INSTABILITY OF EINSTEIN WARPED PRODUCTS AND A ONE-SIDED WEYL CURVATURE GAP

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Abstract. Let $\mathrm{Ric}_g = \Lambda g$, where $\Lambda > 0$. For Einstein warped products of closed manifolds, we prove that the least transverse-traceless Einstein eigenvalue is strictly less than $-(n-2)\Lambda/(n-1)$, without an upper bound on the dimension. The same estimate holds for the smooth spherical-fibre completions specified here, unless the completed metric is round. A trace-free Hessian correction expresses the shifted quadratic form as a negative sum of squared base-curvature tensors and admits an H^1 extension across the collapsing fibres. In dimension four, there is a universal positive gap for $\|(\lambda_{\max}(W^+) - 2\Lambda/3)_+\|_{L^2}$ on closed simply connected oriented positive Einstein manifolds. Below this gap the excess vanishes, and the metric is anti-self-dual or Kähler–Einstein in the given orientation. The proof uses Einstein orbifold compactness, extension of parallel real line bundles through anti-self-dual ALE limits, and LeBrun–Ozuch's Kähler–Einstein desingularization theorem. For circle-warped four-metrics, the corrected quadratic form equals a negative multiple of either chiral Weyl energy. The small-excess TT-semistable metrics are the round sphere and the Fubini–Study plane.

Contents

1. Introduction and statements

Two problems for positive Einstein metrics are treated. The first concerns the sign of the Einstein quadratic form on a global warped product. The second concerns a scale-invariant integral rigidity condition for one chiral curvature block in dimension four. For a circle-warped Einstein four-metric, the trace-free Ricci tensor of the three-dimensional base determines the spectra of both Weyl blocks and the energy of the corrected relative-factor tensor.

For warped products, a trace-free Hessian correction converts the critically shifted quadratic form into a negative sum of squared curvature tensors. In dimension four the relevant quantity is the L^2 norm of the positive part of $\lambda_{\max}(W^+) - 2\Lambda/3$. The smallness assumption yields uniform volume and topological bounds. Scale invariance implies $W^+ = 0$ on every Ricci-flat ALE bubble.

2020 *Mathematics Subject Classification.* 53C25, 58J50, 53C23.

Key words and phrases. Einstein metric, warped product, transverse-traceless tensor, Weyl curvature, integral pinching, orbifold degeneration.

The simple positive eigenline of the limiting W^+ is extended through the finite ALE reconstruction and shown to be globally orientable before the Kähler rigidity theorem is invoked.

1.1. **Conventions.** All manifolds, metrics and tensors are smooth unless another regularity class is specified. A closed manifold is compact and has no boundary. Unless an orientation is specified, integrals use the Riemannian density. On an Einstein n-manifold with $\mathrm{Ric}_g = \Lambda g$, $\Lambda > 0$, put

$$\Delta = -\operatorname{div}\nabla, \qquad (\delta T)_j = -\nabla^i T_{ij}, \qquad L = \nabla^*\nabla + 2\Lambda - 2\mathring{R}, \qquad E = L - 2\Lambda. \tag{1.1}$$

We take $R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$, so sectional curvature on an orthonormal pair is $\langle R(X,Y)Y, X\rangle$. Write $R_{ijkl} = \langle R(e_i,e_j)e_l, e_k\rangle$; thus R_{ijij} is sectional curvature. The curvature action is

$$(\mathring{R}T)(X,Y) = \sum_i T(R(e_i,X)Y, e_i), \qquad \mathring{R}g = \mathrm{Ric}_g\,. \tag{1.2}$$

A tensor is TT if $\mathrm{tr}_g\, T = 0$ and $\delta T = 0$. Define

$$\lambda_{\mathrm{TT}}(L) = \inf_{0\neq H\in C^\infty(S^2T^*M)\cap \mathrm{TT}} \frac{\langle LH, H\rangle_{L^2}}{\|H\|^2_{L^2}}, \qquad \lambda_{\mathrm{TT}}(E) = \lambda_{\mathrm{TT}}(L) - 2\Lambda. \tag{1.3}$$

The convention for an empty test space is $+\infty$. The self-adjoint realization used in this definition is proved in Lemma 2.2. We call g TT-semistable when $\lambda_{\mathrm{TT}}(E) \geq 0$. All instability conclusions in this paper are TT conclusions; no scalar spectral-gap hypothesis is imposed.

1.2. **Warped products.** Let (B^k, b) and (F^m, γ) be connected manifolds and let

$$g = b + f^2\gamma, \qquad n = k + m. \tag{1.4}$$

For the first theorem both factors are closed.

Theorem A. *Suppose B and F are closed, $k, m \geq 2$, $f \in C^\infty(B)$ is positive, and the metric* (1.4) *satisfies* $\mathrm{Ric}_g = \Lambda g$, $\Lambda > 0$. *Then*

$$\lambda_{\mathrm{TT}}(L) < \frac{n}{n-1}\Lambda, \qquad \lambda_{\mathrm{TT}}(E) < -\frac{n-2}{n-1}\Lambda. \tag{1.5}$$

The function f is allowed to be constant.

For the next theorem, let (B^k, b) be compact and connected with nonempty smooth boundary, let γ_m be the unit-round metric on S^m, and let $f \in C^\infty(B)$. Assume $f > 0$ on B° and $f = 0$ on ∂B. In each boundary collar, with inward distance r, assume

$$b = dr^2 + h(y, r^2), \qquad f = r\, a(y, r^2), \qquad a(y, 0) = 1, \tag{1.6}$$

where h and a are smooth and $a > 0$. Collapse each copy of S^m over ∂B to one point. The resulting space is a smooth closed manifold: in the normal coordinates $x = r\omega \in \mathbb{R}^{m+1}$ its metric is

$$h(y, |x|^2) + a(y, |x|^2)^2 \sum_j dx_j^2 + \frac{1 - a(y,|x|^2)^2}{|x|^2}\Big(\sum_j x_j dx_j\Big)^2. \tag{1.7}$$

Put $A(y,s) = a(y,s)^2$. Since $A(y,0) = 1$,

$$\frac{1 - A(y,s)}{s} = -\int_0^1 \partial_s A(y, ts)\, dt.$$

This identity extends the last coefficient smoothly to $s = 0$. At $x = 0$ the normal block is the Euclidean metric, and the tangential block is $h(y, 0)$. These charts define the smooth closed completion used in Theorem B. On the complement of the collapsed set the completed metric equals the original Einstein metric. In the Cartesian coordinates of (1.7), the Ricci tensor is a smooth function of the metric, its inverse, and its derivatives of orders at most two. Hence the identity $\mathrm{Ric}_g = \Lambda g$, valid for $x \neq 0$, extends to $x = 0$ by continuity.

Theorem B. *Let $k \geq 2$, $m \geq 1$, and suppose* (1.6) *holds. If $b + f^2\gamma_m$ is positive Einstein on $B^\circ \times S^m$, its smooth completion is either the round sphere of curvature $\Lambda/(n-1)$ or satisfies* (1.5).

The case $m = 1$ is proved in Eqs. (3.27) and (4.10); the identities containing $(m-1)^{-1}$ are restricted to $m \geq 2$.

1.3. **Relative-factor variations and the spectral conclusion.** The inverse-power relative-factor variations of Gibbons–Hartnoll–Pope [13, Sections 3.2–3.3] are antecedents of the tensor used here. Batat–Hall–Murphy [6, Theorem A] prove Ricci-flow instability of compact Einstein warped products in total dimension at most six. Their proof also covers a three-dimensional base with a four-dimensional fibre. Their Theorems B and C treat, respectively, families converging to a nontrivial Ricci soliton as the fibre dimension increases, and a sufficient spectral condition on the Einstein fibre. None of these statements asserts instability for every compact Einstein warped product in arbitrary dimension. Theorems A and B give instead the TT estimate (1.5) for the global geometries stated above, without a dimensional upper bound or a convergence assumption. The quasi-Einstein equations and their boundary interpretation are studied in He–Petersen–Wylie [15].

Let $g_V = f^2\gamma$ and $S_0 = b/k - g_V/m$. The tensor is

$$S = f^{-1}S_0 + \frac{n}{km}Pf, \qquad Pu = \nabla^2 u + \frac{\Delta u}{n}g. \tag{1.8}$$

The critically shifted form $\mathcal{F}_n$ is defined in (2.17). It vanishes on the scalar-generated tensors Pu, and its negative index equals the number of TT eigenvalues of L below $n\Lambda/(n-1)$; see Proposition 2.4 and Lemma 2.5. For $m \geq 2$, after normalizing the fibre Einstein constant to $m-1$, put

$$Z = \mathrm{Ric}_b - \frac{R_b}{k}b, \qquad U = R_b - \frac{k(k-1)}{n-1}\Lambda.$$

The correction (1.8) has the equivalent expression (3.22), and its energy is

$$\mathcal{F}_n(S) = -\frac{n^2}{k^2m^3}\int_M \left(|Z|^2 + \frac{n-1}{k(m-1)}U^2\right) dv_g. \tag{1.9}$$

At a collapsed fibre, S is interpreted as an H^1 tensor, obtained as the limit of smooth cutoffs. The boundary integrals are estimated before passage to the limit. Equality in (1.9) reduces to the concircular Hessian equation on the base and yields the round completion. For $m = 1$, the separate identity is (4.10).

The sharp divergence-penalty formulation is taken from the author's supplied manuscript [26, Section 2]; the identities needed here are reproved in Section 2. The conclusions $\lambda_{\mathrm{TT}}(L) < 0$ in that manuscript have additional dimensional hypotheses. They are not asserted in all dimensions here. Verger [31, Theorem A] proves that the negative TT index of the full Lichnerowicz operator tends to infinity along the compact Böhm sequences in his stated dimensional range. That is an asymptotic index theorem, whereas (1.5) is an estimate for each metric satisfying the present hypotheses, at a different spectral threshold. Kröncke's stable noncompact Einstein warped products [21] concern other geometries and form domains. In particular, closedness and the specified completion in the present statements cannot be omitted.

1.4. **A one-sided integral gap in dimension four.** Now let M be oriented and four-dimensional. The Weyl tensors are endomorphisms of $\Lambda^\pm$, and their norms are Hilbert–Schmidt endomorphism norms. Set

$$B_\pm = W^\pm + \frac{\Lambda}{3}I, \quad q_\pm = \left(\lambda_{\max}(B_\pm) - \Lambda\right)_+, \quad \Theta_\pm(g) = \left(\int_M q_\pm^2\, dv_g\right)^{1/2}. \tag{1.10}$$

Here $s_+ = \max\{s, 0\}$. The function $q_\pm$ is continuous and locally Lipschitz. For self-adjoint endomorphisms A, C of one Euclidean space, the variational characterization gives

$$|\lambda_{\max}(A) - \lambda_{\max}(C)| \leq \|A - C\|_{\mathrm{op}}. \tag{1.11}$$

Apply this estimate in a smooth local orthonormal frame. No derivative of $q_\pm$ is used. The quantities $\Theta_\pm$ are unchanged by constant rescaling of the metric.

Theorem C. *There exists a universal $\varepsilon_* > 0$ with the following property. Let (M^4, g) be closed, connected, oriented and simply connected, with $\mathrm{Ric}_g = \Lambda g$ and $\Lambda > 0$. If $\Theta_+(g) < \varepsilon_*$, then $q_+ \equiv 0$. In that case either $W^+ \equiv 0$ or g is Kähler–Einstein for an orientation-compatible complex structure. The corresponding assertion holds with the two chiralities exchanged.*

The constant in theorem C is independent of the topology, volume and Einstein constant. Its existence follows from compactness; no explicit value is obtained. We use the Kähler rigidity theorem of LeBrun–Ozuch [23, Theorem A], stated as Theorem 9.2; its hypotheses are verified in Proposition 9.3 and Section 10.

Corollary 1.1. *Under the hypotheses of theorem C, if g is TT-semistable and $\min\{\Theta_+(g), \Theta_-(g)\} < \varepsilon_*$, then g is isometric, up to scale, to the round metric on S^4 or the Fubini–Study metric on $\mathbb{C}P^2$.*

1.5. **Integral and pointwise chiral rigidity.** Derdziński [12] relates harmonic self-dual Weyl curvature with two eigenvalues to conformally Kähler geometry and proves the Weitzenböck identity used below. Gursky–LeBrun [14, Theorem 1] prove, for a compact positive Einstein four-manifold,

$$W^+ \equiv 0 \quad \text{or} \quad \int_M |W^+|^2 \, dv_g \geq \frac{2}{3}\Lambda^2 \,\mathrm{Vol}(M, g). \tag{1.12}$$

Their quantity is the norm of the whole Weyl component. Theorem C concerns only its largest-eigenvalue excess above $2\Lambda/3$. These hypotheses are not interchangeable. In particular, a Kähler–Einstein surface with positive Einstein constant has $\Theta_+ = 0$ in its complex orientation although $W^+ \neq 0$.

Wu [32, Theorems 1.1 and 1.3] gives conformally Kähler rigidity under positivity or nonnegativity of $\det W^+$, with the corresponding nonvanishing alternatives. LeBrun [22] gives another proof of the positive-determinant result and further pointwise criteria. Catino–Dameno [11, Theorems 1.1 and 1.4] assume that the largest self-dual Weyl eigenvalue is everywhere simple, or satisfies their uniform-simplicity condition. These pointwise conditions are not assumed in Theorem C. No assertion that a small critical L^2 potential has a pointwise bound is used in its proof. The zero-excess classification follows from the Weitzenböck identity and eigenvalue algebra in Lemma 6.1; the additional assertion is a universal lower bound for nonzero Θ_+.

For singular limits, the external rigidity input is LeBrun–Ozuch's Theorem A [23]. It assumes a *globally* Kähler–Einstein orbifold limit and an admissible Einstein sequence, meaning that every Ricci-flat ALE limit at every level is anti-self-dual in the induced orientation. It then gives eventual Kähler–Einstein metrics. These hypotheses are established before any complex structure on the limit is used. The small-excess estimate first bounds the volume and the two Betti numbers. It next gives anti-self-duality at all rescaling levels and the uniform integrability estimate needed at the singular points. A parallel real eigenline is then extended through the finite ALE reconstruction. Simple connectivity of the smooth approximating manifolds makes the resulting real line bundle orientable. The resulting statement is Proposition 9.3.

1.6. **Circle warping and chiral Weyl energy.** For a circle-warped Einstein metric $g = b + f^2 d\theta^2$, the base has dimension three and scalar curvature 2Λ. Put $Z = \mathrm{Ric}_b - \frac{2\Lambda}{3} b$. In these conventions,

$$\mathrm{spec}(W^+) = \mathrm{spec}(W^-) = -\,\mathrm{spec}(Z). \tag{1.13}$$

For the corrected tensor constructed below, the critical quadratic form then satisfies

$$\mathcal{F}_4(S) = -\frac{16}{9}\int_M |Z|^2 \, dv_g = -\frac{16}{9}\int_M |W^\pm|^2 \, dv_g \leq -\frac{8}{3}\Theta_\pm(g)^2. \tag{1.14}$$

The equality of the two chiral curvature blocks for an Einstein 3 + 1 warped metric is explicitly obtained by Hughes–Jamal Beek–Kusmartsev [17, Section 3.3]. Accordingly, (1.13) is included with its direct calculation in our curvature convention, not as a new chiral classification. Its use here is the energy identity (1.14), including the form-domain statement for S at a collapsed fibre. The same base tensor therefore determines both the destabilizing quadratic form and the excess in theorem C. No conclusion about arbitrary locally warped charts or twisted fibrations is drawn from this computation.

1.7. **Dependencies and scope.** Sections 2–4 prove the warped-product theorems. Their proof is independent of the integral gap. Section 5 establishes the circle-warped identities with the fixed norm and curvature conventions. Sections 6–7 give the estimates for a sequence with vanishing excess. The Sobolev inequality is the Einstein equality case of Ilias' inequality [18], also presented by Petean [28, Theorem A]; the subsequent heat estimates are derived in the text. The compactness and finite ALE reconstruction input is stated as Theorem 9.1; its ingredients are due to [1, 2, 4, 5, 24, 27] and are summarized in [23, Section 1]. Sections 8–10 prove the eigenline descent and the gap, with the external rigidity statement Theorem 9.2 invoked only at the end.

The semistability conclusion is a consequence of the gap and Biquard–Ozuch's conformally Kähler instability theorem [8, Theorem A], followed by the positive half-conformally-flat classification [16][7, Theorem 13.30]. These external classification results are not reproved here. The special circle-warped classification in Proposition 11.1 is proved from parallelism of both Weyl components; it is consistent with Brendle's local-symmetry theorem under nonnegative isotropic curvature [10]. Zero excess alone does not imply stability: the equal-curvature $S^2 \times S^2$ has $\Theta_+ = \Theta_- = 0$ and the parallel TT tensor $g_1 - g_2$ has Einstein eigenvalue -2Λ.

The small-excess hypothesis is part of Theorem C. It neither proves that every TT-semistable metric satisfies that hypothesis nor classifies all positive Einstein four-manifolds. Likewise, (1.5) is a bound below a positive shift of L, not an all-dimensional negative-eigenvalue theorem for L.

2. The critically shifted quadratic form

Throughout this section $n \geq 3$, M is closed and connected, and $\mathrm{Ric} = \Lambda g$, $\Lambda > 0$. Put

$$\vartheta = \frac{n-1}{n}, \quad c_n = \vartheta^{-1}, \quad Pu = \nabla^2 u + \frac{\Delta u}{n} g, \quad \mathcal{K}\omega = \delta^* \omega + \frac{\delta\omega}{n} g, \tag{2.1}$$

where $(\delta^*\omega)_{ij} = \frac{1}{2}(\nabla_i \omega_j + \nabla_j \omega_i)$. Both Pu and $\mathcal{K}\omega$ are trace-free and $\mathcal{K} du = Pu$.

Lemma 2.1. *The following differential identities hold:*

$$\delta P u = d(\vartheta\Delta - \Lambda)u, \qquad LPu = P\Delta u, \tag{2.2}$$

$$\mathrm{tr}(LT) = \Delta \,\mathrm{tr}\, T, \qquad \delta LT = \Delta_H \delta T, \tag{2.3}$$

where $\Delta_H = \nabla^*\nabla + \Lambda$ *on one-forms. Moreover,*

$$\|Pu\|_{L^2}^2 = \vartheta \|\Delta u\|_{L^2}^2 - \Lambda \|du\|_{L^2}^2. \tag{2.4}$$

Proof. Work at a point in a geodesic orthonormal frame; repeated indices are summed. For a one-form ω and a symmetric tensor T, the covector commutator gives

$$\sum_k ([\nabla_k, \nabla_i]\omega)_k = \Lambda \omega_i, \qquad \sum_k ([\nabla_k, \nabla_i] T)_{jk} = \Lambda T_{ij} - (\mathring{R}T)_{ij}. \tag{2.5}$$

For the second identity, the two contractions are $-\sum_{k,p} R_{kipj} T_{pk} = -(\mathring{R}T)_{ij}$ and $-\sum_{k,p} R_{kipk} T_{jp} = \Lambda T_{ij}$. Apply the first identity to du and use $\nabla_i \nabla_j u = \nabla_j \nabla_i u$. Hence

$$\nabla^*\nabla du = d\Delta u - \Lambda du, \qquad 2\delta\delta^*\omega = \nabla^*\nabla\omega + d\delta\omega - \Lambda\omega. \tag{2.6}$$

Since $\delta(vg) = -dv$, the first formula of (2.2) follows.

For a metric variation with derivative T, differentiating the Koszul formula gives

$$2\dot{\Gamma}_{ij}^k = \nabla_i T_j{}^k + \nabla_j T_i{}^k - \nabla^k T_{ij}. \tag{2.7}$$

Consequently,

$$\begin{aligned} 2(D\,\mathrm{Ric})_g(T)_{ij} &= \nabla_k \nabla_i T_j{}^k + \nabla_k \nabla_j T_i{}^k - \nabla_k \nabla^k T_{ij} - \nabla_j \nabla_i \,\mathrm{tr}\, T \\ &= (LT)_{ij} - 2(\delta^*\delta T)_{ij} - (\nabla^2 \,\mathrm{tr}\, T)_{ij}. \end{aligned} \tag{2.8}$$

The second equality uses the two contractions in (2.5). For $T = \delta^*\omega = \frac{1}{2}\mathcal{L}_{\omega^\sharp}g$, diffeomorphism invariance of Ricci gives $(D\,\mathrm{Ric})_g(T) = \frac{1}{2}\mathcal{L}_{\omega^\sharp}(\Lambda g) = \Lambda T$ and $\operatorname{tr} T = -\delta\omega$. Substitution into (2.8) gives

$$\begin{aligned} LT &= 2\Lambda T + 2\delta^*\delta\delta^*\omega - \nabla^2\delta\omega \\ &= \delta^*(\nabla^*\nabla\omega + \Lambda\omega) = \delta^*\Delta_H\omega. \end{aligned} \tag{2.9}$$

In the second line, the $\delta^* d\delta\omega$ and $\nabla^2\delta\omega$ terms cancel. The identities $L(ug) = (\Delta u)g$ and $\Delta_H du = d\Delta u$ now imply $LPu = P\Delta u$. Taking formal adjoints of $L\delta^* = \delta^*\Delta_H$ and $L(ug) = (\Delta u)g$ proves (2.3). Finally,

$$\begin{aligned} \|\nabla^2 u\|_2^2 &= \langle \nabla^*\nabla du, du\rangle_{L^2} = \|\Delta u\|_2^2 - \Lambda\|du\|_2^2, \\ |Pu|^2 &= |\nabla^2 u|^2 - \frac{1}{n}(\Delta u)^2. \end{aligned}$$

Integrating the second equality proves (2.4). □

Lemma 2.2. *The restriction of L to the closed L^2 space of distributionally TT tensors is self-adjoint with compact resolvent and smooth eigentensors. Its domain is $H^2 \cap \mathrm{TT}$. Its quadratic-form domain is $H^1 \cap \mathrm{TT}$. Finite sums of smooth TT eigentensors are dense in this form domain.*

Proof. Trace and divergence are continuous from L^2 to L^2 and H^{-1}, respectively; hence the TT subspace $\mathcal{T}$ is closed. The full operator L on $H^2(S^2T^*M)$ is a formally self-adjoint elliptic operator with scalar positive principal symbol. The closed-manifold elliptic spectral theorem gives a self-adjoint realization with compact resolvent [30]. Choose $c > 0$ such that $L + c$ is positive and invertible. If $U \in \mathcal{T}$ and $V = (L + c)^{-1}U$, then (2.3) gives

$$(\Delta + c)\operatorname{tr} V = 0, \qquad (\Delta_H + c)\delta V = 0. \tag{2.10}$$

The first equation holds weakly in L^2; the second holds in H^{-1}, since $V \in H^2$ and $\delta V \in H^1$. Their weak pairings give

$$\|d\operatorname{tr} V\|_2^2 + c\|\operatorname{tr} V\|_2^2 = 0, \qquad \|d\delta V\|_2^2 + \|\delta\delta V\|_2^2 + c\|\delta V\|_2^2 = 0.$$

Thus $\operatorname{tr} V = 0$ and $\delta V = 0$. The resolvent therefore preserves $\mathcal{T}$ and its orthogonal complement. Its restriction to $\mathcal{T}$ is compact, positive, self-adjoint and injective; its range is dense because the orthogonal complement of the range equals its kernel. The compact spectral theorem gives an orthonormal basis of $\mathcal{T}$ consisting of eigenvectors of this resolvent. Each is a weak eigentensor of L; elliptic regularity gives smoothness. The domain of the restricted operator is $\mathrm{Dom}(L) \cap \mathcal{T} = H^2 \cap \mathcal{T}$. Increase c, if necessary, so that for every smooth tensor T

$$C^{-1}\|T\|_{H^1}^2 \le \langle (L + c)T, T\rangle_{L^2} \le C\|T\|_{H^1}^2.$$

These inequalities follow by bounding the zeroth-order curvature endomorphism on compact M. The square-root domain of $L + c$ is therefore H^1. Restriction to the reducing subspace $\mathcal{T}$ gives the stated form domain. Truncation of the spectral expansion converges in that form norm. The Rayleigh and min–max principles now apply on $H^1 \cap \mathcal{T}$. □

Lemma 2.3. *Every smooth trace-free tensor has a decomposition*

$$T = H + P\psi + V, \qquad H \in \mathrm{TT}, \qquad V = \delta^*\eta, \quad \delta\eta = 0. \tag{2.11}$$

The three tensor summands are unique, linear in T, and mutually orthogonal for both L^2 and L. Their divergences in the last two summands are orthogonal. Furthermore,

$$\langle LP\psi, P\psi\rangle_{L^2} = c_n\|\delta P\psi\|_{L^2}^2 + c_n\Lambda\|P\psi\|_{L^2}^2, \tag{2.12}$$

$$\langle LV, V\rangle_{L^2} = 2\|\delta V\|_{L^2}^2 + 2\Lambda\|V\|_{L^2}^2. \tag{2.13}$$

Proof. On trace-free tensors, $\mathcal{K}^* = \delta$. Equation (2.6) gives

$$\delta\mathcal{K} = \frac{1}{2}\Delta_H - \Lambda + \left(\frac{1}{2} - \frac{1}{n}\right) d\delta. \tag{2.14}$$

Its principal symbol at $\xi \neq 0$ is $\frac{1}{2}|\xi|^2 I + (\frac{1}{2} - \frac{1}{n})\xi \otimes \xi$, which is positive definite. Also $\delta\mathcal{K} = \mathcal{K}^*\mathcal{K}$, so its kernel is $\ker\mathcal{K}$. For $\zeta \in \ker\mathcal{K}$, $\langle \delta T, \zeta\rangle_{L^2} = \langle T, \mathcal{K}\zeta\rangle_{L^2} = 0$. The self-adjoint elliptic operator $\delta\mathcal{K}$ has closed range equal to $(\ker\mathcal{K})^\perp$. Hence the displayed orthogonality gives a unique solution $\omega \in H^2 \cap (\ker\mathcal{K})^\perp$. Since δT is smooth, iteration of the elliptic estimate gives $\omega \in C^\infty$. Then $H = T - \mathcal{K}\omega$ is TT and orthogonal to $\operatorname{im}\mathcal{K}$.

Since $\int_M \delta\omega \, dv_g = 0$, solve $\Delta\psi = \delta\omega$ with $\int_M \psi \, dv_g = 0$ and put $\eta = \omega - d\psi$. Then $\delta\eta = 0$ and $\mathcal{K}\omega = P\psi + \delta^*\eta$. Equation (2.14) preserves exact and coclosed one-forms: $\Delta_H d = d\Delta$ and $\delta\Delta_H = \Delta\delta$. In particular,

$$\langle P\psi, \delta^*\eta\rangle_{L^2} = \langle d(\vartheta\Delta - \Lambda)\psi, \eta\rangle_{L^2} = 0, \qquad \delta(\delta^*\eta) = (\tfrac{1}{2}\Delta_H - \Lambda)\eta$$

is coclosed. Hence the two divergences are orthogonal. Equation (2.9) and $LP = P\Delta$ show that L preserves the two summands. They are orthogonal to H because $\langle H, \mathcal{K}\xi\rangle = \langle \delta H, \xi\rangle = 0$. Formal self-adjointness gives the corresponding L-orthogonality. If two such decompositions are subtracted, their three differences remain in these mutually orthogonal subspaces. The squared norm of their sum is the sum of their squared norms. A zero sum therefore has three zero summands. This proves uniqueness of $H, P\psi, V$; linearity follows by adding decompositions and applying uniqueness. The potentials themselves are not claimed unique.

Set $u = (\vartheta\Delta - \Lambda)\psi$. Integration by parts gives

$$\|P\psi\|_{L^2}^2 = \langle du, d\psi\rangle_{L^2}, \qquad \langle LP\psi, P\psi\rangle_{L^2} = \langle du, d\Delta\psi\rangle_{L^2}. \tag{2.15}$$

Since $d\Delta\psi = c_n(du + \Lambda d\psi)$, these equations prove (2.12). On coclosed one-forms put $\mathcal{B} = \frac{1}{2}\Delta_H - \Lambda$. Then $\delta V = \mathcal{B}\eta$, $\|V\|_{L^2}^2 = \langle \eta, \mathcal{B}\eta\rangle_{L^2}$, and

$$\langle LV, V\rangle_{L^2} = \langle \Delta_H\eta, \mathcal{B}\eta\rangle_{L^2} = 2\|\mathcal{B}\eta\|_{L^2}^2 + 2\Lambda\langle \eta, \mathcal{B}\eta\rangle_{L^2}. \tag{2.16}$$

This is (2.13). □

Define on smooth trace-free tensors

$$\mathcal{F}_n(T) = \langle LT, T\rangle_{L^2} - c_n\|\delta T\|_{L^2}^2 - c_n\Lambda\|T\|_{L^2}^2. \tag{2.17}$$

For trace-free T, S, integration by parts expresses its polarization as

$$\mathcal{F}_n(T, S) = \int_M \left(\langle \nabla T, \nabla S\rangle - c_n\langle \delta T, \delta S\rangle + (2 - c_n)\Lambda\langle T, S\rangle - 2\langle \mathring{R}T, S\rangle\right) dv_g. \tag{2.18}$$

The inequalities $|\delta T|^2 \leq n|\nabla T|^2$ and $\sup_M \|\mathring{R}\|_{\mathrm{op}} < \infty$ imply

$$|\mathcal{F}_n(T, S)| \leq C_g\|T\|_{H^1}\|S\|_{H^1}.$$

Thus (2.18) defines its continuous extension to $H^1(S_0^2 T^*M)$.

Proposition 2.4. *For* (2.11),

$$\mathcal{F}_n(T) = \langle (L - c_n\Lambda)H, H\rangle_{L^2} + (2 - c_n)\|\delta V\|_{L^2}^2 + (2 - c_n)\Lambda\|V\|_{L^2}^2. \tag{2.19}$$

Hence $\mathcal{F}_n(T) < 0$ implies (1.5). *The negative index of $\mathcal{F}_n$ is the supremum of the dimensions of linear subspaces of smooth trace-free tensors on which $\mathcal{F}_n$ is negative definite. It equals the number of TT eigenvalues of L below $c_n\Lambda$, with multiplicities.*

Proof. Subtract the last two terms in (2.17) from the orthogonal energy decomposition and use (2.12) and (2.13). The scalar contribution cancels and gives (2.19). Let U be a subspace on which $\mathcal{F}_n$ is negative definite. For $0 \neq T \in U$, (2.19) and $2 - c_n = (n-2)/(n-1) > 0$ give

$$\langle (L - c_n\Lambda)\Pi_{\mathrm{TT}}T, \Pi_{\mathrm{TT}}T\rangle \leq \mathcal{F}_n(T) < 0.$$

In particular $\Pi_{\mathrm{TT}} T \neq 0$; hence projection is injective on U. The dimension of U is at most the negative index of $L - c_n\Lambda$ on TT tensors. Conversely, the sum of TT eigenspaces with eigenvalues below $c_n\Lambda$ is a $\mathcal{F}_n$-negative subspace. There are finitely many such eigenvalues by Lemma 2.2. This proves the index equality. The coefficient c_n is optimal for a finite-index identity of this type. Let $a > c_n$ and let $\Delta u_j = \sigma_j u_j$ with $\sigma_j \to \infty$. Set $\alpha_j = \vartheta\sigma_j - \Lambda$. Then $\alpha_j > 0$ for large j. Equations (2.2) and (2.4) give

$$\|Pu_j\|_2^2 = \sigma_j\alpha_j\|u_j\|_2^2, \qquad \|\delta Pu_j\|_2^2 = \sigma_j\alpha_j^2\|u_j\|_2^2, \qquad \langle LPu_j, Pu_j\rangle = \sigma_j\|Pu_j\|_2^2.$$

Thus $Pu_j \neq 0$, and

$$\begin{aligned} &\langle LPu_j, Pu_j\rangle - a\|\delta Pu_j\|_2^2 - a\Lambda\|Pu_j\|_2^2 \\ &\quad = \sigma_j\alpha_j[\sigma_j - a\alpha_j - a\Lambda]\|u_j\|_2^2 = \sigma_j^2\alpha_j(1 - a\vartheta)\|u_j\|_2^2 < 0. \end{aligned} \tag{2.20}$$

Choose an orthogonal scalar eigenbasis. Its nonzero images under P are orthogonal for all three forms in (2.20), by (2.2) and (2.4). The penalized form with coefficient a therefore has infinite negative index, whereas $L - a\Lambda$ has finite negative index on TT tensors. For one negative test tensor at coefficient c_n, the Rayleigh inequality gives $\lambda_{\mathrm{TT}}(L) < c_n\Lambda$; subtraction of 2Λ gives the second inequality in (1.5). □

Lemma 2.5. *On local smooth trace-free tensors define*

$$\mathscr{A} = L - c_n\mathcal{K}\delta - c_n\Lambda I. \tag{2.21}$$

Then $\mathscr{A}Pu = 0$ for every local smooth function u. On a closed manifold, $\mathcal{F}_n(T) = \langle\mathscr{A}T, T\rangle_{L^2}$ and

$$\mathcal{F}_n(T + Pu) = \mathcal{F}_n(T) \tag{2.22}$$

for globally smooth T, u.

Proof. Equations (2.2) give

$$\mathscr{A}Pu = P\Delta u - c_nP(\vartheta\Delta - \Lambda)u - c_n\Lambda Pu = 0.$$

For global smooth sections, $\mathcal{K}^* = \delta$ gives $\mathcal{F}_n(T, S) = \langle\mathscr{A}T, S\rangle$. Formal self-adjointness implies $\mathcal{F}_n(T, Pu) = \langle T, \mathscr{A}Pu\rangle = 0$ and $\mathcal{F}_n(Pu, Pu) = 0$. Expanding $\mathcal{F}_n(T + Pu)$ proves (2.22). Only the differential equation $\mathscr{A}Pu = 0$ is used when u is defined on a regular subset. □

3. The Hessian correction on a warped product

Let $g = b + f^2\gamma$ be Einstein on $B^k \times F^m$, with $f > 0$, $k \geq 2$, $m \geq 1$, and $n = k + m$. All identities in this section are local on the regular product. Write $g_V = f^2\gamma$, $D_b = \operatorname{div}_b \nabla_b$, $H = \nabla_b^2 f$ and $W = |df|_b^2$.

3.1. The Einstein equations.

For horizontal lifts X, Y and vertical lifts U, V, the Koszul formula gives

$$\begin{aligned} &\nabla_X Y = \nabla_X^b Y, && \nabla_X U = \nabla_U X = X(\log f)U, \\ &\nabla_U V = \nabla_U^\gamma V - g(U, V)\nabla_b \log f. && \end{aligned} \tag{3.1}$$

With X, Y horizontal and U, V vertical, curvature contraction is computed in a frame consisting of a b-orthonormal horizontal frame and f^{-1} times a γ-orthonormal vertical frame. Its horizontal, mixed and vertical Ricci blocks are

$$\begin{aligned} &\operatorname{Ric}_g|_{TB} = \operatorname{Ric}_b - mf^{-1}H, && \operatorname{Ric}_g(TB, TF) = 0, \\ &\operatorname{Ric}_g|_{TF} = \operatorname{Ric}_\gamma - \big(fD_bf + (m-1)W\big)\gamma. && \end{aligned} \tag{3.2}$$

For any fixed base point x_0, the vertical equation identifies Ric_γ with a scalar multiple $\mu\gamma$. The same tensor Ric_γ occurs at every base point, so that scalar is independent of $x \in B$. Thus μ is constant, and

$$\mathrm{Ric}_b - mf^{-1}H = \Lambda b, \tag{3.3}$$

$$\mu = fD_b f + (m-1)W + \Lambda f^2. \tag{3.4}$$

For $m = 1$, $\mu = 0$. The trace, Hessian and volume identities used below are

$$R_b = k\Lambda + mf^{-1}D_b f, \qquad \nabla_g^2 f = H + \frac{W}{f} g_V,$$

$$\Delta_g f = -D_b f - \frac{mW}{f}, \qquad dv_g = f^m\, dv_b\, dv_\gamma. \tag{3.5}$$

For example, a horizontal unit vector X and a vertical unit vector U satisfy $K_g(X, U) = -f^{-1}H(X, X)$; this contraction gives the horizontal term in (3.2). The vertical contraction adds $-(m-1)W\gamma$ from vertical planes and $-fD_b f\gamma$ from mixed planes. These calculations also specify the conventions in the warped formulas of [6, 15].

Lemma 3.1. *If B is closed, then $m = 1$ is impossible. If $m \geq 2$, the fibre and warping function may be normalized, without changing g, so that $\mu = m - 1$.*

Proof. For $m = 1$, (3.4) gives $D_b f = -\Lambda f$. Its integral over the closed base is a contradiction to $f > 0$. For $m \geq 2$, at a minimum of f one has $df = 0$ and $D_b f \geq 0$, so (3.4) gives $\mu \geq \Lambda \min(f)^2 > 0$. Replace γ by $\frac{\mu}{m-1}\gamma$ and f by $\sqrt{(m-1)/\mu}\, f$. Their product $f^2\gamma$ is unchanged and the new fibre Einstein constant is $m - 1$. □

Set

$$K = \frac{\Lambda}{n-1}, \qquad Z = \mathrm{Ric}_b - \frac{R_b}{k} b, \qquad U = R_b - k(k-1)K, \qquad S_0 = \frac{1}{k} b - \frac{1}{m} g_V. \tag{3.6}$$

We regard Z as a horizontal tensor on the total space. Then

$$\mathrm{tr}_g S_0 = 0, \qquad |S_0|^2 = \frac{n}{km}, \qquad \langle Z, S_0 \rangle = 0. \tag{3.7}$$

Equations (3.3)–(3.5) become

$$H = \frac{f}{m} Z + \left(\frac{fU}{km} - Kf \right) b, \qquad D_b f = \frac{fU}{m} - kKf. \tag{3.8}$$

Lemma 3.2. *On the regular product,*

$$\mathscr{A}(f^{-1}S_0) = -\frac{n}{kmf}(Z + US_0). \tag{3.9}$$

Proof. Write $T = f^{-1}S_0$. Let e_a $(1 \leq a \leq k)$ and e_μ $(1 \leq \mu \leq m)$ be horizontal and vertical orthonormal vectors. For a horizontal vector X, the divergence of ϕS_0 is

$$-\frac{1}{k}\, d\phi(X) - m\left(\frac{1}{k} + \frac{1}{m}\right) \phi\, d\log f(X).$$

The first term is the horizontal contraction. For each vertical index, (3.1) contributes $-(1/k + 1/m)\phi\, d\log f(X)$. The vertical divergence is zero, since ϕ, f are base functions and S_0 is scalar on each factor. Consequently

$$\delta(\phi S_0) = -\frac{1}{k}\big(d\phi + n\phi\, d\log f\big), \qquad \delta T = \frac{n-1}{k} d(f^{-1}). \tag{3.10}$$

Compute $D\,\mathrm{Ric}(T)$ using the variation

$$b_t = \left(1 + \frac{t}{kf}\right) b, \qquad f_t = f\left(1 - \frac{t}{mf}\right)^{1/2}, \qquad g_t = b_t + f_t^2\gamma. \tag{3.11}$$

This variation is used on a relatively compact regular set, where it is positive for sufficiently small $|t|$. Its derivative is T, $\dot f = -1/(2m)$, and b_t has infinitesimal conformal factor $u = 1/(2kf)$. Differentiation of the base conformal connection gives

$$\begin{aligned}(D\,\mathrm{Ric})_b(2ub) &= -(k-2)\nabla_b^2 u - (D_b u)b,\\ \frac{d}{dt}\bigg|_0 \nabla^2_{b_t} f_t &= -du\otimes df - df\otimes du + \langle du, df\rangle b = \frac{df\otimes df}{kf^2} - \frac{W}{2kf^2}b.\end{aligned} \tag{3.12}$$

Substitute these equations and $\dot f = -1/(2m)$ into the horizontal part of (3.2). The horizontal derivative is

$$2(D\,\mathrm{Ric})_g(T)|_{TB} = -\frac{2H}{kf^2} - \frac{2(n-2)}{kf^3}df\otimes df + \left(\frac{D_b f}{kf^2} + \frac{(m-2)W}{kf^3}\right)b. \tag{3.13}$$

Also

$$(\delta^*\delta T)|_{TB} = \frac{n-1}{k}\left(-\frac{H}{f^2} + \frac{2df\otimes df}{f^3}\right), \tag{3.14}$$

$$\delta\delta T = \frac{n-1}{k}\Delta_g(f^{-1}) = \frac{n-1}{k}\left(\frac{D_b f}{f^2} + \frac{(m-2)W}{f^3}\right). \tag{3.15}$$

By (2.8) and $\operatorname{tr} T = 0$,

$$\mathscr{A}T = 2(D\,\mathrm{Ric})_g(T) + (2-c_n)\delta^*\delta T - \frac{c_n}{n}(\delta\delta T)g - c_n\Lambda T. \tag{3.16}$$

Since $2 - c_n = (n-2)/(n-1)$, the $df\otimes df$ terms in (3.13) and (3.14) cancel. The scalar terms involving $D_b f$ and W cancel with (3.15). The remaining horizontal block is

$$(\mathscr{A}T)|_{TB} = -\frac{nH}{kf^2} - \frac{c_n\Lambda}{kf}b = -\frac{n}{kmf}Z - \frac{nU}{k^2mf}b. \tag{3.17}$$

For the vertical block, differentiating $D_{b_t}f_t$ and $|df_t|^2_{b_t}$ gives

$$\frac{d}{dt}\bigg|_0 D_{b_t}f_t = -\frac{D_b f}{kf} - \frac{k-2}{2kf^2}W, \qquad \frac{d}{dt}\bigg|_0 |df_t|^2_{b_t} = -\frac{W}{kf}. \tag{3.18}$$

Since γ is fixed, (3.2) implies

$$2(D\,\mathrm{Ric})_g(T)|_{TF} = \left(\frac{k+2m}{kmf^2}D_b f + \frac{k+2m-4}{kf^3}W\right)g_V. \tag{3.19}$$

Also $(\delta^*\delta T)|_{TF} = -(n-1)Wg_V/(kf^3)$. Substituting into (3.16) cancels its W-coefficient and gives

$$(\mathscr{A}T)|_{TF} = \left(\frac{nD_b f}{kmf^2} + \frac{c_n\Lambda}{mf}\right)g_V = \frac{nU}{km^2 f}g_V. \tag{3.20}$$

The mixed block of every term in (3.16) is zero, by (3.2) and the Hessian of a base function. Equations (3.17) and (3.20) prove (3.9). ☐

3.2. **Fibres of dimension at least two.** Assume $m \geq 2$ and $\mu = m - 1$. Define

$$S = f^{-1}S_0 + \frac{n}{km}Pf. \tag{3.21}$$

Proposition 3.3. *The tensor S has the alternative expression*

$$S = f\left(\frac{n}{km^2}Z + \frac{n-1}{km(m-1)}US_0\right). \tag{3.22}$$

Pointwise on the regular product,

$$\langle\mathscr{A}S, S\rangle = -\frac{n^2}{k^2m^3}\left(|Z|^2 + \frac{n-1}{k(m-1)}U^2\right). \tag{3.23}$$

Proof. Equations (3.4) and (3.8) give

$$1 - W - Kf^2 = \frac{f^2}{m(m-1)} U. \tag{3.24}$$

By (3.5), the vertical coefficient of Pf is

$$\frac{W}{f} - \frac{D_b f + mW/f}{n} = \frac{kW}{nf} - \frac{D_b f}{n}.$$

The vertical coefficient of (3.21) is therefore

$$\begin{aligned} -\frac{1}{mf} + \frac{W}{mf} - \frac{D_b f}{km} &= -\frac{1 - W - Kf^2}{mf} - \frac{fU}{km^2} \\ &= -\frac{(n-1)fU}{km^2(m-1)}. \end{aligned}$$

The horizontal trace-free part is $nfZ/(km^2)$ by (3.8). If the horizontal scalar coefficient is a_H and the vertical coefficient is a_V, trace-freeness gives $ka_H + ma_V = 0$. Hence

$$a_H = \frac{(n-1)fU}{k^2 m(m-1)}, \qquad a_V = -\frac{(n-1)fU}{km^2(m-1)},$$

which proves (3.22). By Lemmas 2.5 and 3.2, $\mathscr{A}S = -n(Z + US_0)/(kmf)$. Pair with (3.22) and use (3.7). The cross terms vanish by $\langle Z, S_0\rangle = 0$. The coefficients of $|Z|^2$ and U^2 are, respectively,

$$-\frac{n}{kmf}\frac{nf}{km^2} = -\frac{n^2}{k^2m^3}, \qquad -\frac{n}{kmf}\frac{(n-1)f}{km(m-1)}\frac{n}{km} = -\frac{n^2(n-1)}{k^3m^3(m-1)}.$$

These are the coefficients in (3.23). □

3.3. **Circle fibres.** Assume $m = 1$, so $n = k + 1$. The fibre equation and its trace give

$$D_b f = -\Lambda f, \qquad R_b = (k-1)\Lambda, \qquad U = 0. \tag{3.25}$$

Set

$$\zeta = 1 - |df|^2 - \frac{\Lambda}{k} f^2, \qquad S = f^{-1} S_0 + \frac{n}{k} Pf. \tag{3.26}$$

The horizontal and vertical blocks of Pf are

$$(Pf)|_{TB} = fZ + \left(-\frac{\Lambda f}{k(k+1)} - \frac{W}{(k+1)f}\right) b, \qquad (Pf)|_{TF} = \left(\frac{kW}{(k+1)f} + \frac{\Lambda f}{k+1}\right) g_V.$$

Multiplication by $(k+1)/k$ and addition of $f^{-1}S_0$ give

$$S = \frac{n}{k} fZ + \frac{\zeta}{f} S_0, \qquad \mathscr{A}S = -\frac{n}{kf} Z, \qquad \langle \mathscr{A}S, S\rangle = -\frac{n^2}{k^2}|Z|^2. \tag{3.27}$$

The second equation follows from (3.9) with $U = 0$, and the last uses $\langle Z, S_0\rangle = 0$. No division by $m - 1$ occurs.

4. Integration, smooth collapse and equality

4.1. **Closed bases.** If the base is closed, $f > 0$ makes (3.21) a globally smooth tensor. Proposition 3.3 and integration by parts give

$$\mathcal{F}_n(S) = -\frac{n^2}{k^2m^3} \int_M \left(|Z|^2 + \frac{n-1}{k(m-1)} U^2\right) dv_g. \tag{4.1}$$

All coefficients in the integrand are positive. If its integral vanished, continuity of Z, U would give $Z = U = 0$ on the regular product, and hence on B. Then (3.8) would imply $D_b f = -kKf$. Its integral over the closed base is $0 = -kK\int_B f\, dv_b$, a contradiction. The right side of (4.1) is therefore negative. Proposition 2.4 proves theorem A, including constant warping.

4.2. **Admissibility at the collapsing fibres.** Assume (1.6). The base tensors Z, U and their first base covariant derivatives are bounded up to ∂B. For $m \geq 2$, (3.22) implies

$$|S| = O(r), \qquad |\nabla_g S| + |\delta_g S| = O(1), \qquad dv_g \asymp r^m \, dr \, dv_{\partial B} \, dv_{S^m}. \tag{4.2}$$

The constants depend on the fixed metric. If Q is a lifted base tensor and π_H, π_V are the two orthogonal projections, (3.1) gives on a sufficiently small collar

$$|\nabla \pi_H| + |\nabla \pi_V| \leq C\frac{|df|}{f}, \qquad |\nabla_g Q| \leq |\nabla_b Q| + C\frac{|df|}{f}|Q|. \tag{4.3}$$

Here $cr \leq f \leq Cr$ and $|df| \leq C$. Each coefficient of the projection tensors in (3.22) is $O(r)$, with bounded first derivative. Applying (4.3) to $Q = Z$ and the product rule to fZ and fUS_0 gives the derivative estimate in (4.2).

For $m = 1$, write $a(y, r^2) = 1 + r^2 a_1(y, r^2)$ in (1.6). Hence

$$f = r + O(r^3), \quad \partial_r f = 1 + O(r^2), \quad d_y f = O(r^3), \quad \zeta = O(r^2), \quad |d\zeta|_b = O(r).$$

Consequently $\zeta/f = O(r)$ and $|d(\zeta/f)|_b \leq C$. Apply (4.3) to the two summands of (3.27). This proves (4.2) also for circle fibres. All these estimates are uniform on the finitely many boundary collars.

Choose $\chi \in C^\infty([0,\infty))$ with $0 \leq \chi \leq 1$, $\chi = 0$ on $[0,1]$ and $\chi = 1$ on $[2,\infty)$. On the collars set $S_\epsilon = \chi(r/\epsilon)S$, and put $S_\epsilon = S$ away from them. This is a globally smooth trace-free tensor on the completed manifold, extended by zero near its collapsing set. For $0 < \delta < \epsilon$, the product rule and (4.2) give

$$\begin{aligned} \|S_\epsilon - S_\delta\|_{H^1}^2 &\leq C\int_0^{2\epsilon} r^m \, dr + C\epsilon^{-2}\int_\epsilon^{2\epsilon} r^{m+2} \, dr + C\delta^{-2}\int_\delta^{2\delta} r^{m+2} \, dr \\ &\leq C'(\epsilon^{m+1} + \delta^{m+1}). \end{aligned} \tag{4.4}$$

The smooth tensors S_ϵ are Cauchy in the complete space $H^1(S_0^2 T^*M)$. Their L^2 limit equals the original tensor on the regular set; its value on the collapsed set is immaterial, since that set has measure zero. Denote this H^1 tensor by S. Letting $\delta \downarrow 0$ in (4.4) yields

$$\|S_\epsilon - S\|_{H^1}^2 \leq C'\epsilon^{m+1}, \qquad \lim_{\epsilon\downarrow 0} \mathcal{F}_n(S_\epsilon) = \mathcal{F}_n(S). \tag{4.5}$$

For a compact regular domain Ω with smooth boundary, define

$$\mathcal{F}_{n,\Omega}(T) = \int_\Omega \left(|\nabla T|^2 + (2 - c_n)\Lambda|T|^2 - 2\langle \mathring{R}T, T\rangle - c_n|\delta T|^2\right) dv_g. \tag{4.6}$$

This is the first-order integral form, not the expression $\int_\Omega \langle LT, T\rangle - c_n|\delta T|^2 - c_n\Lambda|T|^2$; the two expressions differ by a boundary term. Let M_ϵ be the complement of the radius-ϵ collars, with outward normal ν. Integration by parts gives

$$\mathcal{F}_{n,M_\epsilon}(S) = \int_{M_\epsilon} \langle \mathscr{A}S, S\rangle \, dv_g + \int_{\partial M_\epsilon} \left(\langle \nabla_\nu S, S\rangle + c_n S(\nu, (\delta S)^\sharp)\right) dA. \tag{4.7}$$

For a trace-free tensor T and a one-form ω, the two formulas are

$$\begin{aligned} \int_\Omega \langle \nabla^*\nabla T, T\rangle &= \int_\Omega |\nabla T|^2 - \int_{\partial\Omega} \langle \nabla_\nu T, T\rangle, \\ \int_\Omega \langle \mathcal{K}\omega, T\rangle &= \int_\Omega \langle \omega, \delta T\rangle + \int_{\partial\Omega} T(\nu, \omega^\sharp). \end{aligned} \tag{4.8}$$

Substituting $T = S$ and $\omega = \delta S$ proves (4.7). Its boundary integral is bounded in absolute value by

$$C\int_{\partial M_\epsilon} |S|\left(|\nabla S| + |\delta S|\right) dA \leq C'\epsilon^{m+1}. \tag{4.9}$$

By (4.2), every summand in the first-order integrand (4.6) belongs to $L^1(M)$; the potential is bounded because the completed metric is smooth. Hence $\mathcal{F}_{n,M_\epsilon}(S) \to \mathcal{F}_n(S)$ by absolute continuity of the integral. The pairing $\langle \mathscr{A} S, S\rangle$ equals the bounded function in (3.23) or (3.27). Its omitted collar integral tends to zero. Together with (4.9), these two limits prove (4.1) for $m \geq 2$, and

$$\mathcal{F}_n(S) = -\frac{n^2}{k^2}\int_M |Z|^2\, dv_g \qquad (m = 1). \tag{4.10}$$

The equation $\mathscr{A} S \in L^2$ is not required; in the circle case it need not hold. Only the displayed pairing and boundary estimate are used.

If either energy is negative, then $\mathcal{F}_n(S_\epsilon) < 0$ for all sufficiently small positive ϵ. Applying Proposition 2.4 to these smooth tensors proves (1.5).

4.3. **Equality.**

Lemma 4.1. *Let (B^k, b) be compact and connected, $k \geq 2$, with nonempty smooth boundary. Suppose $f > 0$ on B°, $f = 0$ and $|df| = 1$ on ∂B, and*

$$\nabla_b^2 f = -Kfb, \qquad K > 0. \tag{4.11}$$

Then B is isometric to the round hemisphere of curvature K, and $f = K^{-1/2}\cos(\sqrt{K}\, r)$, where r is distance from its centre.

Proof. Equation (4.11) implies $d(|df|^2 + Kf^2) = 0$. Connectedness and the boundary values give

$$|df|^2 + Kf^2 = 1. \tag{4.12}$$

Let $C = \{df = 0\}$. A positive maximum of f is attained in B°, so $C \neq \emptyset$. At every $p \in C$,

$$f(p) = K^{-1/2}, \qquad \nabla^2 f(p) = -\sqrt{K}\, b_p.$$

Thus C is a discrete compact subset of B°, hence finite. The set $B^\circ \setminus C$ is connected: a path in B° can be modified inside finitely many coordinate balls to avoid C, since $k \geq 2$.

On $B \setminus C$ put $r = K^{-1/2}\arccos(\sqrt{K}f)$ and $R = \pi/(2\sqrt{K})$. Differentiating $f = K^{-1/2}\cos(\sqrt{K}r)$ and using (4.11) gives

$$|dr| = 1, \qquad \nabla^2 r = \sqrt{K}\cot(\sqrt{K}r)(b - dr^2), \qquad 0 < r \leq R. \tag{4.13}$$

Let $x \in B^\circ \setminus C$ and let γ_x be the integral curve of $-\nabla r$ with $\gamma_x(0) = x$. Then

$$r(\gamma_x(t)) = r(x) - t, \qquad |\dot{\gamma}_x| = 1 \quad (0 \leq t < r(x)).$$

For t in a compact subinterval of $[0, r(x))$, the curve lies in a compact subset disjoint from C and from ∂B. The ordinary differential equation therefore exists up to time $r(x)$. Moreover $d_b(\gamma_x(s), \gamma_x(t)) \leq |s - t|$; metric completeness of compact B gives a limit p_x as $t \uparrow r(x)$. Continuity of f implies $p_x \in C$.

For $p \in C$, along a unit geodesic η starting at p,

$$(f \circ \eta)'' = -K(f \circ \eta), \qquad (f \circ \eta)(0) = K^{-1/2}, \qquad (f \circ \eta)'(0) = 0.$$

Thus $f \circ \eta = K^{-1/2}\cos(\sqrt{K}t)$ on a normal ball about p, and r equals geodesic distance there. Choose disjoint normal balls about the finitely many points of C. If $p_x = p$, choose $a > 0$ so small that the terminal segment of γ_x with $0 < r \leq a$ lies in the normal ball about p. At $r = a$, the vector field $-\nabla r$ is transverse to the level, with derivative -1 on r. Existence and continuous dependence of its flow on the compact segment from x to that level show that every initial point sufficiently close to x reaches the same normal ball. There r is distance from p, and the integral curves of $-\nabla r$ are the inward radial geodesics. Their endpoint is p. Thus $\{x \in B^\circ \setminus C : p_x = p\}$ is open. These sets are disjoint, nonempty, and cover the connected set $B^\circ \setminus C$. Hence $C = \{p\}$.

For sufficiently small $a > 0$, the flow of ∇r carries the geodesic sphere $r = a$ diffeomorphically onto every level $a \leq r \leq R$. Existence follows on each compact regular strip. Near $r = R$, the smooth metric and the function r extend to a boundary collar, and $dr \neq 0$; the flow therefore extends to the terminal level. Its inverse is the flow of $-\nabla r$. Every regular point lies on one such curve. The flow is orthogonal to the levels. Therefore the polar coordinates about p cover $B \setminus \{p\}$ without identifications. Writing $b = dr^2 + b_r$, (4.13) implies

$$\partial_r b_r = 2\sqrt{K}\cot(\sqrt{K}r)b_r, \qquad \partial_r\left(\frac{b_r}{\sin^2(\sqrt{K}r)}\right) = 0.$$

Smoothness at p gives $r^{-2}b_r \to \gamma_{k-1}$, so $b_r = K^{-1}\sin^2(\sqrt{K}r)\gamma_{k-1}$. Since $f = 0$ exactly on ∂B, the last level is $r = R$. This is the round hemisphere and the asserted formula for f. □

If (4.1) vanishes, then $Z = U = 0$ and (3.8) gives (4.11). If (4.10) vanishes, then $Z = 0$ and (3.25) gives the same equation with $K = \Lambda/k$. The boundary data in (1.6) give $|df| = 1$. In either case Lemma 4.1 applies. The completed total metric is

$$dr^2 + K^{-1}\sin^2(\sqrt{K}r)\gamma_{k-1} + K^{-1}\cos^2(\sqrt{K}r)\gamma_m, \qquad 0 \le r \le \frac{\pi}{2\sqrt{K}}, \tag{4.14}$$

which is the induced metric on the sphere of radius $K^{-1/2}$ in $\mathbb{R}^k \oplus \mathbb{R}^{m+1}$ under $(r, x, y) \mapsto K^{-1/2}(\sin(\sqrt{K}r)x, \cos(\sqrt{K}r)y)$. This proves the equality case and completes theorem B.

5. Circle warping and the two Weyl spectra

The chiral block equality in this section has the antecedent [17, Section 3.3]. We compute it with the conventions (1.1)–(1.2) in order to apply (4.10). Let $g = b + f^2 d\theta^2$ be Einstein in dimension four. The local calculation in this section requires no global completion. By (3.25), $R_b = 2\Lambda$, and

$$Z = \mathrm{Ric}_b - \frac{2\Lambda}{3}b, \qquad \nabla_b^2 f = f\left(Z - \frac{\Lambda}{3}b\right). \tag{5.1}$$

Proposition 5.1. *At every regular point,*

$$\mathrm{spec}(W^+) = \mathrm{spec}(W^-) = -\,\mathrm{spec}(Z), \qquad |W^+|^2 = |W^-|^2 = |Z|^2. \tag{5.2}$$

Consequently $q_+ = q_-$ and, in a smooth completion covered by theorem B,

$$\mathcal{F}_4(S) = -\frac{16}{9}\int_M |W^\pm|^2\, dv_g \le -\frac{8}{3}\Theta_\pm(g)^2. \tag{5.3}$$

Proof. At a fixed regular point choose a b-orthonormal eigenbasis e_1, e_2, e_3 of Z, with eigenvalues $z_1 + z_2 + z_3 = 0$, and put $e_4 = f^{-1}\partial_\theta$. For the base curvature, with horizontal indices, the three-dimensional identity is

$$R^b_{abcd} = (\mathrm{Ric}_b)_{ac}\delta_{bd} + (\mathrm{Ric}_b)_{bd}\delta_{ac} - (\mathrm{Ric}_b)_{ad}\delta_{bc} - (\mathrm{Ric}_b)_{bc}\delta_{ad} - \frac{R_b}{2}(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}).$$

Thus, for cyclic i, j, k,

$$K_g(e_j, e_k) = \mathrm{Ric}_b(e_j, e_j) + \mathrm{Ric}_b(e_k, e_k) - \tfrac{1}{2}R_b = \frac{\Lambda}{3} - z_i,$$
$$K_g(e_i, e_4) = -f^{-1}\nabla_b^2 f(e_i, e_i) = \frac{\Lambda}{3} - z_i.$$

Off-diagonal components in these three two-plane pairs vanish: in the base the Ricci tensor is diagonal and determines curvature; in the mixed block the Hessian is diagonal; components with exactly one vertical index vanish by (3.1). Thus the curvature operator on the pair $\mathrm{span}\{e_j \wedge e_k, e_i \wedge e_4\}$ is $(\Lambda/3 - z_i)I$. The self-dual and anti-self-dual combinations have these same three eigenvalues. Subtracting $\Lambda/3$ proves (5.2). The calculation is pointwise; it applies also at repeated eigenvalues.

If $z_1 = \min z_i$, then $z_2 + z_3 = -z_1$ and $z_2^2 + z_3^2 \ge z_1^2/2$. Hence $z_1^2 \le 2|Z|^2/3$, and

$$q_+ = q_- = \left(-\lambda_{\min}(Z) - \frac{2\Lambda}{3}\right)_+, \qquad q_\pm^2 \le \frac{2}{3}|Z|^2. \tag{5.4}$$

Equations (4.10) and (5.4) give (5.3). Curvature is smooth in the completed metric; therefore the spectral identities, expressed as continuous invariant functions, extend across the collapsed set. □

6. Chiral curvature identities and a Sobolev inequality

In the remaining sections all four-manifolds and orbifolds are oriented. For an Einstein metric $\mathrm{Ric} = \Lambda g$, the curvature operator on two-forms has the decomposition

$$\mathcal{R} = \begin{pmatrix} B_+ & 0 \\ 0 & B_- \end{pmatrix}, \qquad B_\pm = W^\pm + \frac{\Lambda}{3} I, \qquad \operatorname{tr} B_\pm = \Lambda. \tag{6.1}$$

The operator on two-forms is normalized by $(\mathcal{R}\alpha)_{ij} = \frac{1}{2} R_{ij}{}^{kl}\alpha_{kl}$, with the curvature components as specified before (1.2). In particular it acts as KI at constant sectional curvature K. Two-forms have norm $|\alpha|^2 = \frac{1}{2}\alpha_{ij}\alpha^{ij}$. The endomorphism norms used in (1.10) satisfy

$$|W|^2_{\text{tensor}} = 4(|W^+|^2 + |W^-|^2), \qquad |\operatorname{Rm}|^2_{\text{tensor}} = 4(|W^+|^2 + |W^-|^2) + \frac{8}{3}\Lambda^2. \tag{6.2}$$

The characteristic identities on a closed manifold are

$$8\pi^2\chi(M) = \int_M \left(|W^+|^2 + |W^-|^2 + \frac{2}{3}\Lambda^2 \right) dv_g, \tag{6.3}$$

$$12\pi^2\tau(M) = \int_M (|W^+|^2 - |W^-|^2)\, dv_g, \tag{6.4}$$

$$2\chi(M) + 3\tau(M) = \frac{1}{2\pi^2}\int_M |B_+|^2\, dv_g. \tag{6.5}$$

The last equation follows by adding twice (6.3) divided by $8\pi^2$ to three times (6.4) divided by $12\pi^2$, and using $|B_+|^2 = |W^+|^2 + \Lambda^2/3$. These are the Einstein specializations of the Chern–Gauss–Bonnet and signature formulas [7, Chapter 13].

The Hodge and Weyl Weitzenböck identities in our sign convention are

$$\Delta_H|_{\Lambda^\pm} = \nabla^*\nabla + 2\Lambda I - 2B_\pm, \tag{6.6}$$

$$\frac{1}{2}\Delta|W^\pm|^2 = 18\det W^\pm - 2\Lambda|W^\pm|^2 - |\nabla W^\pm|^2. \tag{6.7}$$

Equation (6.7) is Derdziński's identity [12]; [32, Remark 1.2] records it with the opposite scalar Laplacian. The change of sign gives exactly (6.7). The chiral energy estimate of Gursky–LeBrun is

$$W^+ \not\equiv 0 \quad\Longrightarrow\quad \int_M |W^+|^2\, dv_g \geq \frac{2}{3}\Lambda^2 \operatorname{Vol}(M, g). \tag{6.8}$$

This is Gursky–LeBrun [14, Theorem 1]; the formulation for either chirality and the endomorphism norm is also recorded in [32, Remark 1.2]. Its assumptions are closedness and the positive Einstein equation.

If $\mathrm{Ric}_g = g$ and $V = \operatorname{Vol}(M, g)$, then

$$\|u\|_4^2 \leq V^{-1/2}\left(\frac{3}{2}\|du\|_2^2 + \|u\|_2^2 \right), \qquad u \in H^1(M). \tag{6.9}$$

Set $g_0 = g/3$. Then $\mathrm{Ric}_{g_0} = 3g_0$ and $\operatorname{Vol}(g_0) = V/9$. The Ilias–Petean inequality [18][28, Theorem A] gives $Y(M,[g]) = Y(M,[g_0]) \geq 12\sqrt{V/9} = 4\sqrt{V}$. The constant function in the Yamabe quotient for g, whose scalar curvature is 4, gives the reverse inequality. Thus $Y(M,[g]) = 4\sqrt{V}$. Substitution into $Y(M,[g])\|u\|_4^2 \leq 6\|du\|_2^2 + 4\|u\|_2^2$ proves (6.9). Density extends the smooth inequality to H^1.

Lemma 6.1. *Let (X^4, g) be a compact connected oriented Einstein orbifold with isolated quotient singularities, $\mathrm{Ric}_g = g$. If $B_+ \leq I$, then W^+ is parallel. Either it vanishes identically or its spectrum is $\{2/3, -1/3, -1/3\}$ at every point.*

Proof. Let ℓ, u, v be the eigenvalues of W^+ at a point, with ℓ the largest. Trace-freeness gives $\ell \geq 0$ and $u + v = -\ell$; hence

$$3\ell|W^+|^2 - 18\det W^+ = 6\ell(u - v)^2. \tag{6.10}$$

Substitute this identity in (6.7). Since $\ell \le 2/3$,

$$-\tfrac{1}{2}\Delta|W^+|^2 = |\nabla W^+|^2 + (2-3\ell)|W^+|^2 + 6\ell(u-v)^2 \ge 0. \tag{6.11}$$

The eigenvalues are continuous by (1.11). Since X is compact, the three nonnegative summands are integrable. To integrate the left side, remove radius-r orbifold balls about the finitely many singular points. In each uniformizing chart $\nabla|W^+|^2$ is bounded and the boundary area is $O(r^3)$. The resulting boundary integrals tend to zero. Therefore the integral of every nonnegative term in (6.11) is zero. In particular $\nabla W^+ = 0$. If $W^+ \not\equiv 0$, its parallel norm is a positive constant. The second term in (6.11) then gives $\ell = 2/3$; the third gives $u = v$. Since $u + v = -2/3$, both equal $-1/3$. ☐

7. Uniform estimates from small chiral excess

Let M be a closed connected oriented Einstein four-manifold. Normalize $\mathrm{Ric}_g = g$ and write $q = q_+$, $\theta = \|q\|_2$, $V = \mathrm{Vol}(M, g)$. Assume throughout this section

$$\theta \le \frac{\pi}{12}. \tag{7.1}$$

This threshold is used only for preliminary estimates. It is not a value of the constant in theorem C.

Lemma 7.1. *There are bounds independent of the metric and topology:*

$$\frac{\pi^2}{4} \le V \le 24\pi^2, \qquad \mathrm{diam}(M, g) \le \pi\sqrt{3}. \tag{7.2}$$

Proof. Let b_1, b_2, b_3 be the eigenvalues of B_+. They sum to one and are at most $1 + q$. For $c = 1 + q$ put $x_j = c - b_j \ge 0$; then $\sum_j x_j = 3c - 1$. Since $x_j \ge 0$, $\sum_j x_j^2 \le (\sum_j x_j)^2 = (3c-1)^2$. Therefore

$$|B_+|^2 = 3c^2 - 2c(3c-1) + \sum_j x_j^2 \le 3c^2 - 2c(3c-1) + (3c-1)^2 = 3 + 8q + 6q^2. \tag{7.3}$$

By (6.5), $2\chi + 3\tau$ is positive because $\mathrm{tr}\, B_+ = 1$. It is an integer, hence at least one. Integrating (7.3) and using Cauchy–Schwarz gives

$$2\pi^2 \le 3V + 8\sqrt{V}\,\theta + 6\theta^2 \le 4V + 22\theta^2. \tag{7.4}$$

The second inequality is $(\sqrt{V} - 4\theta)^2 \ge 0$. Under (7.1), $4V \ge (133/72)\pi^2$, and therefore $V \ge \pi^2/4$. The comparison metric with $\mathrm{Ric} = g$ is the round sphere of sectional curvature $1/3$. Myers' diameter theorem gives $\mathrm{diam} \le \pi\sqrt{3}$, and Bishop comparison gives its volume $24\pi^2$ as an upper bound. These comparison theorems are applied only on the smooth closed manifold [29]. ☐

Lemma 7.2. *Every harmonic self-dual two-form satisfies*

$$\|\nabla\alpha\|_2^2 \le \frac{2}{3}\|\alpha\|_2^2. \tag{7.5}$$

Proof. Equation (6.6) and $B_+ - I \le qI$ give

$$\|\nabla\alpha\|_2^2 = 2\int_M \langle (B_+ - I)\alpha, \alpha\rangle\, dv_g \le 2\theta\|\alpha\|_4^2.$$

The norm of a smooth tensor belongs to H^1, with $|d|\alpha|| \le |\nabla\alpha|$ almost everywhere. Apply (6.9) to this norm. For $d = \theta/\sqrt{V} \le 1/6$, the result is

$$(1 - 3d)\|\nabla\alpha\|_2^2 \le 2d\|\alpha\|_2^2. \tag{7.6}$$

Since $2d/(1-3d) \le 2/3$, this proves (7.5). ☐

Lemma 7.3. *There is a universal constant K_0 such that the scalar heat kernel satisfies $p(1, x, x) \le K_0$. If $D_+ = \nabla^*\nabla$ on Λ^+, then*

$$\mathrm{Tr}(e^{-D_+}) \le 3\,\mathrm{Tr}(e^{-\Delta}). \tag{7.7}$$

Proof. By (7.2) and (6.9), one may take $C_S = 3/\pi$ in $\|u\|_4^2 \le C_S(\|du\|_2^2 + \|u\|_2^2)$. Interpolation between L^1 and L^4 gives

$$\|u\|_2^3 \le C_S\|u\|_1(\|du\|_2^2 + \|u\|_2^2). \tag{7.8}$$

For smooth f set $u(t) = e^{-t(\Delta+1)}f$ and $F(t) = \|u(t)\|_2^2$. Scalar positivity gives the L^1 bound $\|u(t)\|_1 \le \|f\|_1$. Thus

$$F'(t) = -2(\|du(t)\|_2^2 + \|u(t)\|_2^2) \le -\frac{2F(t)^{3/2}}{C_S\|f\|_1}.$$

For $f \neq 0$, injectivity of the scalar heat semigroup implies $F(t) > 0$. Integration gives

$$F(t)^{-1/2} \ge F(0)^{-1/2} + \frac{t}{C_S\|f\|_1} \ge \frac{t}{C_S\|f\|_1}.$$

Consequently $\|e^{-t(\Delta+1)}f\|_2 \le C_S\|f\|_1/t$. The inequality extends from smooth f by density. Self-adjointness and the semigroup property, with time split into two halves, give

$$\|e^{-t\Delta}\|_{1\to\infty} \le 4e^t C_S^2 t^{-2}. \tag{7.9}$$

The smooth scalar heat kernel therefore satisfies the same bound. One can take $K_0 = 4eC_S^2$.

For a solution $s(t)$ of $\partial_t s = -D_+ s$, the function $v_\epsilon = (|s|^2 + \epsilon^2)^{1/2}$ satisfies $(\partial_t + \Delta)v_\epsilon \le 0$. In fact $(\partial_t + \Delta)|s|^2 = -2|\nabla s|^2$, and the chain rule yields

$$(\partial_t + \Delta)v_\epsilon = -\frac{|\nabla s|^2}{v_\epsilon} + \frac{|d|s|^2|^2}{4v_\epsilon^3} \le 0.$$

The initial value obeys $v_\epsilon(0) \le |\alpha| + \epsilon$. The scalar maximum principle, with $e^{-t\Delta}1 = 1$, gives $v_\epsilon(t) \le e^{-t\Delta}|\alpha| + \epsilon$. Letting $\epsilon \downarrow 0$ proves $|e^{-tD_+}\alpha| \le e^{-t\Delta}|\alpha|$. Fix y, a unit vector $v \in \Lambda_y^+$, and nonnegative approximate identities φ_j supported in a trivializing neighborhood of y. Extend v to a local orthonormal trivialization and apply the preceding inequality to $\varphi_j v$. Smoothness of both heat kernels for $t > 0$ and passage to the limit give

$$|K_{D_+}(t,x,y)v| \le p(t,x,y).$$

Thus $\|K_{D_+}(t,x,y)\|_{\mathrm{op}} \le p(t,x,y)$. The semigroup identity and self-adjointness imply

$$K_{D_+}(t,x,x) = \int_M K_{D_+}(t/2,x,y)K_{D_+}(t/2,x,y)^*\,dv_g(y),$$

so this endomorphism is nonnegative. Moreover,

$$\|K_{D_+}(t,x,x)\|_{\mathrm{op}} \le \int_M p(t/2,x,y)^2\,dv_g(y) = p(t,x,x),$$

where the last equality is the scalar semigroup identity. Since the rank of Λ^+ is three, $\operatorname{tr} K_{D_+}(t,x,x) \le 3p(t,x,x)$. Integration over M proves (7.7). ☐

Proposition 7.4. *Under* (7.1)*, if M is simply connected there is a universal integer N_0 such that*

$$b_2^+ \le N_0, \qquad b_2^- \le 3 + 5N_0, \qquad \chi(M) \le 5 + 6N_0. \tag{7.10}$$

In particular $\int_M |\operatorname{Rm}|^2_{\mathrm{tensor}}\,dv_g$ is universally bounded.

Proof. The estimate (7.5) holds on the entire harmonic self-dual space. Min–max for D_+ gives at least b_2^+ eigenvalues at most $2/3$. Hence

$$b_2^+ e^{-2/3} \le \operatorname{Tr}(e^{-D_+}) \le 3\operatorname{Tr}(e^{-\Delta}) \le 3K_0 V \le 72\pi^2 K_0. \tag{7.11}$$

Since $K_0 = 36e/\pi^2$, one may take $N_0 = \lceil 2592e^{5/3}\rceil$. Simple connectivity and Poincaré duality give $\chi = 2 + b_2^+ + b_2^-$ and $\tau = b_2^+ - b_2^-$. Thus

$$2\chi + 3\tau = 4 + 5b_2^+ - b_2^- > 0. \tag{7.12}$$

Its integrality implies the remaining bounds in (7.10). Finally (6.3) and (6.2) give

$$\int_M |\operatorname{Rm}|^2_{\text{tensor}}\, dv_g = 32\pi^2\chi(M). \tag{7.13}$$

Equation (7.13) is the asserted curvature-energy bound. □

8. Flat real line bundles on ALE orbifolds

Orbifold coverings are understood in the local subgroup sense of [19, Section 2.2], with the corrections in [20]. A local isotropy homomorphism need not be surjective. The surjectivity in Lemma 8.1 is a global conclusion about the inclusion of an ALE end, and is proved below rather than assumed from the orbifold covering definition.

An ALE four-orbifold in this section is connected, complete, and has only isolated quotient singularities. Outside a compact subset it has one end $(R_0,\infty)\times S^3/\Gamma$, where $\Gamma < SO(4)$ is finite and acts freely on S^3. In lifted end coordinates its metric satisfies

$$g_{ij} = \delta_{ij} + O(r^{-\tau}), \qquad \partial_k g_{ij} = O(r^{-1-\tau}), \qquad \tau > 0. \tag{8.1}$$

Only this first-derivative decay is used in the line-bundle argument. The ALE metrics in the Einstein compactness theorem satisfy these conditions [5, 4, 27].

All fundamental groups in this section are orbifold fundamental groups when the space is singular. This is not the fundamental group of the underlying topological space. The existence and subgroup description of orbifold coverings are as in [19, Section 2.2].

Lemma 8.1. *Let Y be a complete Ricci-flat ALE four-orbifold. Inclusion of a cross-section of its end induces a surjection*

$$\pi_1(S^3/\Gamma) \longrightarrow \pi_1^{\mathrm{orb}}(Y). \tag{8.2}$$

Proof. Let $p : \widetilde{Y} \to Y$ be the universal orbifold covering with its lifted complete Ricci-flat metric, and let $U = (R_0,\infty)\times S^3/\Gamma$ be an end neighborhood. The covering is a local isometry in the orbifold sense. The lifted metric is complete: a finite-length geodesic projects to a geodesic on complete Y, and its continuation lifts in the local covering charts. The metric space underlying the cover is locally compact; Hopf–Rinow therefore makes it proper and geodesic. These completeness statements concern orbifold geodesics with their local lifts, as in [19, Section 2.5] and [20]. No manifold universal cover is assumed. Each component U_j of $p^{-1}(U)$ is an ordinary connected covering of the smooth manifold U. Since $\pi_1(U) = \Gamma$ is finite, its degree is finite, and

$$U_j \cong (R_0,\infty)\times S^3/\Gamma_j \quad \text{for a subgroup } \Gamma_j < \Gamma.$$

Its cross-sections are compact. The lifted ALE metric is uniformly comparable to the flat cone for $r \geq R_1$; in particular the distance from radius r to radius R_1 is at least $c(r - R_1)$.

Suppose U_1, U_2 are distinct components. Choose $R_1 > R_0$ and let $\Sigma_j = \{r = R_1\} \subset U_j$, $j = 1, 2$. The component $\{r > R_1\} \subset U_j$ can meet its complement only through Σ_j: the covering extends over a product collar of this cross-section, whose finite-degree closure is compact. Hence removing $\Sigma_1 \cup \Sigma_2$ leaves two distinct unbounded components. Choose $x_i \in U_1$ and $y_i \in U_2$ with $r(x_i), r(y_i) \to \infty$. Every minimizing geodesic from x_i to y_i meets the fixed compact set $\Sigma_1 \cup \Sigma_2$. Parameterize it with such an intersection at time zero. The ALE lower radial-length estimate makes both endpoint parameters tend to infinity in absolute value. On every bounded parameter interval these unit-speed geodesics remain in a fixed compact metric ball. Arzelà–Ascoli and a diagonal subsequence give $\gamma : \mathbb{R} \to \widetilde{Y}$. Taking limits of distances gives

$$d(\gamma(s), \gamma(t)) = t - s \qquad (s < t),$$

so γ is a line. The orbifold Ricci splitting theorem [9, Theorem 1] applies to this complete connected orbifold: its Ricci curvature is zero and it contains the line γ. Thus $\widetilde{Y} \cong \mathbb{R}\times Z$, where Z is a complete connected Riemannian orbifold with nonnegative Ricci curvature. The version needed here is the

Ricci-curvature theorem of [9], also identified in [19, Remark 3.4]; a sectional curvature lower bound is not assumed.

If Z were noncompact, choose $z_0 \in Z$. For every $R > 0$ the set

$$\big((\mathbb{R} \setminus [-R, R]) \times Z\big) \ \cup\ \big(\mathbb{R} \times (Z \setminus \overline{B}_R(z_0))\big)$$

is connected: the two first-factor ends are joined at any point of $Z \setminus \overline{B}_R(z_0)$, and within either first-factor end one can move in connected Z. It is the complement of a compact product. These products exhaust $\mathbb{R} \times Z$, which would therefore have one end. Hence Z is compact. If Z is compact, the projection of a metric ball of radius R to $\mathbb{R}$ has length at most $2R$, and therefore $\operatorname{Vol} B_R \le 2R \operatorname{Vol}(Z)$. On the other hand, fix a point o on a lifted cross-section. The annulus $A_R = \{R < r < 2R\}$ in one lifted ALE end satisfies, for $R \ge R_2$,

$$\operatorname{Vol}(A_R) \ge c_1 R^4, \qquad A_R \subset B_{c_2 R}(o). \tag{8.3}$$

The first inequality follows by integrating the uniformly Euclidean volume density over a finite spherical quotient. For the inclusion, join o to a point of the inner cross-section and then follow a radial curve; their lengths are bounded by a fixed constant plus $c_2 R$. The two estimates imply $c_1 R^4 \le 2c_2 R \operatorname{Vol}(Z)$ for arbitrarily large R, a contradiction.

Thus $p^{-1}(U)$ is connected. By the subgroup description of orbifold coverings [19, Section 2.2], its components correspond to the cosets

$$\pi_1^{\mathrm{orb}}(Y) \big/ \operatorname{im}\big(\pi_1(U) \longrightarrow \pi_1^{\mathrm{orb}}(Y)\big).$$

There is one coset; the homomorphism is surjective. □

Lemma 8.2. *Suppose Y satisfies the hypotheses of Lemma 8.1 and $W_Y^+ = 0$. A real line in $\Lambda^+\mathbb{R}^4$ invariant under the end group Γ extends, under the asymptotic identification, to a parallel real orbifold line subbundle of Λ_Y^+. Its sign character on the end is the restriction of its sign character on Y.*

Proof. The curvature of the induced connection on Λ_Y^+ consists of $W_Y^+ + \mathrm{Scal}_Y I/12$ and the trace-free Ricci block. Both are zero. Let

$$\rho_Y : \pi_1^{\mathrm{orb}}(Y) \longrightarrow SO(3) \tag{8.4}$$

be its flat holonomy, and let $j : \Gamma = \pi_1(U) \to \pi_1^{\mathrm{orb}}(Y)$ be the end homomorphism. Let $\rho_+ : SO(4) \to SO(\Lambda^+\mathbb{R}^4)$ be the chiral representation. Its restriction to Γ is the action of the asymptotic cone group.

Choose a lifted radial ray and orthonormalize the asymptotic coordinate frame by the positive square root of the metric matrix. By (8.1), the induced Λ^+-connection matrix in this frame is bounded by $Cr^{-1-\tau}$. Let $Q_r \in SO(3)$ compare this frame with a frame transported radially from a fixed basepoint. The transport equation gives, for $s \ge r \ge R_1$,

$$\|Q_s - Q_r\| \le C \int_r^s t^{-1-\tau}\, dt \le \frac{C}{\tau} r^{-\tau}. \tag{8.5}$$

Thus Q_r converges to some $Q \in SO(3)$, with the same error bound. For each $\gamma \in \Gamma$, choose a fixed smooth path on S^3 joining the ray direction to its γ-translate, and radially dilate this path to radius r. It projects to a loop representing γ, of length at most $C_\gamma r$. If P is its transport matrix, the integral equation $P(t) = I - \int_0^t A(\dot\gamma)P$ and Gronwall's inequality give $\|P - I\| \le \exp(C_\gamma r^{-\tau}) - 1 = O(r^{-\tau})$ in the lifted frame. The endpoint identification is the action $\rho_+(\gamma)$. Flatness identifies the radially based holonomy with $\rho_Y(j(\gamma))$. Consequently

$$Q_r \rho_Y(j(\gamma)) Q_r^{-1} = \rho_+(\gamma) + O(r^{-\tau}) \qquad (\gamma \in \Gamma). \tag{8.6}$$

The same matrices Q_r are used for all $\gamma \in \Gamma$. Since Γ is finite, the errors in (8.6) are uniform over this set. Passage to the limit established in (8.5) gives

$$Q \rho_Y(j(\gamma)) Q^{-1} = \rho_+(\gamma) \quad \text{for all } \gamma \in \Gamma.$$

Let ℓ be the prescribed $\rho_+(\Gamma)$-invariant line. Then $Q^{-1}\ell$ is invariant under $\rho_Y(j(\Gamma))$. Lemma 8.1 gives $j(\Gamma) = \pi_1^{\mathrm{orb}}(Y)$. Parallel transport therefore defines a line subbundle of Λ_Y^+ with fibre $Q^{-1}\ell$

at the basepoint. Its holonomy on this real metric line is a homomorphism to $O(1) = \{\pm 1\}$, whose restriction is the prescribed end character. No orientation of the line is required. ☐

Lemma 8.3. *The extension property in Lemma 8.2 is preserved when a spherical end marking is changed by an orientation-preserving diffeomorphism of its round link.*

Proof. Let $\varphi : S^3/\Gamma \to S^3/\Gamma$ preserve orientation. By [3, Theorem 1.3], φ is isotopic to an orientation-preserving isometry a. Choose a lift $A \in SO(4)$ of a; then $A\Gamma A^{-1} = \Gamma$. On end fundamental groups, $a_*(\gamma) = A\gamma A^{-1}$, up to an inner automorphism. For a $\rho_+(\Gamma)$-invariant line ℓ, let χ_ℓ be its sign character. Then

$$\chi_\ell(A\gamma A^{-1}) = \chi_{\rho_+(A)^{-1}\ell}(\gamma). \tag{8.7}$$

The line on the right is invariant under $\rho_+(\Gamma)$, so its character extends by Lemma 8.2. Isotopy changes an induced based fundamental-group homomorphism by conjugation only. Since $\{\pm 1\}$ is abelian, the same extension applies to $\chi_\ell \circ \varphi_*$. If two links are first identified by an oriented cone isometry, the identical argument applies after that identification. ☐

9. Einstein orbifold limits and the eigenline bundle

9.1. Compactness and finite ALE decompositions.

Theorem 9.1 (Einstein compactness and finite ALE reconstruction). *Let (M_i^4, g_i) be closed connected oriented Einstein manifolds with $\mathrm{Ric}_{g_i} = g_i$. Assume*

$$\inf_i \mathrm{Vol}(M_i, g_i) > 0, \qquad \sup_i \chi(M_i) < \infty.$$

After passage to a subsequence there exist a compact connected Einstein orbifold (X, g_∞) with isolated quotient singularities and a finite rooted tree of Ricci-flat ALE manifolds or orbifolds such that:

- (i) *(M_i, g_i) converges to (X, g_∞) in the Gromov–Hausdorff sense and smoothly on compact subsets of X_{reg};*
- (ii) *after a further subsequence, all M_i have one oriented diffeomorphism type;*
- (iii) *every singular region is resolved by finitely many rescalings, each nonterminal rescaling limit being a Ricci-flat ALE orbifold and each terminal limit a smooth Ricci-flat ALE manifold;*
- (iv) *for sufficiently large i, M_i is obtained by gluing a compact truncation of X_{reg} to compact truncations of the ALE pieces, recursively along the corresponding spherical space-form links, by the actual boundary diffeomorphisms arising from the convergence.*

The orbifold compactness and ALE asymptotics are due to [1, 4, 5, 24]; diffeomorphism finiteness is supplied by [2]; the multiscale neck and reconstruction statements are those of [27]. The form of Theorem 9.1 used here, including the finite hierarchy of bubbles and reconstruction of the smooth manifold, is summarized in [23, Section 1]. This theorem is used as an external input.

Each edge of this tree identifies two copies of the same oriented flat cone. The link orientations used below are their radial cross-section orientations. The outward boundary orientations of the two truncated pieces are opposite. A gluing that reverses the latter therefore preserves the former. This distinction fixes which chiral representation is used.

9.2. Kähler rigidity of admissible limits.

Theorem 9.2 (LeBrun–Ozuch, Theorem A). *Let a sequence of closed oriented Einstein four-manifolds of fixed positive Einstein constant converge to a compact Kähler–Einstein orbifold. If every Ricci-flat ALE bubble is anti-self-dual in the induced orientation, then the sequence is eventually Kähler–Einstein.*

The reference is [23, Theorem A], arXiv:2601.19215v2, 5 February 2026. The approximating metrics in this theorem solve the Einstein equation. The limit is allowed to be smooth. The complex structure on the limit is global and to induce the specified orientation.

Proposition 9.3. *Let X be a compact oriented Einstein orbifold with Einstein constant one and isolated quotient singularities. Suppose W_X^+ is parallel and has spectrum $\{2/3, -1/3, -1/3\}$. Suppose it is an orbifold limit of smooth simply connected oriented Einstein four-manifolds, and all the ALE pieces in their finite bubble tree satisfy $W^+ = 0$ in the induced orientation. Then X is globally Kähler–Einstein in that orientation.*

Proof. The simple positive eigenspaces of W_X^+ form a parallel real orbifold line bundle $\mathcal{L}_X \subset \Lambda_X^+$. On the regular part its metric holonomy is a character

$$\chi_X : \pi_1(X_{\mathrm{reg}}) \longrightarrow \{\pm 1\}. \tag{9.1}$$

Remove disjoint small orbifold balls about the singular points and call the remaining compact manifold X_0. In a uniformizing ball at any removed point, the parallel eigenline lifts smoothly. Its value at the centre is a line in $\Lambda^+\mathbb{R}^4$ invariant under the local group. Its representation on the spherical link is the restriction of (9.1). This follows either from radial parallel transport in the ball or by evaluating its equivariant parallel lift.

Index the finite rooted tree by vertices v, with root 0 representing X. Write Y_v for a nonroot ALE orbifold and Y_v° for its compact regular truncation, with small neighborhoods of its child singularities removed. For each nonroot vertex, denote its parent by $p(v)$ and its incoming boundary link by L_v.

At the root take the sign character of $\mathcal{L}_X|_{X_0}$. Suppose characters have been constructed through depth d, and let v have depth $d+1$. At the corresponding parent singularity, the parallel line has a local invariant value in $\Lambda^+\mathbb{R}^4$. Its action on the link is the boundary sign character of the parent line bundle. Lemma 8.3 transfers this character through the specified attaching map. Lemma 8.2 extends it to a parallel orbifold line in $\Lambda_{Y_v}^+$. At each child singularity this line again has an invariant value. Induction over the finite depths defines line bundles $\mathcal{L}_v$ on every regular truncated piece.

Let $\varphi_v : L_v \to L_{p(v),v}$ be the attaching map into the corresponding parent boundary. If χ_v and $\chi_{p(v),v}$ denote the boundary sign characters, the construction gives

$$\chi_v = \chi_{p(v),v} \circ (\varphi_v)_*. \tag{9.2}$$

On a connected manifold L, a flat real metric line bundle with character χ is $\widetilde{L} \times_\chi \mathbb{R}$, where $\gamma \cdot (x,t) = (\gamma x, \chi(\gamma)t)$. Equation (9.2) therefore supplies an isomorphism $\mathcal{L}_v|_{L_v} \cong \varphi_v^*(\mathcal{L}_{p(v)}|_{L_{p(v),v}})$. Extend each isomorphism over a collar by the collar projection and glue the bundles using these isomorphisms. There are no triple intersections between the truncated pieces. The resulting real line bundle $\mathcal{L}_i \to M_i$ satisfies

$$\mathcal{L}_i|_{X_0} \cong \mathcal{L}_X|_{X_0}, \qquad w_1(\mathcal{L}_X|_{X_0}) = \iota^* w_1(\mathcal{L}_i). \tag{9.3}$$

Here $\iota : X_0 \hookrightarrow M_i$ is the inclusion supplied by the reconstruction. The bundle $\mathcal{L}_i$ is used only as a real line bundle; no embedding into $\Lambda_{g_i}^+$ and no parallel connection for g_i are asserted. By the universal coefficient theorem,

$$H^1(M_i; \mathbb{Z}/2) = \mathrm{Hom}(H_1(M_i;\mathbb{Z}), \mathbb{Z}/2) = 0,$$

since $\pi_1(M_i) = 1$. The classification of real line bundles by $w_1 \in H^1(-;\mathbb{Z}/2)$ implies $w_1(\mathcal{L}_i) = 0$. Equation (9.3) therefore gives $w_1(\mathcal{L}_X|_{X_0}) = 0$. Consequently $\mathcal{L}_X|_{X_0}$ is orientable and its sign holonomy is trivial. The punctured orbifold balls retract onto their links; hence X_{reg} retracts onto X_0. The character (9.1) is therefore trivial on the whole regular part.

Choose a global parallel unit section ω of $\mathcal{L}_X$ there. Its lift extends across the centre of every uniformizing ball as a parallel section of the smooth eigenline. The local isotropy group acts trivially on that section, because its sign action agrees with the now trivial boundary character. Thus ω is a global orbifold form. For a self-dual two-form of norm one the associated skew endomorphism squares to $-I/2$. The form $\sqrt{2}\,\omega$ therefore defines an orthogonal almost complex structure J. It is parallel. For vector fields U, V and a torsion-free connection,

$$N_J(U,V) = (\nabla_{JU}J)V - (\nabla_{JV}J)U + J(\nabla_V J)U - J(\nabla_U J)V = 0.$$

The Newlander–Nirenberg theorem [25] applies to the smooth almost complex structure on each uniformizing chart, since its Nijenhuis tensor vanishes. The local group preserves J, so these complex charts define an orbifold complex structure. The metric is consequently an orbifold Kähler metric, and it is already Einstein. □

10. Proof of the critical integral gap

Proof of theorem C. For $\widehat{g} = \Lambda g$, one has $\mathrm{Ric}_{\widehat{g}} = \widehat{g}$, $q_+(\widehat{g}) = \Lambda^{-1} q_+(g)$ and $dv_{\widehat{g}} = \Lambda^2 dv_g$. Thus $\Theta_+(\widehat{g}) = \Theta_+(g)$; it suffices to treat Einstein constant one. If there were no universal positive gap, there would be a sequence of closed connected oriented simply connected Einstein manifolds such that

$$\mathrm{Ric}_{g_i} = g_i, \qquad 0 < \theta_i := \Theta_+(g_i) \longrightarrow 0. \tag{10.1}$$

After discarding finitely many indices, (7.1) holds. Lemma 7.1 and Proposition 7.4 provide a uniform positive volume lower bound, diameter bound, Euler-characteristic bound and total curvature-energy bound. Since Euler characteristics are integers, pass to a subsequence with fixed Euler characteristic. The external compactness theorem Theorem 9.1 gives an Einstein orbifold limit (X, g_∞) and its finite ALE tree, after further subsequences as required.

10.1. **Anti-self-duality at every rescaling.** At any bubble scale $\rho_i \downarrow 0$, put $k_i = \rho_i^{-2} g_i$ in the corresponding pointed coordinate region. Its Einstein constant is ρ_i^2, its self-dual Weyl operator scales by ρ_i^2, and

$$q_+(k_i) = \rho_i^2 q_+(g_i), \qquad \int_A q_+(k_i)^2 \, dv_{k_i} = \int_A q_+(g_i)^2 \, dv_{g_i} \leq \theta_i^2. \tag{10.2}$$

For a compact subset of the regular part of a limiting ALE orbifold, smooth convergence and (10.2) show that $(\lambda_{\max}(W^+))_+$ vanishes. The limit is Ricci-flat, so the Einstein constant in this expression is zero. A trace-free self-adjoint endomorphism whose maximum eigenvalue is nonpositive is zero. It follows that

$$W^+ = 0 \tag{10.3}$$

on the regular part of that bubble and, by smooth orbifold extension, on the whole bubble. The same argument uses its absolute scale relative to g_i at every subsequent level; the upper bound remains θ_i^2. Thus (10.3) holds for every node of the finite tree.

10.2. **Uniform integrability of the self-dual curvature.** Since $|W_i^+|^2 = |B_{+,i}|^2 - 1/3$, (7.3) gives, for every measurable set $A \subset M_i$,

$$\int_A |W_i^+|^2 \, dv_{g_i} \leq \frac{8}{3} \mathrm{Vol}(A, g_i) + 8 \, \mathrm{Vol}(A, g_i)^{1/2} \theta_i + 6\theta_i^2. \tag{10.4}$$

Suppose $W_\infty^+ \equiv 0$. Let $s_1, \dots, s_N$ be the singular points of X. For each fixed sufficiently small $r > 0$, smooth convergence on the compact regular set $X \setminus \bigcup_a B_r(s_a)$ gives convergence to zero of the integral there. If $N = 0$, this is already the whole integral. Under the compactness identifications the omitted regions lie, for large i, in the union of a fixed finite number of balls of radius $2r$ in M_i. Ricci nonnegativity and Bishop volume comparison bound their total volume by $C_X r^4$. Equation (10.4) gives

$$\limsup_{i \to \infty} \int_{M_i} |W_i^+|^2 \, dv_{g_i} \leq \frac{8}{3} C_X r^4. \tag{10.5}$$

Let $r \downarrow 0$. The integrals are nonnegative, so

$$\int_{M_i} |W_i^+|^2 \, dv_{g_i} \longrightarrow 0. \tag{10.6}$$

But $\theta_i > 0$ implies $W_i^+ \not\equiv 0$. The estimate (6.8) and (7.2) give

$$\int_{M_i} |W_i^+|^2 \, dv_{g_i} \geq \frac{2}{3} \mathrm{Vol}(M_i, g_i) \geq \frac{\pi^2}{6}, \tag{10.7}$$

contradicting (10.6). Therefore W^+_∞ is nonzero.

10.3. **The limiting geometry.** On every compact regular subset, smooth convergence and $\theta_i \to 0$ imply

$$B_{+,\infty} \le I. \tag{10.8}$$

If the maximum eigenvalue exceeded one at a regular point, continuity would give $a > 0$ and a relatively compact regular ball U on which $q_{+,\infty} \ge 2a$. Smooth convergence and (1.11) would then give $q_{+,i} \ge a$ on the corresponding set U_i and $\mathrm{Vol}(U_i, g_i) \ge \mathrm{Vol}(U, g_\infty)/2$ for large i. Hence $\theta_i^2 \ge a^2 \mathrm{Vol}(U, g_\infty)/2 > 0$, a contradiction. Continuity in orbifold charts extends (10.8) to singular points. By Lemma 6.1 and nonvanishing,

$$\nabla W^+_\infty = 0, \qquad \mathrm{spec}(W^+_\infty) = \{2/3, -1/3, -1/3\}. \tag{10.9}$$

Equations (10.3) and (10.9) verify the hypotheses of Proposition 9.3. Hence X is a globally Kähler–Einstein orbifold in the given orientation. If X is smooth, the same proposition applies with an empty tree; the approximants are then diffeomorphic to X and its parallel eigenline is oriented by simple connectivity.

The sequence has a fixed oriented diffeomorphism type after passage to a subsequence, consists of Einstein metrics with Einstein constant one, and satisfies (10.3). We may therefore apply Theorem 9.2. For all sufficiently large i, g_i is Kähler–Einstein in the corresponding orientation. Its chiral spectrum is then

$$\mathrm{spec}(W_i^+) = \{2/3, -1/3, -1/3\}, \qquad \mathrm{spec}(B_{+,i}) = \{1, 0, 0\}. \tag{10.10}$$

Thus $q_+(g_i) \equiv 0$, contrary to (10.1). Therefore there exists $\varepsilon_* > 0$ satisfying theorem C.

It remains to classify zero excess on the original smooth manifold. If $q_+ \equiv 0$, first apply Lemma 6.1 to $\widehat{g} = \Lambda g$. Rescaling back preserves parallelism and multiplies the Weyl eigenvalues by Λ. Thus either $W^+ \equiv 0$, or the parallel simple eigenline in Λ^+ has eigenvalue $2\Lambda/3$. On a simply connected manifold every real line bundle is orientable; its metric connection on a line has only sign holonomy and is then trivial. A global parallel unit eigenform, multiplied by $\sqrt{2}$, defines a parallel complex structure as in Proposition 9.3. This establishes the remaining assertion of theorem C. Reversing orientation proves its other-chirality version. □

11. Semistability and circle-warped metrics

Proof of Corollary 1.1. Reverse orientation if needed and apply theorem C. In the half-conformally-flat case the compact simply connected positive Einstein classification gives the round sphere or the Fubini–Study plane [16][7, Theorem 13.30]. In the Kähler–Einstein case, Biquard–Ozuch [8, Theorem A] give a negative TT direction unless the metric is half-conformally flat. Their operator is $E/2$ under (1.1), so the sign is the sign excluded by TT-semistability. The half-flat classification again applies. □

Proposition 11.1. *Let (M^4, g) be simply connected and belong to the circle-collapse class of theorem B. If $\Theta_+(g) < \varepsilon_*$ or $\Theta_-(g) < \varepsilon_*$, then g is round or isometric, up to scale, to the product of two round two-spheres with equal Einstein constant. If g is TT-semistable, only the round case occurs.*

Proof. By Proposition 5.1, $q_+ = q_-$ and their L^2 norms agree. The gap theorem gives $q_+ = q_- = 0$. The critical-rigidity calculation Lemma 6.1, after rescaling, gives parallel W^+ and W^-. Their norms are equal by (5.2). Thus either both vanish, giving constant sectional curvature $\Lambda/3$, or both have spectrum $\{2\Lambda/3, -\Lambda/3, -\Lambda/3\}$. In the second case simple connectivity gives parallel unit eigenforms of both chiralities. After normalization they define orthogonal complex structures J_+, J_- of opposite orientations. The two chiral skew endomorphism actions commute, so J_+J_- is a parallel symmetric involution. Its trace is zero, since the corresponding two-forms are orthogonal. Its two eigendistributions therefore both have rank two. The de Rham decomposition theorem on the complete simply connected manifold gives a Riemannian product of two complete simply connected surfaces. The Einstein equation restricts to $\mathrm{Ric} = \Lambda g$ on each; its Gaussian curvature is Λ. They are round two-spheres. Finally, on that product $H_0 = g_1 - g_2$ is parallel and TT, and $\mathring{R}H_0 = \Lambda H_0$. Hence $EH_0 = -2\Lambda H_0$, excluding TT-semistability. □

Remark 11.2. On the equal-curvature product $S^2 \times S^2$, both B_+ and B_- have eigenvalues $\{\Lambda, 0, 0\}$. Thus $\Theta_+ = \Theta_- = 0$, although $EH_0 = -2\Lambda H_0$ and $LH_0 = 0$. Zero excess does not imply stability, and negative Einstein spectrum does not mean negative spectrum of the full operator L. On the round four-sphere $W^+ = W^- = 0$. On the Fubini–Study plane with its complex orientation, $W^- = 0$ and B_+ has eigenvalues $\{\Lambda, 0, 0\}$; in particular its two chiral Weyl norms are different. These curvature values agree with its exclusion from Proposition 11.1.

Corollary 11.3. *Let (M^4, g) be closed, simply connected, positive Einstein and TT-semistable. If g is not isometric, up to scale, to the round sphere or the Fubini–Study plane, then $\Theta_+ \geq \varepsilon_*$ and $\Theta_- \geq \varepsilon_*$.*

Proof. A strict inequality in either orientation would contradict Corollary 1.1. □

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