EINSTEIN LINK SPECTRA AND THE MORSE INDEX OF RICCI-FLAT CONES

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Abstract. We study the low transverse-traceless spectrum of positive Einstein manifolds and its effect on the Morse index of Ricci-flat cones. Free quaternionic quotients of four equal round spheres have exactly determined first scalar and tensor eigenvalues. For four round two-spheres, the quotient cone is nine-dimensional, is stable and has restricted holonomy $SO(9)$, whereas its eightfold cover has infinite negative index. For a general Einstein link, its invariant eigenspaces below the Hardy threshold determine the exact index on cone annuli and the leading logarithmic accumulation of negative eigenvalues on complete Ricci-flat manifolds with weighted C^2 convergence to the cone. No convergence rate is required for the leading term. At the threshold, an error of order $o((\log r)^{-2})$ makes cone stability equivalent to finite index. A pointwise comparison for canonically conformally Kähler Einstein four-manifolds gives, for the Chen–LeBrun–Weber metric at Einstein constant 3, the inequality $207/100 < \sup D_k(s^2) < 21/10$, where $g = s^{-2}k$, $s = \mathrm{Scal}_k > 0$ and $D_k = \mathrm{div}_k \nabla$. This proves the Hall–Haslhofer–Siepmann derivative inequality and improves the low tensor eigenvalue bound. Applications include an explicit leading coefficient for the complete Ricci-flat Böhm metrics, an index formula on long Einstein necks, and Euclidean rigidity for finite-index five-dimensional fillings under a uniform half-Weyl eigenvalue inequality.

Contents

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1. Introduction

The stability of a Ricci-flat cone is governed by a finite part of the tensor spectrum of its Einstein link. A finite Riemannian quotient preserves the curvature tensor locally, but retains only the invariant eigenspaces. We study the resulting change of Morse index. The same invariant eigenspaces determine the index on cone annuli and the leading accumulation of negative eigenvalues on complete Ricci-flat ends. Under uniform convergence on long conical annuli, they also give a local index asymptotic and a lower bound for the global Einstein index.

Two classes of links give contrasting applications. For conformally Kähler Einstein four-manifolds, a pointwise comparison produces tensor eigenspaces below the cone threshold. For products of round spheres, a transitive permutation of the factors can eliminate all such invariant eigenspaces. An explicit quaternionic quotient gives a stable cone whose eightfold cover has infinite negative index. The general spectral theorem and the pointwise comparison are developed separately; the neck and filling statements are their geometric consequences. All manifolds and metrics are smooth; closed means compact without boundary. Unless stated otherwise, manifolds are connected. On an Einstein manifold with $\mathrm{Ric}_g = \Lambda g$, put

$$\Delta_g = -\operatorname{div}_g \nabla, \qquad (\delta_g T)_j = -\nabla^i T_{ij}, \qquad L_g = \nabla^*\nabla + 2\Lambda I - 2\mathring{R}_g, \qquad E_g = L_g - 2\Lambda I. \tag{1.1}$$

We use $R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$ and $(\mathring{R}T)(X,Y) = \sum_i T(R(e_i,X)Y, e_i)$. Thus round spheres have positive sectional curvature and $\mathring{R}g = \mathrm{Ric}_g$. The full Lichnerowicz operator is L_g; E_g is the Einstein operator. A symmetric covariant tensor is transverse-traceless, or TT, if $\mathrm{tr}_g\, T = 0$ and $\delta_g T = 0$. Eigenvalues are counted with multiplicity. Integrals on unoriented manifolds use the Riemannian density.

For a quadratic form q, let $\mathrm{ind}^-(q)$ be the supremum of the dimensions of subspaces on which q is negative definite. On a Ricci-flat manifold put

$$\mathfrak{q}_G(T) = \int \left(|\nabla T|_G^2 - 2\langle \mathring{R}_G T, T\rangle_G \right) dv_G. \tag{1.2}$$

A cone is stable when this form is nonnegative on all smooth compactly supported symmetric tensors in the regular cone. No extension across the vertex is assumed.

1.1. **A stable cone with an unstable eightfold cover.** Let γ be the unit-round metric on S^m. On $(S^m)^4$, give each factor the metric $(m-1)\gamma/\Lambda$. In coordinates $x_a \in S^m \subset \mathbb{R}^{m+1}$, define

$$\begin{aligned} I(x_0, x_1, x_2, x_3) &= (-x_1, x_0, -x_3, x_2), \\ J(x_0, x_1, x_2, x_3) &= (-x_2, x_3, x_0, -x_1). \end{aligned} \tag{1.3}$$

Theorem A. *For each $m \geq 2$, four equal round m-spheres admit a free orientation-preserving isometric action of Q_8. For the quotient (Z_m, h_m), normalized by $\mathrm{Ric}_{h_m} = \Lambda h_m$,*

$$\pi_1(Z_m) = Q_8, \qquad \lambda_1^{\mathrm{sc}}(h_m) = \lambda_1^{\mathrm{TT}}(L_{h_m}) = \frac{2m}{m-1}\Lambda, \qquad \lambda_1^{\mathrm{TT}}(E_{h_m}) = \frac{2}{m-1}\Lambda. \tag{1.4}$$

For even m, Z_m is a rational homology sphere. After normalization to $\Lambda = 4m-1$, its cone has restricted holonomy $SO(4m+1)$ and satisfies the sharp full-tensor inequality

$$\mathfrak{q}_C(T) \geq \frac{(4m-1)^2}{4} \int_C r^{-2}|T|^2 \, dv_C. \tag{1.5}$$

For $m = 2$ the covering product cone has infinite negative index, and

$$\mathcal{D}((S^2)^4) = \frac{3\sqrt{7}}{2\pi}, \qquad \mathcal{D}((S^2)^4/Q_8) = 0. \tag{1.6}$$

Here $\mathcal{D}$ is the coefficient defined in (1.9). For $m = 2$, the covering map has degree eight. More explicitly, the covering cone has annular index

$$i_D = 3\#\left\{ \ell \geq 1 \,:\, \left(\frac{\ell\pi}{\log(R/r)} \right)^2 < \frac{7}{4} \right\}, \tag{1.7}$$

whereas every annulus in the quotient cone has index zero. All tensor components vanish on the two annular boundaries. The three unstable directions on the covering product are the relative scalings of the four factors; the Q_8 action has no invariant vector in this space. An estimate on every other tensor component is required and is proved in Section 5.

Relative variations of the factors are the destabilizing tensors of an Einstein product; see Kröncke [14, Section 4]. For compact Einstein stability and its relation to deformation theory, see Schwahn–Semmelmann [25]. Schwahn [24, Introduction] observes that instability of a Riemannian covering need not descend to its quotient. Theorem A gives an explicit free action for which all destabilizing cone eigenspaces have zero invariant part. The lower bound for the remaining spectrum, together with invariant tensors attaining it, gives the equalities in (1.4). The sharp constant in (1.5) is taken over all symmetric tensors. The metrics in this theorem are defined on the regular cones; their vertices are singular. The stable nonflat cone over the Fubini–Study plane was obtained in [10, Theorem 1.3]. No smooth complete filling is asserted in Theorem A.

1.2. **The coefficient determined by the link.** Let (Y^d, h) be closed, $d \geq 3$, with $\mathrm{Ric}_h = (d-1)h$. A finite group Γ acts freely and isometrically on Y; put $Z = Y/\Gamma$. Define

$$\nu_d = \frac{d-1}{2}, \qquad b_d = 2(d-1) - \nu_d^2 = \frac{(d-1)(9-d)}{4}, \qquad \mathcal{H}_\lambda(h) = \ker(L_h - \lambda) \cap \mathrm{TT}(Y), \tag{1.8}$$

$$\mathcal{D}_\Gamma(h) = \frac{1}{\pi} \sum_{\lambda < b_d} \dim \mathcal{H}_\lambda(h)^\Gamma \sqrt{b_d - \lambda}. \tag{1.9}$$

The sum is finite. Write $\mathcal{D}(h)$ for the trivial group and $G_C = dr^2 + r^2 h_Z$ for the cone metric. Proposition 2.2 identifies $\mathcal{D}_\Gamma(h) = \mathcal{D}(h_Z)$ and gives its character formula. For the eight-dimensional link in Theorem A, $b_8 = 7/4$ and the entire subthreshold space is

$$\left\{ \sum_{a=0}^{3} c_a h_a \,:\, \sum_{a=0}^{3} c_a = 0 \right\}, \qquad L_h T = 0.$$

It has dimension three upstairs and no Q_8-invariant vector. Formula (1.9) therefore gives the two values in (1.6). In general, the eigenvalues and their invariant multiplicities enter the following theorem in exactly the same way.

Theorem B. *With the notation above, the following statements hold.*

(i) *The cone over Z is stable if and only if $\mathcal{D}_\Gamma(h) = 0$. On a cone annulus $r < \rho < R$, with every tensor component vanishing at both boundary components, its negative index is*

$$i_D(C(Z); r, R) = \sum_{\lambda < b_d} \dim \mathcal{H}_\lambda(h)^\Gamma \, \# \left\{ \ell \geq 1 \, : \, \left(\frac{\ell \pi}{\log(R/r)} \right)^2 < b_d - \lambda \right\}. \tag{1.10}$$

(ii) *Let (X^{d+1}, G) be smooth, complete, Ricci-flat and without boundary, and let $K \subset X$ be a compact smooth domain. Assume that there is a diffeomorphism of manifolds with boundary*

$$\Phi \, : \, [R_0, \infty) \times Z \longrightarrow X \setminus \operatorname{int} K, \qquad \Phi(\{R_0\} \times Z) = \partial K. \tag{1.11}$$

In particular, $X \setminus K \cong (R_0, \infty) \times Z$. In this identification suppose

$$e(r) := \sup_Z \sum_{j=0}^{2} r^j |\nabla_C^j (G - G_C)|_{G_C}(r, \cdot) \longrightarrow 0. \tag{1.12}$$

For $N_G(\epsilon) = \dim \operatorname{im} \mathbf{1}_{(-\infty, -\epsilon)}(L_G)$,

$$N_G(\epsilon) = \frac{\mathcal{D}_\Gamma(h)}{2} \log \frac{1}{\epsilon} + o\left(\log \frac{1}{\epsilon} \right). \tag{1.13}$$

Every negative eigentensor is smooth and TT.

(iii) *If, in addition, $e(r) = o((\log r)^{-2})$, then*

$$\operatorname{ind}^-(\mathfrak{q}_G) < \infty \quad \Longleftrightarrow \quad \mathcal{D}_\Gamma(h) = 0. \tag{1.14}$$

If $\lambda_1^{\mathrm{TT}}(L_{h_Z}) > b_d$, the implication from right to left requires only (1.12).

Kröncke's cone criterion [15, Theorem 1.2] identifies the TT threshold b_d. The angular decomposition of Kröncke–Szabó [16, Theorem 3.15] includes the scalar and one-form eigenspaces as well as the TT eigenspaces. After the radial change $t = \log r$, each eigenvalue $\lambda < b_d$ contributes the one-dimensional form

$$\int \left(|v'|^2 - (b_d - \lambda)|v|^2 \right) dt.$$

Restriction to invariant angular tensors gives part (i).

Kirsch–Simon [13] studied eigenvalue accumulation for inverse-square potentials. Hassell–Marshall [12, Theorem 1] obtained the angular sum in the leading logarithmic term for scalar Schrödinger operators, including the critical angular threshold under logarithmic decay assumptions. In Theorem B, the perturbation is the full tensor operator of a metric satisfying (1.12). A comparison of quadratic forms proves part (ii) without a prescribed convergence rate. The remainder is $o(\log(1/\epsilon))$; no bounded remainder is asserted under this hypothesis. At the threshold, the radial form on an exterior domain has a positive inner-boundary term. The Hardy inequality in $\log r$ then gives the nonnegative exterior Neumann form required for part (iii). When $\mathcal{D}_\Gamma(h) = 0$, part (ii) alone states only that the eigenvalue count is $o(\log(1/\epsilon))$.

The positive-power decay assumed in [16, Definition 1.4] implies (1.12), but is stronger than it. All end estimates below use the chosen asymptotic chart and the error in (1.12). This coordinate assumption does not assert the existence of Ricci-flat metrics whose decay cannot be improved in another chart. Proposition 3.8 treats an added scalar potential at equality; it does not realize that potential by a Ricci-flat metric perturbation.

1.3. A pointwise estimate for the Chen–LeBrun–Weber link.

Page's Einstein metric on $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$ [22] and the Chen–LeBrun–Weber metric on $\mathbb{CP}^2 \# 2\overline{\mathbb{CP}}^2$ [7] are conformal to Kähler metrics with positive scalar curvature. Derdziński's conformal identities [8] and LeBrun's classification [18, Theorem A] give the following normalization for every closed positive Einstein–Hermitian four-manifold:

$$g = s^{-2} k, \qquad s = \mathrm{Scal}_k > 0, \qquad D_k = \operatorname{div}_k \nabla = -\Delta_k. \tag{1.15}$$

Hall–Haslhofer–Siepmann [10, Theorem 6.3] associate TT tensors to harmonic primitive $(1,1)$-forms and express their quadratic form in terms of $D_k(s^2)$. They prove the Page cone unstable and ask, in Conjecture 7.2, whether $D_k(s^2) < 15/4$ for the Chen–LeBrun–Weber metric normalized by $\mathrm{Ric}_g = 3g$.

Theorem C. *For the Chen–LeBrun–Weber metric with* $\mathrm{Ric}_g = 3g$,

$$\frac{207}{100} < \sup_M D_k(s^2) < \frac{21}{10}, \qquad \lambda_2^{\mathrm{TT}}(L_g) < \frac{21}{10}. \tag{1.16}$$

Equivalently, at Einstein constant $\Lambda > 0$,

$$\lambda_2^{\mathrm{TT}}(L_g) < \frac{7}{10}\Lambda, \qquad \lambda_2^{\mathrm{TT}}(E_g) < -\frac{13}{10}\Lambda. \tag{1.17}$$

For every closed positive Einstein–Hermitian four-manifold,

$$D_k(s^2) < \frac{53}{60}\Lambda. \tag{1.18}$$

If $m = b_2^- > 0$ *in its complex orientation, then* $\lambda_m^{\mathrm{TT}}(L_g) < 53\Lambda/60$.

Theorem 4.1 is the general comparison underlying (1.16). The conformal Einstein equation makes $\mathrm{Hess}_k\, s$ J-invariant and gives an autonomous equation for $D_g s$. At a maximum of $|ds|_g^2/(s^2 Q(s))$, the resulting Hessian relations reduce the Bochner inequality to three inequalities for Q, Q', Q''. No identity of the form $|ds|_g^2 = F(s)$ is assumed. Two polynomial choices of Q give comparisons on different scalar intervals. The sharper comparison proves the derivative inequality in Conjecture 7.2 of [10]. The affine scalar curvature and the defining polynomial of the CLW Kähler class are given in [20, Section 6.1]. Evaluation at one toric fixed point determines the Einstein normalization. Rational polynomial identities then bound the scalar range and establish the required comparison. No expression for the metric on the interior of the moment polygon is required.

Biquard–Ozuch [4] prove compact tensor instability for the non-Kähler Einstein–Hermitian metrics. Hall–Murphy's numerical work [11] concerns conformal variations, which are governed by the scalar Laplacian. In [20, Theorems A–C], the bounds

$$\frac{4\Lambda}{3} < \lambda_1^{\mathrm{sc}}(g_{\mathrm{CLW}}) < \frac{1951\Lambda}{1000}, \qquad \lambda_2^{\mathrm{TT}}(L_{g_{\mathrm{CLW}}}) < \frac{7}{6}\Lambda$$

prove, respectively, Hall–Murphy's Conjecture 5.3 and instability of the CLW cone. Theorem C adds the pointwise estimate and improves the latter tensor bound to $7\Lambda/10$. The Page bound $D_k(s^2) < 53/20$ at Einstein constant 3 is due to Hall–Haslhofer–Siepmann [10, Section 7.1]. Together with LeBrun's classification and the harmonic-form identity, it gives (1.18). The determinant-based cone classification and the anti-holomorphic quotient kernel estimate of [20, Theorem D and Proposition 9.2] are used in Section 6. Thus the scalar estimate and the qualitative CLW cone instability belong to [20]; the pointwise derivative comparison, the quaternionic spectra, and the quantitative conical index statements are the results considered here. The proofs of the attributed harmonic-form identity and Page estimate, with their normalization formulas, are retained in Appendices A and B.

1.4. **Einstein necks and five-dimensional fillings.** The next two statements are applications of the spectral comparison. The first assumes uniform control on a whole annulus; it does not assert that every Einstein degeneration has such annuli.

Corollary D. *Let* (M_i, G_i) *be closed Einstein* $(d+1)$*-manifolds,* $d \geq 3$*, with* $\mathrm{Ric}_{G_i} = \Lambda_i G_i$*. Suppose they contain annuli identified with* $(r_i, R_i) \times Z$*, where* $\mathrm{Ric}_{h_Z} = (d-1)h_Z$*, and*

$$T_i = \log(R_i/r_i) \longrightarrow \infty, \qquad \delta_i := \sup_{(r_i,R_i)\times Z} \sum_{j=0}^{2} r^j |\nabla_C^j (G_i - G_C)|_{G_C} + |\Lambda_i| R_i^2 \longrightarrow 0. \tag{1.19}$$

Then the componentwise tensor-Dirichlet index of the full operator satisfies

$$i_D(G_i; \textit{annulus}) = \mathcal{D}(h_Z) T_i + o(T_i), \qquad \liminf_i \frac{\mathrm{ind}^-(L_{G_i}|_{\mathrm{TT}})}{T_i} \geq \mathcal{D}(h_Z). \tag{1.20}$$

Contributions from finitely many disjoint such annuli add.

Negative variations supported on conical annuli occur in [19, Appendix A]. Verger [26] uses such variations to prove divergence of the index along compact Böhm sequences. Corollary D determines the leading coefficient under uniform convergence on the entire annulus.

Corollary E. *Let (X^5, G) be complete, oriented, Ricci-flat and without boundary, with a compact domain and exterior identification as in* (1.11)*, satisfying* (1.12)*. Let h be its four-dimensional link metric. In one orientation of the link, write $\alpha \geq \beta \geq \gamma$ for the eigenvalues of W_h^+. Assume*

$$\beta \leq \rho_0 \alpha \quad \textit{everywhere}, \qquad \rho_0 < 1. \tag{1.21}$$

Then either (X, G) is Euclidean, or

$$\lim_{\epsilon \downarrow 0} \frac{N_G(\epsilon)}{\log(1/\epsilon)} > \frac{1}{2\pi}\sqrt{\frac{11}{10}}. \tag{1.22}$$

In particular, finite index forces Euclidean isometry.

Yudowitz [27, Theorems 1.5 and 1.10] proves index semicontinuity for Ricci shrinkers under bubble-tree convergence and, under an additional semiboundedness hypothesis, a comparison with the index of an asymptotic cone. Those statements concern weighted shrinker operators and specified convergence hypotheses. Corollary D gives instead an unweighted local asymptotic with an explicit logarithmic coefficient and a global lower bound. The elementary transfer statement in Corollary 6.2 is placed in this context in Section 6.

Catino–Dameno [6, Theorem 1.4] prove that (1.21), when $W^+ \not\equiv 0$, implies conformal Kähler structure after a cover of degree at most two. The determinant classification in [20, Theorem D] and LeBrun's quotient theorem [17, Theorem B] then determine the possible stable links. The boundary link in (1.11) has signature zero when X is oriented and five-dimensional. Thus neither the CLW manifold nor $\mathbb{CP}^2$ is the sole link in Corollary E. This topological restriction, combined with the index formula, gives Euclidean rigidity.

Section 2 gives the angular spectrum, quotient multiplicities and cone annuli. Section 3 treats complete ends and the critical threshold. Section 4 proves the pointwise comparison and the CLW estimate. Section 5 gives the quotient construction, its intermediate covers and the least possible dimension in the stated round-product class. Section 6 proves the neck and curvature consequences. It also applies the full spectral theorem to the complete Ricci-flat Böhm metrics on $\mathbb{R}^{p+1} \times S^q$, yielding

$$N_G(\epsilon) = \frac{\sqrt{(d-1)(9-d)}}{4\pi} \log(1/\epsilon) + o(\log(1/\epsilon)), \qquad d = p + q \leq 8.$$

The existence and asymptotics of these metrics are taken from [5, 2]; infinitude of their negative spectrum is already proved in [2, Main Theorem II]. Appendices A–C contain the conformal tensor computation, the Page normalization and the polynomial identities.

2. Tensor spectra and finite quotients

2.1. **The TT realization.** All L^2 spaces below are real. On a closed Einstein manifold, L is a self-adjoint elliptic operator with domain $H^2(S^2T^*Y)$. Its principal symbol is $|\xi|^2 I$. Its spectrum is discrete, bounded below, and each eigenspace is finite-dimensional; these elliptic realization facts are used in the formulation of [9, Sections 1.3, 1.6]. The TT space is the closed subspace defined distributionally by $\operatorname{tr} T = 0$ and $\delta T = 0$.

The commutations and the closed TT realization used here are [20, Lemmas 2.1–2.2]:

$$\operatorname{tr}(LT) = \Delta \operatorname{tr} T, \qquad \delta(LT) = \Delta_H \delta T. \tag{2.1}$$

The identities require $\nabla \operatorname{Ric} = 0$, not a sign for Λ. To prove that the TT space reduces L, choose real z below the spectra of L, Δ and Δ_H. If $T = (L - z)^{-1} F$ and F is distributionally TT, then (2.1) gives $(\Delta - z) \operatorname{tr} T = 0$ and $(\Delta_H - z)\delta T = 0$. Invertibility implies that T is TT. Self-adjointness of the resolvent makes its orthogonal complement invariant as well. The restricted resolvent is compact because the full resolvent is compact. Thus the same realization applies without a sign restriction on the Einstein constant, and min–max applies to the restricted spectrum. In particular,

$$LT = \mu T, \quad \mu < 0 \quad \Longrightarrow \quad \|d \operatorname{tr} T\|^2 = \mu \|\operatorname{tr} T\|^2, \quad \|d\delta T\|^2 + \|\delta\delta T\|^2 = \mu\|\delta T\|^2.$$

Both left sides are nonnegative; hence every negative eigenvector is TT. The corresponding argument on a complete end, including the cutoff estimate, is given in Lemma 3.2.

Lemma 2.1. *Let $\mathcal{H}$ be a real Hilbert space, let q be a symmetric bilinear form on a linear subspace $\mathcal{V} \subset \mathcal{H}$, and let $v_1, \dots, v_r \in \mathcal{V}$ be linearly independent. Suppose q is negative definite on their span. Put*

$$A = (q(v_i, v_j))_{i,j=1}^r, \qquad B = (\langle v_i, v_j\rangle)_{i,j=1}^r.$$

Choose $\kappa > 0$ and $\beta > 0$ such that $A \le -\kappa I$ and $B \ge \beta I$. Let q' be a symmetric bilinear form on a subspace of a real Hilbert space $\mathcal{H}'$. If vectors $v_1', \dots, v_r'$ in its domain have energy and Gram matrices A', B' satisfying

$$\|A' - A\|_{\mathrm{op}} < \kappa/2, \qquad \|B' - B\|_{\mathrm{op}} < \beta/2, \tag{2.2}$$

then their span has dimension r and q' is negative definite on it.

Proof. For $c = (c_1, \dots, c_r) \in \mathbb{R}^r$, the two inequalities give

$$\Big\| \sum_i c_i v_i' \Big\|^2 = c^{\mathsf{T}} B' c \ge (\beta/2)|c|^2, \qquad q'\Big(\sum_i c_i v_i'\Big) = c^{\mathsf{T}} A' c \le -(\kappa/2)|c|^2.$$

The first inequality proves linear independence; the second proves the asserted sign. For a fixed form q, let $\|\cdot\|_{\mathcal{V}}$ be a norm such that the inclusion $\mathcal{V} \to \mathcal{H}$ and the bilinear map q are continuous. If $v_{i,n} \to v_i$ in this norm, then

$$\begin{aligned} |q(v_{i,n}, v_{j,n}) - q(v_i, v_j)| &\le C\|v_{i,n} - v_i\|_{\mathcal{V}} \|v_{j,n}\|_{\mathcal{V}} \\ &\quad + C\|v_i\|_{\mathcal{V}} \|v_{j,n} - v_j\|_{\mathcal{V}} \longrightarrow 0. \end{aligned}$$

Continuity of the inclusion gives convergence of the Gram matrices. Since the matrices have fixed finite order, entrywise convergence implies convergence in operator norm, and (2.2) holds for large n. When the forms or Hilbert scalar products vary, convergence of the energy and Gram matrices is a separate hypothesis. □

2.2. Invariant eigenspaces.

Proposition 2.2. *Let $p : Y \to Z = Y/\Gamma$ be a finite Riemannian covering. For every λ, pullback identifies*

$$\ker(L_Z - \lambda) \cap \mathrm{TT}(Z) \cong \big(\ker(L_Y - \lambda) \cap \mathrm{TT}(Y)\big)^{\Gamma}. \tag{2.3}$$

If χ_λ is the character of the representation $\gamma \mapsto (\gamma^{-1})^*$ on $\mathcal{H}_\lambda(h)$, then

$$\mathcal{D}_\Gamma(h) = \frac{1}{\pi|\Gamma|} \sum_{\gamma\in\Gamma} \sum_{\lambda < b_d} \chi_\lambda(\gamma)\sqrt{b_d - \lambda}. \tag{2.4}$$

Proof. The map $|\Gamma|^{-1/2} p^*$ is an isometry from $L^2(S^2T^*Z)$ onto the invariant subspace of $L^2(S^2T^*Y)$. Covariant differentiation, curvature contraction, trace, and divergence commute with local isometries. Thus it intertwines L, and preserves the TT equations. Conversely, an invariant smooth tensor has well-defined components in every evenly covered coordinate neighborhood of Z; these components agree on overlaps and define a tensor on Z. This proves (2.3).

On a finite-dimensional eigenspace, the endomorphism

$$\Pi_\Gamma = |\Gamma|^{-1} \sum_{\gamma\in\Gamma} (\gamma^{-1})^*$$

fixes invariant vectors and has invariant image, since multiplication by any group element permutes the sum. Consequently $\Pi_\Gamma^2 = \Pi_\Gamma$, and $\dim \mathcal{H}_\lambda^\Gamma = \operatorname{tr}\Pi_\Gamma = |\Gamma|^{-1}\sum_\gamma \chi_\lambda(\gamma)$. Substitution in (1.9) proves (2.4). □

Proposition 2.3. *Let $V \subset C^\infty(S^2T^*Y) \cap \mathrm{TT}(Y)$ be a finite-dimensional Γ-invariant space, and suppose*

$$\langle L_h v, v\rangle_{L^2} \le c\|v\|_{L^2}^2 \quad (v \in V), \qquad c < b_d. \tag{2.5}$$

If $r = \dim V^\Gamma$, then

$$\mathcal{D}_\Gamma(h) \ge \frac{r}{\pi}\sqrt{b_d - c}. \tag{2.6}$$

If the inequality in (2.5) *is strict for every $v \neq 0$ and $r > 0$, then* (2.6) *is strict.*

Proof. The invariant space descends to an r-dimensional TT space on Z. Both the quadratic form and the squared norm are divided by $|\Gamma|$, so the Rayleigh bound is unchanged. Min–max gives r eigenvalues at most c. Their contributions to (1.9) prove the assertion. In the strict case, the maximum of the Rayleigh quotient on the unit sphere of V is a number $c' < c$, because that sphere is compact. Apply the non-strict assertion with c'. □

2.3. **The full angular operator.** Write $\nu = (d-1)/2$. Radial parallel transport identifies the cone tensor bundle with the pullback of

$$\mathbb{R} \oplus T^*Z \oplus S^2T^*Z.$$

The summands represent the radial, mixed, and tangential components, measured in the orthonormal coframe $dr, r\theta^1, \dots, r\theta^d$.

Lemma 2.4. *There is a self-adjoint elliptic angular operator $\mathcal{A}_h$ such that*

$$L_C = -\partial_r^2 - \frac{d}{r}\partial_r + \frac{1}{r^2}\mathcal{A}_h. \tag{2.7}$$

Put $\mathcal{B}_h = \mathcal{A}_h + \nu^2 I$. Its negative eigenvalues, with multiplicity, are exactly

$$\lambda_j^{\mathrm{TT}}(L_h) - b_d < 0. \tag{2.8}$$

Moreover:

(a) *if $\lambda_1^{\mathrm{TT}}(L_h) > b_d$, then $\mathcal{B}_h$ is strictly positive;*
(b) *if $E_h|_{\mathrm{TT}} \ge 0$ and $\lambda_1^{\mathrm{sc}}(h) \ge 2(d+1)$, then $\mathcal{A}_h \ge 0$.*

The angular tensor corresponding to G_C has $\mathcal{A}_h$-eigenvalue zero.

Proof. We use the complete angular decomposition of [16, Theorem 3.15], including its zero-vector exceptions. Set

$$\eta(a) = a(a+d-1), \qquad \xi_\pm(v) = -\nu \pm \sqrt{\nu^2 + v}.$$

The tangential TT block is E_h. The other angular blocks have eigenvalues among

$$0, \quad 2(d+1), \quad \lambda, \quad \eta(\xi_\pm(\lambda) - 2), \quad \eta(\xi_\pm(\mu+1) - 1), \tag{2.9}$$

where $\lambda > 0$ is a scalar eigenvalue and μ is an eigenvalue of the connection Laplacian on coclosed one-forms. Values for which the tensor constructed in that theorem vanishes are omitted; the formula is not a list of multiplicities before those omissions.

For $\Delta f = \lambda f$, $\lambda > 0$, the integrated scalar Bochner identity is

$$\lambda^2\|f\|^2 = \|\mathrm{Hess}\, f\|^2 + (d-1)\lambda\|f\|^2 \ge \frac{\lambda^2}{d}\|f\|^2 + (d-1)\lambda\|f\|^2.$$

Division by $(d-1)\lambda\|f\|^2 > 0$ gives $\lambda \ge d$. For a coclosed one-form α, integration by parts gives

$$2\|\delta^*\alpha\|^2 = \|\nabla\alpha\|^2 - (d-1)\|\alpha\|^2.$$

For a rough eigensection, the right side is $(\mu - d + 1)\|\alpha\|^2$, so $\mu \ge d-1$. The identity $\eta(a) + \nu^2 = (a+\nu)^2$ gives, in the two generated families,

$$\eta(\xi_\pm(\lambda) - 2) + \nu^2 = (\pm\sqrt{\nu^2 + \lambda} - 2)^2, \quad \eta(\xi_\pm(\mu+1) - 1) + \nu^2 = (\pm\sqrt{\nu^2 + \mu + 1} - 1)^2.$$

The other values in (2.9), after addition of ν^2, are positive. Hence every non-TT value of $\mathcal{B}_h$ is nonnegative. In the TT block,

$$E_h + \nu^2 = L_h - 2(d-1) + \nu^2 = L_h - b_d.$$

This proves (2.8), with the TT multiplicities supplied by the angular decomposition.

For strict positivity, the vector-generated squared terms satisfy $\sqrt{\nu^2+\mu+1} \geq (d+1)/2 > 1$. The scalar-generated squared terms can vanish only if $\sqrt{\nu^2+\lambda} = 2$. For $d \geq 4$, the bound $\lambda \geq d$ gives $\nu^2 + \lambda \geq (d+1)^2/4 > 4$, so equality is impossible. For $d = 3$ it forces $\lambda = 3$. Equality in the scalar bound gives the round three-sphere by Obata's theorem [21]; the corresponding degree-one function extends to a linear function on the flat four-cone, whose Hessian is zero. It is exactly a deleted zero tensor in [16, Theorem 3.15]. Thus the remaining non-TT blocks are strictly positive. Compact angular resolvent then gives a positive lower spectral bound.

For (b), $\lambda \geq 2(d+1)$ implies $\xi_+(\lambda) \geq 2$. Hence $\eta(\xi_+(\lambda)-2) \geq 0$; the minus branch has argument at most $-(d-1)$ and also has nonnegative η. The vector branches are nonnegative because $\mu+1 \geq d$ gives $\xi_+(\mu+1) \geq 1$. Together with $E_h|_{\mathrm{TT}} \geq 0$ and the first three entries of (2.9), this proves $\mathcal{A}_h \geq 0$. Finally, $\nabla G_C = 0$ and $\mathring{R}_C G_C = 0$, so (2.7) applied to its constant parallel coefficients gives the last assertion. ☐

2.4. **Radial quadratic forms.**

Lemma 2.5. *Let k be a link TT eigentensor with $L_h k = \lambda k$. Then $T = r^2 f(r) k$ is TT on the cone and*

$$\mathfrak{q}_C(T) = \|k\|^2_{L^2(h)} \int \left(r^d (f')^2 + [\lambda - 2(d-1)] r^{d-2} f^2 \right) dr. \tag{2.10}$$

For a general tensor let its parallel coefficients be $r^{-\nu} w(t)$, $t = \log r$. On a finite interval $[a, b]$ with zero boundary trace,

$$\mathfrak{q}_C(T) = \int_a^b \left(\|w'\|^2 + \langle \mathcal{B}_h w, w \rangle \right) dt, \tag{2.11}$$

$$\|T\|^2_{L^2(G_C)} = \int_a^b e^{2t} \|w\|^2 dt, \qquad \int r^{-2} |T|^2 dv_C = \int_a^b \|w\|^2 dt. \tag{2.12}$$

On $[a, \infty)$, with free inner boundary trace and compact support at infinity, the first identity is replaced by

$$\mathfrak{q}_C(T) = \int_a^\infty \left(\|w'\|^2 + \langle \mathcal{B}_h w, w \rangle \right) dt + \nu \|w(a)\|^2. \tag{2.13}$$

Proof. For tangential lifts X, Y, the cone connection is

$$\nabla^C_{\partial_r} \partial_r = 0, \qquad \nabla^C_{\partial_r} X = \nabla^C_X \partial_r = r^{-1} X, \qquad \nabla^C_X Y = \nabla^h_X Y - r h(X, Y) \partial_r.$$

Let $(e_a)_{a=1}^d$ be an h-orthonormal frame, normal at the link point under consideration, and use the cone frame $e_0 = \partial_r$, $E_a = r^{-1} e_a$. The nonzero components of T are $T(E_a, E_b) = f k_{ab}$. Its derivative components are

$$\begin{aligned} (\nabla^C_0 T)_{ab} &= f' k_{ab}, & (\nabla^C_c T)_{ab} &= r^{-1} f (\nabla^h_c k)_{ab}, \\ (\nabla^C_c T)_{0b} &= -r^{-1} f k_{cb}, & (\nabla^C_c T)_{a0} &= -r^{-1} f k_{ca}. \end{aligned} \tag{2.14}$$

All remaining components vanish. Contraction gives

$$\operatorname{tr}_C T = f \operatorname{tr}_h k, \qquad (\delta_C T)(X) = f(\delta_h k)(X), \qquad (\delta_C T)(\partial_r) = r^{-1} f \operatorname{tr}_h k.$$

Thus T is TT. For a trace-free k, the same component formulas and the cone curvature formula give

$$|\nabla_C T|^2 = (f')^2 |k|_h^2 + r^{-2} f^2 (|\nabla_h k|_h^2 + 2|k|_h^2), \tag{2.15}$$

$$\langle \mathring{R}_C T, T \rangle_C = r^{-2} f^2 \left(\langle \mathring{R}_h k, k \rangle_h + |k|_h^2 \right). \tag{2.16}$$

Indeed, the tangential cone curvature is r^{-2} times the difference between the link curvature and the curvature tensor of the unit sphere. The latter acts on a trace-free symmetric tensor as $k \mapsto -k$. Subtracting twice (2.16) from (2.15) cancels the two additional $|k|^2$ terms. Integration over the link, where there is no boundary, gives for every smooth link TT tensor

$$\mathfrak{q}_C(r^2 f k) = \|k\|^2 \int r^d (f')^2 \, dr + \left(\langle L_h k, k \rangle - 2(d-1) \|k\|^2 \right) \int r^{d-2} f^2 \, dr. \tag{2.17}$$

For $L_h k = \lambda k$, this is (2.10). No radial boundary term has been discarded in this calculation.

For an arbitrary tensor, integration over the closed link in (2.7) yields the radial form with angular part $\mathcal{A}_h$. The derivative of its parallel coefficients is

$$\partial_r(r^{-\nu} w(\log r)) = r^{-\nu-1}(w' - \nu w).$$

Since $2\nu = d - 1$ and $dr = r\,dt$, the form is

$$\int \left(\|w' - \nu w\|^2 + \langle \mathcal{A}_h w, w\rangle\right) dt = \int \left(\|w'\|^2 + \langle \mathcal{B}_h w, w\rangle\right) dt - \nu [\|w\|^2]_{\text{endpoints}}.$$

Zero traces give (2.11). Compact support at infinity gives (2.13), with its positive inner-boundary term. Finally, $r^d r^{-2\nu} dr = e^{2t} dt$ and $r^{d-2} r^{-2\nu} dr = dt$; these are the two mass identities. □

Proposition 2.6. *The exact annular formula* (1.10) *holds. The cone is stable if and only if $\mathcal{B}_h \geq 0$, equivalently $\mathcal{D}(h) = 0$.*

Proof. Expand w in an orthonormal angular eigenbasis of $\mathcal{B}_h$. The associated form is the orthogonal sum of the scalar forms $\int (|w_j'|^2 + \beta_j |w_j|^2) dt$. If $\beta_j \geq 0$, its index is zero. If $\beta_j = -\sigma_j^2 < 0$, expansion in the sine functions on an interval of length T gives coefficients $(\ell\pi/T)^2 - \sigma_j^2$, $\ell \geq 1$. The positive bounded weight e^{2t} on a fixed annulus does not change the index of a form. Thus the contribution is

$$\#\{\ell \geq 1 \,:\, \ell < T\sigma_j/\pi\} = \lceil T\sigma_j/\pi \rceil - 1.$$

This count differs from $T\sigma_j/\pi$ by at most one. There are finitely many negative angular eigenvalues. Lemma 2.4 and Proposition 2.2 now prove (1.10) and its $O(1)$ remainder.

If $\mathcal{B}_h \geq 0$, (2.11) is nonnegative for every compactly supported tensor. If $\mathcal{B}_h$ has an eigenvalue $-\sigma^2 < 0$, choose a nonzero smooth function χ supported in $(0, 1)$ and replace it by $\chi(t/T)$. Its derivative quotient is $T^{-2}\|\chi'\|^2/\|\chi\|^2 < \sigma^2$ for large T, producing a negative tensor. Arbitrarily many disjoint translates in t give negative spaces of every finite dimension. The last equivalence is the consequence of (2.8). □

3. Negative spectrum at conical infinity

The quadratic forms in this section act on all symmetric tensors. Every complete manifold considered in this section is smooth, connected and without boundary, and has a compact smooth domain with exterior identification (1.11). The error $e(r)$ is measured in that identification. On finite boundary components, tensors are smooth up to the boundary; an inner trace is unrestricted unless Dirichlet data are specified. On a complete Ricci-flat manifold, negative eigentensors are TT by Lemma 3.2.

3.1. **Comparison of quadratic forms.**

Lemma 3.1. *Let $G_C = dr^2 + r^2 h$ on an annulus or exterior, and let G be a smooth metric. Suppose*

$$e(r) = \sup_Z \sum_{j=0}^{2} r^j |\nabla_C^j (G - G_C)|_{G_C} \leq \delta$$

with δ sufficiently small. For compactly supported covariant symmetric tensors, with arbitrary trace on a finite inner boundary, put $Z(T) = \int r^{-2}|T|_C^2 dv_C$. Then

$$|\mathfrak{q}_G(T) - \mathfrak{q}_C(T)| \leq C \int e(r)\left(|\nabla_C T|^2 + r^{-2}|T|^2\right) dv_C, \tag{3.1}$$

$$(1 - C\delta)\|T\|_C^2 \leq \|T\|_G^2 \leq (1 + C\delta)\|T\|_C^2, \tag{3.2}$$

$$(1 - C\delta)\mathfrak{q}_C(T) - C'\delta Z(T) \leq \mathfrak{q}_G(T) \leq (1 + C\delta)\mathfrak{q}_C(T) + C'\delta Z(T). \tag{3.3}$$

Here $\mathfrak{q}_G$ denotes the Einstein-operator form $\int (|\nabla T|^2 - 2\langle \mathring{R}T, T\rangle)$, whether or not G is Ricci-flat. Constants depend only on the fixed link.

Proof. Write $a = G - G_C$. In a G_C-orthonormal frame, the eigenvalues of G belong to $[1 - C\delta, 1 + C\delta]$. The inverse-matrix identity $G^{-1} - G_C^{-1} = -G^{-1} a G_C^{-1}$ and the determinant formula give inverse and density errors at most $Ce(r)$. The connection difference is

$$(\nabla^G - \nabla^C)^k{}_{ij} = \tfrac{1}{2} G^{ka} \big(\nabla_i^C a_{ja} + \nabla_j^C a_{ia} - \nabla_a^C a_{ij} \big),$$

so its norm is at most $Ce(r)/r$. Its first covariant derivative is bounded by $Ce(r)/r^2$, using the bound on $\nabla_C^2 a$ and the derivative of the inverse matrix. Put $S^k{}_{ij} = (\nabla^G - \nabla^C)^k{}_{ij}$. If $R^k{}_{\ell ij}$ denotes the coefficient of $R(\partial_i, \partial_j)\partial_\ell$, then

$$(R^G - R^C)^k{}_{\ell ij} = \nabla_i^C S^k{}_{j\ell} - \nabla_j^C S^k{}_{i\ell} + S^k{}_{ip} S^p{}_{j\ell} - S^k{}_{jp} S^p{}_{i\ell}.$$

Each derivative term is bounded by $Ce(r)r^{-2}$, and each quadratic term by $Ce(r)^2 r^{-2}$. The inverse-metric bounds therefore give an error $Ce(r)r^{-2}$ in the curvature endomorphism. Since $|\mathrm{Rm}_C| \le C/r^2$, expansion of the energy and $2r^{-1}|T||\nabla T| \le |\nabla T|^2 + r^{-2}|T|^2$ prove (3.1). The inverse and density estimates prove (3.2).

Finally,

$$\int |\nabla_C T|^2 \, dv_C \le \mathfrak{q}_C(T) + CZ(T).$$

Insert this inequality and $e \le \delta$ in (3.1). The right side is nonnegative, and the two resulting inequalities are exactly (3.3). The estimates hold for arbitrary boundary trace. □

3.2. **The negative spectrum of the complete operator.**

Lemma 3.2. *Let (X, G) be complete, smooth, connected, Ricci-flat and without boundary. Assume* (1.11) *and $e(r) \to 0$ on this entire exterior. The Friedrichs realization of L_G has no negative essential spectrum. Its negative eigenvalues have finite multiplicity, are finite in number below every $-\epsilon < 0$, and have smooth TT eigenvectors. Its negative spectral dimension equals the index of its quadratic form on $C_c^\infty(S^2 T^* X)$.*

Proof. The curvature is bounded on the compact core and is $O(r^{-2})$ on the end. Thus $L_G = P + V$, where $P = \nabla^* \nabla$ is the nonnegative Friedrichs operator and $V = -2\mathring{R}_G$ is a bounded self-adjoint multiplication operator tending to zero. For a compactly supported cutoff ζ, local elliptic estimates map $(P+1)^{-1} L^2$ boundedly to H^2 on the support of ζ. The compact embedding of this space into L^2 shows that $\zeta V (P+1)^{-1}$ is compact. The norm of $(1-\zeta)V(P+1)^{-1}$ tends to zero along an exhaustion, since the supremum of $|V|$ on the complement tends to zero. Hence $V(P+1)^{-1}$ is compact. The relatively compact perturbation theorem [23, Theorem XIII.14] implies $\mathrm{spec}_{\mathrm{ess}}(L_G) \subset [0, \infty)$.

The form domain is H^1, defined by closure of compactly supported smooth sections. For a fixed base point, composition of its distance function with a Lipschitz function equal to one on $[0,1]$ and zero on $[2, \infty)$ gives cutoffs χ_R with $|d\chi_R| \le C/R$; multiplication by those cutoffs and local smoothing approximate every form-domain tensor. The min–max characterization [23, Theorem XIII.1] identifies the negative spectral dimension with the index on the closed form domain. If a finite-dimensional subspace is negative, approximate a basis by compactly supported smooth sections in the form norm. Lemma 2.1 preserves its dimension and negative definiteness. Hence restricting to C_c^∞ does not change the index. Local elliptic regularity makes each eigentensor smooth.

Let $L_G T = \mu T$, $\mu < 0$. Form membership gives $T, \nabla T \in L^2$, so $u = \mathrm{tr}\, T$ and $v = \delta T$ belong to L^2. The distributional commutations (2.1) give $\Delta u = \mu u$ and $\Delta_H v = \mu v$. Since $\mathrm{Ric}_G = 0$, Δ_H is the rough Laplacian on one-forms. Choose compactly supported Lipschitz cutoffs χ_R with $0 \le \chi_R \le 1$, $\chi_R = 1$ on B_R, and $|d\chi_R| \le C/R$. For either $z = u$ or $z = v$, test its equation with $\chi_R^2 z$ and expand the derivative. This gives

$$\|\nabla(\chi_R z)\|^2 - \mu \|\chi_R z\|^2 = \int |d\chi_R|^2 |z|^2 \, dv_G \le CR^{-2} \|z\|^2. \tag{3.4}$$

The test is legitimate by local smoothness; compact support removes all boundary terms. Fatou's lemma and $-\mu > 0$ imply $z = 0$. Thus T is TT. □

3.3. One-dimensional counting.

Lemma 3.3. *Fix $a \in \mathbb{R}$ and $\sigma > 0$. On $[a, \infty)$ consider*

$$q_\epsilon[v] = \int_a^\infty \left(|v'|^2 - \sigma^2|v|^2 + \epsilon e^{2t}|v|^2\right) dt. \tag{3.5}$$

Impose Dirichlet data at a, or add a fixed nonnegative multiple of $|v(a)|^2$ and allow free data. In either case,

$$\lim_{\epsilon \downarrow 0} \frac{\operatorname{ind}^-(q_\epsilon)}{\log(1/\epsilon)} = \frac{\sigma}{2\pi}. \tag{3.6}$$

If $-\sigma^2$ is replaced by a nonnegative number, the index is zero.

Proof. For small ϵ, set $t_\epsilon = \frac{1}{2}\log(\sigma^2/\epsilon) > a$. Enlarge the form domain by cutting at t_ϵ and allowing independent traces. Subtracting the nonnegative inner boundary term decreases the form. The potential on $[t_\epsilon, \infty)$ is nonnegative, so this piece has zero index. On $[a, t_\epsilon]$, replacing the potential $-\sigma^2 + \epsilon e^{2t}$ by $-\sigma^2$ decreases it further. The Neumann eigenfunctions of $-\partial_t^2$ have eigenvalues $(\ell\pi/(t_\epsilon - a))^2$, $\ell = 0, 1, \ldots$. Thus

$$\operatorname{ind}^-(q_\epsilon) \le \frac{\sigma}{\pi}(t_\epsilon - a) + 1.$$

For $0 < \eta < \sigma^2$, restrict instead to zero-trace functions on $[a, s_\epsilon]$, where $s_\epsilon = \frac{1}{2}\log(\eta/\epsilon)$, and extend them by zero. On this interval the potential is at most $-(\sigma^2 - \eta)$. The Dirichlet sine count gives

$$\operatorname{ind}^-(q_\epsilon) \ge \frac{\sqrt{\sigma^2 - \eta}}{\pi}(s_\epsilon - a) - 1.$$

Division by $\log(1/\epsilon)$ gives upper limit $\sigma/(2\pi)$ and lower limit at least $\sqrt{\sigma^2 - \eta}/(2\pi)$. Let $\eta \downarrow 0$. The last assertion follows directly from the sign of the integrand and boundary term. $\square$

Theorem 3.4. *Under the complete-end hypotheses in Theorem B,*

$$N_G(\epsilon) = \frac{\mathcal{D}(h_Z)}{2}\log(1/\epsilon) + o(\log(1/\epsilon)). \tag{3.7}$$

Here $\mathcal{D}(h_Z) = \mathcal{D}_\Gamma(h)$.

Proof. Cut X at a sufficiently large fixed radius R. Restricting the global form to sections with zero trace on the cut gives a lower bound for its negative count. Allowing independent traces on the compact core and the exterior gives an upper bound. The core Neumann operator has compact resolvent and a finite total negative index; its contribution to $N_G(\epsilon)$ is bounded independently of ϵ.

Let $\delta = \sup_{r \ge R} e(r)$. On the exterior, apply (3.3) and (3.2) to the form $\mathfrak{q}_G + \epsilon\|\cdot\|_G^2$. After division by a positive factor, the upper and lower comparisons are cone forms of the shape

$$\int_{\log R}^\infty \left(\|w'\|^2 + \langle(\mathcal{B}_h + \theta I)w, w\rangle + c_\delta \epsilon e^{2t}\|w\|^2\right) dt + \nu\|w(\log R)\|^2, \tag{3.8}$$

or their Dirichlet restrictions, where $|\theta| \le C\delta$ and $0 < c_- \le c_\delta \le c_+ < \infty$. These bounds follow by dividing $(1 \pm C\delta)\mathfrak{q}_C \pm C'\delta Z + \epsilon(1 \pm C\delta)\|\cdot\|_C^2$ by $1 \pm C\delta$.

Let β_j be the full angular spectrum of $\mathcal{B}_h$, including multiplicity. Only finitely many $\beta_j + \theta$ are negative. Expansion in the angular eigenbasis and Lemma 3.3 show that the coefficient of (3.8) is

$$F(\theta) = \frac{1}{2\pi}\sum_j \sqrt{(-\beta_j - \theta)_+}. \tag{3.9}$$

Multiplying ϵ by c_δ adds only a bounded quantity to its logarithm. For $|\theta| \le 1$, all contributing β_j lie below 1; this is a finite set. Therefore F is continuous at zero, even when $\mathcal{B}_h$ has a kernel.

For fixed R, the compact-core contribution divided by $\log(1/\epsilon)$ tends to zero. The form inequalities and the inclusions of the Dirichlet and Neumann domains give

$$F(C\delta) \le \liminf_{\epsilon \downarrow 0} \frac{N_G(\epsilon)}{\log(1/\epsilon)} \le \limsup_{\epsilon \downarrow 0} \frac{N_G(\epsilon)}{\log(1/\epsilon)} \le F(-C\delta). \tag{3.10}$$

Let $R \to \infty$. Then $\delta \to 0$, and continuity of F makes both outer quantities converge to $F(0) = \mathcal{D}(h_Z)/2$, by Lemma 2.4. Lemma 3.2 identifies this index with the full negative spectral count and makes the negative eigentensors TT. Proposition 2.2 gives the stated quotient coefficient. □

Remark 3.5. When $\mathcal{D}(h_Z) = 0$, (3.7) becomes $N_G(\epsilon) = o(\log(1/\epsilon))$. This asymptotic relation is compatible with infinitely many negative eigenvalues. Theorem 3.7 gives a sufficient decay condition for finiteness.

3.4. **The critical threshold.**

Lemma 3.6. *For $a > 0$ and a smooth compactly supported Hilbert-space-valued function on $[a, \infty)$,*

$$\int_a^\infty \|w'\|^2 dt + \frac{\|w(a)\|^2}{2a} - \frac{1}{4}\int_a^\infty t^{-2}\|w\|^2 dt = \int_a^\infty \left\| w' - \frac{w}{2t}\right\|^2 dt \geq 0. \tag{3.11}$$

Proof. Expand the square. Its mixed term is $-\int_a^\infty \langle w', w\rangle/t\, dt = -\frac{1}{2}\int_a^\infty (\|w\|^2)'/t\, dt$. Integration by parts gives $\|w(a)\|^2/(2a) - \frac{1}{2}\int_a^\infty t^{-2}\|w\|^2 dt$. Combining this with the square term $\frac{1}{4}\int t^{-2}\|w\|^2$ proves the equality. □

Theorem 3.7. *Under the complete-end hypotheses of Theorem B, if $e(r) = o((\log r)^{-2})$, then the complete negative index is finite if and only if the asymptotic cone is stable. If $\lambda_1^{\mathrm{TT}}(L_h) > b_d$, then $e(r) \to 0$ already implies finite negative index.*

Proof. Suppose first that $\mathcal{B}_h \geq 0$, and put $\varepsilon(t) = e(e^t)$. Let $\mathcal{R}_h$ denote the angular curvature endomorphism appearing in the cone form. It is bounded because the link is closed. The angular connection form is $\langle \mathcal{A}_h w, w\rangle + 2\langle \mathcal{R}_h w, w\rangle$, and therefore

$$q_{\nabla,h}(w) \leq \langle \mathcal{B}_h w, w\rangle + C\|w\|^2.$$

Moreover, $\|w' - \nu w\|^2 \leq 2\|w'\|^2 + 2\nu^2\|w\|^2$. Using these inequalities in (3.1), with its variable coefficient $\varepsilon(t)$, gives

$$\mathfrak{q}_G(T) \geq a_0 \int_{t_0}^\infty (\|w'\|^2 + \langle \mathcal{B}_h w, w\rangle)\, dt + \nu\|w(t_0)\|^2 - C\int_{t_0}^\infty \varepsilon(t)\|w\|^2\, dt, \tag{3.12}$$

$$a_0 = 1 - C\sup_{t\geq t_0} \varepsilon(t). \tag{3.13}$$

Here C is enlarged to bound both derivative coefficients. Since $\langle \mathcal{B}_h w, w\rangle \geq 0$, replacing $1 - C\varepsilon(t)$ by its lower bound a_0 preserves the inequality.

Fix $\eta > 0$ with $C\eta < 1/16$. The hypothesis $t^2\varepsilon(t) \to 0$ permits a choice of t_0 for which

$$a_0 \geq \tfrac{1}{2}, \qquad \varepsilon(t) \leq \eta t^{-2}\ (t \geq t_0), \qquad \nu \geq \frac{a_0}{2t_0}.$$

The last condition follows after further increasing t_0, since $a_0 \leq 1$ and $\nu > 0$. Lemma 3.6 now gives the more precise inequality

$$\begin{aligned}\mathfrak{q}_G(T) \geq a_0 \int_{t_0}^\infty \left\|w' - \frac{w}{2t}\right\|^2 dt + a_0\int_{t_0}^\infty \langle \mathcal{B}_h w, w\rangle\, dt \\ + (a_0/4 - C\eta)\int_{t_0}^\infty t^{-2}\|w\|^2 dt + \left(\nu - \frac{a_0}{2t_0}\right)\|w(t_0)\|^2 \geq 0.\end{aligned} \tag{3.14}$$

Every term on the right is nonnegative; no inner boundary value is prescribed. The curvature is bounded on the exterior, so its semibounded form norm is equivalent to the H^1 norm. To pass to the Neumann form domain, take cutoffs η_R equal to one for $r \leq R$ and zero for $r \geq 2R$, with $|d\eta_R|_G \leq C/R$. For $T \in H^1$,

$$\|\eta_R T - T\|_{H^1}^2 \leq C\int_{\{r\geq R\}} \big(|T|^2 + |\nabla T|^2\big)\, dv_G + CR^{-2}\int_{\{r\geq R\}} |T|^2 dv_G \longrightarrow 0.$$

In each smooth boundary chart, extend the components of $\eta_R T$ across the boundary by reflection in the normal coordinate and approximate by convolution; an interior partition of unity and a finite partition on the boundary combine these approximations. They converge in H^1 and impose no zero boundary trace. Continuity of the quadratic form therefore extends (3.14) to the exterior Neumann domain. Cutting X at $r = e^{t_0}$ enlarges the global form domain to the direct sum of the core and exterior Neumann domains. The exterior index is zero. The compact core has a semibounded elliptic Neumann form with compact resolvent, hence finite negative index. Form-domain inclusion and min–max prove finite global index.

If $\mathcal{B}_h \geq cI$ with $c > 0$, (3.12) instead yields

$$\mathfrak{q}_G(T) \geq a_0 \int \|w'\|^2 dt + (a_0 c - C \sup_{t\geq t_0} \varepsilon(t)) \int \|w\|^2 dt + \nu \|w(t_0)\|^2.$$

The middle coefficient is positive for a sufficiently large t_0 whenever $e(r) \to 0$. Lemma 2.4(a) gives this strict angular inequality under the strict TT hypothesis.

Conversely, let $\mathcal{B}_h k = -\sigma^2 k$, $\sigma > 0$, with $\|k\| = 1$. Choose $\chi \in C_c^\infty((0,T))$ such that $\int (|\chi'|^2 - \sigma^2\chi^2) = -\kappa < 0$. For $s > 0$, give a cone tensor T_s the parallel coefficients $r^{-\nu}\chi(\log r - s)k$. Formula (2.11) gives $\mathfrak{q}_C(T_s) = -\kappa$ for every s. The integral $\int (|\nabla_C T_s|^2 + r^{-2}|T_s|^2) dv_C$ equals a fixed finite number M, independent of s, by the same change of variable. Hence

$$\mathfrak{q}_G(T_s) \leq -\kappa + CM \sup_{r\geq e^s} e(r) < -\kappa/2$$

for all sufficiently large s. Choose $s_1, \dots, s_N$ with disjoint translated supports. Locality of the form gives $\mathfrak{q}_G(\sum a_j T_{s_j}) = \sum a_j^2 \mathfrak{q}_G(T_{s_j}) < 0$ for every nonzero coefficient vector. Since N is arbitrary, the index is infinite. □

3.5. **A logarithmic inverse-square potential**. Fix $k \in \ker \mathcal{B}_h$ with $\|k\|_{L^2(h)} = 1$, and put $V = \mathbb{R}k$. The following proposition concerns tensors whose angular component belongs to V.

Proposition 3.8. *On the one-dimensional angular subspace V, add the scalar potential $-c/(r^2(\log r)^2)$ for $r \geq e^a$, $a > 0$, and impose Dirichlet data at $r = e^a$. The form is nonnegative if $c \leq 1/4$. If $c > 1/4$, it has infinite index and*

$$N(\epsilon) = \frac{\sqrt{c - 1/4}}{\pi} \log\log(1/\epsilon) + o(\log\log(1/\epsilon)). \tag{3.15}$$

Proof. The substitution $t = \log r$ gives

$$q[w] = \int_a^\infty (|w'|^2 - ct^{-2}|w|^2)dt, \qquad \|w\|^2 = \int_a^\infty e^{2t}|w|^2 dt, \qquad w(a) = 0.$$

Lemma 3.6 proves $q \geq 0$ when $c \leq 1/4$. For $c > 1/4$, put $s = \log t$, $w = t^{1/2}z(s)$ and $\sigma = \sqrt{c - 1/4}$. The mixed derivative is a total derivative with zero endpoint values. Thus

$$q[w] = \int_{\log a}^\infty (|z'|^2 - \sigma^2|z|^2)ds, \qquad \|w\|^2 = \int_{\log a}^\infty W(s)|z|^2 ds, \quad W(s) = e^{2e^s + 2s}.$$

For every fixed $b > 0$ and sufficiently small ϵ, there is a unique $s_b(\epsilon) > \log a$ with $\epsilon W(s_b) = b$, since $W' > 0$ and $W(s) \to \infty$. Writing $L = \log(1/\epsilon)$, its defining equation is $2e^{s_b} + 2s_b = L + \log b$. For large L this implies $\log(L/4) < s_b < \log L$: substitution at the two endpoints gives values respectively smaller and larger than $L + \log b$. Hence

$$s_b(\epsilon) = \log\log(1/\epsilon) + O_b(1).$$

For an upper bound, cut at s_{σ^2} and allow independent Neumann traces. The outer form has potential $\epsilon W - \sigma^2 \geq 0$. On the inner interval, comparison with the constant potential $-\sigma^2$ gives

$$N(\epsilon) \leq \frac{\sigma}{\pi}(s_{\sigma^2} - \log a) + 1.$$

For $0 < \eta < \sigma^2$, use Dirichlet functions on $[\log a, s_\eta]$ extended by zero. There $\epsilon W - \sigma^2 \leq -(\sigma^2 - \eta)$, so

$$N(\epsilon) \geq \frac{\sqrt{\sigma^2 - \eta}}{\pi}(s_\eta - \log a) - 1.$$

Divide by $\log\log(1/\epsilon)$ and let first $\epsilon \downarrow 0$, then $\eta \downarrow 0$. This proves (3.15). Its positive coefficient implies infinite index; equivalently, disjoint translates of a long negative test function for the constant-potential form give negative spaces of arbitrary dimension.

More generally, let $V_0 \subset \ker \mathcal{B}_h$ have dimension m_0, and impose the same radial potential and boundary condition on each tensor in an orthonormal basis of V_0. Orthogonality gives

$$q_{V_0}[w_1, \dots, w_{m_0}] = \sum_{j=1}^{m_0} q[w_j], \qquad \|(w_1, \dots, w_{m_0})\|^2 = \sum_{j=1}^{m_0} \int_a^\infty e^{2t} |w_j|^2 dt.$$

The operator is the orthogonal direct sum of m_0 identical scalar operators. Hence $N_{V_0}(\epsilon) = m_0 N(\epsilon)$ for every $\epsilon > 0$. In particular, the coefficient in (3.15) is multiplied by m_0 when the whole space V_0 is used. □

4. Pointwise comparison on conformally Kähler four-manifolds

4.1. **The scalar equation.** In this section (M^4, k, J) is closed and Kähler,

$$u = \mathrm{Scal}_k > 0, \qquad g = u^{-2} k, \qquad \mathrm{Ric}_g = 3g, \qquad D_k = \mathrm{div}_k \nabla = -\Delta_k. \tag{4.1}$$

Put $w = |du|_g^2$. The conformal Ricci formula, in the conventions of [3], is

$$\mathrm{Ric}_g = \mathrm{Ric}_k + 2u^{-1} \mathrm{Hess}_k\, u + (u^{-1} D_k u - 3u^{-2} |du|_k^2) k. \tag{4.2}$$

Every tensor except the Hessian on the right is J-invariant. Thus

$$\mathrm{Hess}_k\, u(JX, JY) = \mathrm{Hess}_k\, u(X, Y). \tag{4.3}$$

Tracing (4.2) with k, and using $\mathrm{Scal}_k = u$, gives

$$12 = u^3 + 6u D_k u - 12 |du|_k^2. \tag{4.4}$$

For a scalar function v in dimension four, $D_g v = u^2 D_k v - 2u \langle du, dv \rangle_k$. Hence

$$D_g u = F(u), \qquad\qquad F(u) = 2u - u^4/6, \tag{4.5}$$

$$D_k(u^2) = 4 + 6w/u^2 - u^3/3. \tag{4.6}$$

The second equality follows by expanding $D_k(u^2) = 2u D_k u + 2|du|_k^2$ and using $|du|_k^2 = w/u^2$.

Theorem 4.1. *Let $I \subset (0, \infty)$ be an open interval containing the range of u, and let $Q \in C^2(I)$ be positive. Define*

$$\begin{aligned} A &= 2(Q - uQ')^2 - u^2 Q Q'', \\ B &= (6 - u^3) Q + (u^4/2 - 6u) Q', \\ C &= (2 - u^3/6)^2. \end{aligned} \tag{4.7}$$

If $A > 0$, $A + B + C > 0$, and $2A + B > 0$ on I, then

$$|du|_g^2 < u^2 Q(u). \tag{4.8}$$

Proof. Let $q(u) = u^2 Q(u)$ and $c = \max_M w/q(u)$. Suppose $c \geq 1$, and let x attain the maximum. The function $w - cq(u)$ has maximum zero at x. Since $w(x) = cq(u(x)) > 0$, choose a g-orthonormal frame with $e_1 = \nabla_g u/\sqrt{w}$, $e_2 = Je_1$, and $e_4 = Je_3$. Set $H = \mathrm{Hess}_g\, u$. At x,

$$w = cq, \qquad dw = cq'\, du, \qquad D_g(w - cq) \leq 0. \tag{4.9}$$

As $dw(e_j) = 2H(\nabla u, e_j)$, we have $H_{11} = cq'/2$ and $H_{1j} = 0$ for $j > 1$.

The conformal Hessian relation is

$$H = \mathrm{Hess}_k\, u + 2u^{-1} du \otimes du - u^{-1} w g. \tag{4.10}$$

By (4.3), $H_{22} = H_{11} - 2cq/u$ and $H_{33} = H_{44}$. Also $\mathrm{tr}_g H = F(u)$. Since $|H|^2 = \sum_i H_{ii}^2 + 2 \sum_{i<j} H_{ij}^2$ and $H_{33} = H_{44} = (F - cq' + 2cq/u)/2$, nonnegativity of the off-diagonal squares gives

$$|H|^2 \geq (cq'/2)^2 + (cq'/2 - 2cq/u)^2 + \tfrac{1}{2}(F - cq' + 2cq/u)^2. \tag{4.11}$$

Bochner's identity, $\mathrm{Ric}_g = 3g$, and (4.5) yield

$$\begin{aligned} 0 &\geq D_g(w - cq) \\ &= 2|H|^2 + 2(F' + 3)cq - cq'F - c^2 q'' q \\ &\geq u^2(Ac^2 + Bc + C). \end{aligned} \tag{4.12}$$

Substitute $q' = 2uQ + u^2 Q'$ and $q'' = 2Q + 4uQ' + u^2 Q''$ into (4.11). The coefficients of $c^2, c, 1$, after division by u^2, are respectively

$$\begin{aligned} &2Q^2 - 4uQQ' + 2u^2(Q')^2 - u^2 QQ'', \\ &(6 - u^3)Q + (u^4/2 - 6u)Q', \\ &(u^3 - 12)^2/36. \end{aligned} \tag{4.13}$$

The first is $2(Q - uQ')^2 - u^2 QQ''$, and the third is $(2 - u^3/6)^2$. Thus these coefficients are A, B, C.

For $c \geq 1$,

$$Ac^2 + Bc + C = (A + B + C) + (c - 1)(2A + B) + A(c - 1)^2 > 0. \tag{4.14}$$

This contradicts (4.12). Therefore $c < 1$, proving (4.8). ☐

Remark 4.2. Only the differential relation $dw = cq'\,du$ at a maximum point is used. The argument does not assume that $|du|_g^2$ is globally a function of u, nor does it assume that Q is concave.

4.2. **Two explicit comparisons.** For a polynomial $f(X) = \sum_{j=0}^{n} f_j X^j$ and rational $l < h$, define

$$\mathcal{T}_{l,h} f(z) = (1 + z)^n f\left(\frac{l + hz}{1 + z}\right). \tag{4.15}$$

Its coefficient of z^k is the rational number

$$C_k(f; l, h) = \sum_{j=0}^{n} f_j \sum_{i=0}^{j} \binom{j}{i}\binom{n - j}{k - i} l^{j-i} h^i, \tag{4.16}$$

where an out-of-range binomial coefficient is zero. If all C_k are positive, then $f > 0$ on $[l, h]$: the substitution covers $[l, h)$ for $z \geq 0$, and its leading coefficient is $f(h) > 0$.

Corollary 4.3. *Under* (4.1), *suppose* $7/4 < u < 10/3$. *Then*

$$|du|_g^2 < u^2(u - 7/4)(10/3 - u), \qquad D_k(u^2) < 115/32. \tag{4.17}$$

Proof. Put $Q = (u - 7/4)(10/3 - u)$. On the scalar range $Q > 0$ and $Q'' = -2$, so $A = 2(Q - uQ')^2 + 2u^2 Q > 0$. Direct expansion of (4.7) gives

$$\begin{aligned} P &:= A + B + C = (2u^6 - 183u^4 + 1104u^3 - 2088u^2 + 2668)/72, \\ S &:= 2A + B = (-183u^4 + 1884u^3 - 4608u^2 + 7280)/72. \end{aligned} \tag{4.18}$$

The following coefficient identities use (4.15):

$$\begin{aligned} 107495424\, \mathcal{T}_{7/4,10/3} P &= 793310193 + 2653373376z + 3713014728z^2 \\ &\quad + 4406019840z^3 + 4719015936z^4 \\ &\quad + 2877014016z^5 + 758112256z^6, \\ \frac{497664}{19}\, \mathcal{T}_{7/4,10/3} S &= 563409 + 674352z + 393984z^2 \\ &\quad + 1889280z^3 + 1187840z^4. \end{aligned} \tag{4.19}$$

Every displayed coefficient is positive. Theorem 4.1 therefore proves the first inequality in (4.17). The identity

$$\frac{u^3}{18} - \frac{1}{24} - Q = \frac{(u - 2)^2(u + 4)}{18} + \left(u - \frac{53}{24}\right)^2 + \frac{5}{192} \tag{4.20}$$

and (4.6) imply $D_k(u^2) < 4 + 6Q - u^3/3 \leq 15/4 - 30/192 = 115/32$. ☐

Proposition 4.4. *Under* (4.1), *assume*

$$a_0 := 359/200 < u < b_0 := 3257/1000. \tag{4.21}$$

Then $D_k(u^2) < 21/10$.

Proof. Choose

$$Q(u) = (u - a_0)(b_0 - u)(600u - 277)/2000. \tag{4.22}$$

All three factors are positive on the scalar range. Form A, B, C as in (4.7). The three degree-six polynomials A, $P = A + B + C$, and $S = 2A + B$ have positive coefficients after the transform $\mathcal{T}_{a_0,b_0}$; the full positive vectors and normalization factors are in Appendix C. Formula (4.16) specifies every entry by integer arithmetic. Thus Theorem 4.1 applies.

Let $G(u) = 21/10 - 4 - 6Q(u) + u^3/3$. The two exact identities

$$\begin{aligned} 600000000\,\mathcal{T}_{a_0,2}G &= 16706975 + 14847600z + 18995406z^2 + 31882141z^3, \qquad (4.23)\\ 15000000000\,\mathcal{T}_{2,b_0}G &= 797053525 + 14141200620z \\ &\quad + 94040334084z^2 + 144252077965z^3 \end{aligned}$$

prove $G > 0$ throughout $[a_0, b_0]$. Consequently (4.6) and (4.8) give $D_k(u^2) < 4 + 6Q(u) - u^3/3 < 21/10$. ☐

4.3. **Toric scalar curvature and Einstein normalization.** The CLW metric is conformal to the extremal Kähler metric in the bilaterally symmetric critical class described in [7, Proposition 7, Corollary 8 and Theorem 27]. In the following coordinates, its scalar curvature and the equation of its Kähler class are given by [20, Sections 4.2 and 6.1].

Use angular periods 2π and symplectic potential $\frac{1}{2}\sum l_\alpha \log l_\alpha$ + smooth. The moment polygon is

$$P_a = \operatorname{conv}\{(0,0), (a,0), (a,1), (1,a), (0,a)\}, \qquad a > 1. \tag{4.24}$$

Let $t = x + y$. The extremal scalar curvature is affine and exchange-invariant, hence $s_0 = b + ct$. Write

$$\begin{aligned} D &= a^6 + 6a^5 + 9a^4 + 4a^3 - 3a^2 - 6a + 1,\\ B_0 &= a^5 + 7a^4 + 6a^3 + 2a^2 - 5a - 3,\\ C_0 &= 4(1 - a^3),\\ R &= B_0 + (a+1)C_0, \qquad E = B_0^2 + 2C_0D. \end{aligned} \tag{4.25}$$

Then

$$s_0 = \frac{12}{D}(B_0 + C_0 t), \qquad b = \frac{12B_0}{D}, \quad c = \frac{12C_0}{D}. \tag{4.26}$$

Formula (4.26), with the normalization in (4.24), is the toric scalar formula in [20, Sections 4.2 and 6.1]. The CLW construction [7] selects a parameter $a > 1$ satisfying

$$p(a) = a^{10} + 5a^9 + 6a^8 - 12a^7 - 34a^6 - 42a^5 + 20a^3 + 21a^2 - 7a - 6 = 0. \tag{4.27}$$

The root in $(1, \infty)$ is unique and determines the CLW class by [20, Section 6.1, equation (6.9) and Lemma 6.1].

Lemma 4.5. *For the CLW metric normalized by* $\mathrm{Ric}_g = 3g$, *its canonical scalar satisfies* (4.21) *and*

$$(\min u)^3 < 579/100. \tag{4.28}$$

Proof. By the cited uniqueness of the CLW root, exact endpoint substitution, with numerators recorded in Appendix C, gives

$$a_- := 19577/10000 < a < a_+ := 19578/10000. \tag{4.29}$$

The first four polynomial identities listed below give $D, B_0, R, E > 0$ on $[a_-, a_+]$. In particular all denominators used in the normalization are positive.

At the polygon vertex $(0,0)$, the inverse symplectic Hessian of the actual smooth toric metric has $U^{ij} = 2x_i\delta_{ij} + O(|x|^2)$. Write the Hessian as $\operatorname{diag}(1/(2x_1), 1/(2x_2)) + S(x)$ with S smooth, and invert using $(D_0 + S)^{-1} = D_0^{-1} - D_0^{-1}S(I + D_0^{-1}S)^{-1}D_0^{-1}$. Since $D_{k_0}v = \partial_i(U^{ij}\partial_j v)$ for invariant functions, (4.26) gives $|ds_0|^2_{k_0} = 0$ and $D_{k_0}s_0 = 4c$ at this vertex. The scalar curvature of $g_0 = s_0^{-2}k_0$, which is constant by the CLW theorem, is therefore

$$\kappa = b^3 + 24bc = 1728B_0E/D^3. \tag{4.30}$$

Let $k = (\kappa/12)^{1/3}k_0$ and $u = (12/\kappa)^{1/3}s_0$. Then $u = \mathrm{Scal}_k$ and $u^{-2}k = (\kappa/12)g_0$ has Ricci tensor $3g$. As $C_0 < 0$ and $0 \le t \le a+1$,

$$u^3_{\min} = \frac{12R^3}{B_0E}, \qquad u^3_{\max} = \frac{12B_0^2}{E}. \tag{4.31}$$

The transforms on $[a_-, a_+]$ of the following polynomials have strictly positive coefficients:

$$\begin{gathered} D,\ B_0,\ R,\ E, \\ 12\cdot 200^3R^3 - 359^3B_0E, \qquad 3257^3E - 12\cdot 1000^3B_0^2, \\ 579B_0E - 1200R^3. \end{gathered} \tag{4.32}$$

The coefficient vectors are given in Appendix C; each entry is defined by (4.16). The first four signs justify every division and the positivity of the Einstein scale. The remaining three signs, inserted into (4.31), give (4.21) and (4.28). □

Theorem 4.6. *For the normalized CLW metric,*

$$207/100 < \sup_M D_k(u^2) < 21/10 < 15/4. \tag{4.33}$$

Proof. Lemma 4.5 and Proposition 4.4 give the pointwise upper inequality. The smooth function $D_k(u^2)$ attains its maximum at a point of the compact manifold M; the pointwise strict inequality at that point proves the strict upper bound for the supremum. At a minimum point of u, $w = 0$. Equations (4.6) and (4.28) give $D_k(u^2) = 4 - u^3_{\min}/3 > 4 - 579/300 = 207/100$. □

4.4. **Tensor eigenvalues.** For a real anti-self-dual harmonic two-form η of k, put $\widetilde{h}(X,Y) = \eta(X, JY)$ and $h = u^2\widetilde{h}$. Hall–Haslhofer–Siepmann's identity [10, Theorem 6.3] states that h is TT and

$$\langle L_g h, h\rangle_{L^2(g)} = \int_M D_k(u^2)|h|_g^2\, dv_g.$$

The map $\eta \mapsto h$ is injective. Appendix A proves these statements in the conventions of (1.1). For the Page metric at Einstein constant 3, the bound $D_k(s^2) < 53/20$ of [10, Section 7.1] is proved in Appendix B. Both statements are used in the following consequence.

Theorem 4.7. *For every closed positive Einstein–Hermitian four-manifold with* $\mathrm{Ric}_g = \Lambda g$ *and canonical* $g = s^{-2}k$,

$$D_k(s^2) < 53\Lambda/60. \tag{4.34}$$

If $m = b_2^- > 0$, *then*

$$\lambda_m^{\mathrm{TT}}(L_g) < 53\Lambda/60, \qquad \lambda_m^{\mathrm{TT}}(E_g) < -67\Lambda/60. \tag{4.35}$$

For CLW, these bounds improve to $\lambda_2^{\mathrm{TT}}(L_g) < 7\Lambda/10$ *and* $\lambda_2^{\mathrm{TT}}(E_g) < -13\Lambda/10$.

Proof. LeBrun's classification [18, Theorem A] identifies a compact positive Einstein–Hermitian metric as Kähler–Einstein, Page, or CLW. In the Kähler–Einstein case with $\Lambda = 3$, put $s = 12^{1/3}$, $k = s^2g$; then $\mathrm{Scal}_k = s$ and $D_k(s^2) = 0$. Proposition B.1 and Theorem 4.6 treat the other cases. This proves (4.34) when $\Lambda = 3$.

Hodge theory gives an m-dimensional space of harmonic anti-self-dual forms. Proposition A.1 maps it injectively into TT tensors. The continuous pointwise coefficient has maximum strictly below 53/20, or below 21/10 in the CLW case. The same strict bound holds on the entire trial space, so min–max gives the asserted eigenvalue bound. For CLW, $b_2^- = 2$ in the complex orientation.

Under $k \mapsto ck$, its scalar becomes $c^{-1}s$ and $s^{-2}k \mapsto c^3 g$; both $D_k(s^2)$ and the eigenvalues of L_g scale by c^{-3}. Taking $c^3 = \Lambda/3$ reduces arbitrary Λ to three. Subtracting 2Λ gives the Einstein-operator bounds. ☐

Corollary 4.8. *For the CLW link normalized by* $\mathrm{Ric} = 3g$,

$$\mathcal{D}(g) > \frac{2}{\pi}\sqrt{33/20}. \tag{4.36}$$

Every logarithmic cone annulus of length $T > \pi\sqrt{20/33}$ contains a two-dimensional negative TT space. For a positive Einstein–Hermitian link normalized by $\mathrm{Ric} = 3g$, *every logarithmic annulus of length $T > \pi\sqrt{10/11}$ contains a negative TT space of dimension b_2^-.*

Proof. Theorem 4.7 and Proposition 2.3 give the density bound. On the harmonic-form trial space, Lemma 2.5 yields

$$\mathfrak{q}_C(r^{1/2}\chi(\log r)h) < \|h\|^2 \int \left(|\chi'|^2 - (33/20)|\chi|^2\right) dt.$$

The exponent $1/2$ is $2 - 3/2$ for a covariant tangential lift. The first Dirichlet sine function on length T has derivative quotient $\pi^2/T^2 < 33/20$. Approximate it in H_0^1 by a smooth compactly supported function, preserving this strict inequality. Using this same cutoff on the two-dimensional trial space gives the stated negative space. Replace $33/20$ by $15/4 - 53/20 = 11/10$ for the general Hermitian bound. ☐

5. Finite quotients of Einstein products

5.1. Low tensor modes on a round product.

Let

$$(\widetilde{Y}^d, h) = \prod_{a=1}^{k} (S^{m_a}, h_a), \quad m_a \geq 2, \quad k \geq 2, \quad \mathrm{Ric}_{h_a} = \Lambda h_a, \quad \Lambda > 0, \quad K_a = \Lambda/(m_a - 1). \tag{5.1}$$

Here $h = \sum h_a$ and K_a is the sectional curvature of the ath factor. The tensor bundle splits orthogonally into parallel subbundles

$$S^2 T^* \widetilde{Y} = \bigoplus_a \mathbb{R} h_a \;\oplus\; \bigoplus_a S_0^2 T^* S^{m_a} \;\oplus\; \bigoplus_{a<b} T^* S^{m_a} \odot T^* S^{m_b}. \tag{5.2}$$

The connection and curvature endomorphism preserve the splitting. The product spectral decomposition is treated in [14, Section 4]; the estimates needed here are proved next.

Lemma 5.1. *On a round sphere of dimension $m \geq 2$ and sectional curvature K,*

$$\int |\nabla \alpha|^2 \geq K \int |\alpha|^2 \tag{5.3}$$

for every smooth one-form α.

Proof. Hodge decomposition writes $\alpha = df + \beta$, $\delta\beta = 0$; there is no harmonic one-form by positive Ricci curvature. The decomposition is orthogonal for the rough form, since $\Delta_H = \nabla^*\nabla + (m-1)K$ preserves both summands. On a scalar eigenfunction with eigenvalue $\lambda \geq mK$, the one-form df has rough eigenvalue $\lambda - (m-1)K \geq K$. Spectral expansion proves the estimate on exact forms. For β coclosed, integration by parts gives $2\|\delta^*\beta\|^2 = \|\nabla\beta\|^2 - (m-1)K\|\beta\|^2$. Thus

$$\|\nabla\alpha\|^2 = \|\nabla df\|^2 + \|\nabla\beta\|^2 \geq K\|df\|^2 + (m-1)K\|\beta\|^2 \geq K\|\alpha\|^2,$$

since $m \geq 2$. ☐

Proposition 5.2. *The full Lichnerowicz operator on* (5.1) *is nonnegative. Its kernel is the space of parallel tensors $\sum c_a h_a$. On the orthogonal complement of this space its eigenvalues are bounded below by*

$$c_* = \min\left\{ \min_a m_a K_a, \quad \min_a (2\Lambda + 2K_a), \quad \min_{a<b} (2\Lambda + K_a + K_b) \right\} > 0. \tag{5.4}$$

Proof. For the factor-trace part $\sum f_a h_a$, the metric tensors are parallel and $\mathring{R}(f_a h_a) = \Lambda f_a h_a$, so $L(f_a h_a) = (\Delta f_a)h_a$. After removal of its constant coefficient, f_a has scalar spectral lower bound $\min_b m_b K_b$ on the product; this follows by expansion in product spherical harmonics.

On a within-factor trace-free tensor T_a, the curvature action is $-K_a T_a$. Thus $LT_a = \nabla^*\nabla T_a + (2\Lambda + 2K_a)T_a$. On a mixed block the curvature action is zero. Applying Lemma 5.1 in each of its two one-form factors, with all other coordinates fixed, and then integrating gives the rough bound $K_a + K_b$. Derivatives in the remaining factors have nonnegative energy. Thus the mixed block has the last lower bound in (5.4). Adding these orthogonal inequalities proves the claim and identifies the kernel. □

Theorem 5.3. *Normalize* (5.1) *by* $\Lambda = d - 1$*, and let a finite group* Γ *act freely by isometries. Let* s *be the number of orbits of its permutation action on the de Rham factors. For* $4 \le d \le 8$*, the TT spectrum below* b_d *on* $Y = \widetilde{Y}/\Gamma$ *consists exactly of the eigenvalue zero with multiplicity* $s - 1$*. Consequently*

$$\mathcal{D}(h_Y) = \frac{s-1}{\pi}\sqrt{b_d}, \qquad C(Y) \text{ is stable} \iff s = 1. \tag{5.5}$$

For $d \ge 9$ *every such quotient cone is stable.*

Proof. An isometry permutes the irreducible parallel distributions of the round product, and can permute only isometric factors. This follows from the uniqueness of the de Rham decomposition of a complete simply connected Riemannian manifold. The invariant parallel tensors have one coefficient on each factor orbit. Trace-freeness imposes the single nonzero linear condition $\sum_a m_a c_a = 0$, giving dimension $s - 1$. Parallel tensors are divergence-free.

For $d = 4$, both factors have dimension two, so the smallest positive bound in (5.4) is $6 > 15/4 = b_4$. For $d = 5$, $b_5 = d - 1$ and every term in (5.4) exceeds $d - 1$. For $6 \le d \le 8$, $b_d \le d - 1$, so the same inequality applies. By Proposition 2.2, no new eigenvalues appear on a quotient. This proves the exact low spectrum and (5.5). If $d \ge 9$, $b_d \le 0$ whereas $L \ge 0$ by Proposition 5.2; the cone criterion proves stability. □

The relative-scaling tensor and its compact Einstein instability are discussed in [24, Introduction]. For the cone, the dimension threshold follows from the change of sign of b_d in (1.8), together with the cone criterion [15, Theorem 1.2]. Formula (5.5) gives the invariant multiplicity for each quotient.

5.2. **A free quaternionic action.** Let $m \ge 2$ and put

$$\widetilde{Z}_m = (S^m)^4, \qquad \widetilde{h} = \sum_{a=0}^{3} h_a, \qquad h_a = (m-1)\Lambda^{-1}\gamma_a, \tag{5.6}$$

where γ_a is unit-round. On $x_a \in S^m \subset \mathbb{R}^{m+1}$ define

$$\begin{aligned} I(x_0, x_1, x_2, x_3) &= (-x_1, x_0, -x_3, x_2), \\ J(x_0, x_1, x_2, x_3) &= (-x_2, x_3, x_0, -x_1), \\ A(x_0, x_1, x_2, x_3) &= (-x_0, -x_1, -x_2, -x_3). \end{aligned} \tag{5.7}$$

Proposition 5.4. *The transformations* I, J *generate a free orientation-preserving isometric action of* Q_8*. Its quotient* Z_m *has* $\pi_1(Z_m) = Q_8$*. If* m *is even,* Z_m *is a rational homology sphere.*

Proof. The signed permutation matrices on the four coordinates are

$$M_I = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad M_J = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}. \tag{5.8}$$

Their products satisfy $M_I^2 = M_J^2 = -I_4$ and $M_I M_J = -M_J M_I$. The eight distinct matrices $\pm I_4, \pm M_I, \pm M_J, \pm M_I M_J$ are closed under multiplication, by these identities, and have the quaternion multiplication law. The associated transformations are isometries because all factor metrics coincide.

The transformation A is antipodal in every factor and has no fixed point. Each of the other six nonidentity transformations has square A. A fixed point for one of them would therefore be fixed by A, which is impossible.

Each generator has two transpositions of m-dimensional factors and two antipodal maps. Its degree is $(-1)^{2m}(-1)^{2(m+1)} = 1$. The action is consequently orientation preserving. The simply connected product is its universal cover, so $\pi_1(Z_m) = Q_8$.

Suppose m is even. With $|u_a| = m$, the rational cohomology is $\mathbb{Q}[u_0, u_1, u_2, u_3]/(u_0^2, u_1^2, u_2^2, u_3^2)$. The central element sends every u_a to $-u_a$; therefore its invariant subspaces in degrees m and $3m$ vanish. In degree $2m$, write $v_{ab} = u_a u_b$, $a < b$. Since m is even, these generators commute. The actions are

$$\begin{array}{c|cccccc} & v_{01} & v_{02} & v_{03} & v_{12} & v_{13} & v_{23} \\ \hline I^* & -v_{01} & v_{13} & -v_{12} & -v_{03} & v_{02} & -v_{23} \\ J^* & -v_{23} & -v_{02} & v_{12} & v_{03} & -v_{13} & -v_{01} \end{array} \tag{5.9}$$

The first row gives the fixed subspace $\operatorname{span}\{v_{02} + v_{13}, v_{03} - v_{12}\}$. The second row is minus the identity on that subspace. There are thus no invariant classes of degree $2m$. Constants and the orientation class are invariant, and there are no other degrees.

For the covering p, the rational transfer satisfies $p_*p^* = 8I$ and $p^*p_* = \sum_{\gamma\in Q_8} \gamma^*$. These identities follow by lifting a singular simplex to each of the eight sheets and summing the lifts. They make p^* injective with image exactly the invariant cohomology. Hence $H^*(Z_m;\mathbb{Q})$ is the cohomology of a $4m$-sphere. Finally, the abelianization of the presentation $I^2 = J^2 = A$, $IJ = AJI$ imposes $A = 1$ and leaves two independent commuting elements of order two. Thus $H_1(Z_m;\mathbb{Z}) = (\mathbb{Z}/2)^2$, and Z_m is not an integral homology sphere. □

5.3. **Exact scalar and tensor gaps.** Set $K = \Lambda/(m - 1)$.

Theorem 5.5. *The quotient metric satisfies*

$$\lambda_1^{\mathrm{sc}}(h_m) = \lambda_1^{\mathrm{TT}}(L_{h_m}) = 2mK, \qquad \lambda_1^{\mathrm{TT}}(E_{h_m}) = 2K. \tag{5.10}$$

Proof. A product spherical harmonic of degrees $(\ell_0,\dots,\ell_3)$ has scalar eigenvalue $K\sum_a \ell_a(\ell_a + m - 1)$ and parity $(-1)^{\sum_a \ell_a}$ under A. The only positive eigenvalue below $2mK$ is mK, from degree one on a single factor. It is A-odd. Every mean-zero A-even function therefore has Rayleigh quotient at least $2mK$. To attain the bound by a Q_8-invariant function, let y_a, z_a be two coordinate functions on the ath sphere and put

$$F = y_0z_1 - y_1z_0 - y_2z_3 + y_3z_2. \tag{5.11}$$

Each term has degree one in two factors. Put $F_{ab} = y_az_b - y_bz_a$. Then $I^*F_{01} = F_{01}$, $I^*F_{23} = F_{23}$, $J^*F_{01} = -F_{23}$ and $J^*F_{23} = -F_{01}$. Since $F = F_{01} - F_{23}$, both generators fix F. At $x_0 = e_1$, $x_1 = e_2$, $x_2 = x_3 = e_1$, with $y = x^1$ and $z = x^2$, one has $F = 1$. Hence F descends and has eigenvalue $2mK$.

Let T be the lift to $(S^m)^4$ of a TT tensor on the quotient, and decompose it as in (5.2). In its trace part $\sum f_ah_a$, each f_a is A-even. Transitivity of the factor permutations makes the four integrals of f_a equal, while trace-freeness gives $m\sum_a f_a = 0$ pointwise. Every f_a therefore has zero mean. The preceding scalar estimate bounds this block by $2mK$.

For a within-factor trace-free block, Proposition 5.2 gives $2\Lambda + 2K = 2mK$. For a mixed block, Lemma 5.1 in both factors gives the same lower bound. Orthogonality of the blocks gives $\langle L_hT, T\rangle \geq 2mK\|T\|^2$ for every quotient TT tensor. The proof did not assume that individual blocks are divergence-free.

For equality, let $f_a = x_a^1$ and define

$$H = f_0f_1(h_2 - h_3) - f_2f_3(h_0 - h_1). \tag{5.12}$$

Each summand has zero trace. Its coefficient depends only on factors complementary to the metric factors; contraction of its derivative in a supporting metric factor is therefore zero. Thus $\delta H = 0$. Write $U = f_0f_1(h_2 - h_3)$ and $V = f_2f_3(h_0 - h_1)$, so $H = U - V$. The pullbacks are $I^*U = U$, $I^*V = V$, $J^*U = -V$ and $J^*V = -U$. Thus $I^*H = J^*H = H$. The metric factors are parallel and their curvature term cancels 2Λ, so $LH = 2mKH$. At a point with $f_0 = f_1 = 1$ and $f_2 = f_3 = 0$, it equals $h_2 - h_3 \neq 0$. This proves the TT equality. Subtracting $2\Lambda = 2(m-1)K$ proves the last equality in (5.10). □

5.4. **The quotient cones.** Normalize $\Lambda = 4m-1$, so that $dr^2 + r^2 h_m$ is Ricci-flat.

Theorem 5.6. *For every compactly supported symmetric tensor away from the vertex,*

$$\mathfrak{q}_C(T) \geq \frac{(4m-1)^2}{4} \int_C r^{-2} |T|^2 \, dv_C. \tag{5.13}$$

The constant is sharp on all symmetric tensors. The restricted holonomy is $SO(4m+1)$. For $m = 2$, the quotient cone is stable and its eightfold covering product cone has infinite negative index.

Proof. Theorem 5.5 gives $E_h|_{\mathrm{TT}} > 0$ and

$$\lambda_1^{\mathrm{sc}}(h_m) - 2(4m+1) = \frac{2(2m+1)}{m-1} > 0.$$

Lemma 2.4(b) gives $\mathcal{A}_h \geq 0$. Equation (2.11) then implies (5.13). The constant angular tensor given by the cone metric has $\mathcal{A}_h$-eigenvalue zero. Multiplying it by $r^{-\nu}\chi(t/T)$ makes the derivative quotient tend to zero as $T \to \infty$. This proves sharpness on the full tensor space, without an assertion of TT sharpness.

Restricted holonomy is unchanged by a covering, so it suffices to compute it on $C((S^m)^4)$. At $r = 1$ write the tangent space as $\mathbb{R}\partial_r \oplus V$, $V = \bigoplus_a V_a$. For unit $X \in V_a$, $Y \in V_b$, $a \neq b$, the cone curvature is $R_C(X, Y) = -X \wedge Y$ on V, and kills ∂_r. Here $(X \wedge Y)Z = \langle Y, Z\rangle X - \langle X, Z\rangle Y$. Curvature endomorphisms belong to the holonomy algebra by the Ambrose–Singer theorem [1]. Their commutators give all within-factor rotations: for fixed unit $Y \in V_b$ and $X, X' \in V_a$, the commutator of $X \wedge Y$ and $X' \wedge Y$ is $-X \wedge X'$. Thus the holonomy algebra contains $\mathfrak{so}(V)$.

The identity $R_C(U, V)\partial_r = 0$ holds for all tangent fields. Covariantly differentiating it at $r = 1$ gives $(\nabla_Y R_C)(X, Y)\partial_r = -R_C(X, Y)Y \neq 0$ for mixed-factor X, Y, because $\nabla_Y \partial_r = Y$. Covariant derivatives of curvature also belong to the holonomy algebra: parallel-transport the curvature operators along a curve, take their difference quotient, and use that this finite-dimensional vector space is closed. Subtract the tangential part of this skew endomorphism, already in $\mathfrak{so}(V)$. A nonzero radial–tangential rotation remains. Its commutators with $\mathfrak{so}(V)$ give every radial–tangential rotation. The resulting algebra is $\mathfrak{so}(4m+1)$, proving the restricted holonomy statement.

For $m = 2$, Theorem 5.3 gives three covering zero modes, the trace-free relative scalings of four factors, and no other modes below $b_8 = 7/4$. Their Q_8-invariant space is zero because the permutation action is transitive. Thus the covering coefficient is $3\sqrt{7}/(2\pi)$ and the quotient coefficient is zero. The infinite-index and stability assertions follow from Proposition 2.6. □

The covering cone at $m = 2$ has the exact annular index

$$i_D = 3\#\left\{\ell \geq 1 : (\ell\pi/\log(R/r))^2 < 7/4\right\} = \frac{3\sqrt{7}}{2\pi}\log(R/r) + O(1). \tag{5.14}$$

Let $A = I^2 = J^2$ be the central antipodal map. All subgroups of Q_8 and their coefficients at $m = 2$ are given by

Γ	$\lvert\Gamma\rvert$	$\dim \mathcal{H}_0^\Gamma$	$\mathcal{D}_\Gamma((S^2)^4)$
$\{1\}$	1	3	$3\sqrt{7}/(2\pi)$
$\langle A\rangle$	2	3	$3\sqrt{7}/(2\pi)$
$\langle I\rangle$, $\langle J\rangle$, $\langle IJ\rangle$	4	1	$\sqrt{7}/(2\pi)$
Q_8	8	0	0

(5.15)

The column $|\Gamma|$ is the degree of the covering from $(S^2)^4$ to its quotient by Γ. The central order-two action fixes every factor metric. Each order-four subgroup has two factor orbits, so its invariant trace-free relative scalings have dimension one. Only the quotient by Q_8 has no invariant relative-scaling tensor. There are no other subgroups: every noncentral element has order four, its square is A, and two elements from distinct cyclic order-four subgroups generate Q_8.

Proposition 5.7. *Within finite free orientation-preserving quotients of products of at least two round spheres of dimensions at least two, nine is the least cone dimension for which the quotient cone is stable while the covering cone is unstable.*

Proof. For link dimension $4 \le d \le 7$, stability requires transitivity by Theorem 5.3. All factors then have equal dimension. The possibilities are $S^2 \times S^2$, $S^3 \times S^3$, and $(S^2)^3$.

A factor-interchanging map of $S^m \times S^m$ has the form $g(x, y) = (ay, bx)$ with $a, b \in O(m+1)$. If it acts freely, ab has no fixed point on S^m, since a fixed x would give the fixed pair (x, bx). The orthogonal map ab therefore has no eigenvalue 1. Its nonreal eigenvalues occur in conjugate pairs with product one, and its real eigenvalues are all -1. If there are j real eigenvalues, then $j \equiv m+1 \pmod 2$, so $\det(ab) = (-1)^j = (-1)^{m+1}$. The degree of g is consequently $(-1)^m \det(a)\det(b) = (-1)^m \det(ab) = -1$. Thus no orientation-preserving free group can act transitively on these two factors.

For $(S^2)^3$, transitivity implies $3 \mid |\Gamma|$. On the other hand, the Euler characteristic of the quotient is $8/|\Gamma|$, an integer, so $|\Gamma|$ divides eight. This is impossible. Theorem 5.6 gives the required oriented example at link dimension eight. The proposition concerns this covering phenomenon only; it does not assert a minimum dimension for stable nonflat cones in general. ☐

Corollary 5.8. *A smooth complete connected Ricci-flat manifold without boundary, with exterior* (1.11) *over Z_m satisfying* (1.12)*, has finite negative index. For $m = 2$, any such end over the covering product $(S^2)^4$ instead gives*

$$N_G(\epsilon) = \frac{3\sqrt{7}}{4\pi}\log(1/\epsilon) + o(\log(1/\epsilon)).$$

These are implications for a filling satisfying the stated hypotheses, not existence assertions.

Proof. For Z_m, Theorem 5.5 gives $\lambda_1^{\mathrm{TT}}(L) = 2m(4m-1)/(m-1) > 2(4m-1) > b_{4m}$. Apply the strict case of Theorem 3.7. The product coefficient for $m = 2$ is given by (5.14); Theorem 3.4 divides it by two. ☐

6. Einstein necks and four-dimensional links

6.1. Index on long necks.

Theorem 6.1. *Let (M_i^{d+1}, G_i), $d \ge 3$, be closed Einstein manifolds with $\mathrm{Ric}_{G_i} = \Lambda_i G_i$. Suppose they contain annuli identified with $(r_i, R_i) \times Z$, where $\mathrm{Ric}_h = (d-1)h$, such that*

$$T_i = \log(R_i/r_i) \longrightarrow \infty,$$
$$\delta_i := \sup_{(r_i,R_i)\times Z} \sum_{j=0}^{2} r^j |\nabla_C^j (G_i - G_C)|_{G_C} + |\Lambda_i| R_i^2 \longrightarrow 0. \tag{6.1}$$

For the full Lichnerowicz form with every tensor component vanishing on both annular boundary components,

$$i_D(G_i; (r_i, R_i) \times Z) = \mathcal{D}(h) T_i + o(T_i). \tag{6.2}$$

Moreover

$$\liminf_i \frac{\mathrm{ind}^-(L_{G_i}|_{\mathrm{TT}})}{T_i} \ge \mathcal{D}(h). \tag{6.3}$$

For finitely many disjoint necks the local contributions to the global lower bound add.

Proof. On the neck, $L_{G_i} = E_{G_i} + 2\Lambda_i I$. Since $r \le R_i$,

$$2|\Lambda_i| \int |T|^2 dv_{G_i} \le C|\Lambda_i| R_i^2 \int r^{-2} |T|_C^2 dv_C.$$

Lemma 3.1 therefore bounds the full form above and below by $(1 \pm C\delta_i)\mathfrak{q}_C \pm C'\delta_i Z$. After division by a positive scalar and the logarithmic change, these are the Dirichlet forms with angular operators $\mathcal{B}_h + \theta_i I$, $|\theta_i| \le C\delta_i$.

For every such shift, the exact sine count differs from $T_i\pi^{-1}\sum_j \sqrt{(-\beta_j-\theta_i)_+}$ by at most the number of angular eigenvalues $\beta_j < 1$. This number is finite and independent of large i. The finite sum is continuous at zero, so the two bounding counts are $\mathcal{D}(h)T_i + o(T_i)$. This proves (6.2).

Every finite-dimensional negative Dirichlet space on the annulus extends by zero to $H^1(M_i)$. Choose a basis of the negative Dirichlet space and approximate each basis vector in H^1_0 by a smooth tensor compactly supported in the annulus. The form is continuous in H^1 because the coefficients are smooth on the compact annulus. The energy and Gram matrices therefore satisfy (2.2) for a sufficiently close approximation. Lemma 2.1 preserves the negative dimension. Hence global min–max gives at least the local number of negative full eigenvalues. Equation (2.1) makes those global eigenvectors TT, by pairing their scalar and Hodge eigen-equations with the trace and divergence. For disjoint necks, the extended forms have zero cross terms. Their negative dimensions add, proving the last assertion. □

Corollary 6.2. *Let (M_i^{d+1}, G_i) be closed Einstein manifolds, $d \geq 3$, with $\operatorname{ind}^-(L_{G_i}) \leq I_0$. Suppose there are constants $c_i > 0$, open sets $U_i \subset C(Z) = (0,\infty)\times Z$, and embeddings $\Phi_i : U_i \to M_i$ such that, for every compact subset $A \subset C(Z)$,*

$$A \subset U_i \quad \text{for all sufficiently large } i, \qquad \Phi_i^*(c_iG_i) \longrightarrow G_C \quad \text{in } C^\infty(A). \tag{6.4}$$

Then $C(Z)$ is stable. If the same hypotheses hold with a smooth complete Ricci-flat manifold (X,G) in place of $C(Z)$ in (6.4), then $\operatorname{ind}^-(\mathfrak{q}_G|_{C_c^\infty(S^2T^*X)}) \leq I_0$.

Proof. Write $\widehat{G}_i = c_iG_i$ and $\operatorname{Ric}_{G_i} = \Lambda_i G_i$. On covariant symmetric two-tensors,

$$L_{\widehat{G}_i} = c_i^{-1}L_{G_i}, \qquad \|T\|^2_{L^2(\widehat{G}_i)} = c_i^{(d-3)/2}\|T\|^2_{L^2(G_i)}. \tag{6.5}$$

The factors are positive, so $\operatorname{ind}^-(L_{\widehat{G}_i}) = \operatorname{ind}^-(L_{G_i})$. The Einstein constant of $\widehat{G}_i$ is Λ_i/c_i. Taking the scalar curvature at a fixed point of the limiting manifold in (6.4) gives $(d+1)\Lambda_i/c_i \to 0$.

Suppose that $C(Z)$ is unstable. For $N > I_0$, Proposition 2.6 supplies smooth compactly supported cone tensors $T_1,\ldots,T_N$ spanning a negative space. Their supports lie in the interior of a fixed compact union A of annuli. For sufficiently large i, define $T_{j,i} = (\Phi_i^{-1})^*T_j$ on $\Phi_i(U_i)$ and extend it by zero to M_i. This extension is smooth because T_j vanishes outside a compact subset of U_i. Both the energy and Gram matrices converge: their integrands on A involve the tensors, the metric through second derivatives, and the Einstein shift $2\Lambda_i/c_i$, which tends to zero. The domains of integration and supports are fixed. Lemma 2.1 therefore gives an N-dimensional negative space for $L_{\widehat{G}_i}$ for all sufficiently large i. Min–max contradicts the bound I_0.

For a complete limit, start with any finite-dimensional subspace of $C_c^\infty(S^2T^*X)$ on which $\mathfrak{q}_G$ is negative definite. A finite basis is supported in one compact subset of X. The same construction and convergence of both matrices bound its dimension by I_0. Taking the supremum over these spaces proves the assertion. No existence or smoothness of a rescaling limit is inferred from the index bound. □

The transfer of finite-dimensional negative spaces also occurs in Yudowitz's index semicontinuity theorem for bubble-tree limits of Ricci shrinkers [27, Theorem 1.5]. His weighted operator, entropy and curvature assumptions, and finite bubble-tree hypothesis differ from those above. His Theorem 1.10 concerns a weighted asymptotic-cone comparison under a semiboundedness assumption. Neither theorem is used in the proof of Corollary 6.2.

6.2. **Complete Ricci-flat Böhm metrics.** Let $p,q \geq 2$, $d = p+q \leq 8$, and let γ_p, γ_q be unit-round metrics. Böhm's construction [5, Theorem 6.1] gives a smooth complete Ricci-flat metric

$$G_{p,q} = dr^2 + a(r)^2\gamma_p + b(r)^2\gamma_q \quad \text{on } \mathbb{R}^{p+1}\times S^q, \tag{6.6}$$

with the S^p factor collapsing smoothly at $r=0$. Its scale may be fixed by $b(0)=1$. Put

$$\alpha_p = \sqrt{\frac{p-1}{d-1}}, \qquad \alpha_q = \sqrt{\frac{q-1}{d-1}}, \qquad h = \alpha_p^2\gamma_p + \alpha_q^2\gamma_q.$$

Then $\mathrm{Ric}_h = (d-1)h$.

Corollary 6.3. *The full negative spectrum of* (6.6)*, counted with multiplicity, satisfies*

$$N_{G_{p,q}}(\epsilon) = \frac{\sqrt{(d-1)(9-d)}}{4\pi}\log(1/\epsilon) + o(\log(1/\epsilon)). \tag{6.7}$$

Every eigentensor counted here is smooth and TT.

Proof. The asymptotic estimates in [2, Section 6.4, equation (6.9)] give

$$\begin{aligned} a/(\alpha_p r) &\longrightarrow 1, \quad r(\log a)' \longrightarrow 1, \quad r^2(\log a)'' \longrightarrow -1, \\ b/(\alpha_q r) &\longrightarrow 1, \quad r(\log b)' \longrightarrow 1, \quad r^2(\log b)'' \longrightarrow -1. \end{aligned} \tag{6.8}$$

Set $\xi_1 = \log(a/(\alpha_p r))$, $\xi_2 = \log(b/(\alpha_q r))$ and $\theta_j = e^{2\xi_j} - 1$ for $j = 1, 2$. Equation (6.8) implies

$$\xi_j \to 0, \quad r\xi_j' \to 0, \quad r^2\xi_j'' \to 0, \qquad \theta_j' = 2e^{2\xi_j}\xi_j', \quad \theta_j'' = 2e^{2\xi_j}\xi_j'' + 4e^{2\xi_j}(\xi_j')^2.$$

Consequently $|\theta_j| + r|\theta_j'| + r^2|\theta_j''| \to 0$. For $\Pi_1 = r^2\alpha_p^2\gamma_p$ and $\Pi_2 = r^2\alpha_q^2\gamma_q$, defined by the projections to the first and second sphere factors, respectively,

$$G_{p,q} - G_C = \theta_1\Pi_1 + \theta_2\Pi_2, \qquad |\nabla_C^k \Pi_j|_{G_C} \le C_k r^{-k} \quad (0 \le k \le 2).$$

For the dilation $D_\rho(r, y) = (\rho r, y)$, one has $D_\rho^* G_C = \rho^2 G_C$ and $D_\rho^*\Pi_j = \rho^2\Pi_j$. Naturality of the Levi-Civita connection and invariance under constant rescaling imply

$$|\nabla_C^k\Pi_j|_{G_C}(r, y) = r^{-k}|\nabla_C^k\Pi_j|_{G_C}(1, y).$$

The last factor is bounded on the compact link. The product rule therefore gives

$$\sup_{S^p\times S^q} \sum_{k=0}^{2} r^k|\nabla_C^k(G_{p,q} - G_C)|_{G_C} \le C\sum_{j=1}^{2}(|\theta_j| + r|\theta_j'| + r^2|\theta_j''|) \longrightarrow 0.$$

A truncation $r \le R$ is a compact smooth domain, and its whole complement is the stated conical exterior. Theorem 5.3 with two factors and the trivial group gives exactly one eigenvalue below b_d, namely 0. Hence $\mathcal{D}(h) = \sqrt{b_d}/\pi$. Apply Theorem 3.4 and Lemma 3.2; since $\sqrt{b_d}/(2\pi) = \sqrt{(d-1)(9-d)}/(4\pi)$, this proves (6.7). □

Infinitude of the negative spectrum in this family is established in [2, Main Theorem II], whose operator has the opposite sign. The assertion here specifies the leading coefficient of the full counting function. It makes no assertion that every negative eigentensor is invariant under the product isometry group.

6.3. **Anti-holomorphic quotients.** The kernel calculation used here is already proved in [20, Lemma 9.1 and Proposition 9.2]. If (Y^4, g, J) is closed and positive Kähler–Einstein, then $b_1(Y) = 0$, $b_2^+(Y) = 1$, and

$$\dim\ker(L_g|_{\mathrm{TT}}) \ge b_2^-(Y). \tag{6.9}$$

If a free anti-holomorphic isometric involution has quotient M, then

$$\dim\ker(L_{g_M}|_{\mathrm{TT}}) \ge b_2(M) + 1. \tag{6.10}$$

The zero eigenspace in (6.10) contributes to (1.9) as follows.

Corollary 6.4. *At normalization* $\mathrm{Ric} = 3g$*, the above metrics satisfy*

$$\mathcal{D}(g) \ge \frac{b_2^-(Y)}{\pi}\sqrt{15/4}, \qquad \mathcal{D}(g_M) \ge \frac{b_2(M)+1}{\pi}\sqrt{15/4}. \tag{6.11}$$

Proof. Every independent zero mode contributes $\sqrt{b_4}/\pi$ to (1.9), and $b_4 = 15/4$. Equations (6.9)–(6.10) give the indicated multiplicities. Every remaining summand in the coefficient is nonnegative. □

6.4. **One half-Weyl block.** Let $\alpha \ge \beta \ge \gamma$ denote the eigenvalues of W_g^+ in a fixed orientation. Catino–Dameno [6, Theorem 1.4] prove that a closed oriented Einstein four-manifold with $W^+ \not\equiv 0$ and $\beta \le \rho_0 \alpha$, for some constant $\rho_0 < 1$, has nowhere-vanishing W^+ and is conformal to a positive-scalar-curvature Kähler metric after a covering of degree at most two.

Theorem 6.5. *Let (M^4, g) be closed and oriented,* $\mathrm{Ric}_g = \Lambda g > 0$, *and assume*

$$\beta \le \rho_0 \alpha \quad \text{on } M, \qquad \rho_0 < 1. \tag{6.12}$$

If $W^+ \not\equiv 0$, then α is smooth and positive, and

$$|\nabla \log \alpha|_g^2 < 3\alpha - 27\Lambda/40. \tag{6.13}$$

The normalized cone over $(M, (\Lambda/3)g)$ is stable if and only if its link is the round sphere or the Fubini–Study projective plane, up to isometry and orientation.

Proof. Suppose $W^+ \not\equiv 0$. By [6, Theorem 1.4], the positive largest eigenvalue is simple and W^+ is nowhere zero. In any smooth orthonormal trivialization of Λ^+, the characteristic polynomial of W^+ is smooth in the base point. At its simple root α its derivative in the eigenvalue variable is nonzero. The implicit function theorem gives a smooth local root; ordering makes these roots agree on overlaps. On the conformally Kähler cover choose its canonical scalar and metric. The Weyl eigenvalues under $g = u^{-2}k$ give

$$\alpha = u^3/6, \qquad u = (6\alpha)^{1/3}, \qquad k = u^2 g.$$

Indeed, for the canonical Kähler metric the largest eigenvalue is $u/6$, and conformal change to g multiplies the Weyl endomorphism by u^2. The arbitrary-Λ version of (4.6) is

$$D_k(u^2) = 4\Lambda/3 - 2\alpha + (2/3)|\nabla \log \alpha|^2. \tag{6.14}$$

Theorem 4.7 gives $D_k(u^2) < 53\Lambda/60$. Rearranging proves (6.13). The scalar quantities descend even if the deck involution changes the sign of J.

For the classification, if $W^+ \not\equiv 0$, its eigenvalues on the conformally Kähler cover are $\alpha, -\alpha/2, -\alpha/2$. Thus $\det W^+ = \alpha^3/4 > 0$ on M. If $W^+ \equiv 0$, its determinant is zero. The normalized metric therefore satisfies all hypotheses of [20, Theorem D], which gives the asserted list.

The stability condition in [20, Theorem D] is imposed on divergence-free tensors. It is equivalent to nonnegativity on all compactly supported symmetric tensors on an exact cone. Indeed, if a link TT eigenvalue is below b_d, Lemma 2.5 gives a compactly supported TT variation with negative energy. Otherwise Lemma 2.4 gives $\mathcal{B}_h \ge 0$, and (2.11) is nonnegative for every compactly supported tensor. Thus the cited classification has the same stability convention as the present statement. □

The determinant hypothesis $\det W^+ \ge -c|W^+|^3$, $0 \le c < 1/(3\sqrt{6})$, is another sufficient condition, by [6, Corollary 1.5]. For $c = 0$, the cone classification is [20, Theorem D].

6.5. **Smooth five-dimensional fillings.**

Lemma 6.6. *Let X^5 be smooth and oriented, and assume the compact-domain identification* (1.11), *where Z is closed and oriented by the outward radial direction. Then Z bounds a compact oriented five-manifold and* $\mathrm{sign}(Z) = 0$.

Proof. Choose $R > R_0$ and remove $(R, \infty) \times Z$. The remainder is a compact oriented manifold K^5 with boundary $\{R\} \times Z$, identified with Z. Work over $\mathbb{R}$ and write $i : Z \to K$. Let $V = H^2(Z; \mathbb{R})$ with its nonsingular symmetric intersection form, and let $W = \mathrm{im}(i^* : H^2(K; \mathbb{R}) \to V)$. In the exact sequence of the pair,

$$H^2(K; \mathbb{R}) \xrightarrow{i^*} V \xrightarrow{\partial} H^3(K, Z; \mathbb{R}),$$

$W = \ker \partial$. Poincaré–Lefschetz duality identifies ∂ with the adjoint of i^*: for $a \in H^2(K; \mathbb{R})$ and $b \in V$,

$$\langle i^* a \smile b, [Z] \rangle = \langle a \smile \partial b, [K, Z] \rangle.$$

For closed differential forms representing a, b, extend the boundary representative of b to a collar of Z, multiply it by a cutoff equal to one near Z, and extend by zero to K. Its exterior derivative represents ∂b. Stokes' formula applied to the wedge with a closed representative of a gives the displayed equality. The pairing on the right is nonsingular. Thus $W^{\perp} = \ker \partial = W$. Choose a basis of W and a dual complementary basis of V; the matrix of the form is $\left(\begin{smallmatrix} 0 & I \\ I & B \end{smallmatrix}\right)$ with $B = B^{\mathsf{T}}$. Subtracting half of B times the first basis from the second transforms this matrix into $\left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right)$. It has equally many positive and negative eigenvalues. Therefore $\operatorname{sign}(Z) = 0$. $\square$

Lemma 6.7. *Let (Z^4, h) be closed, oriented, $\mathrm{Ric}_h = 3h$, satisfy* (6.12)*, and have signature zero. Up to isometry and orientation, it is one of*

$$S^4, \qquad \textit{Page}, \qquad S^2 \times S^2, \qquad \mathcal{P} = (S^2 \times S^2)/\langle \mathfrak{a} \times \mathfrak{r} \rangle, \qquad \mathcal{Q} = (S^2 \times S^2)/\langle \mathfrak{a} \times \mathfrak{a} \rangle. \tag{6.15}$$

The product and quotient metrics have equal factor curvature; $\mathfrak{a}$ is antipodal and $\mathfrak{r}$ is a reflection.

Proof. For a simply connected link, the positive Einstein classification in [6, Corollary 1.6], together with [18, Theorem A], consists of the round sphere, the Fubini–Study plane, positive Kähler–Einstein del Pezzo surfaces, Page, and CLW. The signatures of $\mathbb{CP}^2 \# k\overline{\mathbb{CP}^2}$ are $1 - k$. The positive Kähler–Einstein list has $k = 0$ or $3 \le k \le 8$, together with $\mathbb{CP}^1 \times \mathbb{CP}^1$. Page has $k = 1$ and CLW has $k = 2$. Thus zero signature leaves S^4, Page, and $S^2 \times S^2$. The Kähler–Einstein metric on $\mathbb{CP}^1 \times \mathbb{CP}^1$ is invariant under a maximal compact subgroup $SO(3) \times SO(3)$ of its connected complex automorphism group, as explained in [17, Section 2.1, before Proposition 2]. An invariant metric is a sum of constant multiples of the two round factor metrics: the isotropy representation has two inequivalent real two-dimensional summands, so there is no invariant mixed term. Equality of the two Einstein constants makes the factor curvatures equal. This proves the stated isometry description.

In the non-simply-connected case with $W^+ \not\equiv 0$, Theorem 6.5 gives $\det W^+ > 0$. Hence [17, Theorem B] gives precisely $\mathcal{P}$ or $\mathcal{Q} \# k\overline{\mathbb{CP}^2}$, $0 \le k \le 3$. Their signatures are 0 and $-k$; hence $k = 0$. The metric rigidity in [17, Proposition 2] identifies both remaining metrics with the stated product quotients. There is no additional oriented half-conformally-flat quotient. The universal cover is complete and has $\mathrm{Ric} = 3h$; the Bonnet–Myers theorem makes it compact. Its pullback orientation is half-conformally flat. It is therefore round S^4 or Fubini–Study $\mathbb{CP}^2$ by [6, Corollary 1.6]. Positive Ricci curvature gives $b_1(Z) = 0$, hence $\chi(Z) = 2 + b_2(Z) \ge 2$. A covering of degree $q \ge 2$ would give $\chi(Z) = 2/q$ or $3/q < 2$, a contradiction. $\square$

Theorem 6.8. *Let (X^5, G) be complete, connected, smooth, oriented, without boundary, Ricci-flat, and satisfy* (1.11)*, with $e(r) \to 0$ and smooth link (Z, h), $\mathrm{Ric}_h = 3h$. Suppose one orientation of the link satisfies* (6.12)*. Then either X is isometric to Euclidean space or*

$$\lim_{\epsilon \downarrow 0} \frac{N_G(\epsilon)}{\log(1/\epsilon)} > \frac{1}{2\pi}\sqrt{11/10}. \tag{6.16}$$

In particular, finite negative index implies Euclidean isometry.

Proof. Lemma 6.6 gives zero signature, so Lemma 6.7 applies. For Page, Theorem 4.7 gives $\lambda_1^{\mathrm{TT}}(L_h) < 53/20$; hence $\mathcal{D}(h) > \pi^{-1}\sqrt{15/4 - 53/20} = \pi^{-1}\sqrt{11/10}$. On the equal-curvature product, the parallel TT tensor $h_1 - h_2$ has L-eigenvalue zero. Both involutions in (6.15) preserve the two factor metrics, so this tensor descends to their quotients. Thus their coefficients are at least $\pi^{-1}\sqrt{15/4}$, which is strictly larger. Theorem 3.4 proves (6.16) in all these nonround cases.

If the link is the unit round S^4, its volume is the Euclidean unit-sphere volume. The asymptotic volume ratio equals one, as follows. For every $\eta > 0$, outside a fixed compact set the metric lies between $(1 - \eta)G_C$ and $(1 + \eta)G_C$, and its density tends uniformly to the cone density. Radial curves give $d_G(p, (r, z)) \le \sqrt{1 + \eta}\, r + C_\eta$. Conversely, $|dr|_G \le (1 - \eta)^{-1/2}$ on the end gives $d_G(p, (r, z)) \ge \sqrt{1 - \eta}\, r - C_\eta$, by integration along any curve after its last exit from the compact part. These inequalities sandwich large distance balls between coordinate truncations. Integrating the convergent volume density and then letting $\eta \downarrow 0$ gives

$$\lim_{R\to\infty} \frac{\mathrm{Vol}_G B_R(p)}{\omega_5 R^5} = 1.$$

For $\mathrm{Ric}_G = 0$, Bishop–Gromov volume ratios are nonincreasing and tend to one as $R \downarrow 0$. The limit at infinity is also one, so every ratio equals one. To determine the metric, let $c(\theta)$ be the cut time in a unit direction $\theta \in S_pX$, and let $j(r,\theta)$ be the polar volume density before the cut. Jacobi fields normal to the radial geodesic define the shape operator S, with

$$m = \operatorname{tr} S = \partial_r \log j, \qquad m' = -|S|^2 - \mathrm{Ric}(\partial_r, \partial_r) = -|S|^2.$$

Since $|S|^2 \geq m^2/4$, the function $f = m - 4/r$ satisfies $(r^2 f)' \leq -r^2 f^2/4 \leq 0$. Smoothness at p gives $r^2 f \to 0$ as $r \downarrow 0$. Therefore $m \leq 4/r$ and $j(r,\theta) \leq r^4$ before the cut.

Suppose $c(\theta_0) < R$. The radial segment of length R in that direction does not minimize, so $d(p, \exp_p(R\theta_0)) < R$. Continuity of distance and of the exponential map gives the same strict inequality on an open neighborhood of θ_0. All those directions have cut time less than R. The polar integration formula for $B_{R+1}(p)$, together with $j \leq r^4$, would then give $\mathrm{Vol}\, B_{R+1}(p) < \omega_5(R+1)^5$, since a set of directions of positive measure contributes no radial interval $(R, R+1)$. This contradicts the established volume equality. Hence every cut time is infinite.

The polar density is now smooth for $r > 0$. The equality of all ball volumes and $j \leq r^4$ imply $j = r^4$ pointwise. Consequently $m = 4/r$ and $|S|^2 = 4/r^2$. Equality in $|S|^2 \geq m^2/4$ gives $S = r^{-1} I$. If g_r is the angular metric, its equation is $\partial_r g_r = 2r^{-1} g_r$; hence $r^{-2} g_r$ is constant in r. Its limit at p is the unit round metric on S_pX. There are no cut points, and completeness makes every point the endpoint of a minimizing radial geodesic. Thus $\exp_p : T_pX \to X$ is a global isometry with Euclidean five-space. □

Corollary 6.9. *Let (M^4, g) be closed, oriented and positive Einstein. Suppose its normalized cone is stable and g is neither round nor Fubini–Study, up to isometry and scale. Then*

$$W^\pm \not\equiv 0, \qquad \sup_{\{W^\pm \neq 0\}} \frac{\beta_\pm}{\alpha_\pm} = 1$$

in both orientations. The same necessity holds for the link of a nonflat finite-index filling satisfying the asymptotic hypotheses of Theorem 6.8.

Proof. For a nonzero trace-free symmetric endomorphism of a three-dimensional Euclidean space, its largest eigenvalue is positive and its middle-to-largest ratio is at most one. If the displayed supremum were less than one in either orientation, the inequality would extend over the zero set and give (6.12). Apply Theorem 6.5 or Theorem 6.8. If a block were identically zero, those theorems would apply directly. When a block is nowhere zero, compactness forces the supremum to be attained at a positive eigenvalue collision. With zeros present it may instead be approached near the zero set; the proof does not exclude that possibility. □

The signature obstruction excludes a smooth oriented five-dimensional manifold satisfying (1.11) with CLW link, since $\operatorname{sign}(\mathbb{CP}^2 \# 2\overline{\mathbb{CP}}^2) = -1$. For Page, the signature is zero; Theorem 6.8 states a necessary spectral property of any complete Ricci-flat filling with that asymptotic link.

Appendix A. Conformal harmonic forms

Assume (4.1). The following proposition is the identity of [10, Theorem 6.3], expressed for the full operator L_g.

Proposition A.1 (Hall–Haslhofer–Siepmann, Theorem 6.3). *Under* (4.1)*, let η be a real anti-self-dual harmonic two-form of k. Put*

$$\widetilde{h}(X,Y) = \eta(X,JY), \qquad h = u^2\widetilde{h}. \tag{A.1}$$

Then h is TT for g, the map $\eta \mapsto h$ is injective, and

$$\langle L_g h, h\rangle_{L^2(g)} = \int_M D_k(u^2)|h|_g^2\, dv_g. \tag{A.2}$$

Proof. Write $S = \widetilde{h}$, $a = |S|_k^2$ and $\theta = d\log u$. An anti-self-dual two-form on a Kähler surface is primitive of type $(1,1)$. Therefore S is symmetric, J-invariant and trace-free. Since $\nabla^k J = 0$ and $\delta_k\eta = 0$,

$$(\delta_k S)(Y) = -\sum_i (\nabla^k_{e_i}\eta)(e_i, JY) = (\delta_k\eta)(JY) = 0.$$

For a trace-free covariant symmetric tensor T, contraction of the conformal connection gives

$$\delta_g T = u^2\delta_k T + 2uT(\nabla^k u,\cdot).$$

Consequently

$$\delta_g(u^2S) = u^4\delta_k S - 2u^3S(\nabla^k u,\cdot) + 2u^3S(\nabla^k u,\cdot) = 0, \qquad \mathrm{tr}_g(u^2S) = u^4\,\mathrm{tr}_k\, S = 0.$$

Multiplication by u^2 and composition with J are invertible, which proves injectivity.

The conformal identity of [10, Theorem 6.3] uses the following contractions. All contractions in the next four displays use k. A J-invariant trace-free symmetric endomorphism in real dimension four has eigenvalues $\lambda, \lambda, -\lambda, -\lambda$. Hence

$$S_i{}^pS_{pj} = \tfrac{a}{4}k_{ij}, \qquad |S(\theta^\sharp,\cdot)|^2 = \tfrac{a}{4}|\theta|^2. \tag{A.3}$$

The conformal derivative is

$$u^{-2}(\nabla^g_i h)_{jk} = (\nabla^k_i S)_{jk} + C_{ijk}, \qquad C_{ijk} = 4\theta_i S_{jk} + \theta_j S_{ik} + \theta_k S_{ij} - k_{ij}S_{pk}\theta^p - k_{ik}S_{pj}\theta^p. \tag{A.4}$$

Its contractions are

$$|C|^2 = 18a|\theta|^2 + 8|S\theta|^2 = 20a|\theta|^2, \qquad 2\langle\nabla S, C\rangle = 5\langle da, \theta\rangle. \tag{A.5}$$

For the second identity, the term $4\theta_i S_{jk}$ contributes $4\langle da,\theta\rangle$. The two terms with θ_j and θ_k contribute $4\theta_j S_{ik}\nabla_i S_{jk} = \langle da,\theta\rangle$, because $\nabla_i(S_{ik}S_{jk}) = \frac{1}{4}\nabla_j a$ and $\nabla_i S_{ik} = 0$. The remaining terms vanish by this divergence equation.

The conformal curvature formula, before using (A.3), gives

$$\begin{aligned} u^{-6}\langle \mathring{R}_g h, h\rangle_g \frac{dv_g}{dv_k} &= \langle \mathring{R}_k S, S\rangle - 2\langle S^2, \mathrm{Hess}_k \log u\rangle - 2|S\theta|^2 + a|\theta|^2 \\ &= \langle \mathring{R}_k S, S\rangle - \tfrac{a}{2}D_k\log u + \tfrac{a}{2}|\theta|^2. \end{aligned} \tag{A.6}$$

Equations (A.4)–(A.6), with $dv_g = u^{-4}dv_k$, imply

$$\langle E_g h, h\rangle_{L^2(g)} = \int_M u^6\big(|\nabla S|^2 - 2\langle \mathring{R}_k S, S\rangle + 5\langle da,\theta\rangle + 19a|\theta|^2 + aD_k\log u\big)\, dv_k. \tag{A.7}$$

Here the factor u^6 includes the two factors u^2 in h, the inverse metrics in the contraction and the conformal volume density.

The Hodge Laplacian on η, transferred by the parallel endomorphism J, gives

$$\nabla^*\nabla S - 2\mathring{R}_k S + \mathrm{Ric}_k \circ S + S\circ \mathrm{Ric}_k = 0;$$

this is the Weitzenböck identity used in [10, equations (6.5)–(6.8)]. Contracting with S and using (A.3) and $\mathrm{Scal}_k = u$ yields

$$|\nabla S|^2 - 2\langle \mathring{R}_k S, S\rangle = \tfrac{1}{2} D_k a - \tfrac{u}{2} a. \tag{A.8}$$

On the closed manifold, integration by parts gives

$$\tfrac{1}{2}\int u^6 D_k a = -3\int u^6\langle \theta, da\rangle, \qquad 2\int u^6\langle\theta, da\rangle = -2\int u^6 a(D_k \log u + 6|\theta|^2).$$

Inserting these equalities into (A.7) proves

$$\langle E_g h, h\rangle_{L^2(g)} = \int_M u^6 a\big(-u/2 + 7|\theta|^2 - D_k \log u\big)\, dv_k. \tag{A.9}$$

The traced conformal Einstein equation and the product rule are, respectively,

$$\frac{6}{u^2} = \frac{u}{2} + 3D_k \log u - 3|\theta|^2, \qquad \frac{D_k(u^2)}{u^2} = 2D_k \log u + 4|\theta|^2.$$

Their difference identifies the parenthesis in (A.9) with $u^{-2}(D_k(u^2) - 6)$. Since $|h|_g^2 dv_g = u^4 a\, dv_k$,

$$\langle E_g h, h\rangle_{L^2(g)} = \int_M (D_k(u^2) - 6)|h|_g^2\, dv_g.$$

Adding $6\|h\|^2$ proves (A.2). □

Appendix B. The Page normalization

The symplectic potential in [10, Section 7.1] has singular part $\sum_\alpha \ell_\alpha \log \ell_\alpha$, with angular periods 4π. In the CLW normalization of (4.24), the corresponding coefficient is $1/2$ and the angular periods are 2π. Let $a \in (0,1)$ satisfy

$$p_P(a) = 1 - 6a^2 - 16a^3 + 9a^4 = 0. \tag{B.1}$$

Put $D_P = (1-a)(1+4a+a^2)$, $c_1 = 24a/D_P$, $c_2 = 6(1-3a^2)/D_P$, and $s_0 = c_1 t + c_2$, $a \le t \le 1$. Inverting the Page symplectic Hessian gives

$$H(t) = |dt|_{k_0}^2 = \left(\frac{1}{1-t} + \frac{1}{t-a} + \frac{2a(1-a)}{2at^2 + (1+2a-a^2)t + 2a^2}\right)^{-1}. \tag{B.2}$$

The measure on invariant radial functions is proportional to $t\,dt$; hence $D_{k_0} s_0 = c_1(H' + H/t)$. To compute the derivatives, put

$$\begin{aligned} N(a,t) &= (t-a)(1-t)\big(2at^2 + (1+2a-a^2)t + 2a^2\big) \\ &= -2at^4 + (3a^2-1)t^3 + (-a^3 - 3a^2 + 3a + 1)t^2 + (3a^3 - a)t - 2a^3. \end{aligned} \tag{B.3}$$

Combining the three fractions in (B.2) gives

$$H = \frac{N}{D_P t}, \qquad H' + \frac{H}{t} = \frac{\partial_t N}{D_P t}. \tag{B.4}$$

These expressions extend to $t = a, 1$ by continuity. Define

$$J_P = -3a^6 + 24a^5 - 53a^4 - 32a^3 + 15a^2 + 8a + 1. \tag{B.5}$$

The conformal scalar expression expands to

$$s_0^3 + 6s_0 D_{k_0} s_0 - 12c_1^2 H = 216 J_P/D_P^3 - 864a^2 p_P(a)/(tD_P^3). \tag{B.6}$$

At the root (B.1), normalization to $\mathrm{Ric} = 3g$ therefore gives

$$D_k(s^2) = 4 - \frac{P(a,t)}{18 J_P t}, \qquad P(a,t) = \sum_{i=0}^{4} \alpha_i(a) t^i, \tag{B.7}$$

where

$$\begin{aligned}
\alpha_0 &= 6912a^5, \\
\alpha_1 &= 72 - 648a^2 + 3456a^3 + 1944a^4 - 10368a^5 - 1944a^6, \\
\alpha_2 &= 864a - 3456a^2 - 15552a^3 + 10368a^4 + 11232a^5, \\
\alpha_3 &= 6912a^2 - 20736a^4, \qquad \alpha_4 = 11520a^3.
\end{aligned} \tag{B.8}$$

The polynomial identities giving these two formulas are

$$tD_P^3\big(s_0^3 + 6s_0c_1(H' + H/t) - 12c_1^2H\big) = 216J_Pt - 864a^2p_P(a), \tag{B.9}$$

$$72J_Pt - 2D_P^2c_1\big(s_0\partial_tN + c_1N\big) - P(a,t) = 288a^2p_P(a). \tag{B.10}$$

All denominators cancel after (B.4) is substituted. Expanding N by (B.3) gives the coefficients in (B.8). At the root $p_P(a) = 0$, the first identity gives $\kappa = 216J_P/D_P^3$. Under the normalization $k = (\kappa/12)^{1/3}k_0$, the quantity $D_k(s^2)$ is $(12/\kappa)D_{k_0}(s_0^2)$. Since $D_{k_0}(s_0^2) = 2c_1(s_0\partial_tN + c_1N)/(D_Pt)$, the second identity gives (B.7).[1]

Proposition B.1 (Hall–Haslhofer–Siepmann). *For the Page metric with* $\mathrm{Ric}_g = 3g$, $D_k(s^2) < 53/20$. *The estimate is the one recorded in* [10, p. 14].

Proof. Since $p_P' = -12a(1 + 4a - 3a^2) < 0$ on $(0,1)$, the root is unique. Exact signs at

$$a_P^- = 314076/10^6, \qquad a_P^+ = 314077/10^6$$

place it strictly between these numbers. The polynomial J_P is positive on that interval. Put $t = a + (1-a)v$. The bivariate polynomial

$$10P(a, a + (1-a)v) - 243J_P(a)(a + (1-a)v) \tag{B.11}$$

has strictly positive coefficients after the two-variable interval substitution on each of the five rational rectangles in Appendix C. These rectangles cover $[a_P^-, a_P^+] \times [0,1]$. The two-variable version of (4.15) therefore gives $10P - 243J_Pt > 0$ throughout the required parameter rectangle. Divide by $180J_Pt > 0$ in (B.7). The result is $D_k(s^2) < 4 - 243/180 = 53/20$. ☐

Appendix C. Polynomial identities

All entries in this appendix are integers. For each univariate polynomial f and interval $[l,h]$, the table gives a positive rational s_f and the coefficients c_j in

$$s_f(1+z)^n f\left(\frac{l + hz}{1+z}\right) = \sum_{j=0}^{n} c_jz^j. \tag{C.1}$$

For $0 \le j \le n$, the coefficient is $c_j = s_fC_j(f; l, h)$, where C_j is given by (4.16). Every displayed coefficient is strictly positive. The constant coefficient equals $s_ff(l)$ and the leading coefficient equals $s_ff(h)$. Hence each identity proves $f > 0$ on the indicated closed interval.

C.1. Parameter isolation.

$$p\left(\frac{19577}{10000}\right) = \frac{-7489606465272893709991591693345254843 51}{10^{40}},$$

$$p\left(\frac{9789}{5000}\right) = \frac{4982204559197403085863891755192231601}{9765625 \cdot 10^{30}}.$$

$$p_P\left(\frac{78519}{250000}\right) = \frac{26578095850206889}{390625 \cdot 10^{16}},$$

$$p_P\left(\frac{314077}{10^6}\right) = \frac{-584495776626590631}{10^{24}}.$$

[1] In the coefficient list following equation (7.8) of [10, p. 14], the second term of α_3 is printed as $-20736a^5$. The term obtained from (B.2) is $-20736a^4$.

C.2. **CLW range.** The interval is $[a_-, a_+]$ from (4.29). The polynomials are those in (4.32).

$f = D$, $n = 6$, $s_f = 10^{24}$.

j	c_j
0	368799734102842548571919089
1	2212889560030952567008504476
2	5532451797591793954756941660
3	7376906272114955614139564320
4	5532907619560990567905285360
5	2213254217606582568808814016
6	368890898496863678722736704

$f = B_0$, $n = 5$, $s_f = 10^{20}$.

j	c_j
0	17147237116043803651657
1	85739739048139358633490
2	171486585305357348727720
3	171493692787683845048080
4	85750400271634060356560
5	17150790857212009054368

$f = R$, $n = 5$, $s_f = 10^{20}$.

j	c_j
0	9453596828501522011657
1	47269917210104665153490
2	94543700713166848247720
3	94547567163894155768080
4	47275716886198817476560
5	9455530053868366814368

$f = E$, $n = 10$, $s_f = 10^{40}$.

j	c_j
0	102161595240759215651648167016168484043885649
1	1021656470601296316843402400970037386948465860
2	4597636456786093458387311162625033044986524180
3	12260850141533947810942134885051800685782102720
4	21457338730676473222822609786244405237858916640
5	25749827696775155398789434164316113091274119552
6	21459040797594210582893551154909553929208461440
7	12262795360870028350198741192013041467030891520
8	4598730642664067095768550447857506489788980480
9	1021980673824992110932975002268978183422428160
10	102202120643816412183016925967303628571239424

$f = 12 \cdot 200^3 R^3 - 359^3 B_0 E$, $n = 15$, $s_f = 10^{60}$.

j	c_j
0	55503754435909470262415753796524692911517342416950530716320143458336353
1	833368241782217250380061312870031236423744963052041937703317063890466630
2	5839261731249733241557761087385416979914393401245442645944504502755564740
3	25328100772573328003211688844663112320467769547717808324561729256378417560

j	c_j (continued)
4	76058209437111425593747659103808691830278845219098401527044273788973117520
5	167490672502846411975942131993902496434795222806196326278672122062565083616
6	279422167206180641693564640533137224546716888786834118897207869254670111040
7	359605594768610817835423371708706980124754009657829837511395831326428878720
8	359954151876208131087652838597268145897838130211843753100230189562100622080
9	280235467125775693162300313048261852492616212082684746907764035556597813760
10	168303972426177437875322417986046308348263643609264578800324167388914371584
11	76575763937342781310114547335522077786276858487201356395826896313005230080
12	25549909846138856124357042327334705362779403925003310535455796215556567040
13	5901823265537954209512875726807722130516661295078590510175061665375887360
14	843930578885526289209004733891859562765486266741741410736252175162654720
15	56316241918457628788790151934604695617829922687589581560379787399299072

$f = 3257^3 E - 12 \cdot 1000^3 B_0^2,\ n = 10,\ s_f = 10^{40}.$

j	c_j
0	1392684646119145700909573477426020638890635518524857
1	13864395124403917166761753622911063956619216772154980
2	62108732131865888250454604095423271371745343543538740
3	164873790109617035075921742379071186293782577415712960
4	287217445839982842510508168587028016302347732743167520
5	343086827269560214119650727547748493934537782766974336
6	284593863343915448887040906453784838623947004115233920
7	161875410111978461031178081141812875809651545299471360
8	60422143381194520354293550604065407827160695764624640
9	13364665123375500608436445659205668564178085186298880
10	1330218395857282298741947133021733794677818305938432

$f = 579 B_0 E - 1200 R^3,\ n = 15,\ s_f = \frac{10^{60}}{3}.$

j	c_j
0	146252206202674722272745826634633078416355620574680255279508328649
1	2190628032151579268804891814481010515519633960722424963144613988790
2	15312308520568223466775951320017730721056403212022072210954134456420
3	66257613663113809744908643141247698476397934417723120669883976859480
4	198485641589568020073162236816630143107790843381408393099262506238160
5	436036507645150032035304790979509125542108193652485155049426204847328
6	725674231027092292636520479947117929589245292954217883494203416384320
7	931655366583789575826638069564226773193838444032730633131997073957760
8	930300867876625353484786262093428347918459112874433266218678821968640
9	722513734036046184548254452057181127830347325508664982374020919534080
10	432876010638778078684354262406642825009944403351282497601044862774272
11	196474416207247561075549538033354596886577150663492723555586962032640
12	65395659919474061037903501678287638088703674355984176832893621432320
13	15069193359158452286676297457984422319071854086557164877018224762880
14	2149582614693078468768998010267558288443521334804831147409080565760
15	143094866344587788407755051042051038199070762351596238647976165376

C.3. **The cubic comparison.** The interval is $[359/200, 3257/1000]$; Q is (4.22) and A, B, C are (4.7). The two polynomial identities for G are already displayed in (4.23).

$f = A,\ n = 6,\ s_f = \frac{3125 \cdot 10^{15}}{534361}.$

j	c_j
0	12888100000000
1	81059704950000
2	169941671427675

j	c_j (continued)
3	138412608189320
4	127296588000000
5	266877296742000
6	186502750873201

$f = A + B + C$, $n = 6$, $s_f = 9 \cdot 10^{20}$.

j	c_j
0	13052284119806640625
1	143594933113310306250
2	5970601874625628051275
3	15459075724318443833260
4	10160352821559483317175
5	926951109455512965450
6	893408803350539163853

$f = 2A + B$, $n = 6$, $s_f = \frac{25 \cdot 10^{18}}{731}$.

j	c_j
0	39153668606250000
1	426135563921906250
2	1450815508518986175
3	1844542172590398220
4	900156469854899550
5	690840339129402450
6	641526033524061931

C.4. **The Page interval.** The interval is $[314076/10^6, 314077/10^6]$.

$f = J_P$, $n = 6$, $s_f = 10^{36}$.

j	c_j
0	3555598661019619206788542561179062272
1	21333594463153615778464082782181156864
2	53333992400434268501810973761115032320
3	71111998190593300281605404792703947520
4	53334004885416159516763041465699263280
5	21333604451139128544364605053933749224
6	3555601158015997379071534840825495733

For $F(a, v) = 10P(a, a + (1 - a)v) - 243J_P(a)(a + (1 - a)v)$, the separate degrees are $(7, 4)$. On a rectangle $[a_l, a_h] \times [v_l, v_h]$, the coefficient identity is

$$s_F(1 + x)^7(1 + y)^4 F\left(\frac{a_l + a_h x}{1 + x}, \frac{v_l + v_h y}{1 + y}\right) = \sum_{i=0}^{7} \sum_{j=0}^{4} c_{ij} x^i y^j. \tag{C.2}$$

All rectangles have the preceding Page interval in the a coordinate. Their v intervals are $[0, 1/16]$, $[1/16, 1/8]$, $[1/8, 1/4]$, $[1/4, 1/2]$, and $[1/2, 1]$. They cover the required closed rectangle. Apply the binomial formula (4.16) separately in each variable to recover every coefficient below.

Rectangle 1: $[v_l, v_h] = [0, \frac{1}{16}]$, $s_F = \frac{128 \cdot 10^{42}}{9}$.

i	j	c_{ij}
0	0	4054529943844715612720761537115837778788343808
0	1	12498462065588611041263587502304583693077705344
0	2	13814858406951888188302591660466769246403952640
0	3	6456487475210688534295169562489019030056468480
0	4	1087275732512959954992593703025568668001075200
1	0	28381768565704694403181950013548172496333701120
1	1	87489423038169772198903193138766352794791772160
1	2	96704225207313668033590628370572721933040680960
1	3	45195515525239675714833935378210667864007966720
1	4	7610946594341510153690034449515072737056972800
2	0	85145482574223034290186796140241779581846814720
2	1	262468834854526989506955915230962934595470376960
2	2	290113324702114135549053870875977797665685749760
2	3	135586856174784623669641797027526241117259448320
2	4	22832889183961422113106927474597612395921356800
3	0	141909432420109804692710916584161334191408742400
3	1	437449000995689762628600660303201054909572403200
3	2	483523289644181761672939775717975240392323379200
3	3	225978609627702093483449788317850522821527654400
3	4	38054897642638059790207409033598425615096256000
4	0	141909727217737717487391512358928948136064204800
4	1	437449943905283212978696468586811910491297766400
4	2	483524371458542965282465425803593369501452518400
4	3	225979125635381552833745106303833741121495628800
4	4	38054979979814622678824467490249010413679312000
5	0	85146013209953279027183741051449376507902709760
5	1	262470532091795207117032036721561469140933671680
5	2	290115271967964312750791331179258670014006734080
5	3	135587784988607657794644089422059678316600738560
5	4	22833037390879237175792174036691703123085834400
6	0	28382063363332610042148999830265175504653158400
6	1	87490365947763234182171779057142781535223549120
6	2	96705307021674889484101762039918145421049830720
6	3	45196031532919147222580453398229887562691215040
6	4	7611028931518076147597995139953469594660954600
7	0	4054588903370299309371462370154199860614826240
7	1	12498650647507305764551860213693333469771326320
7	2	13815074769824136046601915112149449495221403920
7	3	6456590676746585267334713173958019829366227440
7	4	1087292199948273774832366288020459597023536850

Rectangle 2: $[v_l, v_h] = [\frac{1}{16}, \frac{1}{8}]$, $s_F = \frac{128 \cdot 10^{42}}{9}$.

i	j	c_{ij}
0	0	1087275732512959954992593703025568668001075200
0	1	2241718384892991105645580061715530313952133120
0	2	1170551135998795902353823158146303098090946560
0	3	759453001053978601677428865124623078955417600
0	4	615513589968972889240959228706938349105971200
1	0	7610946594341510153690034449515072737056972800
1	1	15692057229492405514686340217909914032447815680
1	2	8193850320071857433147842889670460438360227840
1	3	5315825071371162755331981894788481550984806400
1	4	4308446261665788801071453848621755197607116800
2	0	22832889183961422113106927474597612395921356800
2	1	47076257296906753235213622769254658050111406080
2	2	24581528068480524245769348101163048464241623040
2	3	15946437432795762729796831297569593296426598400

i	j	c_{ij} (continued)
2	4	129248921873136192506691255557897045031518208
3	0	3805489764263805979020740903359842561509625600
3	1	7846057151340238483820948395093688209924239360
3	2	4096917530128263573721886261723431822546759680
3	3	265756661303007554839807182964325248446945280
3	4	215407426604953490298893684443081321087631360
4	0	3805497997981462267882446749024901041367931200
4	1	7846071420313542859685063361815834218793886720
4	2	4096913716180459257178200774656717270078223360
4	3	265739365837793986860691866952500729430274560
4	4	215399983532469285109090781299284280618270720
5	0	2283303739087923717579217403669170312308583440
5	1	4707651413842623961169330287147394666808593664
5	2	2458145941742005820193897152750147506846232832
5	3	159433242490573997575378410489157105463110272
5	4	129235524342664825180706153595094171862974464
6	0	761102893151807614759799513995346959466095460
6	1	1569219991922546195820350772139786919459642176
6	2	819381218059383369097092500942209031676545088
6	3	53140955248499380640567280228217642583174048
6	4	43077019544174019513678508182994917726309376
7	0	108729219994827377483236628802045959702353685
7	1	224174692283960493132421713020565694682206736
7	2	117054350810319503857042698089236084758892368
7	3	7591070917497600847496336406293366347273928
7	4	6153647285193022531555760802852581940455936

Rectangle 3: $[v_l, v_h] = [\frac{1}{8}, \frac{1}{4}]$, $s_F = \frac{8 \cdot 10^{42}}{9}$.

i	j	c_{ij}
0	0	384695993730608055775599517941836468191232
0	1	3667035673449823417210408133896258769600512
0	2	41493515799565473301856878189864737096351744
0	3	79412379692503722287945994375931258665943040
0	4	41372657823683882393242908383478241786494976
1	0	2692778913541118000669658655388596998504448
1	1	25668565623279462563870926496177562043322368
1	2	290453287280091226867372843316375331429666816
1	3	555885713039822340317728168387930587624386560
1	4	289608392864511925686105024578238872002699264
2	0	8078057617071012031668203472368565314469888
2	1	77003644613857440967772402546460792153105408
2	2	871355891914895425144102337939320814002615296
2	3	1667654304713553282394243995434555647339023360
2	4	868824542894142163761234079802128453247453184
3	0	13462964162809593143680855277692582567976960
3	1	128335987290839173369194365461770334759855360
3	2	1452253203357900423994454669591391509804632320
3	3	2779419117208061680605538322524504832691091200
3	4	1448039845331967841217362434585533321851169280
4	0	13462498970779330319318173831205267538641920
4	1	128332566919627715474231602605400619284918720
4	2	1452246586899680416319453230227700443997160640
4	3	2779414393255530424756297203882637728066542400
4	4	1448038785847752357215740478641700279655922560
5	0	8077220271416551573794134599693385741435904
5	1	76997487945676869188607313885175990714342064
5	2	871343982290099492761056844770677963103097968
5	3	1667645801598997077859044986917504356002132880
5	4	868822635822554306921913333810169216618027872
6	0	2692313721510876219604906761437182357894336

i	j	c_{ij} (continued)
6	1	25665145252068092055187971108718982971835276
6	2	290446670821871354912299900096415257339359612
6	3	555880989087291177790878724808401238719372420
6	4	289607333380296465623814359811814128820087448
7	0	384602955324563908222235050178286371278496
7	1	3666351599207566792729778551352765661932711
7	2	41492192507921526054827988774856448098078707
7	3	79411434901997508447054440671586096473263245
7	4	41372445926840795168651033665179693038060078

Rectangle 4: $[v_l, v_h] = [\frac{1}{4}, \frac{1}{2}]$, $s_F = \frac{4 \cdot 10^{42}}{9}$.

i	j	c_{ij}
0	0	20686328911841941196621454191739120893247488
0	1	168823567249599572071511455924938192053026816
0	2	485337375606062270629758333350260674434580480
0	3	573243317682315533414016110300472882568085504
0	4	237414814422222238024899647335069662617239552
1	0	144804196432255962843052512289119436001349632
1	1	1181764644147249213798901979081502644391809024
1	2	3397361764543603384400027834753824918312734720
1	3	4012704147476417670624001194562566951270035456
1	4	1661904273808147278560210279928282447453052928
2	0	434412271447071081880617039901064226623726592
2	1	3545292952651299700173160483378215072145695744
2	2	10192085699549649730293328871625109039635256320
2	3	12038115213541053556320190877667940596627803136
2	4	4985714539985215464919311235382393547093792768
3	0	724019922665983920608681217292766660925584640
3	1	5908819954783745366698636284988695098415924480
3	2	16986810175806377435407850170271639215371014400
3	3	20063529974440046443844715332706785675919845120
3	4	8309527097581646489572942401877812872221826560
4	0	724019392923876178607870239320850139827961280
4	1	5908818321830983718538145667967563949868992960
4	2	16986810852388900654657224548837568886105348800
4	3	20063534592996959955166473694131399294449426240
4	4	8309529961859598549431389183641085369107221120
5	0	434411317911277153460956666905084608309013936
5	1	3545290013336328763672435015943510943706034352
5	2	10192086917398191571682368820158385570873752560
5	3	12038123526943497907967358260683951679695724688
5	4	4985719695685529180169363888586874194187686944
6	0	144803666690148232811907179905907064410043724
6	1	1181763011194487615952007434062483534201152268
6	2	3397362441126126681549678991836200844346728540
6	3	4012708766033331234059096776733236667046641092
6	4	1661907138086099350926737805073983067100090696
7	0	20686222963420397584325516832589846519030039
7	1	168823240659047262564851761319492061755097223
7	2	485337510922566945639743920465289739964410315
7	3	573244241393698256523702670879745133806350237
7	4	237415387277812654999821301039562132055777706

Rectangle 5: $[v_l, v_h] = [\frac{1}{2}, 1]$, $s_F = \frac{2\cdot 10^{42}}{9}$.

i	j	c_{ij}
0	0	118707407211111119012449823667534831308619776
0	1	851245568851017894735381773709945093135351808
0	2	2221684881471285167205662152043146296421859328
0	3	2491584639651830410437200154484369276995485696
0	4	1013410991886142773880484901696900308992000000
1	0	830952136904073639280105139964141223726526464
1	1	5958721495372466000737260485007127733448282112
1	2	15551801594619424254309722476508173356427579392
1	3	17441101401710919693008048963657832920359481344
1	4	7093880710842605731198581458809903722496000000
2	0	2492857269992607732459655617691196773546896384
2	1	17876172026370239233195676534626420685934953472
2	2	46655427056830635006526343199563378481152689152
2	3	52323330977583787047824957539603303621154417664
2	4	21281653435447447706833881543874438209024000000
3	0	4154763548790823244786471200938906436110913280
3	1	29793632611049832493592939078560091557411114240
3	2	77759082216356792094682707196883154897452739840
3	3	87205596250069204211231232621623572040931290880
3	4	35469441230613149645723970676335506058240000000
4	0	4154764980929799274715694591820542684553610560
4	1	29793645178160631341421861407715112920193900480
4	2	77759119338014322547463693808801849088060831680
4	3	87205640870843275861534338089251399815443645760
4	4	35469460068815239052711727973948450168800000000
5	0	2492859847842764590084681944293437097093843472
5	1	17876194647169677173048825070837293485430396976
5	2	46655493875814189836231338286006809267553530416
5	3	52323411294977116019525662723351094210011421712
5	4	21281687344211208635641636356071195330544000000
6	0	830953569043049675463368902536991533550045348
6	1	5958734062483264871501330053710661735553903084
6	2	15551838716276954731589407730070814496976136044
6	3	17441146022484991345236346667982171608501721508
6	4	7093899549044695131902658217247993510402000000
7	0	118707693638906327499910650519781066027888853
7	1	851248082273177673475225135357627658528315999
7	2	2221692305802791267561338931081050841177666559
7	3	2491593563806644741267898142688745150793074613
7	4	1013414759526560652764564145551177085938000000

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