2.$$

This follows by lifting the invariant quadratic operator to G, where it is the sum of the negative squares of the corresponding left-invariant vector fields. The individual fields need not preserve right-H-invariant functions, but their sum does, by $\mathrm{Ad}(H)$-invariance. Positivity shows that its kernel consists precisely of the vectors fixed by H and annihilated by every $d\rho_\pi(X_a)$, which are exactly the K-fixed vectors.

Since $V \neq 0$, some X_a is nonzero and belongs to $\mathrm{Lie}(K) \setminus \mathfrak{h}$. Thus $\dim K > \dim H$. On V_π^K, the normal Laplacian of G/K is $c_\pi I$. The Peter–Weyl formula proves the equality in (3.4); (2.9) in dimension $\dim(G/K) < n$ proves the estimate. If G/K is a point, its count is identically one. □

3.3. **Boundedness of the cometrics.**

Lemma 3.3. *The set $Z \cap \mathcal{P}$ is bounded in the finite-dimensional cometric space.*

Proof. Suppose it is unbounded. Equip the cometric space with the Frobenius norm. Apply the semialgebraic compactification $C \mapsto C/\sqrt{1+\|C\|^2}$. The closure of the image of $Z \cap \mathcal{P}$ meets the unit sphere at a point V. Analytic curve selection [6, Theorem 1.15] gives a Nash arc through V whose positive branch lies in this image. The inverse map $U \mapsto U/\sqrt{1-\|U\|^2}$ is analytic on the open unit ball. It therefore gives a real analytic semialgebraic arc

$$C(t) \in Z \cap \mathcal{P}, \qquad 0 < t < \delta, \qquad \|C(t)\| \longrightarrow \infty \quad (t \downarrow 0).$$

The compactified arc tends to V, and positivity is preserved in its limit. Hence

$$\frac{C(t)}{\|C(t)\|} \longrightarrow V, \qquad V \geq 0, \quad \|V\| = 1. \tag{3.5}$$

For each π and each positive $\mu \in \Sigma$, define

$$k_{\pi,\mu} = \min_{0<t<\delta} \dim\ker(A_\pi(C(t)) - \mu I).$$

If the largest rank of $A_\pi(C(t)) - \mu I$ is zero, its nullity is constant. Otherwise choose a minor of largest possible size which is nonzero at some $t \in (0,\delta)$. Its determinant is analytic and not identically zero; its zero set is discrete. Outside this set the nullity is $k_{\pi,\mu}$. For fixed μ, the finitely many representations in (3.2) attain their minimum nullities simultaneously outside a finite union of discrete sets. At such a parameter, membership in Z_μ gives

$$\sum_{d_\pi \leq m_\mu} d_\pi k_{\pi,\mu} \geq m_\mu. \tag{3.6}$$

By definition, μ occurs at least $k_{\pi,\mu}$ times in $A_\pi(C(t))$ for every $t \in (0,\delta)$.

Fix π and an interior parameter t_1. Every μ with $k_{\pi,\mu} > 0$ is an eigenvalue of $A_\pi(C(t_1))$. There are therefore finitely many such μ, and their distinct eigenspaces are orthogonal. In particular,

$$K_\pi := \sum_{\mu} k_{\pi,\mu} \leq \dim W_\pi.$$

If $K_\pi > 0$, put $M_\pi = \max\{\mu : k_{\pi,\mu} > 0\}$. At every t, at least K_π eigenvalues of $A_\pi(C(t))$ lie in $(0, M_\pi]$. The first K_π eigenvalues of $A_\pi(C(t))/\|C(t)\|$ therefore lie in $[0, M_\pi/\|C(t)\|]$ and tend to zero. The whole matrix tends to $A_\pi(V)$ by (3.5). Continuity of the ordered eigenvalues gives, also when $K_\pi = 0$,

$$K_\pi \leq \dim\ker A_\pi(V). \tag{3.7}$$

In the sums below, μ runs through the distinct positive values of Σ, and m_μ is their multiplicity. Choose any fixed interior point t_0 and $a_0 > 0$ such that $C(t_0) \geq a_0 I$. If $k_{\pi,\mu} > 0$, then μ is an eigenvalue at t_0, so (3.1) implies $\mu \geq a_0 c_\pi$. Combining (3.6) and (3.7) therefore gives

$$\begin{aligned} N_\Sigma^+(T) &\leq \sum_{\substack{\mu\in\Sigma \\ 0<\mu\leq T}} \sum_{d_\pi \leq m_\mu} d_\pi k_{\pi,\mu} \\ &\leq \sum_{c_\pi \leq T/a_0} d_\pi \dim\ker A_\pi(V) = o(T^{n/2}), \end{aligned}$$

by Lemma 3.2. This contradicts the positive leading coefficient in the n-dimensional reference Weyl law. □

3.4. **Finite determination and the isospectral locus.**

Proof of Theorem 3.1. Put $N_0 = 1 + \max_i b_i$. Every $C \in \mathcal{P}$ with the prescribed first N_0 eigenvalues belongs to Z. By Lemma 3.3, there is $b > 0$ such that $C \leq bI$ for all $C \in Z \cap \mathcal{P}$.

There are $N_1 \geq N_0$ and $a > 0$ such that agreement through N_1 also implies $C \geq aI$. Otherwise, for every $j \geq N_0$ one could choose $C_j \in \mathcal{P}$ with the prescribed first j eigenvalues and $\lambda_{\min}(C_j) < 1/j$. These cometrics belong to Z and are bounded. A subsequence converges to $C_0 \geq 0$ with $\lambda_{\min}(C_0) = 0$. For each fixed k, its indexed scalar eigenvalues tend to σ_k. Lemma 2.11 would give $C_0 > 0$, a contradiction.

After decreasing a if necessary, assume $a \leq b$ and put

$$\mathcal{K} = \{C \in \mathrm{Sym}(\mathfrak{m})^{\mathrm{Ad}(H)} \,:\, aI \leq C \leq bI\}.$$

This is a compact semialgebraic subset of $\mathcal{P}$. The matrices $A_\pi(C)$ have linear entries, their outer multiplicities are d_π, and their scalar union has continuous indexed eigenvalues by Lemma 2.9. Lemma 2.8 gives a determining index N_2 on $\mathcal{K}$. For $N = \max\{N_1, N_2\}$, agreement through N places C in $\mathcal{K}$ and then gives its full spectrum.

Every fully isospectral cometric belongs to $\mathcal{K}$. The same lemma therefore makes $\mathcal{I}_\Sigma$ compact and semialgebraic, with constant block spectra on each connected component. A semialgebraic set has finitely many connected components [6, Proposition 1.6]. □

Corollary 3.4. *The same finite-initial-segment determination holds on any prescribed finite union of fixed compact semisimple homogeneous presentations. On a fixed presentation, it also holds after restricting the cometric cone to a real linear subspace. Its full isospectral locus in that subspace remains compact and semialgebraic.*

Proof. For finitely many presentations take the maximum of their determining indices. Restriction to a linear subspace preserves the determining implication; the restricted isospectral locus is the intersection of a compact semialgebraic set with a closed linear subspace. □

Lemma 3.5. *If a smooth reference spectrum has dimension greater than* $\dim(G/H)$*, some finite initial segment is not realized by any invariant metric on* G/H*.*

Proof. If every initial segment were realized, define Z_μ as in (3.2) with the reference multiplicities. The finite intersection equality (3.3) gives one positive cometric whose spectrum contains the entire positive reference multiset. Its counting function has growth $O(T^{\dim(G/H)/2})$, whereas the contained multiset has the larger reference Weyl exponent. This is a contradiction. If G/H is a point, its absence of positive eigenvalues gives the contradiction directly. □

4. Semisimple presentations with varying isotropy

Theorem 4.1 (Global finite spectral determination and compactness). *Fix $n > 0$ and a scalar spectrum Σ of a closed connected smooth n-manifold. There exists an integer $N = N(n, \Sigma)$ such that, among all closed connected homogeneous Riemannian n-manifolds with finite fundamental group, exact agreement with the first N positive eigenvalues of Σ is equivalent to equality of the full spectrum.*

The realization set is compact in the smooth topology modulo diffeomorphisms and has finitely many diffeomorphism types. More precisely, it is covered, up to isometry, by finitely many compact semialgebraic sets of invariant cometrics on fixed semisimple homogeneous presentations. On each such presentation, only finitely many collections $(\operatorname{Spec} L_\pi^H(C))_{\pi\in\widehat{G}}$ *occur.*

Throughout this section eigenvalues are repeated according to their scalar multiplicities, and a positive initial segment starts with the first eigenvalue after the constant eigenfunction. The reference spectrum is always that of a closed connected smooth manifold of the same dimension as the homogeneous manifolds under consideration. Neither the reference manifold nor its spectrum is required to have a homogeneous realization.

Theorem 4.2 (Finiteness of isotropy types from finitely many eigenvalues). *Let G be a fixed compact connected semisimple Lie group, let $n > 0$, and let Σ be the scalar spectrum of a closed connected smooth n-manifold. There exists an integer $N = N(G,\Sigma)$ with the following property. Among the closed subgroups $H \subset G$ for which $\dim(G/H) = n$ and some G-invariant metric on G/H agrees with the first N positive eigenvalues of Σ, only finitely many G-conjugacy classes occur.*

Proof. Suppose that no such integer exists. For every j, infinitely many conjugacy classes admit a metric with the prescribed first j eigenvalues. Inductively choose H_j from a class different from the previously chosen classes, together with such a metric on G/H_j. Thus $\dim(G/H_j) = n$ and the H_j are pairwise nonconjugate.

By Yu's dimension-datum subsequence theorem [21, Theorem 1.1], after passing to a subsequence and conjugating the subgroups and metrics, there is a closed subgroup $H \subset G$ such that

$$H_j \subset H, \qquad \dim V_\pi^{H_j} \longrightarrow \dim V_\pi^H \quad \text{for every irreducible } \pi \in \widehat{G}.$$

Put $W_\pi = V_\pi^H$. Then

$$V_\pi^{H_j} = W_\pi \qquad \text{for all sufficiently large } j, \text{ for each fixed } \pi. \tag{4.1}$$

Indeed, $H_j \subset H$ gives $W_\pi \subset V_\pi^{H_j}$, and the integer dimensions eventually agree. The index from which equality holds may depend on π.

Fix a bi-invariant inner product Q on $\mathfrak{g}$. Write $\mathfrak{h}_j$ for the Lie algebra of H_j, and extend the jth quotient cometric by zero on $\mathfrak{h}_j$. Denote the resulting symmetric endomorphism of $\mathfrak{g}$ by C_j. It is positive definite on $\mathfrak{h}_j^{\perp_Q}$. All parameters belong to the fixed finite-dimensional real vector space $\operatorname{Sym}(\mathfrak{g})$; they need not be bounded.

For a general C in this space, let D_C be the real quadratic differential operator on G formed from left-invariant vector fields, with the sign chosen so that positive C gives a nonnegative operator. In Peter–Weyl decomposition its multiplicity-space matrices $L_\pi(C)$ on V_π are Hermitian and linear in C. The operator D_{C_j} restricts on right-H_j-invariant functions to the quotient Laplacian.

Consider $E = \operatorname{Sym}^2(\mathfrak{g})$ with the action induced by $\operatorname{Ad}(G)$. Its complexification is a finite direct sum of irreducible representations. Equation (4.1), applied to these finitely many summands, gives $(E\otimes\mathbb{C})^{H_j} = (E\otimes\mathbb{C})^H$ for all sufficiently large j. Taking real parts gives

$$E^{H_j} = E^H. \tag{4.2}$$

The zero extension C_j belongs to E^{H_j}, hence is H-invariant. The quadratic operator D_{C_j} consequently preserves smooth right-H- invariant functions and descends to G/H. Its tangent symbol is the restriction of the quadratic form C_j to $\mathfrak{h}^{\perp_Q}$. For $0 \neq X \in \mathfrak{h}^{\perp_Q}$ one has $X \in \mathfrak{h}_j^{\perp_Q}$ and therefore $Q(C_jX, X) > 0$. Thus the descended symbol is positive definite. It is symmetric for quotient Haar probability and annihilates constants. Lemma 2.1 identifies it with the scalar Laplacian of the descended symbol.

For each distinct positive reference eigenvalue μ, write m_μ for its multiplicity and $d_\pi = \dim V_\pi$. There are only finitely many irreducible representations of G with $d_\pi \le m_\mu$. Define the rectangular maps

$$A_{\pi,\mu}(C) = \big(L_\pi(C) - \mu I\big)|_{W_\pi} \,:\, W_\pi \longrightarrow V_\pi$$

and the subsets of the fixed parameter space

$$Z_\mu = \left\{ C \,:\, \sum_{d_\pi \le m_\mu} d_\pi \dim\ker A_{\pi,\mu}(C) \ge m_\mu \right\}. \tag{4.3}$$

Each Z_μ is real algebraic. Namely, take the finite union over tuples of required nullities whose weighted total is at least m_μ; each tuple is specified by rank-minor equations for the corresponding rectangular matrices. This definition does not require preservation of W_π by $L_\pi(C)$ for arbitrary parameters.

Fix μ and put $b_\mu = \max\{k \,:\, \sigma_k = \mu\}$. For $j \ge b_\mu + 1$, the jth metric has exactly m_μ scalar occurrences of μ. Every contributing representation has $d_\pi \le m_\mu$. Equation (4.1) holds for all these finitely many representations when j is sufficiently large. Thus there is J_μ such that $C_j \in Z_\mu$ for every $j \ge J_\mu$.

Noetherianity of the polynomial ring in the entries of C gives a finite list of original reference values $\mu_1, \dots, \mu_s$ such that

$$\bigcap_{\mu\in\Sigma,\, \mu>0} Z_\mu = \bigcap_{i=1}^{s} Z_{\mu_i}.$$

Choose J larger than all J_{μ_i} and large enough for (4.2). For every $j \ge J$, the tensor C_j is H-invariant and belongs to every Z_μ. The kernels in (4.3) are then eigenspaces of the descended Laplacian on G/H. Formula (2.3) shows that its spectrum contains Σ, with at least the prescribed multiplicities.

Set $d = \dim(G/H)$. If $d = 0$, the quotient is a point because G is connected, and its spectrum cannot contain the infinitely many positive reference eigenvalues. Thus $d > 0$. Fix one j in the common tail. Weyl's law on its descended d-dimensional quotient, together with spectral containment and the n-dimensional reference Weyl law, gives $d \ge n$. On the other hand $H_j \subset H$ gives

$$d \le \dim(G/H_j) = n.$$

It follows that $d = n$ and $\mathfrak{h}_j = \mathfrak{h}$.

Equal Lie algebras of these compact subgroups imply

$$H^0 \subset H_j \subset H.$$

The intermediate subgroups correspond to subgroups of the finite group H/H^0, so only finitely many are possible. This contradicts the pairwise nonconjugate choice of the H_j and proves the theorem. □

Proof of Theorem 4.1. Let (M, g) be a closed connected homogeneous n-manifold with finite fundamental group. Choose a compact connected transitive isometry group K and write $M = K/L$. Such a connected group exists because the full compact isometry group has finitely many components. Its identity component has finitely many orbits in the homogeneous space; each orbit is closed, so its complement is closed and it is also open. Connectedness of M gives one orbit.

The derived subgroup $S = [K, K]$ is compact, connected, and semisimple. The product SL is compact. Its image in the abelian group K/S is normal, so SL is a closed normal subgroup of K. The fibration

$$K/L \longrightarrow K/SL$$

has connected fiber $SL/L = S/(S \cap L)$. Its base is a compact connected abelian Lie group, hence a torus. A loop in the base lifts to a path in M whose endpoints lie in the same fiber. Join its endpoint to its initial point by a path in that connected fiber. The resulting loop shows that $\pi_1(M)$ maps onto the fundamental group of the torus. Finiteness of $\pi_1(M)$ forces the torus to be a point. Therefore S acts transitively on (M, g).

Quotient S by the kernel of its action. The resulting compact connected semisimple group S_{eff} acts effectively. Isotropy at one point injects into $O(n)$: an isometry fixing that point with identity

derivative fixes every geodesic issuing from it. Since every point is joined to it by a geodesic, the isometry is the identity. Consequently,

$$\dim S_{\mathrm{eff}} \le n + \frac{n(n-1)}{2} = \frac{n(n+1)}{2}. \tag{4.4}$$

Pass to the compact simply connected covering group of S_{eff}. The covering group has the same transformation image on M and the same cone of invariant metrics.

There are only finitely many simply connected compact semisimple groups of dimension at most the bound in (4.4). Indeed, the classification of compact simple Lie algebras gives finitely many simple factors below a dimension bound, and their positive dimensions bound the number of factors. Denote a list of these groups by $G_1, \dots, G_q$.

Apply Theorem 4.2 to each G_a and to the same reference Σ, and take the maximum of the resulting initial segment lengths. Only finitely many quotient presentations G_a/H can then occur among homogeneous manifolds with that initial segment. Conjugate subgroups give equivariantly diffeomorphic quotient presentations and identify their invariant-cometric cones. The admissible quotients therefore belong to a finite list of fixed presentations $M_b = G_{a(b)}/H_b$.

If the finite list of admissible isotropy presentations is empty, the initial segment already excludes every realization. Otherwise the following maximum is over a nonempty finite list.

On each M_b apply Theorem 3.1 to the full space of invariant cometrics. Its theorem applies to the prescribed reference spectrum even if that spectrum is not realized on this particular presentation; in that case a sufficiently long initial segment has no realization there. Take the maximum of the finitely many determining indices and the preceding isotropy index. Every metric matching this final initial segment is fully isospectral to Σ. Full spectral equality implies equality of every initial segment, proving the converse.

For each presentation in the finite list the isospectral invariant-cometric set is compact and semialgebraic by Theorem 3.1. A compact subset of the positive-cometric cone has uniform positive lower and finite upper eigenvalue bounds. Inverting the cometric and extending the resulting invariant tensor over a fixed quotient gives a continuous map into the Fréchet space of smooth metric tensors, smooth in the finite-dimensional parameters. Its image is therefore compact in the smooth topology. Equivalently, all smooth seminorms are controlled on the compact parameter set, and every parameter-convergent subsequence gives smooth convergence of the metric tensors.

Taking a finite union over the fixed presentations proves compactness modulo diffeomorphisms and finiteness of diffeomorphism types. In particular, a sequence in the realization set has a subsequence which, after diffeomorphisms onto one fixed M_b, converges smoothly to a metric in the same compact isospectral set. Its limit is again homogeneous and has finite fundamental group.

Finally, Theorem 3.1 makes every $\mathrm{Spec}\, L^H_\pi(C)$ constant on each connected component of a fixed-presentation isospectral set. There are finitely many such components. Hence the collections of these spectra form a finite set. □

Corollary 4.3. *For fixed Σ as in Theorem 4.1, its homogeneous realizations with finite fundamental group satisfy*

$$\mathrm{diam}(g) \le D_\Sigma, \qquad \mathrm{inj}(g) \ge i_\Sigma > 0, \qquad \|\nabla^r \mathrm{Rm}_g\|_\infty \le C_{\Sigma,r} \quad (r \ge 0).$$

Proof. Fix one of the presentations in Theorem 4.1, and write $\mathcal{K}$ for its compact set of invariant metrics. Relative to a fixed metric g_*, compactness gives

$$a g_* \le g \le b\, g_* \qquad (g \in \mathcal{K})$$

with $a, b > 0$. Hence $\mathrm{diam}(g) \le \sqrt{b}\ \mathrm{diam}(g_*)$. The coordinate formulas for the Levi-Civita connection and curvature express $\nabla^r \mathrm{Rm}_g$ in terms of g^{-1} and derivatives of g through order $r+2$. Their coefficients are bounded on $\mathcal{K}$, as are all these derivatives. This gives the curvature bounds.

For the injectivity radius, cover M by finitely many coordinate domains whose closures lie in larger coordinate domains. The geodesic equation depends smoothly on the metric parameters, the initial

point, and the initial vector. In the coordinates of any larger domain, put $E_{g,p}(v) = \exp_p^g(v)$, identifying tangent vectors with their coordinate components. One has $dE_{g,p}(0) = I$. Compactness of $\mathcal{K}$ and of the smaller coordinate closures gives $\rho > 0$ such that every $E_{g,p}$ is defined in the corresponding larger domain for $|v| < \rho$ and

$$\|dE_{g,p}(v) - I\| < \tfrac{1}{2} \qquad (|v| < \rho).$$

Integration on the segment from v to w gives $|E_{g,p}(v) - E_{g,p}(w)| \geq \frac{1}{2}|v - w|$. Thus $E_{g,p}$ is injective there and has invertible derivative. Uniform equivalence of the coordinate norms and the g-norms gives $i > 0$ such that the g-ball of radius i in T_pM lies in $\{|v| < \rho\}$, for all $g \in \mathcal{K}$ and all $p \in M$. Consequently $\mathrm{inj}(g) \geq i$. Taking extrema of the finitely many constants over the presentations proves the result. □

Remark 4.4. Every closed homogeneous manifold with finite fundamental group has a semisimple transitive action. The common dimension hypothesis cannot be dropped: for cyclic groups C_k of increasing order in a maximal torus $T \subset \mathrm{SU}(2)$, the normal quotients $\mathrm{SU}(2)/C_k$ eventually have every prescribed finite initial segment of the spectrum of $\mathrm{SU}(2)/T$. To see this, first bound the Casimir by the last eigenvalue under consideration. Only finitely many representations occur below that bound. For k larger than the absolute value of every nonzero integral T-weight in those representations, C_k fixes exactly the zero-weight spaces. These are the T-fixed spaces, so the spectra and multiplicities below that bound coincide. The quotient dimensions are three and two.

5. Periodic directions and multiplicity bounds

Let K be compact, connected, and semisimple, let $H \subset K$ be closed, and put $\mathfrak{l} = \mathrm{Lie}\, H$, $Q = -B_{\mathfrak{k}}$, $\mathfrak{m} = \mathfrak{l}^{\perp_Q}$, and $d = \dim(K/H)$. The group H is not assumed connected. Fix $r \geq 1$ and commuting $c_1, \dots, c_r \in N_K(H)/H$ with $c_a^q = e$ for one integer $q \geq 1$. On $(K/H) \times \mathbb{T}^r$, where $\mathbb{T}^r = \mathbb{R}^r/(2\pi\mathbb{Z})^r$, let $\Gamma_c \cong (\mathbb{Z}/q)^r$ act by

$$(y, z) \longmapsto (yc_a, z + 2\pi e_a/q). \tag{5.1}$$

The translation part makes this action free, including when the homomorphism defined by c is not injective. Write $M_c = ((K/H) \times \mathbb{T}^r)/\Gamma_c$. The left action of $K \times \mathbb{T}^r$ commutes with Γ_c and descends to a transitive action on M_c. Let H_c be the preimage of $\langle c_1, \dots, c_r \rangle$ in $N_K(H)$. It is a finite extension of H and has Lie algebra $\mathfrak{l}$.

Theorem 5.1. *Fix K, H, r, c and the scalar spectrum $\Sigma = (0 = \sigma_0 < \sigma_1 \leq \sigma_2 \leq \cdots)$ of a closed connected smooth $(d + r)$-manifold. There is an integer N such that every $K \times \mathbb{T}^r$-invariant metric g on the cover which descends to M_c satisfies*

$$\lambda_j(g) = \sigma_j \quad (1 \leq j \leq N) \quad \Longrightarrow \quad \mathrm{Spec}(g) = \Sigma.$$

Moreover, there are an integer N_0, finitely many presentations of the form above, and a compact semialgebraic set of positive invariant cometrics on each presentation, such that every g satisfying $\lambda_j(g) = \sigma_j$ for $1 \leq j \leq N_0$ is isometric to a metric in one of these sets.

Here $\lambda_j = \lambda_j(M_c)$.

5.1. The compact group of invariant vector fields.

Lemma 5.2. *The subspace $\mathfrak{d} = \mathfrak{m}^{H_c}$ is the Lie algebra of a closed connected subgroup $D \subset C_K(H_c)^0$, and $D \cap H_c$ is finite.*

Proof. Put $C = C_K(H_c)^0$, $\mathfrak{c} = \mathrm{Lie}\, C$, and $\mathfrak{z} = \mathfrak{c} \cap \mathfrak{l}$. Averaging the adjoint action of H_c on $\mathfrak{l}$ is the orthogonal projection onto $\mathfrak{z}$. For $X \in \mathfrak{c}$ and $Y \in \mathfrak{l}$ it follows that $Q(X, Y) = Q(X, \mathrm{pr}_{\mathfrak{z}}\, Y)$. Consequently

$$\mathfrak{d} = \mathfrak{c} \cap \mathfrak{z}^{\perp_Q}.$$

The subalgebra $\mathfrak{z}$ is central in $\mathfrak{c}$ and integrates to $Z(H_c)^0$. The space $\mathfrak{d}$ contains $[\mathfrak{c},\mathfrak{c}]$ and its remaining summand is the orthogonal complement of $\mathfrak{z}$ in $\mathfrak{z}(\mathfrak{c})$. Let $\mathfrak{a}$ be the Lie algebra of a torus of K, with lattice $\Lambda_{\mathfrak{a}} = \{X : \exp(2\pi X) = e\}$. If α ranges over its adjoint weights, counted with multiplicity, then

$$Q(X,Y) = \sum_{\alpha} \alpha(X)\alpha(Y), \qquad \alpha(\Lambda_{\mathfrak{a}}) \subset \mathbb{Z}.$$

Thus Q is rational on this lattice. The subspace $\mathfrak{z}$ is rational in the centre of $\mathfrak{c}$, since it is the Lie algebra of the closed torus $Z(H_c)^0$. Its orthogonal complement there is therefore rational and is the Lie algebra of a closed torus $T_1 \subset C$. The group $[C,C]$ is the image of a simply connected compact semisimple group, hence is closed. Consequently $D = [C,C]T_1$ is compact and has Lie algebra $\mathfrak{d}$. Its intersection with H_c has zero Lie algebra and is compact, hence is finite. □

An invariant metric in Theorem 5.1 is described uniquely by Schur parameters

$$A \in \mathrm{Sym}^+(\mathfrak{m})^{H_c}, \qquad B \in \mathrm{Sym}^+(\mathbb{R}^r), \qquad V_a \in \mathfrak{d}. \tag{5.2}$$

In the frame $(X_1,\dots,X_d,Z_1,\dots,Z_r)$ its cometric is

$$C(A,B,V) = \begin{pmatrix} A + VBV^{\mathsf{T}} & -VB \\ -BV^{\mathsf{T}} & B \end{pmatrix}, \qquad \det C = \det_{\mathfrak{m}} A \det B, \tag{5.3}$$

where the columns of V are the V_a. To recover the parameters from $C = \left(\begin{smallmatrix} P & R \\ R^{\mathsf{T}} & B \end{smallmatrix}\right)$, set $V = -RB^{-1}$ and $A = P - RB^{-1}R^{\mathsf{T}}$. The factorization

$$C(A,B,V) = \begin{pmatrix} I & -V \\ 0 & I \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} I & 0 \\ -V^{\mathsf{T}} & I \end{pmatrix}$$

proves that $C > 0$ if and only if $A > 0$ and $B > 0$, and gives its determinant. Invariance makes the columns of V belong to $\mathfrak{m}^{H_c} = \mathfrak{d}$. In particular

$$\mathrm{Vol}(M_c, g) = \frac{(2\pi)^r \,\mathrm{Vol}(K/H, Q)}{q^r \sqrt{\det_{\mathfrak{m}} A \det B}}. \tag{5.4}$$

For $h \in D$, the map $(y,z) \mapsto (yh,z)$ commutes with every transformation in (5.1). Pullback by this map replaces (A, V_a) by $(\mathrm{Ad}(h)A\,\mathrm{Ad}(h)^{-1}, \mathrm{Ad}(h)V_a)$ and leaves B unchanged. These simultaneous conjugations therefore give isometric metrics on M_c.

For $\pi \in \widehat{K}$ put $d_\pi = \dim V_\pi$ and $W_\pi = V_\pi^H$. Representatives of the c_a act unitarily on W_π; their restrictions are independent of the representatives. They commute and have order dividing q. Thus

$$W_\pi = \bigoplus_{s \in (\mathbb{Z}/q)^r} W_{\pi,s}, \qquad \pi(c_a)|_{W_{\pi,s}} = e^{2\pi i s_a/q} I. \tag{5.5}$$

Choose the Fourier convention for which invariance under (5.1) is $m \in -s + q\mathbb{Z}^r$. With $T_\pi(X) = -i\, d\pi(X)$, the scalar blocks are

$$M_{\pi,s,m} = L_\pi(A)|_{W_{\pi,s}} + \sum_{a,b} B_{ab}(m_a I - T_\pi(V_a))(m_b I - T_\pi(V_b))|_{W_{\pi,s}}, \qquad m \in -s + q\mathbb{Z}^r, \tag{5.6}$$

each repeated d_π times. The group D preserves every sector. If $B = R^{\mathsf{T}}R$, the second term is $\sum_a (\sum_b R_{ab}(m_b I - T_\pi(V_b)))^2$; hence it is positive, without any commutation assumption on the V_a. On W_π, every $\mathfrak{l}$ derivative vanishes. Therefore

$$A \geq aI \quad \Longrightarrow \quad L_\pi(A)|_{W_\pi} \geq a c_\pi I, \tag{5.7}$$

where c_π is the Q-Casimir of K. Proposition 2.2, applied on $K \times \mathbb{T}^r$ and restricted to Γ_c-invariant functions, gives (5.6). On a matrix coefficient with Fourier factor $e^{i\langle m,z\rangle}$, the field $Z_a - \mathcal{V}_a$, where $\mathcal{V}_a$ is induced by V_a, acts on the representation vector by $i(m_a I - T_\pi(V_a))$.

5.2. **Integral reduction.**

Lemma 5.3. *For each r there is $b_r > 0$ such that a positive form B satisfying $m^{\mathsf{T}} B m \geq \ell$ for $0 \neq m \in \mathbb{Z}^r$ has a lattice basis in which $B \geq b_r \ell I$.*

Proof. If $\ell \leq 0$, positive definiteness gives $B \geq b_r \ell I$ in every lattice basis. Assume $\ell > 0$. For a lattice basis (b_i), write its Gram–Schmidt expansion as $b_j = b_j^* + \sum_{i<j} \mu_{ij} b_i^*$ and put $h_i = \|b_i^*\|$. Successive integer subtractions make $|\mu_{ij}| \leq 1/2$. If

$$h_{i+1}^2 + \mu_{i,i+1}^2 h_i^2 < \tfrac{3}{4} h_i^2,$$

interchange b_i, b_{i+1} and repeat the subtractions. The product $\prod_{j=1}^r \|b_1 \wedge \cdots \wedge b_j\|^2$ is unchanged by the subtractions. A swap multiplies it by $h_{i+1}^2/h_i^2 + \mu_{i,i+1}^2 < 3/4$. Each exterior lattice is discrete, so its nonzero vectors have norms bounded below by a positive constant. The product has a positive lower bound, and there are only finitely many swaps. At termination,

$$h_1^2 \geq \ell, \qquad h_i^2 \geq \tfrac{1}{2} h_{i-1}^2 \geq 2^{1-r} \ell.$$

The Gram matrix is $R^{\mathsf{T}} \operatorname{diag}(h_i^2) R$, where R is unit triangular and its off-diagonal entries have absolute value at most $1/2$. For $R = I + N$, the identity $R^{-1} = \sum_{j=0}^{r-1} (-N)^j$ bounds $\|R^{-1}\|$ in terms of r. Since $\|N\|_{\mathrm{op}} \leq r/2$, one has

$$\|R^{-1}\|_{\mathrm{op}} \leq \sum_{\nu=0}^{r-1} (r/2)^\nu, \qquad B \geq b_r \ell I, \qquad b_r = 2^{1-r} \left(\sum_{\nu=0}^{r-1} (r/2)^\nu \right)^{-2} > 0.$$

□

The trivial K-representation in (5.6) contains the frequencies $q\mathbb{Z}^r$. Thus a fixed first eigenvalue yields, after an integral change of the torus coordinates,

$$B \geq \beta I, \qquad \beta = b_r \lambda_1(\Sigma)/q^2 > 0. \tag{5.8}$$

The coordinate change replaces c by integral products of its entries. These products depend only on the change-of-basis matrix modulo q; there are finitely many resulting tuples. Subsequences may consequently be taken in one fixed presentation.

5.3. **Finite Fourier labels and polynomial rank conditions.**

Lemma 5.4. *Let D be a compact connected Lie group with an invariant inner product on its Lie algebra $\mathfrak{d}$, and let $\mathfrak{t}$ be a maximal abelian subalgebra. Every commuting tuple in $\mathfrak{d}$ is conjugate into $\mathfrak{t}$. Moreover, for $X \in \mathfrak{d}$ and $H \in \mathfrak{t}$, there is $g \in D$ such that*

$$\operatorname{Ad}(g) X \in \mathfrak{t}, \qquad \|\operatorname{Ad}(g) X - H\| \leq \|X - H\|.$$

Proof. The commuting skew-adjoint operators $\operatorname{ad} H$, $H \in \mathfrak{t}$, are simultaneously diagonalizable over $\mathbb{C}$. Their common kernel is $\mathfrak{t}_{\mathbb{C}}$, since $\mathfrak{t}$ is maximal abelian. A vector $H_0 \in \mathfrak{t}$ outside the finitely many nonzero weight kernels therefore satisfies $\mathfrak{z}_{\mathfrak{d}}(H_0) = \mathfrak{t}$. At a minimum of $\|\operatorname{Ad}(g)X - H_0\|^2$, differentiation along $\exp(uY)$ gives $[\operatorname{Ad}(g)X, H_0] = 0$. Thus every X has a conjugate in $\mathfrak{t}$.

Let $\mathfrak{b}$ be the span of a commuting tuple. The same weight decomposition for $\operatorname{ad} \mathfrak{b}$ gives $X \in \mathfrak{b}$ with $\mathfrak{z}_{\mathfrak{d}}(X) = \mathfrak{z}_{\mathfrak{d}}(\mathfrak{b})$. Conjugate X into $\mathfrak{t}$. Then $\mathfrak{t}$ centralizes the conjugate of $\mathfrak{b}$, so this conjugate lies in $\mathfrak{z}_{\mathfrak{d}}(\mathfrak{t}) = \mathfrak{t}$.

For the last assertion, choose a conjugate X_0 minimizing $\|X_0 - H\|^2$. Differentiation gives $[X_0, H] = 0$. Put $C_D(H) = \{g \in D : \operatorname{Ad}(g)H = H\}$. Its identity component is compact, its Lie algebra contains X_0, and $\mathfrak{t}$ is maximal abelian in that Lie algebra. The first paragraph conjugates X_0 into $\mathfrak{t}$ within this centralizer. This conjugation fixes H and preserves the minimum. □

Fix a maximal torus of D with Lie algebra $\mathfrak{t}$ and lattice $\Lambda = \{X \in \mathfrak{t} : \exp(2\pi X) = e\}$. A subspace U is rational if $U = \operatorname{span}_{\mathbb{R}}(U \cap \Lambda)$. An integral covector belongs to $\Lambda^* = \{\chi \in \mathfrak{t}^* : \chi(\Lambda) \subset \mathbb{Z}\}$. For each a separately write

$$V_a = R_a(k_a + \theta_a), \qquad R_a \in \mathrm{Ad}(D), \quad k_a \in q\Lambda, \tag{5.9}$$

where θ_a is in a fixed bounded parallelotope. Lemma 5.4 applies to each V_a separately; the elements R_a may differ. The image of $\mathrm{Ad}(D)$ on $\mathfrak{d}$ equals that of $\mathrm{Ad}(C_K(H_c)^0)$: the complementary connected subgroup with Lie algebra $\mathfrak{z}$ is central in $C_K(H_c)^0$ and acts trivially on $\mathfrak{d}$. Moreover,

$$\mathrm{Ad}(C_K(H_c)^0) = C_{\mathrm{Aut}(\mathfrak{k})^0}(\mathrm{Ad}\, H_c)^0.$$

Indeed, a path in the centralizer lifts through the finite covering $K \to \mathrm{Aut}(\mathfrak{k})^0$; its commutator with each $h \in H_c$ takes values in the finite centre of K and is therefore constant. Orthogonality and bracket preservation are polynomial equations. Commutation with $\mathrm{Ad}\, H_c$ is equivalent to commutation with a finite basis of its real linear span in $\mathrm{End}(\mathfrak{k})$. Connected components and linear images of real algebraic sets are semialgebraic. The conjugating group in (5.9) is consequently compact and semialgebraic.

Let $\mu > 0$ be an eigenvalue of multiplicity m_μ whose whole cluster and next eigenvalue occur in the prescribed initial segment. Every contributing representation has $d_\pi \le m_\mu$; the Weyl dimension formula makes this a finite list. Positivity and (5.8) imply, for a unit μ-eigenvector,

$$\|(m_a I - T_\pi(V_a))v\| \le \sqrt{\mu/\beta}.$$

For each a some torus weight η_a consequently gives

$$m_a = \eta_a(k_a) + n_a, \qquad |n_a| \le \sqrt{\mu/\beta} + C\sqrt{c_\pi}. \tag{5.10}$$

The last term follows by applying the Q-Casimir to the bounded vector θ_a. Since $k_a \in q\Lambda$, the offset remains in the prescribed congruence class $-s + q\mathbb{Z}^r$.

Lemma 5.5 (Lattice flags). *Let Λ be a full lattice in a Euclidean space $\mathfrak{t}$, and let $k_j \in q\Lambda$. After passage to a subsequence and subtraction of one fixed $k_0 \in q\Lambda$, there are a rational subspace $U \subset \mathfrak{t}$, fixed orthonormal vectors $u_1, \dots, u_p$, and real numbers $t_{i,j}$ such that*

$$k_j - k_0 = \sum_{i=1}^{p} t_{i,j} u_i + s_j, \qquad t_{i,j} \longrightarrow +\infty, \qquad \frac{t_{i+1,j}}{t_{i,j}} \longrightarrow 0, \qquad s_j \longrightarrow s_0. \tag{5.11}$$

The vectors s_j are perpendicular to all u_i, $k_j - k_0 \in U$, and for every integral covector χ,

$$\chi|_U \neq 0 \quad \Longrightarrow \quad |\chi(k_j - k_0)| \longrightarrow \infty. \tag{5.12}$$

If the original sequence is bounded, take $p = 0$ and $U = 0$.

Proof. An unbounded sequence has a subsequence with $\|k_j\| \to \infty$ and $k_j/\|k_j\| \to u_1$, $\|u_1\| = 1$. Put $t_{1,j} = \langle k_j, u_1\rangle$ and $r_{1,j} = k_j - t_{1,j}u_1$. Then $t_{1,j} > 0$ eventually, $t_{1,j} \sim \|k_j\|$, and $\|r_{1,j}\|/t_{1,j} \to 0$. If $r_{1,j}$ is unbounded, pass to a subsequence on which its norm tends to infinity and its normalized direction tends to u_2. Then $u_2 \perp u_1$, $t_{2,j} = \langle r_{1,j}, u_2\rangle > 0$ eventually, and $t_{2,j}/t_{1,j} \to 0$. Repeat on the successive orthogonal residuals. At most $\dim \mathfrak{t}$ steps are possible. At termination the residual is bounded; pass to a convergent subsequence. This gives the displayed flag before subtracting k_0.

Let L be the group of integral covectors vanishing on all u_i. It has a finite integral basis $\chi_1, \dots, \chi_b$. Each $\chi_a(k_j) = \chi_a(s_j)$ is both bounded and integral. Pass to a subsequence on which these finitely many values are constant, and choose k_0 to be one member of that subsequence. Then every $k_j - k_0$ lies in $U = \bigcap_{a=1}^{b} \ker \chi_a$. Subtracting k_0 changes each $t_{i,j}$ by a constant and the bounded residual by a constant perpendicular vector; the flag limits persist. If $\chi|_U \neq 0$, then $\chi \notin L$, so there is a first i with $\chi(u_i) \neq 0$. The flag gives $\chi(k_j - k_0) = t_{i,j}\chi(u_i) + o(t_{i,j})$, proving separation. For bounded k_j, discreteness of the lattice gives a constant subsequence, which can be subtracted entirely. □

Apply the lemma to each column in (5.9). Absorb k_0 into the bounded θ_a and again write the shifted columns as k_a. Label weights only by their restrictions to U_a. For a fixed representation, distinct tuples of these restricted weights and bounded offsets give distinct actual Fourier vectors eventually. Indeed, equality of two such Fourier vectors would make every $(\eta_a - \eta'_a)(k_a) = n'_a - n_a$ bounded; separation forces equality of the restricted weights, and then equality of the offsets.

For a fixed μ, let $\mathcal{G}_\mu$ consist of the tuples $\gamma = (\pi, s, \eta, n)$ for which $d_\pi \le m_\mu$, each η_a is a weight restricted to U_a, and $n \in -s + q\mathbb{Z}^r$ satisfies (5.10). This is a finite set. For $\gamma \in \mathcal{G}_\mu$ put

$$\begin{aligned} D_{\gamma,a} &= n_a I + \eta_a(k_a) I - T_\pi(R_a(k_a + \theta_a)), \\ M_\gamma &= L_\pi(A)|_{W_{\pi,s}} + \sum_{a,b} B_{ab} D_{\gamma,a} D_{\gamma,b}. \end{aligned} \tag{5.13}$$

Choose bases once in $\mathfrak{m}$, $\mathfrak{d}$, and every $W_{\pi,s}$. The real and imaginary parts of these entries are polynomials in the coordinates of (A, B, R, k, θ); this parameter space is independent of μ and γ. Put

$$Z_\mu = \left\{ \sum_{\gamma \in \mathcal{G}_\mu} d_\pi \dim \ker(M_\gamma - \mu I) \ge m_\mu \right\}. \tag{5.14}$$

For every choice of integers $0 \le h_\gamma \le \dim W_{\pi,s}$ with $\sum_\gamma d_\pi h_\gamma \ge m_\mu$, impose $\operatorname{rank}(M_\gamma - \mu I) \le \dim W_{\pi,s} - h_\gamma$ for all γ. These are equations in minors. The union over the finitely many choices of (h_γ) is Z_μ. A finite intersection of real zero sets is the zero set of a sum of squares; a finite union is the zero set of a product. Thus Z_μ is real algebraic. The Hilbert basis theorem gives finitely many eigenvalues $\mu_1, \dots, \mu_b$ such that

$$\bigcap_{\mu \in \{\sigma_j \,:\, j \ge 1\}} Z_\mu = \bigcap_{i=1}^{b} Z_{\mu_i}.$$

A sequence with $\lambda_k(g_j) = \sigma_k$ for $k \le j$ belongs to the right side for all sufficiently large j. Indeed, the chosen initial segments eventually contain each of these finitely many clusters and its successor, and (5.10) includes every contributing Fourier label.

Lemma 5.6 (Nash arcs and limiting eigenspaces). *Let a sequence as above be given, with the flags in* (5.11). *There is a Nash family of positive parameters satisfying every* (5.14), *with $k_a(t) \in U_a$ and all separations* (5.12). *Any prescribed upper bounds $A_j \le bI$ and $B_j \le b'I$ may be imposed on this family. If $\operatorname{tr} A_j \to \infty$ or $\operatorname{tr} B_j \to \infty$, the corresponding trace tends to infinity on the family. If $g = e$ minimizes $\sum_a \| \operatorname{pr}_{\mathfrak{t}^\perp} \operatorname{Ad}(g) V_{a,j} \|^2$ over $g \in D$, the same minimum condition may be imposed; divergence of that minimum is then preserved.*

There are integers $s_{\mu,\gamma} \ge 0$, each equal to the nullity of $M_\gamma(t) - \mu I$ for all sufficiently small $t > 0$, such that

$$\sum_\gamma d_\pi s_{\mu,\gamma} \ge m_\mu, \qquad c_\pi \le C\mu, \qquad |n| \le C\sqrt{\mu} \quad (s_{\mu,\gamma} > 0). \tag{5.15}$$

For any finite collection of labels in one representation, choose orthonormal bases in these kernels and pass to a common convergent subsequence. The limit spaces have the respective dimensions $s_{\mu,\gamma}$. Spaces with distinct restricted weight tuples or distinct sectors are orthogonal, for every choice of offsets. For a fixed tuple, sector, and offset, spaces at distinct eigenvalues are orthogonal.

Proof. After passage to a subsequence, choose compact semialgebraic sets Θ_a and $\mathcal{S}_a$ containing, respectively, all bounded θ_a and all flag residuals s_a. Put $\mathcal{R} = \operatorname{Ad}(D)|_{\mathfrak{d}}$. For A and B introduce

$$\alpha = (1 + \operatorname{tr} A)^{-1}, \quad \widehat{A} = \alpha A, \qquad \rho_B = (1 + \operatorname{tr} B)^{-1}, \quad \widehat{B} = \rho_B B.$$

The domain of the rank equations is the set of these variables and the flag variables satisfying

$$\begin{gathered} \alpha, \rho_B > 0, \qquad \alpha + \operatorname{tr} \widehat{A} = \rho_B + \operatorname{tr} \widehat{B} = 1, \qquad \widehat{A} > 0, \quad \widehat{A} \in \operatorname{Sym}(\mathfrak{m})^{H_c}, \quad \widehat{B} \ge \beta \rho_B I, \\ R_a \in \mathcal{R}, \qquad \theta_a \in \Theta_a, \qquad s_a \in \mathcal{S}_a, \qquad s_a \perp u_{a,i}, \qquad k_a \in U_a, \\ k_a = \sum_{i=1}^{p_a} z_{a,i}^{-1} u_{a,i} + s_a, \quad z_{a,i} > 0, \quad h_{a,i} > 0, \quad h_{a,i} z_{a,i+1} = z_{a,i}, \quad V_a = R_a(k_a + \theta_a). \end{gathered}$$

Here $1 \le a \le r$ and $1 \le i < p_a$ in the equations containing $h_{a,i}$; a bounded column has $p_a = 0$, $U_a = 0$ and $k_a = 0$. Intersect this domain with the finite subintersection defining $\bigcap_\mu Z_\mu$. If $A \le bI$ or $B \le b'I$ is prescribed, also impose $\widehat{A} \le b\,\alpha I$ or $\widehat{B} \le b'\rho_B I$, respectively. In an application with a minimizing Cartan, also impose $V_a = H_a + E_a$, $H_a \in \mathfrak{t}$, $E_a \perp \mathfrak{t}$, and the inequalities

$$\sum_a \|E_a\|^2 \le \sum_a \|\operatorname{pr}_{\mathfrak{t}^\perp} RV_a\|^2 \qquad (R \in \mathcal{R}).$$

Quantifier elimination makes this last condition semialgebraic. If the transverse norm diverges, add $s^2 = \sum_a \|E_a\|^2$, $s > 0$ and its reciprocal as a positive variable.

Each remaining unbounded real vector variable x may be replaced by $\widehat{x} = \xi x$, $\xi > 0$, with $\xi^2 + \|\widehat{x}\|^2 = 1$. The original sequence has a convergent subsequence in these bounded coordinates. Its limit has $z_{a,i} = h_{a,i} = 0$ for each nonempty flag and retains the specified limits of the other bounded coordinates. All rational substitutions have positive denominators on the domain just defined. Multiplication by their products therefore preserves every rank equation there. Nash curve selection at this limit gives an arc in the domain [6, Theorem 1.15]. If the limit lies in the domain and all flags are empty, take the constant arc at that point. The positive reciprocal variables recover $A > 0$, $B \ge \beta I$ and all the displayed constraints. A zero limit of α or ρ_B means that the corresponding trace tends to infinity; a zero limit of s^{-1} means that the minimum tends to infinity. Each nonempty flag has $z_{a,i} \to 0$ and $h_{a,i} \to 0$, so its coefficients tend to infinity and its successive ratios to zero. Hence (5.12) holds for every integral covector. Here the variables k_a belong to U_a; they need not be lattice points.

Each matrix minor is a one-variable Nash function. For a fixed pair (μ, γ) let $r_{\mu,\gamma}$ be the largest rank on the connected arc and put $s_{\mu,\gamma} = \dim W_{\pi,s} - r_{\mu,\gamma}$. A nonzero maximal minor has only finitely many zeros near the endpoint: its zero set is a semialgebraic subset of an interval and has empty interior. Since $\mathcal{G}_\mu$ is finite, there is an interval on which all its matrices attain these ranks. Equation (5.14) on that interval gives the first inequality in (5.15). All minors of order $r_{\mu,\gamma} + 1$ vanish identically on the arc. Consequently this nullity is a lower bound at every interior point, including points exceptional for other matrices. Fix one interior point with $A \ge a_* I > 0$. Positivity and (5.7) give $c_\pi \le \mu / a_*$ for every kernel with $s_{\mu,\gamma} > 0$. By the definition of $\mathcal{G}_\mu$, its offset already satisfies (5.10). Therefore, if C_θ bounds the original $\|\theta_a\|_Q$, then

$$|n| \le \sqrt{r}\big(\beta^{-1/2} + C_\theta a_*^{-1/2}\big)\sqrt{\mu}.$$

Both constants are independent of μ.

Fix finitely many labels and, at every parameter, choose $s_{\mu,\gamma}$ orthonormal vectors in each corresponding kernel. Compactness of the finite product of unit spheres gives one subsequence on which all these vectors converge. Their limits remain orthonormal within each selected kernel, so the limiting subspace has dimension $s_{\mu,\gamma}$. Let $P_{a,\eta'}$ be a weight projection for the fixed torus, restricted to U_a. Choose lifts $g_a \in D$ of R_a along the subsequence, converging in the compact group D, and put $v_a = \pi(g_a)^{-1} v$. The residual estimate and boundedness of θ_a give

$$\big|n_a + (\eta_a - \eta')(k_a)\big| \|P_{a,\eta'} v_a\| \le C_{\pi,\mu}.$$

For $\eta' \ne \eta_a$ the scalar factor tends to infinity by (5.12). Thus every limit belongs to the $R_a(0)$-conjugate η_a-weight space for every a. If two tuples differ, choose one column in which they differ; their limit spaces are orthogonal in that column's weight decomposition. This orthogonality holds for all bounded offsets. Distinct sectors are orthogonal because D preserves them. For identical tuple, sector and offset, the same Hermitian matrix acts before passage to the limit, so eigenspaces at distinct μ are orthogonal. Inner products pass to the limit, so the limiting subspaces are orthogonal. □

Proposition 5.7 (Spectral upper bounds). *Let Σ have dimension $d + r$. There are an integer N and positive constants b, b' such that $\lambda_k(g) = \sigma_k$ for $1 \le k \le N$ implies, after an integral change of torus coordinates,*

$$0 < A \le bI, \qquad \beta I \le B \le b'I. \tag{5.16}$$

The constants depend on the fixed presentation and the reference spectrum.

Proof. Suppose that no finite initial segment bounds $\operatorname{tr} A + \operatorname{tr} B$. For each j, choose parameters with

$$\lambda_k = \sigma_k \quad (1 \le k \le j), \qquad \operatorname{tr} A + \operatorname{tr} B > j.$$

After a subsequence, either $\operatorname{tr} A \to \infty$ or $\operatorname{tr} B \to \infty$. Apply Lemma 5.6 with that boundary condition. If $\operatorname{tr} A \to \infty$, then $A/\operatorname{tr} A$ has a positive semidefinite limit A_∞ of trace one. A unit vector in a kernel with $s_{\mu,\gamma} > 0$ satisfies

$$0 \le \langle v, L_\pi(A/\operatorname{tr} A)v\rangle \le \mu/\operatorname{tr} A.$$

Its limit therefore lies in $\ker L_\pi(A_\infty)$. On W_π this kernel is $V_\pi^{H'}$, where

$$H' = \overline{\langle H, \exp(\operatorname{im} A_\infty)\rangle}.$$

Indeed the quadratic form is a sum of nonnegative squares, so its kernel is the intersection of the kernels of the derivatives in $\operatorname{im} A_\infty$. Since this image is a nonzero horizontal space, $\dim H' > \dim H$. Orthogonality and (5.15) give

$$N_\Sigma(T) - 1 \le C(1+\sqrt{T})^r \sum_{c_\pi \le CT} d_\pi \dim V_\pi^{H'} = O\big(T^{(\dim K - \dim H' + r)/2}\big). \tag{5.17}$$

The sum is the normal quotient counting function; if $H' = K$ it is bounded. Equation (5.17) contradicts the positive $(d+r)$-dimensional Weyl coefficient of Σ.

If $\operatorname{tr} B \to \infty$, one diagonal entry, say B_{11}, diverges. For fixed $\pi, s, \eta, n_2, \dots, n_r$ put

$$E_a = \eta_a(k_a)I - T_\pi(R_a(k_a+\theta_a)), \qquad Q_t = \frac{\sum_a B_{1a}E_a + \sum_{a>1} B_{1a}n_a I}{\sqrt{B_{11}}}.$$

For arbitrary vectors w_a,

$$\Big\| \sum_a B_{1a} w_a \Big\|^2 \le B_{11} \sum_{a,b} B_{ab}\langle w_a, w_b\rangle.$$

Thus a unit vector in one of these μ-kernels satisfies $\|(Q_t + \sqrt{B_{11}}n_1 I)v\| \le \sqrt{\mu}$. Self-adjointness yields, for two different integer offsets,

$$|\langle v, v'\rangle| \le \frac{\sqrt{\mu}+\sqrt{\mu'}}{\sqrt{B_{11}}|n_1 - n_1'|} \longrightarrow 0. \tag{5.18}$$

Combining this with Lemma 5.6, the total limiting dimension for fixed $(\pi, n_2, \dots, n_r)$ is at most $\dim W_\pi$. Consequently

$$N_\Sigma(T) - 1 \le C(1+\sqrt{T})^{r-1} \sum_{c_\pi \le CT} d_\pi \dim W_\pi = O(T^{(d+r-1)/2}),$$

another contradiction. If $d = 0$, only the trivial representation has $W_\pi \neq 0$, and A acts on the zero-dimensional space. □

Lemma 5.8 (Dimension bound). *If Σ has dimension greater than $d + r$, there is an integer N for which no invariant metric on M_c satisfies $\lambda_k(g) = \sigma_k$ for $1 \le k \le N$.*

Proof. Assume that successively longer initial segments are realized. The rank sets (5.14) and the positive parameter domain in Lemma 5.6 are defined for these reference multiplicities in every dimension. Their intersection contains a Nash arc approaching the chosen boundary limit. If the compactified limit is interior, all lattice columns are bounded and become zero after the fixed lattice subtraction; one positive parameter in the intersection then gives a constant arc. For each (π, n) the limiting dimensions sum to at most $\dim W_\pi$. Therefore

$$N_\Sigma(T) - 1 \le C(1+\sqrt{T})^r \sum_{c_\pi \le CT} d_\pi \dim W_\pi = O(T^{(d+r)/2}).$$

This contradicts the larger Weyl exponent of the reference. □

6. Cartan minimization and polynomial matrix identities

All orthogonal projections on Lie algebras are taken for the restriction of Q. The compact Lie algebra $\mathfrak{d}$ may have a nonzero centre.

6.1. **A common kernel for unbounded off-diagonal parts.** For $V = (V_1, \dots, V_r) \in \mathfrak{d}^r$ define

$$E(V) = \min_{g \in D} \sum_a \| \operatorname{pr}_{\mathfrak{t}^\perp} \operatorname{Ad}(g) V_a \|^2. \tag{6.1}$$

The minimum exists because D is compact. Its graph and the set of minimizing tuples are semialgebraic: the equations for $\operatorname{Ad}(D)$ above and the inequality against every $g \in D$ are first-order polynomial conditions, to which quantifier elimination applies.

Lemma 6.1. *Let $V_a(t) = H_a(t) + E_a(t)$ be a Nash family on $0 < t < \delta$, with $H_a(t) \in \mathfrak{t}$, $E_a(t) \perp \mathfrak{t}$, and*

$$s(t) = \left(\sum_a \|E_a(t)\|^2 \right)^{1/2} > 0, \qquad s(t) \longrightarrow \infty \quad (t \downarrow 0).$$

Suppose the fixed Cartan realizes (6.1) *for every t. There exists $0 \neq X \in \mathfrak{d}$, depending only on this family, such that, in every finite-dimensional unitary representation $T = -i\, d\rho$ of D,*

$$\begin{aligned} &\|v(t)\| = 1, \qquad \|(m_a(t) I - T(V_a(t)))v(t)\| = O(1) \quad (1 \le a \le r), \\ &\qquad t_j \longrightarrow 0, \qquad v(t_j) \longrightarrow v_0 \end{aligned} \qquad \Longrightarrow \qquad T(X) v_0 = 0.$$

The numbers $m_a(t)$ may be arbitrary real numbers; the bounds are taken as $t \downarrow 0$.

Proof. Differentiation of the minimizing function along $\exp(uY)$ gives

$$0 = 2 \sum_a \langle [Y, V_a], E_a \rangle = 2 \left\langle Y, \sum_a [H_a, E_a] \right\rangle \qquad (Y \in \mathfrak{d}).$$

Thus

$$\sum_a [H_a, E_a] = 0. \tag{6.2}$$

The bounded Nash vectors E_a/s have limits E_a^0, with $\sum_a \|E_a^0\|^2 = 1$.

Each coordinate of H_a/s has a convergent Puiseux expansion near zero [6, §1.5]. Its principal part is finite. Let U be the smallest subspace rational for the cocharacter lattice of $\mathfrak{t}$ containing every coefficient of negative degree in these expansions, for every a. Put $U = 0$ if all principal parts vanish, and let $S = \exp U$. If an integral covector χ vanishes on all these coefficients, its kernel is a rational subspace containing U. Consequently

$$\chi|_U \neq 0 \quad \Longrightarrow \quad s^{-1} \| \big(\chi(H_a) \big)_a \| \longrightarrow \infty. \tag{6.3}$$

Indeed, some column then has a nonzero negative-power coefficient, and its first such term determines the asymptotic behavior. Set

$$\mathfrak{e} = \mathfrak{z}_{\mathfrak{d}}(U), \qquad Z_a = \operatorname{pr}_U H_a, \qquad L_a = H_a - Z_a.$$

Since projection to $U^\perp$ removes the negative-power coefficients, L_a/s is bounded and has a limit L_a^0. Moreover $Z_a \in \mathfrak{z}(\mathfrak{e})$ and $L_a \in \mathfrak{t} \subset \mathfrak{e}$.

Fix a representation and a sequence $t_j \downarrow 0$ as in the hypothesis. All the following vector limits are taken on subsequences of this sequence. Let P_η denote the S-weight projections. Every S-weight occurring here is the restriction of a weight of the fixed maximal torus of D. A difference of distinct S-weights therefore has an integral extension to $\mathfrak{t}$ which is nonzero on U, so (6.3) applies to it. Since $L_a + E_a = O(s)$, the residual hypothesis implies

$$\sum_\eta \|P_\eta v\|^2 \sum_a |m_a - \eta(Z_a)|^2 = O(s^2). \tag{6.4}$$

Two distinct weights with nonzero components in v_0 would therefore satisfy $\|((\eta-\eta')(Z_a))_a\| = O(s)$. This contradicts (6.3), because $H_a - Z_a = O(s)$. Hence exactly one weight has a nonzero limiting component. Denote it by η. Then

$$v_0 \in P_\eta V, \qquad b_a := (m_a - \eta(Z_a))/s = O(1).$$

Pass to a subsequence on which all b_a converge.

Fix $\eta' \neq \eta$, put $P' = P_{\eta'}$, and set $\delta_a = (\eta' - \eta)(Z_a)$. Equation (6.3) gives $|\delta|/s \to \infty$. The projected residual identity is

$$\begin{aligned} -\frac{\delta_a}{s} P'v = P'T(E_a/s)P_\eta v + (T(L_a/s) - b_a I)P'v \\ + P'T(E_a/s)(I-P_\eta)v + O(s^{-1}). \end{aligned}$$

All terms on the second line, and the second term on the first line, tend to zero. Therefore

$$-\frac{\delta_a}{s} P'v = P'T(E_a^0)P_\eta v_0 + o(1). \tag{6.5}$$

Taking the sum of squared norms over a shows that $u_t = (|\delta|/s)P'v$ is bounded. Pass to a subsequence with $\delta/|\delta| \to h$, $|h| = 1$, and $u_t \to u$. Put $F_a = P'T(E_a^0)P_\eta$. Then (6.5) gives $h_a u = -F_a v_0$.

Apply the representation to (6.2), project from P_η to P', and divide by $|\delta|s$. The terms involving $[L_a, E_a]$ are $O(s/|\delta|)$, while the remaining terms give

$$\sum_a \frac{\delta_a}{|\delta|} P'T(E_a/s)P_\eta \longrightarrow \sum_a h_a F_a = 0.$$

Multiplying $h_a u = -F_a v_0$ by h_a and summing yields $u = 0$. It follows that $F_a v_0 = 0$ for every a. The representation has only finitely many weights, so the necessary subsequences can be chosen simultaneously for all $\eta' \neq \eta$.

Let $E_{a,\perp}^0$ be the component of E_a^0 perpendicular to $\mathfrak{e}$. Its complexification is a sum of nonzero adjoint S-weight components. Each such component shifts an S-weight by a nonzero character, so it has no diagonal weight block. The $\mathfrak{e}$ component preserves every S-weight space. The preceding identities therefore give $T(E_{a,\perp}^0)v_0 = 0$ for every a. If one $E_{a,\perp}^0$ is nonzero, choose the least such index in a fixed ordering and set $X = E_{a,\perp}^0$.

It remains to treat the case $E_a^0 \in \mathfrak{e}$ for every a. Put $W_a = L_a^0 + E_a^0$. The identity component $C_D(S)^0$ of the centralizer of S has Lie algebra $\mathfrak{e}$, and the torus with Lie algebra $\mathfrak{t}$ is a maximal torus of this group. If the W_a commute, Lemma 5.4, applied in $C_D(S)^0$, gives g with $\mathrm{Ad}(g)W_a \in \mathfrak{t}$ for every a. Since $\mathrm{Ad}(g)Z_a = Z_a$,

$$\mathrm{Ad}(g)V_a = Z_a + s\,\mathrm{Ad}(g)W_a + o(s), \qquad \sum_a \|\mathrm{pr}_{\mathfrak{t}^\perp}\mathrm{Ad}(g)V_a\|^2 = o(s^2).$$

This contradicts the minimal value s^2. Thus some $[W_a, W_b] \neq 0$. Choose the first such ordered pair in a fixed ordering and set $X = [W_a, W_b]$. Projection of the residual equations to P_η, followed by division by s and passage to the limit, gives

$$T(W_a)v_0 = \rho_a v_0, \qquad \rho_a = \lim b_a \in \mathbb{R}.$$

Hence

$$T(X)v_0 = i[T(W_a), T(W_b)]v_0 = i(\rho_a\rho_b - \rho_b\rho_a)v_0 = 0.$$

In both cases $X \neq 0$ depends only on E_a^0, L_a^0 and the fixed orderings. These objects depend only on the Nash family. Thus the same X works in every representation. If $U = 0$, there is a single weight space and the last case applies directly. □

Proposition 6.2. *Under* (5.16)*, there are an integer N and $C > 0$ such that*

$$\lambda_k(g) = \sigma_k \quad (1 \le k \le N) \quad \Longrightarrow \quad E(V) \le C.$$

The constants depend only on the presentation, Σ, and the bounds in (5.16).

Proof. If no finite initial segment bounds $E(V)$, choose V_j and the other metric parameters with

$$\lambda_k(g_j) = \sigma_k \quad (1 \le k \le j), \qquad E(V_j) > j.$$

A common conjugation of each tuple makes the fixed Cartan minimize (6.1). Impose this semialgebraic condition, $s^2 = \sum_a \|E_a\|^2$, and the reciprocal boundary condition $s^{-1} \to 0$ in the semialgebraic domain of Lemma 5.6. The resulting positive Nash arc satisfies all rank conditions and separate-column weight flags. Lemma 6.1 gives one nonzero $X \in \mathfrak{d}$ annihilating every limit of these kernel vectors, in every sector and representation. Since $X \in \mathfrak{m}$,

$$H' = \overline{\langle H, \exp(\mathbb{R}X)\rangle} \quad \text{satisfies} \quad \dim H' > \dim H.$$

The original separate-column flags give orthogonality for each fixed (π, n). For fixed (π, n), the subspaces at different reference eigenvalues are orthogonal within each sector and restricted weight tuple; those at different sectors or tuples are orthogonal by Lemma 5.6. All lie in $V_\pi^{H'}$. For every finite set $\mathcal{L}$ of such labels,

$$\sum_{(\mu,s,\eta)\in\mathcal{L}} s_{\mu,(\pi,s,\eta,n)} \le \dim V_\pi^{H'}.$$

The estimates (5.15) give

$$N_\Sigma(T) - 1 \le C(1+\sqrt{T})^r \sum_{c_\pi \le CT} d_\pi \dim V_\pi^{H'} = O(T^{(\dim K - \dim H' + r)/2}),$$

which contradicts the reference Weyl law. □

6.2. **Weighted stationarity and the limiting operator.** For $\beta I \le B \le b' I$, minimize instead

$$\sum_{a,b} B_{ab} \langle \mathrm{pr}_{\mathfrak{t}^\perp} \mathrm{Ad}(g) V_a, \mathrm{pr}_{\mathfrak{t}^\perp} \mathrm{Ad}(g) V_b \rangle.$$

At a minimum, the inequalities $\beta I \le B \le b' I$ give

$$\sum_a \|\mathrm{pr}_{\mathfrak{t}^\perp} \mathrm{Ad}(g) V_a\|^2 \le \frac{b'}{\beta} E(V).$$

Proposition 6.2 therefore gives

$$V_a = H_a + E_a, \quad H_a \in \mathfrak{t}, \quad E_a \perp \mathfrak{t}, \quad \|E_a\| \le C, \qquad \sum_{a,b} B_{ab}[H_a, E_b] = 0. \tag{6.6}$$

Indeed, differentiation at a minimum gives, for every $Y \in \mathfrak{d}$,

$$0 = 2\sum_{a,b} B_{ab} \langle [Y, V_a], E_b \rangle = 2 \left\langle Y, \sum_{a,b} B_{ab}[H_a, E_b] \right\rangle.$$

Here $\sum_{a,b} B_{ab}[E_a, E_b] = 0$ by symmetry of B.

Let g_j be metrics on M_c with $\lambda_k(g_j) = \sigma_k$ for $1 \le k \le j$, normalized by (5.16) and (6.6). Decompose $H_{a,j} = k_{a,j} + \theta_{a,j}$ with $k_{a,j} \in q\Lambda$ and $\theta_{a,j}$ bounded. Applying Lemma 5.5 to the columns, adding each fixed lattice subtraction to $\theta_{a,j}$, and taking the sum of their rational subspaces gives a torus $S = \exp U$ such that

$$k_{a,j} \in q\Lambda \cap U, \qquad \chi|_U \ne 0 \implies \|\big(\chi(k_{a,j})\big)_a\| \longrightarrow \infty. \tag{6.7}$$

Pass to $A_j \to A_0 \ge 0$, $B_j \to B_0 > 0$, $\theta_{a,j} \to \theta_a^0$, and $E_{a,j} \to E_a^0$. Relative to $\mathfrak{e} = \mathfrak{z}_{\mathfrak{d}}(U)$ put

$$\begin{aligned} F_a &= \theta_a^0 + E_{a,\|}^0, \\ C_E &= \sum_{a,b} (B_0)_{ab} E_{a,\perp}^0 \otimes E_{b,\perp}^0, \\ A_{\mathrm{e}} &= \mathrm{Av}_S(A_0 + C_E). \end{aligned} \tag{6.8}$$

Here $X \otimes Y$ sends v to $Q(Y,v)X$ and acts on $\mathfrak{m}$. For $X \in \mathfrak{m}$, put $e(X) = (Q(E^0_{a,\perp}, X))_{a=1}^r$. Then

$$Q(C_E X, X) = e(X)^{\mathsf{T}} B_0 e(X) \geq 0.$$

Thus $C_E \geq 0$. Averaging preserves nonnegativity and H_c-invariance, so $A_{\mathrm{e}} \geq 0$ is H_c-invariant. Each F_a belongs to $\mathfrak{z}_{\mathfrak{d}}(U)$ and therefore centralizes S.

6.3. **Separation of bounded and diverging matrix spectra.**

Lemma 6.3. *Let E, F be finite-dimensional Hermitian spaces, with $r = \dim E$. Let*

$$M_j = \begin{pmatrix} A_j & U_j \\ U_j^* & D_j \end{pmatrix} \quad \text{on } E \oplus F, \qquad A_j = A_j^*, \quad D_j = D_j^* \geq b_j I,$$

where $A_j \to A$, $\|U_j\| \leq u$, and $b_j \to \infty$. Choose a such that $\|A_j\| \leq a$ for every j, and put $\eta_j = u^2/(b_j - a - 1)$ when $b_j > a + 1$. For all sufficiently large j, the eigenvalues in increasing order satisfy

$$\lambda_k(A_j) - \eta_j \leq \lambda_k(M_j) \leq \lambda_k(A_j) \quad (1 \leq k \leq r). \tag{6.9}$$

If $F \neq 0$, then $\lambda_{r+1}(M_j) \geq b_j$. In particular,

$$|\lambda_k(M_j) - \lambda_k(A)| \leq \|A_j - A\| + \eta_j \quad (1 \leq k \leq r).$$

If $A_j = A$ and $b_j \geq cj^2$ eventually for a fixed $c > 0$, the error is $O(j^{-2})$. The case $E = 0$ consists only of eigenvalues at least b_j; if $F = 0$, $M_j = A_j$.

Proof. Assume $E, F \neq 0$. The Rayleigh quotient restricted to $E \oplus 0$ gives $\lambda_k(M_j) \leq \lambda_k(A_j)$ for $k \leq r$. The max–min formula applied to the codimension-r subspace $0 \oplus F$ gives $\lambda_{r+1}(M_j) \geq b_j$. For real $s < b_j$, set $Q_j = D_j - sI$ and

$$S_j(s) = A_j - sI - U_j Q_j^{-1} U_j^*.$$

Multiplication of the three factors gives

$$M_j - sI = \begin{pmatrix} I & U_j Q_j^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} S_j(s) & 0 \\ 0 & Q_j \end{pmatrix} \begin{pmatrix} I & 0 \\ Q_j^{-1} U_j^* & I \end{pmatrix}. \tag{6.10}$$

Both outside matrices are invertible and adjoint to one another. Consequently $M_j - sI$ and $S_j(s) \oplus Q_j$ have the same numbers of positive, zero, and negative eigenvalues. For $|s| \leq a + 1$ one has

$$0 \leq U_j Q_j^{-1} U_j^* \leq \eta_j I.$$

Take j so large that $\eta_j < 1/2$, and fix $k \leq r$. For $s < \lambda_k(A_j) - \eta_j$ sufficiently close to this value, $|s| < a + 1$ and

$$S_j(s) \geq A_j - (s + \eta_j) I.$$

The latter matrix has at most $k - 1$ negative eigenvalues. The same is true of $S_j(s)$: on the span of the A_j-eigenvectors with indices $k, \dots, r$, the displayed lower bound is positive. Since $Q_j > 0$, (6.10) gives $\lambda_k(M_j) \geq s$. Letting s increase proves (6.9). Finally $-\|A_j - A\| I \leq A_j - A \leq \|A_j - A\| I$ and min–max give $|\lambda_k(A_j) - \lambda_k(A)| \leq \|A_j - A\|$. This also proves the stated rate and includes repeated eigenvalues. □

Lemma 6.4. *For fixed π, s, an S-weight η of $W_{\pi,s}$, and $n \in -s + q\mathbb{Z}^r$, set $m_a = \eta(k_{a,j}) + n_a$. Precisely $\dim P_\eta W_{\pi,s}$ eigenvalues of the corresponding matrix remain bounded. They converge with multiplicity to the spectrum of*

$$H_{\pi,s,\eta}(n) = L_\pi(A_{\mathrm{e}})|_{P_\eta W_{\pi,s}} + \sum_{a,b} (B_0)_{ab} (n_a I - T_\pi(F_a))(n_b I - T_\pi(F_b))|_{P_\eta W_{\pi,s}}. \tag{6.11}$$

The other eigenvalues tend to infinity.

Proof. Write $X_a = T_\pi(H_{a,j})$, $Y_a = T_\pi(E_{a,j})$. The potential is

$$\sum_{a,b}(B_j)_{ab}\big((m_a - X_a)(m_b - X_b) - (m_a - X_a)Y_b - Y_b(m_a - X_a) + Y_aY_b\big).$$

For $\eta' \neq \eta$ put $\delta_a = (\eta' - \eta)(k_{a,j})$. The only potentially unbounded term joining P_η to $P_{\eta'}$ is $\sum_{a,b}(B_j)_{ab}\,\delta_a P_{\eta'} Y_b P_\eta$. The represented stationarity equation (6.6) gives exactly

$$\sum_{a,b}(B_j)_{ab}\,\delta_a P_{\eta'} Y_b P_\eta = -\sum_{a,b}(B_j)_{ab} P_{\eta'}[T_\pi(\theta_{a,j}), Y_b]P_\eta. \tag{6.12}$$

Its right side is bounded. Every other off-diagonal term, including $L_\pi(A_j)$, is bounded for fixed π, n. On the complementary weight spaces the positive commuting Cartan potential tends uniformly to infinity, since

$$M_{\pi,s,m} \geq \frac{\beta}{2}\sum_a \big(n_a + \eta(k_{a,j}) - T_\pi(k_{a,j})\big)^2 - C_{\pi,n}I.$$

Write the block matrix as $\left(\begin{smallmatrix} D_j & K_j^* \\ K_j & Q_j \end{smallmatrix}\right)$. Then $\|K_j\| = O(1)$ and $Q_j \geq R_j I$, $R_j \to \infty$. Let $d = \dim P_\eta W_{\pi,s}$, and choose a, u with $\|D_j\| \leq a$ and $\|K_j\| \leq u$. Lemma 6.3 gives

$$\lambda_k(D_j) - \frac{u^2}{R_j - a - 1} \leq \lambda_k(M_{\pi,s,m}) \leq \lambda_k(D_j) \quad (1 \leq k \leq d),$$

for all sufficiently large j. If $d < \dim W_{\pi,s}$, its next eigenvalue is at least R_j. This proves both the number of bounded eigenvalues and their convergence, once $\lim D_j$ is computed.

Decompose $E_{a,j} = E_{a,\|,j} + E_{a,\perp,j}$ relative to $\mathfrak{e}$, and put $G_{a,j} = \theta_{a,j} + E_{a,\|,j}$. Then $T_\pi(G_{a,j})$ preserves every P_η, whereas $P_\eta T_\pi(E_{a,\perp,j})P_\eta = 0$. Compression of the square gives

$$\begin{aligned} D_j = {} & P_\eta L_\pi(A_j)P_\eta + \sum_{a,b}(B_j)_{ab}(n_aI - T_\pi(G_{a,j}))(n_bI - T_\pi(G_{b,j}))|_{P_\eta} \\ & + \sum_{a,b}(B_j)_{ab} P_\eta T_\pi(E_{a,\perp,j})T_\pi(E_{b,\perp,j})P_\eta. \end{aligned}$$

All mixed terms with one perpendicular factor vanish, since the other factor preserves P_η. As $j \to \infty$, the last sum tends to $P_\eta L_\pi(C_E)P_\eta$ and $G_{a,j} \to F_a$. For every $s \in S$, the operator $\pi(s)$ is scalar on P_η. Consequently compression is unchanged by conjugation through $\pi(s)$, and integration over S gives

$$P_\eta L_\pi(A_0 + C_E)P_\eta = L_\pi\big(\mathrm{Av}_S(A_0 + C_E)\big)|_{P_\eta}.$$

This is (6.11). □

Lemma 6.5. *The direct sum of* (6.11)*, with the stated sectors, Fourier cosets, and outer multiplicities, has exactly spectrum* Σ*. Moreover* $A_{\mathrm{e}} > 0$*.*

Proof. Write $\gamma = (\pi, s, \eta, n)$ for a limiting block label, with $n \in -s + q\mathbb{Z}^r$. If π, s are fixed, its actual Fourier label is $m_{a,j} = \eta(k_{a,j}) + n_a$. Distinct pairs (η, n) give distinct actual Fourier labels eventually. Indeed, for $\eta \neq \eta'$ the norm of $((\eta - \eta')(k_{a,j}))_a$ tends to infinity by (6.7), whereas $n' - n$ is fixed. For $\eta = \eta'$ equality of the Fourier labels is equivalent to $n = n'$. Distinct representations and sectors already give orthogonal summands.

Choose any finite multiset of limiting block eigenvalues, counting each with its matrix multiplicity and the factor d_π. Lemma 6.4 realizes these occurrences as limits of actual eigenvalues in distinct orthogonal summands. Choose N with σ_N strictly larger than every chosen limiting value. For all sufficiently large j, the corresponding actual eigenvalues lie below $\lambda_N(g_j) = \sigma_N$ and belong to the fixed finite list $\sigma_0, \dots, \sigma_{N-1}$. Every chosen limiting value therefore belongs to this list. Moreover, an interval containing one reference value and no other contains exactly its allowed number of scalar eigenvalues for large j. No finite selection of limiting blocks can exceed that multiplicity. In particular

the constant block accounts for the only possible occurrence of zero. Thus the limiting direct sum has no eigenvalue or multiplicity in excess of Σ.

For the reverse containment, fix a positive reference value μ and include its complete cluster and its successor in the matched initial segment. Every contributing representation satisfies $d_\pi \le m_\mu$, and hence belongs to a finite list. The quadratic form bound $B_j \ge \beta I$, the nonnegativity of $L_\pi(A_j)$, and the boundedness of $\theta_{a,j} + E_{a,j}$ give, for every unit μ-eigenvector,

$$\sum_a \|(m_a I - T_\pi(k_{a,j}))v\|^2 \le \frac{2\mu}{\beta} + 2\sum_a \|T_\pi(\theta_{a,j} + E_{a,j})\|^2 \le C_{\pi,\mu}.$$

The operators $T_\pi(k_{a,j})$ commute, so their joint weight decomposition gives

$$\sum_\eta \|P_\eta v\|^2 \sum_a |m_a - \eta(k_{a,j})|^2 \le C_{\pi,\mu}.$$

For some weight η, the inner sum is at most $C_{\pi,\mu}$. Set $n_a = m_a - \eta(k_{a,j})$. Because $k_{a,j} \in q\Lambda$, $n \in -s + q\mathbb{Z}^r$. The finite representation list and the bound on n leave only finitely many labels γ for this cluster. The preceding separation argument makes their actual blocks distinct for all sufficiently large j.

Pass to a subsequence on which the multiplicity h_γ of μ in each of these finitely many actual blocks is constant. Then

$$\sum_\gamma d_\pi h_\gamma = m_\mu.$$

By Lemma 6.4, the nonbounded eigenvalues of each actual block tend to infinity, and its remaining ordered eigenvalues converge to those of the limiting block. Thus the latter has at least h_γ occurrences of μ. Summation gives at least m_μ occurrences in the limiting direct sum. The opposite inequality was proved for every finite multiset above. Since μ was arbitrary, the direct sum has exactly spectrum Σ.

For $\varepsilon > 0$, the Schur parameters $(A_{\mathrm{e}} + \varepsilon I, B_0, F)$ define a positive Riemannian cometric on M_c. The S-invariance of A_{e} and the centralization identities for F_a preserve the weight spaces. Its block with label γ is therefore

$$H_{\pi,s,\eta}(n) + \varepsilon c_\pi I.$$

Every block eigenvalue is at least its unregularized value. Hence $N_\varepsilon(T) \le N_\Sigma(T)$ for every T, where N_ε is the counting function of this positive metric. For each fixed ε, divide this counting inequality by $T^{(d+r)/2}$ and let $T \to \infty$. The Weyl law and (5.4) give

$$\frac{(2\pi)^r \operatorname{Vol}(K/H, Q)}{q^r \sqrt{\det{}_{\mathfrak{m}}(A_{\mathrm{e}} + \varepsilon I) \det B_0}} \le \operatorname{Vol}(\Sigma). \tag{6.13}$$

If A_{e} were singular, the left side would tend to infinity as $\varepsilon \downarrow 0$. Therefore $A_{\mathrm{e}} > 0$. □

6.4. **Polynomial factors and reducing projections.**

Lemma 6.6. *Let $M(m)$ and $H_\eta(n)$ be Hermitian matrix polynomials in r real variables, with $\sum_\eta \dim H_\eta = \dim M$. Suppose that for every n in a full-rank affine lattice the spectrum of $H_\eta(n)$ is contained with multiplicity in that of $M(n + c_\eta)$. Then*

$$f_\eta(m, z) = \det(zI - H_\eta(m - c_\eta)) \quad \textit{divides} \quad F(m, z) = \det(zI - M(m)).$$

If the f_η are pairwise coprime in $\mathbb{R}(m)[z]$, then $F = \prod_\eta f_\eta$.

Proof. The characteristic coefficients are real polynomials, since the matrices are Hermitian for real parameters. Fix η and divide $F(m, z)$ by the monic polynomial $f_\eta(m, z)$ in the variable z. Monic polynomial division takes place in $\mathbb{R}[m_1, \dots, m_r][z]$:

$$F = Q_\eta f_\eta + R_\eta, \qquad R_\eta(m, z) = \sum_{j < \deg_z f_\eta} r_j(m) z^j.$$

At every $m = n + c_\eta$ allowed by the hypothesis, spectral containment with multiplicity says that $f_\eta(m, \cdot)$ divides $F(m, \cdot)$. Uniqueness of the remainder after evaluation gives $r_j(n + c_\eta) = 0$ for every coefficient r_j.

An invertible affine change sends this full-rank affine lattice to $\mathbb{Z}^r$. A polynomial p vanishing on $\mathbb{Z}^r$ is zero. For $r = 0$ it is a constant, with $p = p(0) = 0$. For $r = 1$ it is the finite-root theorem for a nonzero polynomial. For $r > 1$, write $p(x', x_r) = \sum_k p_k(x') x_r^k$. At each $x' \in \mathbb{Z}^{r-1}$, the one-variable polynomial vanishes at every integer, so every $p_k(x') = 0$. Induction gives $p_k = 0$ for every k. Thus all r_j vanish identically and f_η divides F.

In the field $\mathbb{R}(m_1, \dots, m_r)$, pairwise coprime divisors have a product dividing F. The product and F are monic of the same degree in z. Their quotient is therefore 1. □

Lemma 6.7. *Let $C, D_1, \dots, D_r$ be $m \times m$ Hermitian matrices, let $q \in \mathbb{R}^r$, and let $1 \le s < m$. Suppose that $D(q) = \sum_a q_a D_a$ has a strict gap between its s largest eigenvalues and the rest, and let P be its top rank-s spectral projection. For every $p \in \mathbb{R}^r$, suppose that the sum of the s largest eigenvalues of $A_p + tD(q)$, where $A_p = C + \sum_a p_a D_a$, is affine for $t > T_p$. Then P reduces C and every D_a.*

Proof. Fix p and $t_0 > T_p$. Choose a rank-s orthogonal projection E maximizing $\operatorname{Tr}(E(A_p + t_0 D(q)))$. For a Hermitian matrix T with eigenvalues $\tau_1 \ge \cdots \ge \tau_m$, a rank-$s$ orthogonal projection E' satisfies

$$\operatorname{Tr}(E'T) = \sum_i \tau_i \|E' e_i\|^2 \le \sum_{i=1}^{s} \tau_i,$$

because $0 \le \|E' e_i\|^2 \le 1$ and their sum is s. Projection onto the first s eigenvectors attains equality. This is the finite-dimensional variational principle of [9, Theorem 1]. In particular,

$$F_p(t) = \max_{E' = E'^* = E'^2,\ \operatorname{rank} E' = s} \operatorname{Tr}(E'(A_p + tD(q))).$$

The affine function $\operatorname{Tr}(E(A_p + tD(q)))$ lies below the affine $F_p(t)$ on (T_p, ∞) and agrees at the interior point t_0. Their affine difference is nonnegative and vanishes in the interior; it is therefore identically zero. If the eigenvalues are ordered decreasingly, min–max gives, for $t > 0$,

$$\left| t^{-1} \lambda_i(A_p + tD(q)) - \lambda_i(D(q)) \right| \le t^{-1} \|A_p\|.$$

Consequently $t^{-1} F_p(t) \to \sum_{i=1}^{s} d_i$, where $d_1 \ge \cdots \ge d_m$ are the eigenvalues of $D(q)$. Dividing the identical affine functions by t gives $\operatorname{Tr}(ED(q)) = \sum_{i=1}^{s} d_i$.

In an eigenbasis (e_i) put $e_i' = \langle E e_i, e_i \rangle$. Then $0 \le e_i' \le 1$ and $\sum_i e_i' = s$, whence

$$\sum_{i \le s} d_i - \operatorname{Tr}(ED(q)) \ge (d_s - d_{s+1}) \sum_{i > s} e_i'.$$

The strict gap forces $e_i' = 0$ for $i > s$ and $e_i' = 1$ for $i \le s$. Since E is an orthogonal projection, $\|E e_i\|^2 = e_i'$; these identities give $E = P$.

Thus P maximizes the variational expression at t_0. For every skew-Hermitian K, differentiate at zero the expression $\operatorname{Tr}(e^{uK} P e^{-uK}(A_p + t_0 D(q)))$. Its derivative is zero, so $\operatorname{Tr}(K[P, A_p + t_0 D(q)]) = 0$. The commutator is skew-Hermitian; taking $K = -[P, A_p + t_0 D(q)]$ proves that it vanishes. Since P commutes with $D(q)$, it commutes with A_p. Take first $p = 0$ and then p equal to each coordinate vector. □

Lemma 6.8. *There is a finite set $\mathcal{P} \subset \widehat{K}$ for which both linear maps*

$$\mathfrak{d} \longrightarrow \bigoplus_{\substack{\pi \in \mathcal{P} \\ s \in (\mathbb{Z}/q)^r}} \operatorname{End}(W_{\pi,s}), \qquad X \longmapsto \left(T_\pi(X)|_{W_{\pi,s}} \right)_{\pi,s},$$

$$\operatorname{Sym}(\mathfrak{m})^{H_c} \longrightarrow \bigoplus_{\substack{\pi \in \mathcal{P} \\ s \in (\mathbb{Z}/q)^r}} \operatorname{End}(W_{\pi,s}), \qquad A \longmapsto \left(L_\pi(A)|_{W_{\pi,s}} \right)_{\pi,s}$$

are injective.

Proof. If all such represented vector fields vanish, they vanish on every matrix coefficient on K/H. If all represented quadratic expressions vanish, the associated differential operator vanishes on the same coefficients. The eigenspaces of the normal homogeneous Laplacian are finite sums of these matrix-coefficient spaces. Lemma 2.5 therefore gives convergence in C^∞ by finite sums of the same coefficients. A differential operator is continuous from C^∞ to C^∞, so each operator vanishes on every smooth function. Its first or second principal symbol is respectively the specified vector or symmetric tensor, so these vanish. Apply Lemma 2.10 to the two domains $\mathfrak{d}$ and $\mathrm{Sym}(\mathfrak{m})^{H_c}$, with maps respectively given by the represented vector fields and the quadratic expressions. Each common kernel is zero by these principal-symbol identities. The union $\mathcal{P}$ of the selected representation indices satisfies

$$|\mathcal{P}| \le \dim \mathfrak{d} + \dim \mathrm{Sym}(\mathfrak{m})^{H_c}.$$

It is independent of $X \in \mathfrak{d}$ and $A \in \mathrm{Sym}(\mathfrak{m})^{H_c}$. For each selected representation all finite sectors are included, since their orthogonal direct sum is W_π. □

Lemma 6.9 (Finite separating directions). *Let $r, k \ge 1$, put $M = k(r-1)+1$, and define*

$$w_j = (1, j, \dots, j^{r-1}), \qquad u_j = \frac{w_j}{\|w_j\|} \quad (0 \le j < M), \qquad c_{r,k} = r^{-r} M^{-r(r-1)}.$$

For any nonzero $\delta_1, \dots, \delta_k \in \mathbb{R}^r$, at least one of these M directions satisfies

$$|\langle u_j, \delta_\ell \rangle| \ge c_{r,k} \|\delta_\ell\| \qquad (1 \le \ell \le k). \tag{6.14}$$

Proof. For $r = 1$, the set of directions is $\{1\}$ and the estimate is an equality. Suppose $r \ge 2$. Choose any r of the directions and form the matrix V having them as rows. The Vandermonde determinant gives

$$|\det V| = \frac{\prod_{a<b} |j_b - j_a|}{\prod_a \|w_{j_a}\|} \ge r^{-r/2} M^{-r(r-1)}, \qquad \|V\|_{\mathrm{HS}} = \sqrt{r}.$$

Here the indices are distinct integers, and $\|w_j\| \le \sqrt{r}\, M^{r-1}$. All singular values except the smallest are at most $\sqrt{r}$. Hence

$$s_{\min}(V) \ge r^{-(2r-1)/2} M^{-r(r-1)} = \sqrt{r}\, c_{r,k}.$$

For a unit vector v, there cannot be r indices with $|\langle u_j, v\rangle| < c_{r,k}$: their matrix would give $\|Vv\| < \sqrt{r}\, c_{r,k} \le s_{\min}(V)$. For each ℓ put

$$J_\ell = \{ j \,:\, |\langle u_j, \delta_\ell \rangle| < c_{r,k} \|\delta_\ell\| \}.$$

The preceding estimate gives $|J_\ell| \le r-1$, hence $|\bigcup_{\ell=1}^k J_\ell| \le k(r-1) < M$. Any index outside this union satisfies (6.14).

□

Proposition 6.10. *Let $U \subset \mathfrak{t}$ be rational and put $S = \exp U$. Suppose $0 < A_j \le bI$, $\beta I \le B_j \le b_B I$, with $\beta > 0$, and $V_{a,j} = k_{a,j} + E_{a,j}$, where $k_{a,j} \in q\Lambda \cap U$ and $E_{a,j}$ are bounded. Let $A_* > 0$ belong to $\mathrm{Sym}(\mathfrak{m})^{H_c}$ and be S-invariant, and let $F_a \in \mathfrak{d}$ centralize S. On a finite family as in Lemma 6.8, suppose*

$$\det(zI - M_{\pi,s,j}(m)) = \prod_\eta \det\big(zI - H_{\pi,s,\eta,j}(m - \eta(k_j))\big), \tag{6.15}$$

where $H_{\pi,s,\eta,j}$ is formed from (A_*, B_j, F) by (6.11). Assume that the distinct weight differences in this collection diverge jointly as in (6.7). For all sufficiently large j, one common D-conjugation makes A'_j S-invariant, makes all $V'_{a,j}$ centralize S, and leaves $V'_{a,j} - k_{a,j}$ bounded.

Proof. List the differences of distinct S-weights within all sectors of the finite family in Lemma 6.8. For each such pair put

$$\delta_{\eta,\eta',j} = \big((\eta - \eta')(k_{a,j})\big)_{a=1}^r.$$

Their norms tend to infinity, uniformly over this finite list. If its size is $k > 0$, Lemma 6.9, applied to $B_j\delta_{\eta,\eta',j}$, gives a unit vector u_j in one fixed set of $k(r-1)+1$ directions such that

$$\left|\sum_a (B_j u_j)_a(\eta-\eta')(k_{a,j})\right| \geq c_{r,k}\beta\|\delta_{\eta,\eta',j}\| \longrightarrow \infty. \tag{6.16}$$

If the list is empty, choose one fixed unit vector instead. Put

$$K_j = \sum_a (B_j u_j)_a k_{a,j}, \qquad V_j = \sum_a (B_j u_j)_a V_{a,j}.$$

The bounds on B_j and $E_{a,j}$ give $\|V_j - K_j\| \leq C$.

Lemma 5.4, applied to V_j and K_j, gives one $h_j \in D$, independent of the representation, with $\mathrm{Ad}(h_j)V_j = K_j + R_j \in \mathfrak{t}$ and $\|R_j\| \leq C$. Conjugate only the actual matrices in (6.15).

After subtraction of $m^{\mathsf{T}}B_j mI$, the actual pencil is

$$C_{\pi,s,j} - 2\sum_a m_a D_{\pi,s,a,j}, \qquad D_{\pi,s,a,j} = \sum_b (B_j)_{ab}\, T_\pi(V'_{b,j}),$$

where $C_{\pi,s,j} = L_\pi(A'_j) + \sum_{a,b}(B_j)_{ab}\, T_\pi(V'_{a,j})T_\pi(V'_{b,j})$. On the line $m = -tu_j + p$ the slope is $2T_\pi(K_j+R_j)$. For comparison, on $P_\eta W_{\pi,s}$ the reference factor satisfies

$$H_{\pi,s,\eta,j}(m-\eta(k_j)) - m^{\mathsf{T}}B_j mI = C^*_{\eta,j} - 2\sum_{a,b} m_a (B_j)_{ab}\big(\eta(k_{b,j})I + T_\pi(F_b)\big),$$

where $C^*_{\eta,j} = H_{\pi,s,\eta,j}(-\eta(k_j))$ is independent of m. Its slope on the same line is

$$2\eta(K_j)I + 2\sum_a (B_j u_j)_a T_\pi(F_a)|_{P_\eta W_{\pi,s}}.$$

Choose C_0 bounding both $\|2T_\pi(R_j)\|$ and $\|2\sum_a (B_j u_j)_a T_\pi(F_a)\|$ on the finite representation family of Lemma 6.8, for every j. Such a bound follows from the bounds on R_j, B_j and $\|u_j\| = 1$. The intervals

$$I_{\eta,j} = [2\eta(K_j) - C_0,\ 2\eta(K_j) + C_0] \tag{6.17}$$

are pairwise disjoint within each fixed sector for large j, by (6.16). Decompose a fixed sector into its full $\mathfrak{t}$-weight spaces W_χ. On W_χ the actual slope $2T_\pi(K_j+R_j)$ has eigenvalue $2\chi(K_j) + 2\chi(R_j)$. Since $K_j \in U$, the first term is $2\eta(K_j)$ precisely when $\chi|_U = \eta$; the second term has absolute value at most C_0. The projection onto $I_{\eta,j}$ is therefore exactly

$$\bigoplus_{\chi|_U=\eta} I_{W_\chi} = P_\eta.$$

On the reference side, $T_\pi(F_a)$ preserves P_η and its bounded part has norm at most C_0. Its slope has the same interval projections and the same ordering. The cumulative projections are therefore the same sums of P_η on both sides.

Fix such a j and any $p \in \mathbb{R}^r$. For sufficiently large t, the top cumulative eigenvalue sums on the reference side are traces of complete η-blocks, hence affine in t. Equation (6.15) makes the corresponding cumulative eigenvalue sums affine in t for the actual pencil. Lemma 6.7, applied to each cumulative group and every p, shows that each P_η reduces $C_{\pi,s,j}$ and all $D_{\pi,s,a,j}$. Since B_j is invertible, each P_η also reduces $T_\pi(V'_{a,j})$ and $L_\pi(A'_j)$. For every $X \in U$, each $T_\pi(X)$ is scalar on the spaces P_η, so the reducing identities give

$$[T_\pi(X), T_\pi(V'_{a,j})] = 0, \qquad [T_\pi(X), L_\pi(A'_j)] = 0$$

on all sectors of the finite family. Differentiating equivariance with $T_\pi = -i\,d\pi$ gives the exact identities

$$i[T_\pi(X), T_\pi(V'_{a,j})] = T_\pi([X, V'_{a,j}]), \qquad i[T_\pi(X), L_\pi(A'_j)] = L_\pi([\mathrm{ad}\,X|_{\mathfrak{m}}, A'_j]).$$

Since X centralizes H_c, the vectors $[X, V'_{a,j}]$ belong to $\mathfrak{d}$. The operator $\operatorname{ad} X|_{\mathfrak{m}}$ is skew-adjoint and commutes with $\mathrm{Ad}(H_c)$; hence its commutator with A'_j belongs to $\mathrm{Sym}(\mathfrak{m})^{H_c}$. Injectivity in Lemma 6.8 gives

$$[V'_{a,j}, U] = 0, \qquad [A'_j, \operatorname{ad} U|_{\mathfrak{m}}] = 0.$$

Exponentiating these identities gives right-S invariance of the cometric parameters.

Fix this j and a bounded nonempty open subset of $u_j^{\perp}$. On its closure the constant terms of the actual and reference pencils are bounded. Their slope intervals have positive gaps. Hence, for all sufficiently large t, the η-clusters are separated in the same order on both sides of (6.15), uniformly in p in this set. Since the full multisets agree, the multisets within each η-cluster agree. The corresponding characteristic polynomials are therefore equal for $m = -tu_j + p$ in an open subset of $\mathbb{R}^r$, and hence are identical polynomials. Take $m = tB_j^{-1}e_a$, subtract its common quadratic scalar, divide by $-2t$, and let $t \to \infty$. Continuity of Hermitian spectra gives

$$\mathrm{Spec}\big(T_\pi(V'_{a,j})|_{P_\eta W_{\pi,s}}\big) = \eta(k_{a,j}) + \mathrm{Spec}\big(T_\pi(F_a)|_{P_\eta W_{\pi,s}}\big).$$

All these weight spaces reduce $T_\pi(V'_{a,j} - k_{a,j})$. The norm of that Hermitian operator is consequently bounded by the fixed norm of $T_\pi(F_a)$ on these sectors. The first linear map in Lemma 6.8 is injective, so Lemma 2.10 bounds $\|V'_{a,j} - k_{a,j}\|$ uniformly. □

6.5. **Positive limits after changes of coordinates.**

Proof of Theorem 5.1. Take a sequence matching successively longer exact initial segments. By Propositions 5.7 and 6.2 and (6.6), its normalized parameters have the limits in (6.8). Lemma 6.5 gives the positive effective metric with blocks (6.11). Fix π, s, η, n, and let μ have multiplicity h in its limiting block. By Lemma 6.5, $\mu \in \Sigma$. Choose an interval about μ containing no other reference value, and choose a reference eigenvalue strictly above its closure. Lemma 6.4 gives h actual roots converging to μ. On a sufficiently long matched initial segment these roots belong to the reference list and lie in the chosen interval. Thus all h roots equal μ exactly. If $w = \dim W_{\pi,s}$, every $(w-h+1)\times(w-h+1)$ minor of $M_{\pi,s,j}(n+\eta(k_j)) - \mu I$ consequently vanishes eventually.

The real and imaginary parts of these minors are polynomials in the single finite-dimensional parameter vector $(A_j, B_j, k_j, \theta_j, E_j)$. Their coefficients are real and imaginary parts of the fixed representation matrices, together with the limiting eigenvalues. The indices π, s, η, n, μ vary, but the polynomial ring does not. Each individual polynomial vanishes on a tail of the sequence; Lemma 2.6 gives one index after which all of them vanish. Consequently every limiting block spectrum is contained, with its multiplicity, in the corresponding actual matrix spectrum on this common tail.

In the trivial representation the block is the scalar $n^{\mathsf{T}}B_j n$. Taking $n = qe_a$ fixes $q^2(B_j)_{aa}$, and taking $n = q(e_a + e_b)$ fixes $q^2((B_j)_{aa} + 2(B_j)_{ab} + (B_j)_{bb})$. Their exact limiting values give $B_j = B_0$ for all sufficiently large j.

Lemma 6.6 gives the weight factors over each allowed Fourier coset. In the finite family of Lemma 6.8, choose a separating direction as in Proposition 6.10. For a fixed sufficiently large j, the slope intervals of distinct weight factors are disjoint. At a sufficiently far point of this real line their roots in z are disjoint. Their resultant in z is therefore nonzero at that point and hence is a nonzero polynomial in m. Thus each pair is coprime over $\mathbb{R}(m_1, \dots, m_r)$. Their degrees sum to the whole block dimension. Thus (6.15) holds. Proposition 6.10 gives exact right-S invariance after one common conjugation and bounded residuals. Let g'_j denote the metric after this common conjugation. Define

$$\Phi_j(y, z) = \left(y \exp\Big(\sum_a z_a k_{a,j}\Big), z \right). \tag{6.18}$$

For $b \in \mathbb{Z}^r$, put $R_b(y, z) = (y, z + 2\pi b)$. Then $\Phi_j R_b = R_b \Phi_j$, so its first component is periodic in z. Moreover $\exp(2\pi k_{a,j}/q) = e$ and D centralizes H_c. Consequently Φ_j and its inverse commute with every deck transformation in (5.1), and induce diffeomorphisms of M_c. Set $\widetilde{g}_j = (\Phi_j^{-1})^* g'_j$. The exact right-$S$ invariance gives constant transformed coefficients. In the invariant horizontal and periodic

frames, the differential of Φ_j has matrix $\left(\begin{smallmatrix} I & k_j \\ 0 & I \end{smallmatrix}\right)$, up to the right-$S$ adjoint action, which preserves A'_j and the $V'_{a,j}$. The congruence transformation of the Schur matrix (5.3) by this differential and its transpose gives

$$\widetilde{A}_j = A'_j, \qquad \widetilde{B}_j = B_0, \qquad \widetilde{V}_{a,j} = V'_{a,j} - k_{a,j}.$$

These parameters satisfy

$$0 < \widetilde{A}_j \le bI, \qquad \widetilde{B}_j = B_0 > 0, \qquad \|\widetilde{V}_{a,j}\| \le C.$$

The full Schur cometrics of these actual representatives are bounded: $\widetilde{A}_j$, $\widetilde{B}_j$, and the new drift columns are all bounded. Pass to a limit of these full cometrics. Their indexed eigenvalues converge to Σ, since the transformations were actual isometries and the matched initial segments grow without bound. Lemma 2.11 makes the full limit positive definite. Since $\widetilde{B}_j = B_0 > 0$, its Schur complement is positive, so $\widetilde{A}_0 > 0$. For $d > 0$, let $\mathcal{R}(g)$ be the nonempty set of representatives of the isometry class of g on the finite list of presentations satisfying $0 < A \le bI$ and $\beta I \le B \le b'I$. For metrics matching the fixed initial segment from Proposition 5.7, define

$$f(g) = \inf_{(A,B,V)\in\mathcal{R}(g)} \left(\lambda_{\min}(A)^{-1} + \max_a \|V_a\|\right).$$

If no finite initial segment bounds f, there are metrics g_j matching the first j reference eigenvalues with $f(g_j) > j$. The positive convergent representatives just obtained belong to $\mathcal{R}(g_j)$ on a subsequence, so their displayed quantities are bounded, a contradiction. Hence there are N_0 and $M > 0$ such that every matching metric has a representative with

$$(M+1)^{-1}I \le A \le bI, \qquad \beta I \le B \le b'I, \qquad \max_a \|V_a\| \le M+1.$$

This parameter set is compact and semialgebraic. Its image under $C(A, B, V)$ has the same properties, since this map is polynomial. The factorization after (5.3) gives

$$\|C(A,B,V)^{-1}\| \le (1+\|V\|)^2 \max\{\|A^{-1}\|, \|B^{-1}\|\}.$$

Thus the image consists of uniformly positive cometrics. If $d = 0$, the parameter set is $\beta I \le B \le b'I$, and take N_0 to be the index supplied by Proposition 5.7.

On each compact positive parameter set, the representation and Fourier matrices have polynomial entries in the full cometric coordinates. Their spectra, with the prescribed outer multiplicities, form the scalar spectrum. The comparison in Lemma 2.9 gives continuity of every ordered eigenvalue. Lemma 2.8 therefore gives a determining index N_i on each of the finitely many parameter sets, indexed by $1 \le i \le b$. Put $N = \max\{N_0, N_1, \dots, N_b\}$. Agreement through N first gives a representative in one of these sets and then gives its full spectrum. The full isospectral subset is closed by continuity; smooth dependence of an invariant metric on its positive cometric proves compactness of its representatives. □

Corollary 6.11. *On every fixed compact semisimple group times an arbitrary torus, exact finite-initial-segment determination and smooth isospectral compactness hold for all left-invariant metrics, including all noncommuting mixed directions. The same is true for every fixed homogeneous product* $(K/H) \times \mathbb{T}^r$ *and its fixed finite normalizer quotients above.*

Proof. Take $H = \{e\}$ and $q = 1$ for the group product, and $q = 1$ for a homogeneous product, in Theorem 5.1. If the torus has dimension zero, Theorem 3.1 gives finite determination and compact positive representatives. □

7. Normalizer holonomy and varying isotropy

7.1. Commuting holonomies.

Lemma 7.1. *Let a compact Lie group J act linearly on a finite-dimensional real space P. For fixed r, the groups*

$$C(p,\tau) = \{g \in J \,:\, gp = p,\ g\tau_a = \tau_a g\ (1 \le a \le r)\}$$

have uniformly bounded numbers of connected components, as $p \in P$ and $\tau \in J^r$ vary.

Proof. Choose a faithful orthogonal representation $J \subset O(N)$ containing the given representation on P. For $x \in O(N) \setminus J$, the compact sets J and Jx are disjoint. Stone–Weierstrass approximation gives a polynomial q_x in the matrix entries such that

$$|q_x| < \tfrac{1}{4} \quad \text{on } J, \qquad |q_x - 1| < \tfrac{1}{4} \quad \text{on } Jx.$$

For Haar probability measure dh on J, put

$$p_x(X) = \int_J q_x(hX)\, dh - \int_J q_x(h)\, dh.$$

Integration changes only the finitely many coefficients, so p_x is a polynomial. Right invariance of Haar measure gives $p_x|_J = 0$, whereas $p_x(x) > 1/2$. Hence

$$J = \{X \in O(N) \,:\, p_x(X) = 0 \text{ for every } x \in O(N) \setminus J\}.$$

The Hilbert basis theorem reduces these equations to a finite family; thus J is real algebraic. The conditions defining $C(p,\tau)$ are polynomial in these matrix coordinates. They define a semialgebraic family. Hardt's triviality theorem partitions the parameter space into finitely many semialgebraic sets on each of which the fibres are homeomorphic [13]. Their numbers of connected components have a finite maximum. □

Lemma 7.2. *Let F and J be compact Lie groups. Every J-conjugacy class in* $\mathrm{Hom}(F,J)$ *is open for the uniform topology.*

Proof. Averaging a positive inner product on Lie J over J gives a bi-invariant metric on J^0 invariant under conjugation by J. If $J^0 = \{e\}$, sufficiently close maps $F \to J$ are equal. Assume $\dim J^0 > 0$, and fix $\phi \in \mathrm{Hom}(F,J)$. Choose $\varepsilon > 0$ such that $\overline{B}_{3\varepsilon}(e)$ is geodesically convex and

$$\mathrm{Hess}_x\big(\tfrac{1}{2}d(x,y)^2\big) \ge \tfrac{1}{2}g_x \quad \big(x \in \overline{B}_{3\varepsilon}(e),\ y \in \overline{B}_{\varepsilon}(e)\big).$$

Such an ε exists because squared distance is smooth near the diagonal and its Hessian at (e,e) is the metric. For ψ sufficiently close to ϕ in the uniform topology, put $p_s = \psi(s)\phi(s)^{-1} \in B_\varepsilon(e)$, $s \in F$. The maps

$$a_s(x) = \psi(s)x\phi(s)^{-1} = p_s\big(\phi(s)x\phi(s)^{-1}\big)$$

are isometries of J^0 and satisfy $a_s a_t = a_{st}$. For Haar probability measure ds on F, put

$$f(x) = \int_F d(x,p_s)^2\, ds.$$

The function f attains its minimum on the compact group J^0. One has $f(e) < \varepsilon^2$, whereas $d(x,e) \ge 3\varepsilon$ gives $f(x) \ge 4\varepsilon^2$. Every minimizer lies in $B_{3\varepsilon}(e)$. On this ball differentiation under the integral is legitimate: the squared-distance function is smooth on a neighborhood of the compact set $\overline{B}_{3\varepsilon}(e)\times\overline{B}_{\varepsilon}(e)$. The Hessian inequality makes f strictly convex there, so its global minimizer x is unique. The identities

$$a_t(p_s) = \psi(t)\psi(s)\phi(s)^{-1}\phi(t)^{-1} = p_{ts}$$

and left invariance of ds imply $f \circ a_t = f$. Hence $a_t(x) = x$, which is equivalent to $\psi(t) = x\phi(t)x^{-1}$ for all $t \in F$. Every sufficiently close homomorphism is therefore conjugate to ϕ. □

Corollary 7.3. *For a finite group F and a compact Lie group J, the set* $\mathrm{Hom}(F, J)/J$ *is finite.*

Proof. The equations $\phi(st) = \phi(s)\phi(t)$ define a closed subset of the compact finite product J^F. By Lemma 7.2, its conjugacy classes are open. A finite subcover by these disjoint classes contains all of them. □

Theorem 7.4. *For fixed J, P, r as above there are $q \geq 1$ and finitely many simultaneous conjugacy classes of commuting tuples $(c_1, \dots, c_r)$, with $c_a^q = e$, such that every commuting tuple $\tau_1, \dots, \tau_r$ fixing $p \in P$ can be written*

$$\tau_a = c'_a \exp X_a.$$

The $X_a \in \mathfrak{j}$ commute, their one-parameter subgroups fix p and commute with every τ_b, and the tuple c' is conjugate to one of the fixed tuples.

Proof. Let M bound the component counts in Lemma 7.1, put $q = M!$, and set $C = C(p, \tau)$. Each τ_a is central in C and $\tau_a^q \in C^0$. These powers are central in C^0. Every element of a compact connected group belongs to a maximal torus; since all maximal tori are conjugate, a central element belongs to every maximal torus. Fix one such torus $T_C \subset C^0$. Choose commuting $X_a \in \mathrm{Lie}\, T_C$ with $\exp(qX_a) = \tau_a^q$. Then $c'_a = \tau_a \exp(-X_a)$ commute and satisfy $(c'_a)^q = e$. They form a homomorphism $(\mathbb{Z}/q)^r \to J$; apply Corollary 7.3. Since $\exp(tX_a) \in T_C \subset C(p, \tau)$ for every $t \in \mathbb{R}$, one has

$$\exp(tX_a)p = p, \qquad \exp(tX_a)\tau_b = \tau_b \exp(tX_a) \qquad (a, b = 1, \dots, r).$$

□

For the rest of this section, K is compact, connected, and semisimple, $H \subset K$ is closed, and Q is an invariant positive inner product on $\mathfrak{k}$. Put $Y = K/H$, $\mathfrak{m} = \mathfrak{h}^{\perp_Q}$, and

$$J = N_K(H)/H, \qquad P = \mathrm{Sym}(\mathfrak{m})^H \oplus (\mathfrak{m}^H)^r.$$

The action of J on P is well-defined. For a metric with parameters $p = (A, V_1, \dots, V_r)$ and $B > 0$, its stabilizer consists exactly of the normalizer transformations whose product action on $Y \times \mathbb{R}^r$ preserves the metric. For commuting $\tau_a \in J$ fixing p write

$$M_\tau = (Y \times \mathbb{R}^r)/\langle (y, z) \mapsto (y\tau_a, z + 2\pi e_a) \rangle. \tag{7.1}$$

Corollary 7.5. *For fixed K, H, r, every metric* (7.1) *has an isometric representative on one of finitely many fixed models M_c as in Section 5, with $M_c = K/H$ when $r = 0$. For every smooth reference spectrum of dimension* $\dim(K/H) + r$*, a finite exact initial segment determines the full spectrum on the class with this fixed fibre and arbitrary commuting metric-preserving normalizer holonomy.*

Proof. For $r = 0$, the mapping torus is K/H and Theorem 3.1 gives the finite determining initial segment and compact positive representatives. Assume $r > 0$. By Theorem 7.4, $\tau_a = c'_a \exp X_a$. The quotient map $\mathrm{Lie}\, N_K(H) \to \mathrm{Lie}\, J$ restricts to an isomorphism on $\mathfrak{m}^H$. Identify X_a with the corresponding K-invariant vector field on Y. Put

$$F(y, z) = \left(y \exp\left(-\frac{1}{2\pi} \sum_a z_a X_a \right), z \right).$$

Commutation gives

$$F(y\tau_a, z + 2\pi e_a) = \big(F_1(y, z)c'_a, z + 2\pi e_a\big),$$

so F transports the deck tuple τ to c'. Each $\exp(tX_a)$ preserves A and every V_b, and $F_*(Z_a - V_a) = Z_a - (V_a + X_a/(2\pi))$. The transported metric has constant invariant Schur parameters $A, B, V_a + X_a/(2\pi)$. A constant normalizer conjugation puts c' in the fixed finite list. Denote the resulting parameters again by (A, B, V). The subgroup $q\mathbb{Z}^r$ of the deck group acts only by translations. Its quotient is $Y \times (\mathbb{R}/2\pi q\mathbb{Z})^r$. Rescale $z = qw$; this replaces B by B/q^2 and V by qV, and gives exactly (5.1). The residual deck group has order q^r because of its translation part. Apply Theorem 5.1 to the finite list and take the largest determining index. □

7.2. **Normalizer equations and descent.**

Lemma 7.6. *For a fixed closed $H \subset K$ there are finitely many irreducible representations π such that*

$$a \in N_K(H) \quad \Longleftrightarrow \quad \pi(a)V_\pi^H = V_\pi^H \quad \text{for all representations in this list.}$$

Proof. Suppose first that the fixed spaces are preserved in every irreducible representation. Right translations by a and a^{-1} preserve all right-H-invariant matrix coefficients. These coefficients are uniformly dense in $C(K/H)$, so both translations preserve $C(K/H)$. For $h \in H$ one obtains $f(xha) = f(xa)$ for every such function. Since these functions separate cosets, $a^{-1}ha \in H$; the inverse condition gives equality. The converse follows directly from normalizer membership.

Choose faithful unitary matrix coordinates on K. Every irreducible occurs in a tensor product of this representation and its dual: the coefficient algebra is self-adjoint and separates points, so it is dense by Stone–Weierstrass, and Peter–Weyl orthogonality excludes an absent irreducible. Thus every coefficient of $\pi(a)$ is polynomial in these coordinates and their conjugates. If P_π projects to V_π^H, the entries of $(I - P_\pi)\pi(a)P_\pi$ give real polynomial equations. By the finite-generation argument in the proof of Lemma 2.6, finitely many of these equations have the same zero set. They belong to finitely many representations. Unitarity makes inclusion of the fixed space equivalent to equality. □

Proposition 7.7. *Let K be compact, connected, and semisimple. Suppose $H_j \subset H \subset K$ are closed subgroups and, for every fixed π, $V_\pi^{H_j} = V_\pi^H$ eventually. Suppose metrics on mapping tori with fibre K/H_j and commuting metric-preserving normalizer holonomies match successively longer initial segments of Σ. On a tail their operators descend to metrics on mapping tori with fibre K/H, retaining successively longer initial segments. On each eventually identified fixed space, the holonomy operators are unchanged.*

Proof. Choose lifts $a_{b,j} \in N_K(H_j)$ of the holonomies. They preserve $V_\pi^{H_j}$. The finite list of Lemma 7.6 gives a common tail with $a_{b,j} \in N_K(H)$ for every b. Their commutators lie in $H_j \subset H$, so their images modulo H commute.

Let

$$E = \mathrm{Sym}^2(\mathfrak{k} \oplus \mathbb{R}^r),$$

with the adjoint action on $\mathfrak{k}$ and the trivial action on $\mathbb{R}^r$. This is one fixed finite-dimensional real K-representation. Its complexification is a finite direct sum of irreducible K-representations. The assumed eventual fixed-space equality in each of these finitely many irreducibles gives, for all sufficiently large j,

$$E^{H_j} = E^H. \tag{7.2}$$

Extend the quotient cometric by zero on $\mathfrak{h}_j$ with the Q-orthogonal decomposition of $\mathfrak{k}$, and denote the resulting tensor by $C_j \in E$. Both its horizontal part and its zero extension are H_j-invariant. Hence (7.2) gives $C_j \in E^H$ exactly.

The quadratic differential operator of C_j on $K \times \mathbb{R}^r$ therefore commutes with right translation by H. It preserves right-H-invariant smooth functions and descends to $K/H \times \mathbb{R}^r$. The descended principal symbol is

$$\overline{C}_j = C_j|_{\mathfrak{h}^{\perp_Q} \oplus \mathbb{R}^r}.$$

Since $\mathfrak{h}_j \subset \mathfrak{h}$, this is a restriction of the positive definite horizontal symbol for $K/H_j \times \mathbb{R}^r$ and is positive definite. Haar probability on K/H_j pushes forward to Haar probability on K/H; consequently the descended operator is symmetric for quotient Haar measure. It is real and annihilates constants. The invariant metric of $\overline{C}_j$ has volume density proportional to quotient Haar measure. Lemma 2.1 therefore identifies the descended operator with its scalar Laplacian. The original right holonomies preserve this descended operator, since they preserve the original operator and normalize H. They consequently preserve its principal symbol and are isometries of the descended metric.

For normalized invariant probability measures, pullback from the descended mapping torus identifies its L^2 space isometrically with a closed invariant subspace of the original one. It therefore cannot increase any spectral multiplicity. Fix a positive reference eigenvalue μ, of multiplicity m_μ,

and take j large enough that the matched segment includes the complete μ-cluster and the next eigenvalue. The μ-eigenspace is a finite-dimensional K-module. Every irreducible summand has $d_\pi \le m_\mu$; the set of such types is finite by Lemma 2.3. For every type in this finite set,

$$V_\pi^{H_j} = V_\pi^H$$

once j is sufficiently large. On these spaces the holonomy operators $\pi(a_{b,j})$ are identical before and after descent. Hence their joint eigenspaces and all corresponding Fourier frequencies agree. The complete K-isotypic spaces containing the μ-eigenspace are therefore contained in the descended function space. Its μ-multiplicity remains m_μ. For any fixed finite number of complete reference clusters, one common tail has these properties for all of them. Since restriction introduces no new eigenvalue, the descended spectra agree with successively longer initial segments of Σ. □

Proposition 7.8. *For fixed compact semisimple K, $r \ge 0$, reference dimension n, and smooth Σ of dimension n, some finite exact initial segment leaves only finitely many conjugacy classes of fibre subgroups $H \subset K$ among mapping tori with $\dim(K/H) + r = n$, including arbitrary commuting metric-preserving normalizer holonomies.*

Proof. Otherwise choose pairwise nonconjugate H_j and metrics matching successively longer initial segments. Yu's subsequence theorem [21, Theorem 1.1] gives, after conjugation, $H_j \subset H$ and convergence of their dimension data to that of this same H. Integer-valued convergence and inclusion give $V_\pi^{H_j} = V_\pi^H$ eventually for each fixed π. By Proposition 7.7, the spectra descend to fibre K/H with the same increasing exact initial segments. Its dimension plus r is at most n. If it is smaller and $r > 0$, Corollary 7.5 gives finitely many isometric finite-holonomy presentations of this fixed fibre. This finite list depends only on K, H, r and the decomposition in Theorem 7.4. Each presentation has dimension $\dim(K/H) + r < n$, so Lemma 5.8 excludes a sufficiently long reference initial segment on every member of this finite list. When $r = 0$, the semisimple dimension obstruction in Lemma 3.5 gives the same exclusion of sufficiently long reference initial segments. Hence the dimensions are equal. Then $\mathfrak{h}_j = \mathfrak{h}$ and $H^0 \subset H_j \subset H$. The finite group H/H^0 has finitely many subgroups, contradicting the choice of pairwise nonconjugate H_j. □

7.3. **All closed globally homogeneous manifolds.**

Lemma 7.9. *Every closed connected globally homogeneous Riemannian n-manifold has an isometric presentation* (7.1)*, where K is compact, simply connected, and semisimple,* $\dim K \le n(n+1)/2$*, and $0 \le r \le n$. The covering metric is $K \times \mathbb{R}^r$-invariant, and the holonomies are commuting metric-preserving elements of $N_K(H)/H$.*

Proof. Let $G = \mathrm{Isom}(M, g)^0$. The full compact isometry group has finitely many components. Its identity-component orbits in the connected homogeneous space form a finite collection of closed, hence open, subsets. Thus G is transitive. Its isotropy acts faithfully on a tangent space, so $\dim G \le n + n(n-1)/2$. The universal covering group is $\widetilde{G} = K \times \mathbb{R}^z$ with K compact simply connected semisimple. If $\widetilde{L}$ is the full inverse image of the isotropy, then $M = (K \times \mathbb{R}^z)/\widetilde{L}$. Projection $K \times \mathbb{R}^z \to \mathbb{R}^z$ is proper because K is compact. Thus $P = \mathrm{pr}_{\mathbb{R}^z} \widetilde{L}$ is closed. The map

$$(K \times \mathbb{R}^z)/\widetilde{L} \longrightarrow \mathbb{R}^z/P, \qquad (k, z)\widetilde{L} \longmapsto z + P,$$

is continuous and onto, so $\mathbb{R}^z/P$ is compact. The identity component $V = P^0$ is a vector subspace. For an auxiliary Euclidean inner product on $\mathbb{R}^z$, put $U = V^\perp$. The subgroup $\Lambda = P \cap U$ is discrete, $P = V \oplus \Lambda$, and compactness of $U/\Lambda \cong \mathbb{R}^z/P$ makes Λ a full lattice in U. For $v \in V$, $u \in U$, choose $(a, v) \in \widetilde{L}$. Then

$$(k, v + u)(a, v)^{-1} = (ka^{-1}, u) \in K \times U.$$

Thus every coset has a representative in $K \times U$, and this subgroup is transitive. Its isotropy $L' = \widetilde{L} \cap (K \times U)$ projects onto Λ with kernel $H = \widetilde{L} \cap K$. Choose a lattice basis and lifts $(a_b, \lambda_b) \in L'$.

Conjugation by (a_b, λ_b) preserves H, and $[(a_b, \lambda_b), (a_c, \lambda_c)] \in H$; hence $a_b \in N_K(H)$ and $[a_b, a_c] \in H$. The covering

$$(K \times U)/H \longrightarrow (K \times U)/L' = M$$

has deck transformations $(kH, u) \mapsto (ka_bH, u + \lambda_b)$. A linear change of coordinates with $\lambda_b = 2\pi e_b$ gives (7.1). The pullback metric is $K \times U$-invariant and each deck transformation is an isometry. Finally $n = \dim(K/H) + \dim U$ and $\dim K \leq \dim G$. □

Proof of Theorem A. By Lemma 7.9, every homogeneous n-manifold has a mapping-torus presentation with K compact simply connected semisimple, $\dim K \leq n(n+1)/2$ and $0 \leq r \leq n$. There are finitely many compact simply connected semisimple groups with dimension at most $n(n+1)/2$, by the classification of simple compact Lie algebras and the bound on the number of their summands. There are finitely many choices $0 \leq r \leq n$. For each K, r, Proposition 7.8 gives an initial segment leaving finitely many fibre isotropies. Take the largest of these cutoffs. For each admissible triple K, H, r with $r > 0$, Corollary 7.5 gives finitely many fixed finite-monodromy presentations, and Theorem 5.1 gives the required exact determining initial segment and compact positive representatives. For $r = 0$, Theorem 3.1 gives finite determination and compact positive representatives. Taking the maximum over this finite list gives an exact determining initial segment whose isospectral realization set has compact positive representatives. If a list is empty, its initial spectral segment excludes the corresponding part of the class.

Every full realization matches this initial segment. Any sequence therefore has representatives in one compact positive parameter set on a subsequence. The metrics converge smoothly, and eigenvalue continuity preserves Σ. The finite list of fixed manifolds gives finitely many diffeomorphism types.

For each compact representative set choose $0 < a < b$ such that every full cometric lies in $aI \leq C \leq bI$, and intersect this interval with the fixed linear space of invariant cometrics on its presentation. The resulting parameter set is compact, positive and semialgebraic. The Fourier and representation matrix blocks have polynomial entries in C; Lemmas 2.7 and 2.9 give indexed spectral continuity and finite spectral support on this interval. Lemma 2.8 therefore makes its full Σ-isospectral subset compact and semialgebraic. These subsets contain representatives of every realization, and every member has spectrum Σ. □

Corollary 7.10. *The homogeneous realization space of Σ is the continuous image, up to isometry, of finitely many compact semialgebraic sets of invariant positive cometrics on fixed compact manifolds. It has finitely many path components in the smooth quotient topology. Each realization can be joined isospectrally to one of finitely many representatives by a path which, on its fixed presentation, is piecewise real analytic. On each fixed compact semialgebraic parameter set, a fixed scalar-isospectral fibre has only finitely many representation-and-Fourier-marked spectra.*

Proof. Let $\boldsymbol{\mathcal{K}}_1, \dots, \boldsymbol{\mathcal{K}}_b$ be the compact semialgebraic sets of positive invariant cometrics given by Theorem A. The metric tensor depends smoothly on each positive cometric, so their maps to the smooth quotient by isometries are continuous. Each $\boldsymbol{\mathcal{K}}_i$ has finitely many connected components, and any two points in one component can be joined by a continuous semialgebraic path [6, Proposition 1.6 and Theorem 1.10]. Such a path is Nash on the open intervals of a finite partition. On each piece, choose a positive integer p divisible by the denominators of its endpoint Puiseux exponents. After rescaling the interval to $[0, 1]$, compose with

$$\rho(t) = \frac{t^p}{t^p + (1-t)^p}.$$

The resulting path is real analytic at both endpoints as well as on $(0, 1)$, since ρ vanishes to order p at 0 and $1 - \rho$ to order p at 1. Choose one point in each of the finitely many components of each $\boldsymbol{\mathcal{K}}_i$. Every parameter is joined to one of these points by such a path. Their images in the smooth quotient are path connected, and there are only finitely many of them. For each fixed matrix block, its ordered eigenvalues vary continuously in the discrete set Σ, and are constant on each connected component of

that compact semialgebraic set. The finite number of these components therefore bounds the number of complete marked spectra on that parameter set. □

Corollary 7.11. *For a sufficiently long exact reference initial segment there are constants $i_\Sigma > 0$, $D_\Sigma < \infty$, and $C_{\Sigma,k} < \infty$ such that all homogeneous realizations of that initial segment in the reference dimension satisfy*

$$\operatorname{inj}(g) \geq i_\Sigma, \qquad \operatorname{diam}(g) \leq D_\Sigma, \qquad \|\nabla^k \operatorname{Rm}_g\|_\infty \leq C_{\Sigma,k} \quad (k \geq 0).$$

Proof. By Theorem A, every realization is isometric to a metric in one of finitely many compact positive parameter sets on fixed closed manifolds. The diameter, curvature and injectivity estimates in the proof of Corollary 4.3 apply to each such set: their hypotheses are compactness of the metric parameters and smooth dependence on those parameters. Take the minimum of the positive injectivity bounds and the maxima of the other bounds over this finite family. These quantities are invariant under diffeomorphisms. □

Remark 7.12. The determining index in Theorem A depends on the full reference spectrum and the fixed dimension. To see the role of the dimension condition, let g be a homogeneous metric and let S^1_ℓ be a circle of length ℓ. If $4\pi^2/\ell^2 > \lambda_N(g)$, then

$$\lambda_j(g \times S^1_\ell) = \lambda_j(g) \quad (0 \leq j \leq N), \qquad \dim(M \times S^1_\ell) = \dim M + 1.$$

Indeed, the product eigenvalues are $\lambda_j(g) + 4\pi^2 m^2/\ell^2$, $m \in \mathbb{Z}$, and every term with $m \neq 0$ exceeds $\lambda_N(g)$.

8. Local metric estimates for naturally reductive metrics

Throughout this section, M is a smooth closed connected manifold of dimension $n > 0$. A metric g is *globally naturally reductive* if a connected Lie group A acts transitively by isometries and, for the stabilizer H of $o \in M$, there is an $\mathrm{Ad}(H)$-invariant decomposition $\mathfrak{a} = \mathfrak{h} \oplus \mathfrak{m}$ such that

$$\langle [X, Y]_\mathfrak{m}, Z\rangle + \langle Y, [X, Z]_\mathfrak{m}\rangle = 0, \qquad X, Y, Z \in \mathfrak{m}. \tag{8.1}$$

Here the metric on $\mathfrak{m}$ is the metric at o under the evaluation isomorphism. The isotropy subgroup H may be disconnected.

Write $\lambda_j(g)$ for the nonnegative scalar eigenvalues in order, including multiplicity, with $\lambda_0(g) = 0$, and set

$$\delta_N(g, g_0) = \max_{1 \leq j \leq N} |\lambda_j(g) - \lambda_j(g_0)|. \tag{8.2}$$

All tensor norms in this section are defined by the reference metric and its Levi–Civita connection.

Theorem 8.1 (Local estimate from a finite initial segment). *Let g_0 be a globally naturally reductive metric on M. There exist $N \geq 1$, an open C^∞ neighborhood $\mathcal{U}$ of the orbit $\operatorname{Diff}(M) \cdot g_0$, and constants $C_r > 0$, $r = 0, 1, \ldots$, with the following property. For every globally naturally reductive metric $g \in \mathcal{U}$, there is one smooth diffeomorphism ϕ such that, simultaneously for every $r \geq 0$,*

$$\|\phi^* g - g_0\|_{C^r(g_0)} \leq C_r \delta_N(g, g_0). \tag{8.3}$$

The presenting group, isotropy subgroup, and naturally reductive complement may vary.

The integer N, the neighborhood $\mathcal{U}$, and the constants C_r depend on g_0.

8.1. Central quadratic operators.

Definition 8.2. For a compact connected transitive group B acting on a closed manifold M, let

$$\mathcal{E}_B = \operatorname{image}\left((\operatorname{Sym}^2 \mathfrak{b})^B \longrightarrow \operatorname{Diff}^2(M)\right), \qquad q \longmapsto D_q = -\sum q_{ab} X^\#_a X^\#_b.$$

Here (X_a) is a basis of $\mathfrak{b}$, and $X^\#_a$ is its fundamental vector field. The tensor coefficients q_{ab} are taken in that basis. The symbol convention is $\sigma(-X^2) = X \otimes X$. Let $\Omega_B \subset \mathcal{E}_B$ be the open cone of positive definite principal symbols.

Lemma 8.3. *Every $D \in \Omega_B$ is the scalar Laplacian of a unique B-invariant metric $g(D)$. The map $D \mapsto g(D)$ is smooth. The principal symbol at one point determines D, so*

$$\dim \mathcal{E}_B \le \frac{n(n+1)}{2}.$$

Proof. Let μ be the invariant probability measure. The flow of each fundamental field $X_a^\#$ preserves μ, so $(X_a^\#)^* = -X_a^\#$. Since $q_{ab} = q_{ba}$,

$$D_q^* = -\sum_{a,b} q_{ab} X_b^\# X_a^\# = D_q, \qquad D_q 1 = 0.$$

The positive symbol defines a B-invariant cometric g^{-1}. Its normalized volume density is μ, by uniqueness of invariant probability measure on a compact homogeneous space. Thus D_q and Δ_g have the same principal symbol and are formally self-adjoint for the same measure.

Lemma 2.1 gives $D_q = \Delta_g$. Applied to an element of $\mathcal{E}_B$ with zero principal symbol and to the zero operator, it also proves that the symbol map is injective. A B-invariant tensor is determined by its value at one point. This gives the dimension bound, and inversion on the positive cone gives smoothness of $D \mapsto g(D)$. □

8.2. **Naturally reductive realizations.** For a compact connected transitive action B, let σ denote the principal-symbol map on $\mathcal{E}_B$, and put

$$W_B = \sigma(\mathcal{E}_B). \tag{8.4}$$

By Lemma 8.3, the symbol map is injective; W_B is a finite-dimensional closed subspace of continuous symmetric contravariant tensors. A positive $u \in W_B$ corresponds to the unique operator $\Delta_{u^{-1}} \in \mathcal{E}_B$. All these operators are real, formally self-adjoint for the same invariant probability measure, and annihilate constants. Their inducing tensors on $\mathfrak{b}$ need not be positive definite.

Lemma 8.4 (Central realization). *A metric is globally naturally reductive if and only if its Laplacian belongs to $\mathcal{E}_B$ for some compact connected transitive action B. For a given connected naturally reductive presenting group A, one may take B to be any compact connected group of isometries containing the image of A; in particular, $B = \mathrm{Isom}(M,g)^0$. Every such metric has a compact connected naturally reductive presentation. The inducing tensor on $\mathfrak{b}$ may be indefinite or degenerate.*

Proof. Let A/H be a globally naturally reductive presentation, with A connected. The kernel of the action is a closed normal subgroup contained in H. Dividing by it sends $\mathfrak{m}$ isomorphically to a naturally reductive complement in the quotient Lie algebra, with the same metric. Thus the action may be assumed effective. Let K be a compact connected group of isometries containing the image of A. Such a group exists: $\mathrm{Isom}(M,g)^0$ is compact and contains that image. Since A is transitive, so is K. The infinitesimal action embeds $\mathfrak{a}$ into $\mathfrak{k}$; identify $\mathfrak{a}$ with this image. The immersed subgroup $A \subset K$ may be nonclosed.

The isotropy representation is faithful. Indeed, an isometry fixing o with identity derivative fixes every geodesic from o, hence every point of the complete connected manifold M. The decomposition $\mathfrak{a} = \mathfrak{h} \oplus \mathfrak{m}$ is therefore effective. Kostant's theorem [15, Theorem 4], in the formulation of [19, Theorem 2.12], gives the ideal

$$\mathfrak{s} = [\mathfrak{m},\mathfrak{m}]_{\mathfrak{h}} \oplus \mathfrak{m} \subset \mathfrak{a}$$

and a nondegenerate $\mathrm{ad}(\mathfrak{s})$-invariant symmetric form Q on $\mathfrak{s}$ such that

$$Q|_{\mathfrak{m}} = g_o, \qquad \mathfrak{m} \perp_Q (\mathfrak{s} \cap \mathfrak{h}).$$

Regard $q_s = Q^{-1}$ as an element of $\mathrm{Sym}^2 \mathfrak{k}$ through $\mathfrak{s} \subset \mathfrak{a} \subset \mathfrak{k}$. The isotropy terms vanish in its symbol at o, and the remaining terms give g_o^{-1}.

Let $S \subset K$ be the connected immersed subgroup with Lie algebra $\mathfrak{s}$. Since $\mathfrak{s}$ is an ideal in $\mathfrak{a}$ and A is connected, $\mathrm{Ad}(a)\mathfrak{s} = \mathfrak{s}$ for $a \in A$. For $x = a \cdot o$, evaluation therefore gives

$$\mathrm{ev}_x(\mathfrak{s}) = da\big(\mathrm{ev}_o(\mathfrak{s})\big) = T_x M.$$

Every S-orbit is open; connectedness of M makes S transitive. The tensor q_s is S-invariant, so its principal symbol is g^{-1} on all of M. Each field in $\mathfrak{k}$ is divergence-free for the volume density of g. Hence D_{q_s} is formally self-adjoint and annihilates constants. Lemma 2.1 gives $D_{q_s} = \Delta_g$.

For Haar probability measure dk on K, put

$$q = \int_K \mathrm{Ad}(k) q_s \, dk. \tag{8.5}$$

The assignment $q \mapsto D_q$ is K-equivariant, and K preserves Δ_g. Consequently

$$q \in (\mathrm{Sym}^2 \mathfrak{k})^K, \qquad D_q = \Delta_g,$$

which proves $\Delta_g \in \mathcal{E}_K$.

Conversely suppose $q \in (\mathrm{Sym}^2 \mathfrak{b})^B$ has positive tangent symbol and write $M = B/H$. Let $\mathfrak{h}^0 \subset \mathfrak{b}^*$ be the annihilator of $\mathfrak{h}$, and put

$$\mathfrak{m} = q^\sharp(\mathfrak{h}^0), \qquad \langle q^\sharp \alpha, q^\sharp \beta \rangle = q(\alpha, \beta).$$

Positivity on $\mathfrak{h}^0$ shows that $q^\sharp$ is injective there. If $q^\sharp \alpha \in \mathfrak{h}$, then $q(\alpha, \alpha) = \alpha(q^\sharp \alpha) = 0$, so $\alpha = 0$. Dimension counting gives $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{m}$. Equivariance of $q^\sharp$ makes the complement and its metric $\mathrm{Ad}(H)$-invariant, including the disconnected components of H.

For $X = q^\sharp \alpha$, $Y = q^\sharp \beta$, and $Z = q^\sharp \gamma$, the metric pairing with Z agrees on $\mathfrak{m}$ with γ. Since γ annihilates $\mathfrak{h}$, the left side of (8.1) is

$$\gamma([X, q^\sharp \beta]) + \beta([X, q^\sharp \gamma]) = 0.$$

The equality is precisely infinitesimal invariance of q. The induced tangent cometric is the restriction of q to $\mathfrak{h}^0$, hence the original positive symbol. Therefore this is a naturally reductive presentation of g. □

8.3. **A fixed central action.** Fix a compact connected transitive action $B \curvearrowright M$ with $\Omega_B \neq \emptyset$. For $q \in (\mathrm{Sym}^2 \mathfrak{b})^B$, the symmetrized quadratic element $z_q \in U(\mathfrak{b}_{\mathbb{C}})$ defining D_q is central. On an isotypic summand $V_\pi \otimes \mathbb{C}^{a_\pi}$ it acts as $d\pi(z_q) \otimes I$. Schur's lemma gives $d\pi(z_q) = \ell_\pi(D_q) I$; formal self-adjointness makes this scalar real. Thus every occurring irreducible π determines a real linear functional ℓ_π on $\mathcal{E}_B$, with

$$D|_{L^2(M)_\pi} = \ell_\pi(D)\ \mathrm{id}\,. \tag{8.6}$$

Write m_π for the dimension of its full isotypic summand, including all copies. For $D \in \Omega_B$,

$$\mathrm{Spec}^+(D) = \bigsqcup_{\pi \neq \mathbf{1}} m_\pi \{\ell_\pi(D)\}. \tag{8.7}$$

For $M = B/H$, Peter–Weyl decomposition gives $m_\pi = (\dim \pi) \dim(V_\pi^H)$, with the dual representation if required by the coefficient convention. The trivial isotypic summand consists of the constants. Connectedness of M makes every other summand have positive eigenvalue for $D \in \Omega_B$.

The functionals ℓ_π, $\pi \neq \mathbf{1}$, separate $\mathcal{E}_B$. Indeed, if they all vanish on D, then D annihilates every B-finite function, including constants. Choose $D^0 \in \Omega_B$. By Lemma 8.3, it is the Laplacian of a smooth metric on M. Every eigenspace of D^0 is finite-dimensional and B-invariant, so its elements are B-finite. Lemma 2.5 and continuity of the differential operator $D : C^\infty(M) \to C^\infty(M)$ give $D = 0$. Since $\mathcal{E}_B$ is finite-dimensional, finitely many of these functionals form a basis of $\mathcal{E}_B^*$.

Proposition 8.5. *Fix a compact connected transitive action B, put $g(u) = u^{-1}$ for positive $u \in W_B$, and let g_0 be a metric with $D_0 = \Delta_{g_0} \in \mathcal{E}_B$. There exist $N_B \geq 1$, a C^0 neighborhood of g_0^{-1} in W_B, and constants $c_{B,r}$ such that every positive u in that neighborhood satisfies*

$$\|g(u) - g_0\|_{C^r(g_0)} \leq c_{B,r} \delta_{N_B}(g(u), g_0), \qquad r \geq 0. \tag{8.8}$$

The neighborhood is independent of r.

Proof. Let μ be the invariant probability measure. Let $d = \dim \mathcal{E}_B$. Since D_0 is a nonzero Laplacian, $d \geq 1$. Choose d nontrivial occurring types $\pi_1, \dots, \pi_d$ whose functionals form a basis of $\mathcal{E}_B^*$. Endow $\mathcal{E}_B$ with the norm $\|D\| = \|\sigma(D)\|_{C^0(g_0)}$, which is a norm by Lemma 8.3. Compactness of the unit sphere gives

$$\kappa_B = \min_{\|D\|=1} \max_{1\leq a\leq d} |\ell_{\pi_a}(D)| > 0, \qquad \|D\| \leq \kappa_B^{-1} \max_a |\ell_{\pi_a}(D)|. \tag{8.9}$$

Put $\mu_a = \ell_{\pi_a}(D_0) > 0$. List the distinct values among them as $\nu_1 < \cdots < \nu_t$. Put

$$I_b = \{j \geq 1 : \lambda_j(g_0) = \nu_b\} = \{p_b, \dots, q_b\}, \qquad N_B = q_t.$$

Choose disjoint intervals $J_b = (\alpha_b, \beta_b)$ with

$$\lambda_{p_b-1}(g_0) < \alpha_b < \nu_b < \beta_b < \lambda_{q_b+1}(g_0).$$

For $p_b = 1$, the preceding eigenvalue is $\lambda_0(g_0) = 0$.

If u is close to $u_0 = g_0^{-1}$ in C^0, then, for a prescribed sufficiently small $\varepsilon > 0$,

$$(1-\varepsilon)u_0 \leq u \leq (1+\varepsilon)u_0.$$

Both associated metrics have the same normalized volume density μ. Their Rayleigh quotients and the min–max principle therefore give the simultaneous bounds

$$(1-\varepsilon)\lambda_j(g_0) \leq \lambda_j(g(u)) \leq (1+\varepsilon)\lambda_j(g_0), \qquad j \geq 0. \tag{8.10}$$

Choose $\varepsilon > 0$ so small that, for every b,

$$(1+\varepsilon)\lambda_{p_b-1}(g_0) < \alpha_b < (1-\varepsilon)\nu_b,$$
$$(1+\varepsilon)\nu_b < \beta_b < (1-\varepsilon)\lambda_{q_b+1}(g_0).$$

All four inequalities are strict at $\varepsilon = 0$, so one positive ε satisfies them for the finite list of clusters. Equation (8.10) then puts every index in I_b inside J_b, every smaller index below J_b, and every larger index above J_b. Let $\|\ell_{\pi_a}\|_*$ be the dual norm for the operator norm in (8.9). Choose $\rho > 0$ such that

$$\rho \max_a \|\ell_{\pi_a}\|_* < \min_b\{\nu_b - \alpha_b, \beta_b - \nu_b\}.$$

For $\|D - D_0\| < \rho$, $|\ell_{\pi_a}(D) - \mu_a| \leq \|\ell_{\pi_a}\|_* \|D - D_0\|$. Consequently $\ell_{\pi_a}(D) \in J_b$ whenever $\mu_a = \nu_b$.

For such a positive $D \in \mathcal{E}_B$, the number $\ell_{\pi_a}(D)$ is an actual eigenvalue belonging to the block I_b. All the reference eigenvalues in that block equal μ_a. Consequently

$$|\ell_{\pi_a}(D - D_0)| \leq \max_{j\in I_b} |\lambda_j(g(u)) - \lambda_j(g_0)| \leq \delta_{N_B}(g(u), g_0). \tag{8.11}$$

Combining (8.9) and (8.11) gives

$$\|u - u_0\|_{C^0(g_0)} \leq \kappa_B^{-1} \delta_{N_B}(g(u), g_0). \tag{8.12}$$

Choose a closed finite-dimensional ball $\mathcal{B} \subset W_B$ about u_0, small enough that every element is uniformly positive and all the preceding cluster conditions hold. For each r, inversion $u \mapsto u^{-1}$ is a smooth map from a neighborhood of $\mathcal{B}$ to $C^r(\mathrm{Sym}^2\, T^*M)$. Its derivative is

$$d(u \mapsto u^{-1})_u[v] = -u^{-1} v u^{-1}.$$

The operator norm of this derivative, for the C^0 norm on the finite-dimensional domain and the C^r norm on the target, has a finite bound on $\mathcal{B}$. This follows from finite-dimensional norm equivalence and compactness of $\mathcal{B}$; all of its elements and their derivatives are bounded in every fixed order. Integrating the derivative along the segment from u_0 to u gives

$$\|u^{-1} - u_0^{-1}\|_{C^r(g_0)} \leq L_{B,r} \|u - u_0\|_{C^0(g_0)}.$$

Together with (8.12), this proves (8.8) on the interior of $\mathcal{B}$, for every $r \geq 0$. The ball is independent of r. □

8.4. **Semisimple subgroups and central enlargements.**

Lemma 8.6. *Let S be a compact connected semisimple Lie group and J a compact Lie group. The space* $\mathrm{Hom}(S, J)$ *of continuous homomorphisms is compact in the uniform topology, and it has finitely many J-conjugacy classes.*

Proof. If $J^0 = \{e\}$, connectedness of S makes every homomorphism trivial, so $\mathrm{Hom}(S, J)$ is a singleton. Otherwise choose a faithful unitary representation $\iota : J \hookrightarrow U(N)$ and a positive invariant inner product Q_S on $\mathfrak{s}$. Continuous homomorphisms between Lie groups are smooth. Every $\iota \circ \phi$, $\phi \in \mathrm{Hom}(S, J)$, is a direct sum of irreducible representations of dimension at most N. There are only finitely many such irreducibles by Lemma 2.3. If c_* is the maximum of their Q_S-Casimir eigenvalues, the Casimir estimate gives

$$\|d(\iota \circ \phi)(X)\|_{\mathrm{op}} \le \sqrt{c_*}\, \|X\|_{Q_S} \qquad (X \in \mathfrak{s}).$$

The Hilbert–Schmidt norm is at most $\sqrt{N}$ times this norm. Give J^0 the metric induced by the unitary embedding. Since S is connected, every $\phi(S)$ lies in J^0, and the displayed estimate gives a common Lipschitz constant for all ϕ. Arzelà–Ascoli therefore gives a uniformly convergent subsequence of every sequence in $\mathrm{Hom}(S, J)$. The uniform limit is continuous and satisfies the homomorphism identity. Thus $\mathrm{Hom}(S, J)$ is compact.

By Lemma 7.2, every conjugacy class in this compact space is open. A finite subcover by these classes proves finiteness. □

Corollary 8.7 (Gordon–Sutton). *For a compact Lie group J, only finitely many J-conjugacy classes of subgroups ST occur when $S \subset J$ is connected semisimple, possibly trivial, and T is a maximal torus of* $C_J(S)$ [12, Proposition 3.6(2)].

Proof. The classification of compact simple Lie algebras leaves only finitely many simply connected compact semisimple groups of dimension at most $\dim J$. Every connected semisimple subgroup of J is the image of a homomorphism from one of these groups. The preceding lemma therefore gives finitely many conjugacy classes of such S. For each fixed S, all maximal tori of $C_J(S)^0$ are conjugate in $C_J(S)^0$. Conjugating by this centralizer fixes S and conjugates the products ST. The case $S = \{e\}$ is the conjugacy theorem for maximal tori of J^0. □

8.5. **Finitely many actions near a reference metric.** Let $\mathrm{Met}(M)$ denote the space of smooth Riemannian metrics with its C^∞ topology. Fix $g_0 \in \mathrm{Met}(M)$, and put $K = \mathrm{Isom}(M, g_0)$. The slice theorem gives an open neighborhood $\mathcal{V}$ of g_0 and a continuous map $s : \mathcal{V} \to \mathrm{Met}(M)$ such that

$$s(g_0) = g_0, \qquad s(g) = \psi_g^* g \quad \text{for some } \psi_g \in \mathrm{Diff}(M),$$

and

$$\mathrm{Isom}(M, s(g)) \subset K. \tag{8.13}$$

Here every $s(g)$ and ψ_g is smooth; continuity concerns the C^∞ topology. This form follows from the local product coordinates and the isotropy inclusion in [5, Theorem B and Proposition 4.1]; see also [8, Theorems 7.4 and 8.1]. Shrinking $\mathcal{V}$ makes $s(g)$ lie in any prescribed smooth neighborhood of g_0. In particular, $g_j \to g_0$ implies $s(g_j) \to g_0$.

Lemma 8.8. *Let g_0 be a smooth metric on M, and put $K = \mathrm{Isom}(M, g_0)$. There is a finite list, possibly empty, of compact connected transitive subgroups $B_1, \dots, B_s \subset K^0$ with the following property. For every smooth neighborhood $\mathcal{V}$ of g_0, there is a smooth neighborhood $\mathcal{U}$ of g_0 such that each globally naturally reductive $g \in \mathcal{U}$ admits a diffeomorphism ϕ satisfying*

$$\phi^* g \in \mathcal{V}, \qquad \Delta_{\phi^* g} \in \mathcal{E}_{B_i} \quad \textit{for some } 1 \le i \le s.$$

The diffeomorphism ϕ can be taken as the composition of a slice pullback and an isometry of g_0.

Proof. Choose $\widehat{g} = s(g)$ as in (8.13), and put $A = \mathrm{Isom}(M, \widehat{g})^0 \subset K^0$. This compact connected group is transitive. By Lemma 8.4, there is $q \in (\mathrm{Sym}^2 \mathfrak{a})^A$ with $D_q = \Delta_{\widehat{g}}$.

Set $S = [A, A]$, a compact connected semisimple group, allowing $S = \{e\}$. Its centralizer in K^0 contains the torus $Z(A)^0$. Choose a maximal torus T of that centralizer containing $Z(A)^0$, and set $B = ST$. Because S and T commute and are compact and connected, B is a compact connected subgroup. It contains $A = SZ(A)^0$ and therefore acts transitively.

Every element of T commutes with S and with its subtorus $Z(A)^0$. Since $A = SZ(A)^0$, the group T centralizes A and acts trivially on $\mathfrak{a}$. It therefore fixes q. The tensor q is also S-invariant because it is A-invariant. Consequently

$$q \in (\mathrm{Sym}^2 \mathfrak{b})^B, \qquad \Delta_{\widehat{g}} = D_q \in \mathcal{E}_B.$$

Its principal symbol is B-invariant, so B acts by isometries of $\widehat{g}$.

Apply the finiteness of conjugacy classes of ST to the compact connected group K^0, and choose representatives for those groups ST which act transitively. A conjugation in K^0 puts B into one representative B_i. Pulling $\widehat{g}$ back by the corresponding reference isometry puts its Laplacian in $\mathcal{E}_{B_i}$. Such an isometry preserves every $C^r(g_0)$ norm of the difference from g_0. Choose a neighborhood of g_0 defined by finitely many such norm bounds and contained in $\mathcal{V}$. Continuity of the slice map makes $\widehat{g}$ belong to this smaller neighborhood when g is sufficiently close to g_0. The conjugation preserves these inequalities. □

Proof of Theorem 8.1. Let $B_1, \dots, B_s$ be the groups in Lemma 8.8, and put

$$u_0 = g_0^{-1}, \qquad I = \{i : u_0 \in W_{B_i}\}.$$

Each W_{B_i} is finite-dimensional and hence closed in the C^0 norm. For $i \notin I$, the distance from u_0 to W_{B_i} is positive. There is therefore a C^0 ball $\mathcal{B}_0$ about u_0 disjoint from every W_{B_i} with $i \notin I$. The set I is nonempty: apply the construction of Lemma 8.8 to g_0 itself. Its slice pullback is g_0, and the subsequent conjugation is an isometry of g_0.

For $i \in I$, Lemma 8.3 gives $\Delta_{g_0} \in \mathcal{E}_{B_i}$. Proposition 8.5 provides an integer N_i, constants $c_{B_i,r}$, and a neighborhood $\mathcal{B}_i$ of u_0 in W_{B_i} on which its estimate holds. Choose a C^0 ball $\mathcal{B} \subset \mathcal{B}_0$ about u_0 such that $\mathcal{B} \cap W_{B_i} \subset \mathcal{B}_i$ for every $i \in I$. By Lemma 8.8 and continuity of inversion, there is a smooth neighborhood $\mathcal{U}_0$ of g_0 such that every globally naturally reductive $g \in \mathcal{U}_0$ has a pullback $\phi^* g$ with cometric in $\mathcal{B} \cap W_{B_i}$ for some $i \in I$. Put

$$N = \max_{i \in I} N_i, \qquad C_r = \max_{i \in I} c_{B_i,r}.$$

The estimate for B_i and $\delta_{N_i} \le \delta_N$ give (8.3). The diffeomorphism ϕ and the ball $\mathcal{B}$ are independent of r, so the same pullback satisfies all the inequalities.

Let $\mathcal{U}$ consist of metrics g for which $\psi^* g \in \mathcal{U}_0$ for some $\psi \in \mathrm{Diff}(M)$. Every pullback is a homeomorphism of $\mathrm{Met}(M)$, so $\mathcal{U}$ is open and contains the orbit of g_0. Apply the preceding estimate to $\psi^* g$. Since $(\psi \circ \phi)^* g = \phi^*(\psi^* g)$ and eigenvalues are invariant under pullback, the same N and C_r give the asserted estimates on $\mathcal{U}$. □

8.6. **Smooth limits and finite isospectral families.**

Corollary 8.9 (Local isolation from a finite initial segment). *Let $\mathcal{U}$ and N be as in Theorem 8.1. For every globally naturally reductive metric $g \in \mathcal{U}$, equality of the first N positive eigenvalues of g and g_0 implies that the two metrics are isometric.*

Proof. The right side of (8.3) is zero. Its $r = 0$ case gives $\phi^* g = g_0$. □

Proposition 8.10 (Smooth closure). *Suppose globally naturally reductive metrics g_j on a fixed closed connected M converge smoothly to a positive definite metric g_∞. Then g_∞ is globally naturally reductive.*

Proof. At g_∞, the slice map gives $\widehat{g}_j = s(g_j) \to g_\infty$ and $\mathrm{Isom}(M, \widehat{g}_j) \subset K = \mathrm{Isom}(M, g_\infty)$. The construction in Lemma 8.8 gives finitely many groups $B_i \subset K^0$ and elements $k_j \in K^0$ such that $\Delta_{k_j^* \widehat{g}_j} \in \mathcal{E}_{B_{i(j)}}$. For every r,

$$\|k_j^* \widehat{g}_j - g_\infty\|_{C^r(g_\infty)} = \|\widehat{g}_j - g_\infty\|_{C^r(g_\infty)} \longrightarrow 0.$$

Pass to a subsequence with $i(j) = i$. Its cometrics belong to the closed finite-dimensional space W_{B_i} and converge to g_∞^{-1}, so $g_\infty^{-1} \in W_{B_i}$. By Lemma 8.3, $\Delta_{g_\infty} \in \mathcal{E}_{B_i}$. Lemma 8.4 gives a compact naturally reductive presentation of g_∞. □

Corollary 8.11 (Finiteness under smooth precompactness). *Let $\mathscr{F}$ be a family of closed connected globally naturally reductive Riemannian manifolds with one fixed scalar spectrum. Assume it is smoothly precompact modulo diffeomorphism in the following sense: every sequence in $\mathscr{F}$ has a subsequence that, after pullback to one closed manifold, converges smoothly to a positive definite metric. Then $\mathscr{F}$ contains only finitely many isometry classes.*

Proof. If there were infinitely many classes, choose pairwise nonisometric members. Smooth precompactness gives a convergent subsequence after pullback to a fixed manifold. Their smooth limit is globally naturally reductive by Proposition 8.10. Continuity of each scalar eigenvalue under smooth metric convergence makes the limit isospectral to every member of the sequence. For all sufficiently large indices, Theorem 8.1 at the limit applies with zero initial segment error. It makes those members isometric to the limit, a contradiction. □

Corollary 8.12 (Naturally reductive finiteness). *Fix $n > 0$ and the scalar spectrum $\Sigma = (\sigma_j)_{j \geq 0}$ of a closed connected smooth n-manifold. There is an integer $N = N(n, \Sigma)$ such that the globally naturally reductive n-manifolds satisfying*

$$\lambda_j = \sigma_j \qquad (1 \leq j \leq N)$$

form finitely many isometry classes. They are exactly the globally naturally reductive realizations of Σ. Their topology, transitive presentation, isotropy, and centre rank may vary.

Proof. Theorem A gives an integer N for which initial agreement is equivalent to full isospectrality in the homogeneous class. It also makes the full realization family smoothly precompact modulo diffeomorphisms, with positive definite limits. Apply Corollary 8.11 to its globally naturally reductive members. □

9. Relative spectral convergence with curvature divergence

Let Q_H be the unit round metric on $\mathrm{SU}(2) = S^3$, let $S^1 = \mathbb{R}/2\pi\mathbb{Z}$, and put $Q = Q_H + d\theta^2$.

Proposition 9.1. *Let $\delta_i > 0$ tend to zero and let $A_i > 0$ tend to infinity. There are left-invariant metrics g_i on $\mathrm{SU}(2) \times S^1$ and diffeomorphisms F_i such that*

$$\mathrm{Vol}(g_i) = \mathrm{Vol}(Q), \qquad d_{\mathrm{sp}}(g_i, Q) < \delta_i, \qquad F_i^* g_i \longrightarrow Q \quad \text{in } C^0,$$

whereas $\mathrm{Scal}(g_i) < -A_i$ everywhere. No subsequence converges in C^2 to a positive definite metric modulo diffeomorphisms.

Proof. Choose a Q_H-orthonormal basis X, Y, Z of $\mathfrak{su}(2)$ with

$$[X, Y] = 2Z, \qquad [Y, Z] = 2X, \qquad [Z, X] = 2Y, \qquad \exp(2\pi Z) = e.$$

Let $T = \partial_\theta$ be the unit generator of the circle. For an integer $p \geq 1$ and $0 < \varepsilon < 1$, define $g_{p,\varepsilon}$ by declaring

$$E_1 = \sqrt{1+\varepsilon}\, X, \qquad E_2 = \sqrt{1-\varepsilon}\, Y, \qquad E_3 = Z, \qquad E_4 = T + pZ \tag{9.1}$$

orthonormal. Its cometric in the basis (X, Y, Z, T) is

$$g_{p,\varepsilon}^{-1} = \begin{pmatrix} 1+\varepsilon & 0 & 0 & 0 \\ 0 & 1-\varepsilon & 0 & 0 \\ 0 & 0 & 1+p^2 & p \\ 0 & 0 & p & 1 \end{pmatrix}, \qquad \det(g_{p,\varepsilon}^{-1}) = 1-\varepsilon^2.$$

Consequently

$$\mathrm{Vol}(g_{p,\varepsilon}) = (1-\varepsilon^2)^{-1/2}\,\mathrm{Vol}(Q).$$

Set

$$c_\varepsilon = (1-\varepsilon^2)^{1/4}, \qquad \widehat{g}_{p,\varepsilon} = c_\varepsilon g_{p,\varepsilon}. \tag{9.2}$$

In dimension four this scaling multiplies volume by c_ε^2, so $\mathrm{Vol}(\widehat{g}_{p,\varepsilon}) = \mathrm{Vol}(Q)$.

The integral shear

$$F_p(x,\theta) = (x\exp(p\theta Z), \theta) \tag{9.3}$$

is a well-defined diffeomorphism because $\exp(2\pi pZ) = e$. In left trivialization,

$$dF_p(T) = T + pZ, \qquad dF_p(U) = \mathrm{Ad}(\exp(-p\theta Z))U \quad (U \in \mathfrak{su}(2)).$$

Thus $F_p^* g_{p,\varepsilon}$ has no mixed circle terms. Its horizontal cometric is an orthogonal conjugate of $\mathrm{diag}(1+\varepsilon, 1-\varepsilon, 1)$. It follows that

$$\frac{1-\varepsilon}{c_\varepsilon} Q^{-1} \le (F_p^* \widehat{g}_{p,\varepsilon})^{-1} \le \frac{1+\varepsilon}{c_\varepsilon} Q^{-1}. \tag{9.4}$$

The diffeomorphism F_p preserves product Haar measure, by right invariance on each SU(2) fibre. The volume measures of $\widehat{g}_{p,\varepsilon}$ and Q agree because both are left-invariant and have the same total mass. Hence the two Dirichlet forms in (9.4) have the same L^2 measure. The min–max principle gives, for every $j \ge 1$,

$$\frac{1-\varepsilon}{(1-\varepsilon^2)^{1/4}} \le \frac{\lambda_j(\widehat{g}_{p,\varepsilon})}{\lambda_j(Q)} \le \frac{1+\varepsilon}{(1-\varepsilon^2)^{1/4}}. \tag{9.5}$$

The lower ratio in (9.5) is less than one, and the upper ratio is greater than one. The absolute value of the logarithm of the lower ratio exceeds the logarithm of the upper ratio by $-\frac{1}{2}\log(1-\varepsilon^2) > 0$. Therefore

$$d_{\mathrm{sp}}(\widehat{g}_{p,\varepsilon}, Q) \le b_{\mathrm{sp}}(\varepsilon) := -\frac{3}{4}\log(1-\varepsilon) + \frac{1}{4}\log(1+\varepsilon) = \varepsilon + O(\varepsilon^2). \tag{9.6}$$

Inverting (9.4) also gives

$$\frac{c_\varepsilon}{1+\varepsilon} Q \le F_p^* \widehat{g}_{p,\varepsilon} \le \frac{c_\varepsilon}{1-\varepsilon} Q.$$

Thus these pullbacks converge to Q in C^0 as $\varepsilon \to 0$, with no restriction on p.

Put $a = \sqrt{1+\varepsilon}$ and $b = \sqrt{1-\varepsilon}$. The nonzero brackets in (9.1), up to skew-symmetry, are

$$\begin{aligned} &[E_1, E_2] = 2ab E_3, \quad [E_2, E_3] = \frac{2b}{a} E_1, \qquad [E_3, E_1] = \frac{2a}{b} E_2, \\ &[E_4, E_1] = \frac{2pa}{b} E_2, \quad [E_4, E_2] = -\frac{2pb}{a} E_1. \end{aligned} \tag{9.7}$$

Write $[E_i, E_j] = \sum_k f_{ij}^k E_k$. With the curvature convention $R(U,V)W = \nabla_U \nabla_V W - \nabla_V \nabla_U W - \nabla_{[U,V]} W$, Koszul's formula gives

$$\Gamma_{ij}^k = \langle \nabla_{E_i} E_j, E_k \rangle = \frac{1}{2}(f_{ij}^k - f_{jk}^i + f_{ki}^j).$$

Put

$$\alpha = 2ab, \quad \beta = 2b/a, \quad \gamma = 2a/b, \qquad u = \frac{\alpha - \beta + \gamma}{2}, \quad v = \frac{\beta - \gamma + \alpha}{2}, \quad w = \frac{\gamma - \alpha + \beta}{2},$$

and put $d = (\beta - \gamma)/2$, $s = (\beta + \gamma)/2$. Substitution of (9.7) gives $\nabla_{E_i} E_i = 0$ for every i and

$$\begin{aligned} &\nabla_{E_1} E_2 = uE_3 - pdE_4, && \nabla_{E_2} E_1 = -vE_3 - pdE_4, \\ &\nabla_{E_1} E_3 = -uE_2, && \nabla_{E_2} E_3 = vE_1, \\ &\nabla_{E_3} E_1 = wE_2, && \nabla_{E_3} E_2 = -wE_1, \\ &\nabla_{E_1} E_4 = pdE_2, && \nabla_{E_2} E_4 = pdE_1, \\ &\nabla_{E_4} E_1 = psE_2, && \nabla_{E_4} E_2 = -psE_1. \end{aligned} \tag{9.8}$$

All omitted derivatives are zero. For $K_{ij} = \langle R(E_i, E_j)E_j, E_i\rangle$, the coefficients are constant, so

$$K_{ij} = -\sum_m \Gamma_{ij}^m \Gamma_{jm}^i - \sum_m f_{ij}^m \Gamma_{mj}^i.$$

Equations (9.7) and (9.8) give

$$\begin{aligned} &K_{12} = \alpha w - uv + p^2 d^2, && K_{13} = \gamma v - uw, && K_{23} = \beta u - vw, \\ &K_{14} = p^2 d(s + \gamma), && K_{24} = -p^2 d(s + \beta), && K_{34} = 0. \end{aligned} \tag{9.9}$$

For example, $-\nabla_{E_2}(uE_3 - pdE_4) - \alpha \nabla_{E_3} E_2 = (\alpha w - uv + p^2 d^2)E_1$ is the E_1 component of $R(E_1, E_2)E_2$. Since $s + \gamma - (s + \beta) = -2d$, summation gives

$$\begin{aligned} \operatorname{Scal}(g_{p,\varepsilon}) &= 2 \sum_{i<j} K_{ij} \\ &= 2(\alpha w + \gamma v + \beta u - uv - uw - vw) - 2p^2 d^2 \\ &= \alpha\beta + \beta\gamma + \gamma\alpha - \tfrac{1}{2}(\alpha^2 + \beta^2 + \gamma^2) - \tfrac{1}{2} p^2 (\beta - \gamma)^2. \end{aligned} \tag{9.10}$$

This is also the scalar-curvature formula in [17] for the brackets (9.7). Here

$$\alpha\beta + \beta\gamma + \gamma\alpha = 12, \qquad \alpha^2 = 4(1 - \varepsilon^2), \qquad \beta^2 + \gamma^2 = \frac{8(1 + \varepsilon^2)}{1 - \varepsilon^2}, \qquad (\beta - \gamma)^2 = \frac{16\varepsilon^2}{1 - \varepsilon^2}.$$

Thus

$$\operatorname{Scal}(g_{p,\varepsilon}) = \frac{6 - 12\varepsilon^2 - 2\varepsilon^4 - 8p^2\varepsilon^2}{1 - \varepsilon^2}.$$

Scalar curvature scales by $\operatorname{Scal}(cg) = c^{-1} \operatorname{Scal}(g)$. Hence (9.2) gives

$$\operatorname{Scal}(\widehat{g}_{p,\varepsilon}) = \frac{6 - 12\varepsilon^2 - 2\varepsilon^4 - 8p^2\varepsilon^2}{(1 - \varepsilon^2)^{5/4}}. \tag{9.11}$$

Choose $0 < \varepsilon_i < 1/2$ with $\varepsilon_i \to 0$ and $b_{\mathrm{sp}}(\varepsilon_i) < \delta_i$. Choose an integer $p_i \geq 1$ such that $8p_i^2\varepsilon_i^2 > 6 + A_i$. Set $g_i = \widehat{g}_{p_i,\varepsilon_i}$ and $F_i = F_{p_i}$. Equations (9.6) and (9.11) give all the asserted bounds. Scalar curvature is invariant under pullback and continuous under C^2 convergence to a positive definite metric. Its uniform divergence to $-\infty$ excludes such convergence after every choice of diffeomorphisms. □

References

[1] I. Agricola and J. Henkel, *The Laplace–Beltrami spectrum on naturally reductive homogeneous spaces*, arXiv:2503.21416v2, revised 13 May 2026.

[2] N. Bourbaki, *Lie Groups and Lie Algebras, Chapters 4–6*, Elements of Mathematics, Springer, Berlin, 2002.

[3] N. Bourbaki, *Lie Groups and Lie Algebras, Chapters 7–9*, Elements of Mathematics, Springer, Berlin, 2005.

[4] T. Bröcker and T. tom Dieck, *Representations of Compact Lie Groups*, Graduate Texts in Mathematics, vol. 98, Springer, New York, 1985.

[5] D. Corro and J.-B. Kordaß, *Short survey on the existence of slices for the space of Riemannian metrics*, in Mexican Mathematicians in the World: Trends and Recent Contributions, Contemp. Math., vol. 775, American Mathematical Society, 2021, 65–84, doi:10.1090/conm/775/15589; arXiv:1904.07031v2.

[6] M. Coste, *Real Algebraic Sets*, lecture notes, 23 March 2005, `https://indico.ictp.it/event/a04204/session/10/contribution/7/material/0/0.pdf`.

[7] X. Dai and G. Wei, *Finite part of spectrum and isospectrality*, in Geometry of the Spectrum, Contemp. Math., vol. 173, American Mathematical Society, Providence, RI, 1994, 99–107, doi:10.1090/conm/173/01818.
[8] D. G. Ebin, *The manifold of Riemannian metrics*, in Global Analysis (Berkeley, 1968), Proc. Sympos. Pure Math., vol. 15, American Mathematical Society, Providence, RI, 1970, 11–40.
[9] K. Fan, *On a theorem of Weyl concerning eigenvalues of linear transformations. I*, Proc. Natl. Acad. Sci. USA **35** (1949), 652–655, doi:10.1073/pnas.35.11.652.
[10] P. B. Gilkey, *Invariance Theory, the Heat Equation, and the Atiyah–Singer Index Theorem*, Publish or Perish, Wilmington, DE, 1984; electronic reprint, 1996, `https://pages.uoregon.edu/gilkey/dirPDF/InvarianceTheory1Ed.pdf`.
[11] C. S. Gordon, D. Schueth, and C. J. Sutton, *Spectral isolation of bi-invariant metrics on compact Lie groups*, Ann. Inst. Fourier **60** (2010), no. 5, 1617–1628.
[12] C. S. Gordon and C. J. Sutton, *Spectral isolation of naturally reductive metrics on simple Lie groups*, Math. Z. **266** (2010), 979–995.
[13] R. M. Hardt, *Semi-algebraic local-triviality in semi-algebraic mappings*, Amer. J. Math. **102** (1980), 291–302, doi:10.2307/2374240.
[14] L. Hörmander, *The spectral function of an elliptic operator*, Acta Math. **121** (1968), 193–218, doi:10.1007/BF02391913.
[15] B. Kostant, *On differential geometry and homogeneous spaces. II*, Proc. Natl. Acad. Sci. USA **42** (1956), no. 6, 354–357, doi:10.1073/pnas.42.6.354.
[16] Y. Li, Y. Wu, and J. Zhou, *Shortest closed geodesics on three-dimensional isospectral manifolds*, preprint, arXiv:2609.28380v1, 23 September 2026.
[17] J. Milnor, *Curvatures of left invariant metrics on Lie groups*, Adv. Math. **21** (1976), 293–329.
[18] D. Schueth, *Isospectral manifolds with different local geometries*, J. Reine Angew. Math. **534** (2001), 41–94; arXiv:math/0102123.
[19] R. Storm, *Structure theory of naturally reductive spaces*, Differential Geom. Appl. **64** (2019), 174–200.
[20] C. J. Sutton, *Isospectral simply-connected homogeneous spaces and the spectral rigidity of group actions*, Comment. Math. Helv. **77** (2002), 701–717, doi:10.1007/PL00012438.
[21] J. Yu, *A compactness result for dimension datum*, Int. Math. Res. Not. IMRN (2015), no. 19, 9438–9449, doi:10.1093/imrn/rnu229.
[22] J. Yu, *Some new results on dimension datum*, Bull. London Math. Soc. **54** (2022), no. 3, 929–948, doi:10.1112/blms.12605; arXiv:2008.00378v2, 5 February 2021.