EINSTEIN METRICS AT CONICAL LIMITS: STABILITY, DISCONNECTED MODULI, AND ENTROPY RIGIDITY

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Abstract. We study three variational problems for Einstein metrics with conical limits. For every $k \geq 5$, the complete Ricci-flat Böhm metric on $\mathbb{R}^{k+1} \times S^k$ has holonomy $SO(2k+1)$ and a homogeneous coercive estimate on all symmetric two-tensors. It attracts sufficiently small smooth $L^2 \cap C_b^{2,\eta}$ perturbations under the fixed-background Ricci–DeTurck flow, with convergence in L^2 and every uniform C^j norm. Uniformly small ancient perturbations with bounded L^2 norm are stationary. For $p, q \geq 2$ and $p + q \leq 8$, we construct distinct normalized Einstein volumes on S^{p+q+1}, on $S^{p+1} \times Q$ for every closed connected positive Einstein q-manifold Q, and on the specified identity doubles of disk bundles with vanishing O'Neill tensor. Their full Einstein moduli spaces have infinitely many connected components. The volumes approach the sine-cone value from above; specified subsequences have geometric asymptotics. On a marked conical end of rate greater than $(n-2)/2$, with absolutely integrable scalar curvature, a large-scale upper bound for Perelman's μ-functional has coefficient $\lambda_{\mathrm{nc}} - \mathfrak{m}_C$. No spin or scalar-curvature sign is assumed in this comparison. A parallel spinor compatible with the end induced in the universal cover gives mass–energy rigidity and characterizes equality in the global cone entropy bound by Ricci-flatness. Equality also makes the end homomorphism surjective with finite image and extends the compatible cone spinors. An ancient Ricci flow with one such slice is stationary when its finite backward Nash entropy is at least the cone value.

Contents

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1. Introduction and main statements

The Einstein equation admits both compact degenerations to sine-cones and complete solutions asymptotic to Ricci-flat cones. The limiting link prescribes the leading geometry, but it does not determine the sign of the quadratic form on a smooth filling, the connected component of a compact Einstein metric, or the possibility of extending asymptotic spinorial data. We treat these questions separately, using the scalar-curvature variation and its boundary terms.

Böhm constructed the compact inhomogeneous families and the complete cohomogeneity-one Ricci-flat metrics used in the first two parts [8, 9]. For the complete metrics, we prove a quadratic-form estimate valid through the smooth singular orbit and on the entire end. For the compact metrics, we differentiate the cap parameter, match the two caps at a minimal orbit, and calculate the leading difference from the limiting volume. The third part concerns arbitrary marked conical ends. It compares a Gaussian entropy competitor with the mass flux and then uses a spinor on an induced cover of the end to determine the equality case.

The resulting assertions have different hypotheses. The full-tensor coercivity theorem implies a nonlinear stability theorem. The volume calculation separates connected components of the full compact Einstein moduli space without an infinitesimal rigidity assumption. The entropy–mass comparison has no spin hypothesis; its equality case requires the compatible lifted-end condition stated below. A product cone in the first two parts has no parallel spinor, so the third part is not a spinorial proof of the first two.

1.1. Complete Ricci-flat metrics and nonlinear stability.

For a Ricci-flat metric g, put

$$E_g = \nabla^*\nabla - 2\mathring{R}_g, \qquad Q_g(h) = \int_M \left(|\nabla h|^2 - 2\langle \mathring{R}_g h, h\rangle\right) dv_g. \tag{1.1}$$

Linear stability on all symmetric tensors means $Q_g(h) \geq 0$ for $h \in C_c^\infty(S^2T^*M)$. It implies the corresponding transverse-traceless assertion.

For $k \geq 5$, let γ_k be the unit-round metric on S^k and put

$$M_k = \mathbb{R}^{k+1} \times S^k, \qquad g_k = dt^2 + a(t)^2\gamma_k + b(t)^2\gamma_k, \tag{1.2}$$

where t is distance from the singular orbit $\{0\} \times S^k$ and

$$a(0) = 0, \qquad a'(0) = 1, \qquad b(0) = 1, \qquad b'(0) = 0. \tag{1.3}$$

The notation g_k refers to the complete Ricci-flat Böhm metric with this normalization. The functions a, b and the variables $u = a'/a$, $v = b'/b$ are used on $t > 0$; the metric extends smoothly across $t = 0$.

Theorem A. *For every integer $k \geq 5$, the normalized complete Ricci-flat Böhm metric g_k on $\mathbb{R}^{k+1} \times S^k$ has holonomy $SO(2k+1)$. There is $c > 0$ such that*

$$Q_{g_k}(h) \geq c \int_{\mathbb{R}^{k+1}\times S^k} \left(|\nabla h|^2 + (1+t^2)^{-1}|h|^2\right) dv_{g_k} \qquad (h \in C_c^\infty(S^2T^*M_k)). \tag{1.4}$$

The Friedrichs realization of E_{g_k} satisfies

$$\ker_{L^2} E_{g_k} = 0, \qquad \inf \sigma(E_{g_k}) = 0. \tag{1.5}$$

For $k = 5$, the more precise estimate is

$$Q_{g_5}(h) \geq \frac{1}{93} \int \left(|\nabla h|^2 + uv|h|^2\right) dv_{g_5}, \qquad u = a'/a, \quad v = b'/b. \tag{1.6}$$

The existence of g_k is Böhm's theorem, including the case of one noncollapsing factor in [9, Sections 6 and 11]. The new estimate concerns the smooth complete metric. Stability of its singular cone is a different assertion. Hall–Haslhofer–Siepmann analyze Ricci-flat cone stability [23]; Kröncke gives a criterion in terms of the Einstein spectrum of the link [29, Theorem 1.2]. Neither conclusion supplies the estimate through the compact core in (1.4).

The sign changes with dimension in the equal-factor family. Angenent–Knopf prove that complete Ricci-flat Böhm metrics with $p, q \geq 2$ and $p + q \leq 8$ have infinitely many negative L^2 modes [3, Main Theorem II]. The metrics in Theorem A have dimension $2k + 1 \geq 11$. Their proof uses the global inequalities

$$0 < w < T, \qquad w^2 < TD^2, \qquad D = a'/a - b'/b, \quad T = a'b'/(ab), \quad w = b''/b,$$

and the positive function $(ab)^{-(2k-1)/4}$. Sections 4 and 5 compute the curvature action on every tensor component and its comparison with the scalar Laplacian of that function.

Parallel spinors yield another, established proof of Ricci-flat stability [15]. Artacho obtains an extension under a parallel twisted pure spinor hypothesis [4, Theorem 1]. The tangent endomorphisms in that hypothesis square to $-\operatorname{Id}$ [4, Definition 2]; hence it requires even tangent dimension. It does not apply to Theorem A. Full orthogonal holonomy therefore coexists here with complete-metric linear stability, as opposed to the question recorded in [7, Question 1, arXiv version 3]. No compact example is constructed, and the compact special-holonomy conjecture of [1] is not addressed.

The nonlinear statements are Theorems 6.1 and 6.2. They concern $\mathcal{X}_k = L^2(S^2T^*M_k, g_k) \cap C^{2,\eta}_b(S^2T^*M_k, g_k)$, $0 < \eta < 1$, in a fixed background gauge. A localized energy inequality bounds the L^2 integral outside a compact set uniformly for all positive times. This bound and the weighted dissipation give strong L^2 convergence; local smoothing gives convergence of every uniform spatial derivative. The same localized inequality proves ancient rigidity. Perturbations need not be invariant or transverse-traceless.

Deruelle–Kröncke prove nonlinear stability for integrable ALE Ricci-flat metrics [40]. Kröncke–Petersen obtain further convergence and scalar-curvature rigidity results in an ALE parallel-spinor setting [41]; Stolarski–Waldron give additional ALE stability results [42]. The nonflat conical end here is not an ALE end. Only the local complete bounded-curvature result [40, Lemma 3.2] is used in the nonlinear proof. Neither a positive L^2 spectral gap, a uniform time-decay rate, nor convergence of the DeTurck diffeomorphisms is asserted.

1.2. **Compact Einstein volumes and connected components.** For a closed smooth manifold M and $d > 0$, define

$$\mathcal{E}_d(M) = \{g \in C^\infty(S^2T^*M) \,:\, g > 0,\ \operatorname{Ric}_g = dg\}/\operatorname{Diff}(M), \tag{1.7}$$

with the quotient of the smooth topology. Integration uses the Riemannian density when an orientation is not specified. On page 146 of [8], Böhm writes:

> "We conjecture that the moduli space of Einstein metrics of these manifolds consists of infinitely many connected components."

The theorems below concern the infinite families in [8, Theorems 3.3, 3.4 and 3.6].

Write $\omega_j = \operatorname{Vol}(S^j, \gamma_j)$ for the unit-round sphere volume. Throughout the general compact construction, put

$$p, q \geq 2, \qquad d = p + q \leq 8, \qquad \alpha = \frac{d-1}{2}, \qquad \beta = \frac{\sqrt{(d-1)(9-d)}}{2} > 0. \tag{1.8}$$

For a closed connected Einstein manifold (Q^q, h_Q) with $\operatorname{Ric}_{h_Q} = (q-1)h_Q$, define

$$\Omega = \omega_p \operatorname{Vol}(Q, h_Q), \quad v_0 = \left(\frac{p-1}{d-1}\right)^{p/2} \left(\frac{q-1}{d-1}\right)^{q/2}, \quad V_\infty = \Omega v_0 \int_0^\pi \sin^d t\, dt. \tag{1.9}$$

For the round fibre, write

$$V_{p,q} = \omega_p \omega_q \left(\frac{p-1}{d-1}\right)^{p/2} \left(\frac{q-1}{d-1}\right)^{q/2} \int_0^\pi \sin^d t\, dt. \tag{1.10}$$

Theorem B. *For every pair in* (1.8) *and every* (Q, h_Q) *as above,* $S^{p+1} \times Q$ *admits smooth metrics* g_j *satisfying*

$$\mathrm{Ric}_{g_j} = d g_j, \qquad \mathrm{Vol}(g_j) > V_\infty, \qquad \mathrm{Vol}(g_j) \longrightarrow V_\infty, \tag{1.11}$$

with pairwise distinct volumes. When $Q = S^q$ *is unit round, the same assertion holds on* S^{d+1} *with limiting volume* $V_{p,q}$*. Consequently the full Einstein moduli spaces of these manifolds have infinitely many connected components. In particular, this holds on* S^5, S^6, S^7, S^8, S^9.

Theorem 1.1 (Doubles of disk bundles). *Let* $\mathscr{E}$ *satisfy the hypotheses of Böhm's two-summand construction* [8, Section 3, Theorem 3.3]. *Suppose its sphere fibre has dimension* $p \geq 2$*, its singular orbit has dimension* q*, its base Einstein constant is positive, its O'Neill tensor vanishes, and* $p + q \leq 8$*. Then* $\mathcal{E}_{p+q}(\mathscr{E} \cup_{\partial \mathscr{E}} \mathscr{E})$ *has infinitely many connected components. The double uses the identity identification of the two boundaries.*

Theorem 1.2 (Even-sphere volume asymptotics). *For each pair*

$$(p, q) \in \{(2, 3), (2, 5), (3, 4)\}, \tag{1.12}$$

there is a spherical sequence in Theorem B and a constant $C_{p,q} > 0$ *such that*

$$\mathrm{Vol}(g_j) = V_{p,q} + C_{p,q} e^{-2\pi\alpha j/\beta}(1 + o(1)). \tag{1.13}$$

After deleting and reindexing finitely many terms, its volumes are strictly decreasing. The limits are

$$V_{2,3} = \frac{16\pi^3}{15\sqrt{2}}, \qquad V_{2,5} = \frac{256\pi^4}{945}\sqrt{\frac{2}{3}}, \qquad V_{3,4} = \frac{128\pi^4}{315\sqrt{3}}. \tag{1.14}$$

Reflection doubles in every dimension of (1.8) *satisfy the geometric asymptotic in Corollary 12.5.*

Böhm's construction already gives infinitely many compact Einstein metrics [8]. The conclusion needed here is different: their normalized volumes distinguish connected components of the full moduli space. Gibbons–Hartnoll–Pope give numerical volume data for selected metrics [20, arXiv version, Section 5.2, p. 33, tables following (113)–(114)]. Liu–Sano–Tasin use normalized volume to distinguish Sasaki–Einstein families on odd spheres [32, Section 2.4]. Neither existence of infinitely many metrics nor numerical ordering of finitely many volumes proves the assertion above.

Eschenburg–Wang study the general singular initial-value problem [17]; that theorem does not give uniqueness for arbitrary invariant initial data. In our two-function ansatz, Lemma 8.1 constructs a unique normalized analytic cap and its parameter dependence. Proposition 10.2 transfers the nonzero oscillatory coefficient of its Ricci-flat rescaling to the minimal orbit, together with its logarithmic parameter derivative. The Wronskian computation in Lemma 11.1 then applies in all dimensions $4 \leq d \leq 8$. The transfer matrix may have unequal diagonal entries. The corresponding mixed term cancels under reversal of the normal in the sum of the two cap-volume variations.

Opposite winding gives sphere matchings; intersections with $y = 0$ give reflection doubles. The positive leading volume excess holds for every sufficiently small matched pair considered in the proof. A sequence of pairwise distinct volumes follows by extraction. Geometric spacing is proved separately for reflection doubles and for the even-sphere sequences in Theorem 1.2; it is not inferred for every sphere matching.

The local analytic description of Einstein moduli is due to Koiso; see [28, 6]. Local constancy of the normalized Einstein–Hilbert functional, in the form [36, Theorem 3.6], identifies different fixed-Einstein- constant volumes with different components of the full moduli space. No vanishing of the infinitesimal Einstein kernel is required. Negative-mode theorems, including [37, 35], concern a different question and do not enter this implication.

Theorems B and 1.1 concern the infinite families in [8, Theorems 3.3, 3.4 and 3.6]. The separate existence assertion for $\mathbb{HP}^2\#\mathbb{HP}^2$ in [8, Theorem 3.5] is not included. There is no assertion for $d \geq 9$, or that the constructed points of moduli are isolated.

1.3. **Conical mass, entropy, and equality rigidity.** Let Z^{n-1} be closed, connected and oriented, $n \geq 3$, with $\mathrm{Ric}_{h_Z} = (n-2)h_Z$. A marked conical end is a specified orientation-preserving diffeomorphism of an exterior of M with $(R_0, \infty) \times Z$. Put

$$g_C = dr^2 + r^2 h_Z, \quad e = g - g_C, \quad \beta_C(e) = \mathrm{div}_C\, e - d\,\mathrm{tr}_C\, e, \quad \mathfrak{m}_C(g) = \lim_{R\to\infty} \int_{\{r=R\}} \beta_C(e)(\partial_r)\, dA_C. \tag{1.15}$$

The trace and divergence in this formula use the cone metric; no dimensional normalizing factor is included in $\mathfrak{m}_C$. Assume that M is complete, connected, oriented, smooth, without boundary, and has exactly one such end, and that

$$|\nabla_C^j e| = O(r^{-\tau-j}) \quad (j \geq 0), \qquad \tau > \frac{n-2}{2}, \qquad R_g \in L^1(M, dv_g). \tag{1.16}$$

The last condition means absolute integrability. Define

$$Q_a(w) = \int_M (a|dw|^2 + R_g w^2)\, dv_g, \qquad \lambda_{\mathrm{nc}}(g) = \inf_{v \in C_c^\infty(M)} Q_4(1+v). \tag{1.17}$$

In particular $\lambda_{\mathrm{nc}} \in [-\infty, \infty)$, since $Q_4(1) = \int R_g$ is finite. For a real $u \in C_c^\infty(M)$ with $\|u\|_2 = 1$ and $T > 0$, put

$$\mathcal{W}(g, u, T) = \int_M \left[T(4|du|^2 + R_g u^2) - u^2 \log u^2\right] dv_g - n - \frac{n}{2}\log(4\pi T), \tag{1.18}$$

where $0 \log 0 = 0$, and set

$$\mu(g, T) = \inf_{\substack{u \in C_c^\infty(M;\mathbb{R}) \\ \|u\|_2 = 1}} \mathcal{W}(g, u, T), \qquad \nu(g) = \inf_{T>0} \mu(g, T). \tag{1.19}$$

These are the global versions of Perelman's functionals [51, Section 3], defined here using compactly supported amplitudes. No compact-manifold minimizer theorem is invoked. Write

$$\theta = \frac{\mathrm{Vol}(Z, h_Z)}{\mathrm{Vol}(S^{n-1}, \gamma_{n-1})}, \qquad N_T = \theta(4\pi T)^{n/2}. \tag{1.20}$$

Theorem 1.3 (Large-scale entropy–mass comparison). *Under* (1.16)*,*

$$\limsup_{T\to\infty} \frac{N_T}{T}\left(\mu(g, T) - \log\theta\right) \leq \lambda_{\mathrm{nc}}(g) - \mathfrak{m}_C(g). \tag{1.21}$$

In particular $\nu(g) \leq \log\theta$. Neither a spin condition nor a scalar-curvature sign is required.

The comparison is one-sided. It asserts neither an asymptotic equality for the infimum μ, nor existence of a minimizer, nor a uniform power estimate for its remainder. The proof uses the given marked coordinates, without a radial gauge assumption.

For the rigidity statement let

$$\Gamma = \pi_1(Z), \qquad G = \pi_1(M), \qquad K = \ker(i_* : \Gamma \to G), \qquad \widehat{Z} = \widetilde{Z}/K, \qquad d_\Gamma = |\Gamma/K| = |i_*(\Gamma)|. \tag{1.22}$$

Here tildes denote universal covers, and i includes a large end cross-section. The group Γ is finite by Bonnet–Myers. A selected component of the end lifted to $\widetilde{M}$ has link $\widehat{Z}$ and degree d_Γ over the original end.

Definition 1.4 (Compatible lifted end). Condition (SC) means that $\widetilde{M}$ is spin and that $C(\widehat{Z})$ has a nonzero parallel spinor for the spin structure induced on the selected lifted end by $\widetilde{M}$ and its marked identification. The spin structure is part of the condition; it is not an arbitrary choice of spin structure on the cone.

One sufficient set of hypotheses for (SC) is

$$C(\widetilde{Z}) \text{ has a nonzero parallel spinor}, \quad \widetilde{M} \text{ is spin}, \quad i_* : \Gamma \to G \text{ is injective.} \tag{1.23}$$

They imply (SC): then $K = 1$ and the lifted end is simply connected, so its spin structure is unique. Condition (SC) also permits $K \neq 1$ when the required spinor descends for the induced spin structure.

Theorem C. *Assume* (1.16) *and* (SC). *The mass in* (1.15) *exists, and* $\lambda_{\mathrm{nc}}(g) \leq \mathfrak{m}_C(g)$. *If* $R_g \geq 0$, *then* $\mathfrak{m}_C(g) \geq 0$. *Zero mass under* $R_g \geq 0$, *or equality in* $\lambda_{\mathrm{nc}}(g) \leq \mathfrak{m}_C(g)$, *forces*

$$\mathrm{Ric}_g = 0, \qquad i_* : \Gamma \twoheadrightarrow G, \qquad |G| = d_\Gamma < \infty. \tag{1.24}$$

In the mass–energy equality case both sides vanish. Moreover,

$$\nu(g) \leq \log\theta, \qquad \nu(g) = \log\theta \iff \mathrm{Ric}_g = 0. \tag{1.25}$$

Entropy equality implies $\mathfrak{m}_C(g) = \lambda_{\mathrm{nc}}(g) = 0$ *and* (1.24). *In any of these equality cases, every compatible parallel spinor on* $C(\widehat{Z})$ *extends to* $\widetilde{M}$ *by a complex-linear isometry, in the asymptotic sense of Proposition 15.5. In these equality cases, the additional injectivity hypothesis makes* i_* *an isomorphism.*

The boundary identity underlying the mass statement has spinorial antecedents. Witten expresses the asymptotically flat mass by harmonic spinor energy [38]; Bartnik establishes the associated mass and coordinate theory [5]. Dahl identifies the spin compatibility required at an ALE end [13]. The Dirac variation used below is that of Bourguignon–Gauduchon [10]. Here the harmonic correction is constructed on the universal cover, with its reference spinor supported on one selected lifted end. The filling itself need not be spin.

Nonflat conical mass also has antecedents. Kröncke–Szabó study coordinates, decay, and mass for Ricci-flat conifolds [30]. Hein–LeBrun express Kähler ALE mass by scalar curvature and cohomological data [26]. Yao proves mass formulas, positive-mass results, and scalar-flat asymptotic expansions for the specified Kähler conical resolutions [39, Theorems B–D in the specified preprint]. Ghosh treats ALE almost Kähler geometry by a spin^c method [19]. Theorem C assumes none of these complex or symplectic structures, but its compatible lifted-end hypothesis is not a hypothesis of all those results. There is no inclusion of all of their metric classes in ours.

Several other mass theorems involve different meanings of conical geometry. Dai–Sun–Wang permit isolated interior conical singularities while keeping an asymptotically flat end [14]. Chen–Liu–Shi–Zhu treat incompressibility and ends with a fixed flat fibre [12]. Alaee–Khuri–Kunduri compare masses of toric four-manifolds and gravitational instantons with fixed asymptotic and rod data; their Theorem 1.7 includes conical singularities of the torus action [49]. These are not the arbitrary smooth nonflat conical ends of (1.16). The end hypothesis must be compared before any priority claim about mass or rigidity is made.

For the entropy comparison, Haslhofer proves an asymptotically flat mass–energy inequality [24]. Li proves $\mu(g, T) \to 0$ for asymptotically Euclidean manifolds with strictly positive scalar curvature [50, Theorem 3.4 in the specified preprint]. This is a leading large-scale limit, not a coefficient of its difference from the Euclidean value. Lopez–Ozuch obtain an ALE large-scale μ-estimate with a quantified remainder; their scalar-curvature extensions use a renormalized functional [33, Theorem 1.7 and Remark 11 of arXiv:2407.18438v1]. The comparison in Theorem 1.3 allows nonflat Ricci-flat cone links and signed absolutely integrable scalar curvature at every strict rate $\tau > (n-2)/2$. It asserts an upper limit only.

Its trial probability measure agrees exactly with the cone Gaussian measure outside a compact set. In the original marked coordinates, the first-order metric terms are an Abel average of the mass flux; the quadratic terms are bounded by an integrable multiple of $r^{-2\tau-2}$. No radial gauge, entropy minimizer, or power expansion for the minimizing value is required. The lower entropy bound for a Ricci-flat filling is instead a consequence of Kristály's sharp logarithmic Sobolev inequality [44, Theorem 1.2].

The equality argument is global. The spinor correction belongs to homogeneous energy and to $L^{2n/(n-2)}$ on the universal cover. At zero mass, the corrected spinor is nonzero and parallel. A second lifted end would contradict the correction's integrability, so the end homomorphism is surjective. It is an isomorphism only when injectivity is additionally assumed. The exact conformal identity (16.13) removes the scalar-curvature sign from the mass–energy comparison. Condition (SC) uses the actual induced end cover and permits a nontrivial kernel of i_*.

Combining the equality case with the backward-entropy identity of Chan–Ma–Zhang [45, Theorem 1.14] and Kotschwar's backward uniqueness theorem [46] gives the stationary ancient-flow result in Theorem 19.1. Its finite backward Nash entropy and its lower bound by $\log\theta$ are hypotheses; neither follows from the spatial asymptotics of one slice.

Hall–Haslhofer–Siepmann relate the noncompact energy to an explicitly admissible class of instantaneous smoothings [23, Section 8]. Lopez–Ozuch already exclude the corresponding compatible spin ALE expanders [33, Theorem 1.4 in the published article]. Our consequences retain the stated topological and flow assumptions. Incompressibility and spin universal covers also occur in scalar-curvature obstruction theory [11, Theorem 1.5], under different hypersurface and end hypotheses. No conclusion about an arbitrary weak Ricci flow or a totally nonspin filling is inferred.

1.4. **Structure of the proofs and limitations.** Sections 3–5 prove Theorem A; Sections 6 and 7 prove its nonlinear consequences. Sections 8–12 prove Theorem B, Theorem 1.1, and the quantitative refinement Theorem 1.2. Sections 13–16 prove the mass statements of Theorem C. Sections 17–18 prove its entropy statements, and Section 19 treats ancient flows. The expander and instantaneous-smoothing applications are in Section 20. The mass is defined relative to a conical marking and is unchanged under the asymptotic coordinate changes in Definition 13.2. We do not claim a classification of all possible conical coordinate transitions.

The oscillatory Ricci-flat asymptotic used in the compact argument is also established in [34, Section 3.1, Lemma 3.3]; Lemma 9.3 gives the proof in our variables. No later coercivity theorem from that manuscript is imported. Its dimension restriction therefore does not enter Theorem A. The flow applications use precisely the decay theorem of Deruelle and the admissibility theorem of Hall–Haslhofer–Siepmann [16, 23], not a smoothing assertion for arbitrary weak Ricci flows.

The classes of cones in the three parts do not coincide. A nontrivial product of round spheres gives a Ricci-flat cone with no nonzero parallel spinor; Proposition 21.1 proves this directly. In particular, the mass theorem is not a spinorial proof of the stability theorem, and the compact volume theorem does not follow from either noncompact result.

2. Conventions and boundary variation

The positive scalar Laplacian is $\Delta = -\operatorname{div}\nabla$. On covariant tensors, $\nabla^*\nabla$ is the nonnegative connection Laplacian. We take

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, \quad R_{ijkl} = \langle R(e_i,e_j)e_l, e_k\rangle, \quad (\mathring{R}h)_{ik} = \sum_{j,l} R_{ijkl}h_{jl}. \tag{2.1}$$

Thus R_{ijij} is sectional curvature. In a real orthonormal frame, $\mathring{R}h = (\operatorname{tr}h)g - h$ on a unit sphere. For $\operatorname{Ric} = \Lambda g$, the full Lichnerowicz operator is $L = E + 2\Lambda$. A TT tensor satisfies $\operatorname{tr}h = 0$ and $\delta h = -\nabla^i h_{ij} = 0$. Integrals on nonorientable manifolds, when used, refer to the Riemannian density.

Lemma 2.1. *Let $g(s)$ be a smooth family on a fixed compact manifold with boundary, with $\operatorname{Ric}_{g(s)} = \Lambda g(s)$ and fixed $\Lambda \neq 0$. Let $h = \partial_s g|_0$. If every boundary is minimal, then*

$$2\Lambda\,\partial_s \operatorname{Vol}(g(s))\big|_0 = -\int_{\partial M} \langle \mathrm{II}, h^\top\rangle\, dA. \tag{2.2}$$

The second fundamental form uses the outward normal.

Proof. Choose the signed distance t so that the outward normal is ∂_t. For each parameter value, use its normal exponential map on a fixed collar. The transition maps are the identity on the boundary. Their pullbacks satisfy

$$g(s) = dt^2 + \gamma_s(t), \qquad h_{tt} = h_{ta} = 0.$$

Pullback does not change the volume or the induced boundary tensors. The scalar variation is

$$DR_g(h) = -\langle \mathrm{Ric}, h\rangle + \mathrm{div}(\mathrm{div}\, h - d\,\mathrm{tr}\, h). \tag{2.3}$$

Let e_a be an orthonormal frame tangent to the boundary. Since $\nabla_{e_a}\nu = \mathrm{II}_a{}^b e_b$,

$$(\nabla_{e_a} h)(e_a, \nu) = -\mathrm{II}_a{}^b h_{ab}, \qquad (\nabla_\nu h)(\nu,\nu) = 0.$$

Moreover,

$$H_s = \tfrac{1}{2}\gamma_s^{ab}\partial_t(\gamma_s)_{ab}, \qquad \partial_s H_s\big|_0 = \tfrac{1}{2}\gamma^{ab}\partial_t h_{ab} - \langle \mathrm{II}, h^\top\rangle,$$

whereas $\partial_t \gamma^{ab} = -2\mathrm{II}^{ab}$. Therefore

$$(\mathrm{div}\, h)(\nu) = -\langle \mathrm{II}, h^\top\rangle, \qquad \partial_\nu\, \mathrm{tr}\, h = 2\partial_s H_s\big|_0. \tag{2.4}$$

Since $R_{g(s)} = n\Lambda$ and $H_s = 0$, integration of (2.3) gives

$$0 = -\Lambda \int_M \mathrm{tr}\, h\, dv_g - \int_{\partial M} \langle \mathrm{II}, h^\top\rangle\, dA.$$

Finally $\partial_s dv_{g(s)}|_0 = \frac{1}{2}\,\mathrm{tr}\, h\, dv_g$. Substitution proves (2.2). □

Part 1. Complete Ricci-flat metrics

3. The equal-factor equations and a global barrier

Fix $k \geq 5$ and the complete metric in Theorem A. Its existence is the one-noncollapsing-factor case of [9, Theorem 6.1 and Section 11]; both round factors have dimension at least two. Put

$$u = a'/a, \quad v = b'/b, \quad S = u+v, \quad D = u - v, \quad T = uv, \quad w = b''/b = -a''/a. \tag{3.1}$$

The last equality follows from $\mathrm{Ric}_{tt} = 0$.

Lemma 3.1. *On $t > 0$,*

$$u' = -u^2 - w, \qquad v' = -v^2 + w, \tag{3.2}$$

$$S' = -\tfrac{1}{2}(S^2 + D^2), \qquad D' = -SD - 2w, \tag{3.3}$$

$$T' = -ST + Dw, \qquad w' = kDT - kSw. \tag{3.4}$$

Moreover $a', b' > 0$, and, for $\beta_0 = (k-1)/(k+1)$,

$$a(t) = t - \frac{\beta_0}{6}t^3 + O(t^5), \qquad b(t) = 1 + \frac{\beta_0}{2}t^2 + O(t^4). \tag{3.5}$$

Proof. Use indices i, j on the first S^k and A, B on the second. Besides the two intrinsic spherical connections, the connection coefficients are

$$\Gamma^0_{ij} = -aa'\gamma_{k,ij}, \quad \Gamma^i_{0j} = u\delta^i_j, \quad \Gamma^0_{AB} = -bb'\gamma_{k,AB}, \quad \Gamma^A_{0B} = v\delta^A_B. \tag{3.6}$$

Their curvature contractions give

$$\mathrm{Ric}_{00} = -k\frac{a''}{a} - k\frac{b''}{b},$$

$$\frac{a''}{a} = \frac{k-1}{a^2} - (k-1)u^2 - kT, \qquad w = \frac{k-1}{b^2} - (k-1)v^2 - kT. \tag{3.7}$$

The identity $\mathrm{Ric}_{00} = 0$ gives $a''/a = -w$ and hence $u' = -w - u^2$, $v' = w - v^2$. Addition, subtraction and the product rule yield

$$S' = -(u^2+v^2) = -\tfrac{1}{2}(S^2 + D^2), \quad D' = -(u^2 - v^2) - 2w = -SD - 2w,$$

$$T' = (-u^2 - w)v + u(-v^2 + w) = -ST + Dw.$$

To compute the last derivative, use $(k-1)b^{-2} = w + (k-1)v^2 + kT$:

$$\begin{aligned} w' &= -2(k-1)vb^{-2} - 2(k-1)v(-v^2 + w) - k(-ST + Dw) \\ &= k(S - 2v)T - k(D + 2v)w = kDT - kSw. \end{aligned} \tag{3.8}$$

Smoothness at the zero section gives $a = t + a_3 t^3 + O(t^5)$ and $b = 1 + b_2 t^2 + O(t^4)$. The limits of $\mathrm{Ric}_{00} = 0$ and of the second equation in (3.7) give

$$6a_3 + 2b_2 = 0, \qquad 2b_2 = (k-1) - 2kb_2.$$

Thus $b_2 = (k-1)/(2k+2)$ and $a_3 = -(k-1)/(6k+6)$. This proves (3.5) and shows that $a', b' > 0$ near zero. If $t_* > 0$ were a first zero of a', then $a'(t) > 0$ for $0 < t < t_*$ would imply $a''(t_*) \le 0$. But (3.7) gives $a''(t_*) = (k-1)/a(t_*) > 0$. The contradiction for a first zero of b' is $b''(t_*) \le 0 < (k-1)/b(t_*)$. The given complete solution has $a, b > 0$ on every regular finite interval, so these arguments apply for all $t > 0$. □

Lemma 3.2. *For all $t > 0$,*

$$D > 0, \qquad 0 < w < T, \qquad w^2 < TD^2. \tag{3.9}$$

Proof. The core expansions give $D \sim t^{-1}$ and $w, T \to \beta_0 > 0$. On the maximal initial interval on which $D, w > 0$, put $F = w/(D\sqrt{T})$ and $z = D/\sqrt{T}$. Since $S^2 = D^2 + 4T$,

$$\frac{F'}{F} = \frac{kDT - kSw}{w} - \frac{-SD - 2w}{D} - \frac{-ST + Dw}{2T}.$$

Substitution of $w = FD\sqrt{T}$ gives

$$F' = \sqrt{T}\left[k - \left(k - \frac{3}{2}\right)\sqrt{z^2 + 4}\,F + \left(2 - \frac{z^2}{2}\right)F^2\right]. \tag{3.10}$$

Initially $F \to 0$. If F first reached one at t_* while $D, w > 0$, then $F'(t_*) \ge 0$. Instead (3.10) gives

$$\frac{F'(t_*)}{\sqrt{T(t_*)}} \le k - 2(k - \tfrac{3}{2}) + 2 - \tfrac{1}{2}z(t_*)^2 = 5 - k - \tfrac{1}{2}z(t_*)^2 < 0.$$

Hence $F < 1$ on that interval. In particular

$$D' \ge -(S + 2\sqrt{T})D, \qquad D(t) \ge D(t_0)\exp\left(-\int_{t_0}^{t}(S + 2\sqrt{T})\right) > 0 \tag{3.11}$$

for $0 < t_0 \le t$ before a finite endpoint. The coefficients are bounded near every finite regular endpoint, so D cannot vanish there. A first zero of w would then have $w' \le 0$, whereas $w' = kDT > 0$. The maximal interval is consequently $(0, \infty)$.

The equations for T, w also give

$$(w - T)' + ((k+1)u + (k-1)v)(w - T) = -2(k-1)vT. \tag{3.12}$$

Let $I = a^{k+1}b^{k-1}$. Then $I'/I = (k+1)u + (k-1)v$ and $\lim_{t\downarrow 0} I(t)(w(t) - T(t)) = 0$. Thus

$$I(t)(w(t) - T(t)) = -2(k-1)\int_0^t I(s)v(s)T(s)\,ds < 0.$$

This proves $0 < w < T$ and, together with $F < 1$, all three assertions. □

4. The full curvature form

Denote the two spherical tangent subspaces by P, Q and the radial direction by 0. The curvature tensor determined by (3.6) is diagonal on two-planes of these types:

$$K_{0P} = w, \quad K_{0Q} = -w, \quad K_{PQ} = -T, \quad C_P := K_{PP} = \frac{kT - w}{k-1}, \quad C_Q := K_{QQ} = \frac{kT + w}{k-1}. \tag{4.1}$$

There are no additional Codazzi components: the orbit shape operator is a constant scalar on each parallel factor of the product connection.

Lemma 4.1. *Every symmetric two-tensor satisfies the pointwise bounds*

$$2\langle \mathring{R}h, h\rangle \le \left(4kT + \frac{2k+1}{k}D^2\right)|h|^2, \tag{4.2}$$

$$2\langle \mathring{R}h, h\rangle \le (4k+2)T|h|^2. \tag{4.3}$$

Proof. The trace-free PP and QQ subspaces have curvature-action eigenvalues $-C_P, -C_Q$, both negative by Lemma 3.2. The off-diagonal $PQ, 0P, 0Q$ subspaces have eigenvalues $T, -w, w$, respectively. The vector g has eigenvalue zero. The mutually orthogonal subspaces listed here, together with the two-dimensional trace-free diagonal subspace, exhaust S^2T^*M: their dimensions satisfy

$$2\left(\frac{k(k+1)}{2} - 1\right) + k^2 + 2k + 1 + 2 = (2k+1)(k+1) = \dim S^2(\mathbb{R}^{2k+1})^*. \tag{4.4}$$

The remaining trace-free diagonal tensors can be written

$$h_{00} = -2kx, \qquad h_P = (x+y)I_P, \qquad h_Q = (x-y)I_Q. \tag{4.5}$$

For a diagonal tensor with components $h_{00} = \xi$, $h_P = \eta I_P$, and $h_Q = \zeta I_Q$, the sectional curvatures give

$$\begin{aligned}
(\mathring{R}h)_{00} &= kw(\eta - \zeta),\\
(\mathring{R}h)|_P &= \big(w\xi + (k-1)C_P\eta - kT\zeta\big)I_P,\\
(\mathring{R}h)|_Q &= \big(-w\xi - kT\eta + (k-1)C_Q\zeta\big)I_Q.
\end{aligned} \tag{4.6}$$

For an off-diagonal component in the i, j plane, the action is $h_{ij} \mapsto -K_{ij}h_{ij}$; this also verifies the three mixed blocks listed above. Substituting $\xi = -2kx$, $\eta = x + y$, $\zeta = x - y$ in (4.6) gives $x \mapsto -wy$ and $y \mapsto -(2k+1)wx + 2kTy$. The squared norm in (4.5) is $2k((2k+1)x^2 + y^2)$. Thus in the coordinates $(\sqrt{2k+1}\,x, y)$ the matrix is

$$\begin{pmatrix} 0 & -\sqrt{2k+1}\,w \\ -\sqrt{2k+1}\,w & 2kT \end{pmatrix}. \tag{4.7}$$

Its characteristic polynomial is $\lambda^2 - 2kT\lambda - (2k+1)w^2$. Its largest root is

$$\lambda_+ = kT + \sqrt{k^2T^2 + (2k+1)w^2}. \tag{4.8}$$

It dominates every preceding eigenvalue, since $0 < w < T$ and $\lambda_+ \ge 2kT$. For $x > 0$ and $y \ge 0$, $\sqrt{x^2 + y} - x = y/(\sqrt{x^2+y} + x) \le y/(2x)$. Consequently $2\lambda_+ \le 4kT + (2k+1)w^2/(kT)$. Use $w^2 < TD^2$ for (4.2). For (4.3), use $w < T$ directly in (4.8) and $\sqrt{k^2 + 2k + 1} = k + 1$. $\square$

5. Ground-state inequality and holonomy

Lemma 5.1. *If $\phi > 0$ is smooth and h has compact support in its domain, then*

$$\int |\nabla h|^2 = \int \phi^2 |\nabla(\phi^{-1}h)|^2 + \int \frac{\Delta\phi}{\phi}|h|^2. \tag{5.1}$$

Proof. Writing $h = \phi v$, expand $|\nabla h|^2 = \phi^2|\nabla v|^2 + |d\phi|^2|v|^2 + \phi\langle d\phi, d|v|^2\rangle$. The last two terms integrate to $\int(-\phi \operatorname{div}\nabla\phi)|v|^2 = \int \phi\Delta\phi|v|^2$. There is no boundary term because h is compactly supported. $\square$

Proposition 5.2. *Let* $\gamma = (2k-1)/4$, $\phi = (ab)^{-\gamma}$, *and*

$$c_k = k(k-5) + \tfrac{1}{4}, \qquad d_k = \frac{(2k-1)(2k-3)}{16} - \frac{2k+1}{k}. \tag{5.2}$$

For every compactly supported smooth tensor,

$$Q_g(h) \geq \int \phi^2 |\nabla(\phi^{-1}h)|^2 + c_k \int T|h|^2 + d_k \int D^2|h|^2. \tag{5.3}$$

Both constants are positive. In addition,

$$Q_g(h) \geq \frac{c_k}{c_k + 4k + 3} \int \big(|\nabla h|^2 + T|h|^2\big). \tag{5.4}$$

Proof. The radial volume density is $a^k b^k$, so $\Delta\phi = -\phi'' - kS\phi'$. Since $\phi'/\phi = -\gamma S$,

$$\frac{\Delta\phi}{\phi} = \gamma S' + (k\gamma - \gamma^2)S^2 = \gamma(k - \gamma - \tfrac{1}{2})S^2 - \frac{\gamma}{2}D^2.$$

Use $S^2 = D^2 + 4T$ and $\gamma = (2k-1)/4$ to obtain

$$\frac{\Delta\phi}{\phi} = \frac{(2k-1)^2}{4}T + \frac{(2k-1)(2k-3)}{16}D^2. \tag{5.5}$$

For support away from $t = 0$, subtract (4.2) from (5.1) to obtain (5.3). The constants obey

$$c_k > 0, \qquad d_5 = \frac{139}{80}, \qquad d_k - d_5 = (k-5)\left(\frac{k+3}{4} + \frac{1}{5k}\right) \geq 0. \tag{5.6}$$

Near the zero section, $dv_g \asymp t^k dt\, dv_{S^k}\, dv_{S^k}$, $D^2 = O(t^{-2})$, and $d\log\phi = O(t^{-1})$. For smooth h, both $D^2|h|^2$ and $\phi^2|\nabla(\phi^{-1}h)|^2$ are integrable there. Choose $\chi_\varepsilon = 0$ on $t \leq \varepsilon$, $\chi_\varepsilon = 1$ on $t \geq 2\varepsilon$, and $|\chi_\varepsilon'| \leq C/\varepsilon$. Then

$$\int |d\chi_\varepsilon|^2 |h|^2\, dv_g = O(\varepsilon^{k-1}) \longrightarrow 0. \tag{5.7}$$

Thus $\chi_\varepsilon h \to h$ in H^1. Curvature is bounded near the smooth core; the left side of (5.3) converges. Fatou's lemma applied to its nonnegative right side proves the assertion across the core.

By (4.3), $\int |\nabla h|^2 \leq Q_g(h) + (4k+2)\int T|h|^2$. By (5.3), $\int T|h|^2 \leq Q_g(h)/c_k$. Adding these inequalities gives (5.4). □

Lemma 5.3. *The functions satisfy*

$$tu, tv \longrightarrow 1, \quad t^2 T \longrightarrow 1, \quad \frac{a}{t}, \frac{b}{t} \longrightarrow \sqrt{\frac{k-1}{2k-1}}. \tag{5.8}$$

In particular $T \asymp (1+t^2)^{-1}$ *and* $|\mathrm{Rm}_g| \leq C(1+t^2)^{-1}$.

Proof. Set $\vartheta = D/S$. Positivity of u, v, D gives $0 < \vartheta < 1$, and

$$\vartheta' = -\tfrac{1}{2}\vartheta S(1-\vartheta^2) - 2w/S < 0, \qquad (1/S)' = (1+\vartheta^2)/2. \tag{5.9}$$

For $t \geq t_0 > 0$,

$$\frac{1}{S(t_0)} + \frac{t - t_0}{2} \leq \frac{1}{S(t)} \leq \frac{1}{S(t_0)} + (t - t_0).$$

Thus $S \asymp t^{-1}$ and $\int_{t_0}^\infty S\, dt = \infty$. If $\vartheta \to \ell > 0$, fix $t_0 > 0$. For $t \geq t_0$, $\vartheta \geq \ell$ and $1 - \vartheta^2 \geq 1 - \vartheta(t_0)^2 > 0$; (5.9) would give $\vartheta' \leq -cS$, a contradiction after integration. Thus $\vartheta \to 0$ and $(1/S)' \to 1/2$, giving $tS \to 2$ and $tu, tv \to 1$. Since $T = (S^2 - D^2)/4$, $t^2 T \to 1$. The barrier gives $w/T \leq D/\sqrt{T} \to 0$. Multiplying (3.7) by t^2 now gives $(k-1)t^2/b^2 \to 2k-1$; its first equation gives the assertion for a. The curvature formulas (4.1) then give quadratic decay. All quantities are smooth and bounded on a compact neighborhood of the core, and $T(0) = \beta_0 > 0$. This proves the global comparisons. □

Proposition 5.4. *The assertions* (1.5) *and* $\mathrm{Hol}(g_k) = SO(2k+1)$ *hold.*

Proof. Let C_0 bound the operator norm of $2\mathring{R}$. The norm

$$\left(Q_g(h) + (C_0+1)\|h\|_2^2\right)^{1/2}$$

is equivalent to the H^1 norm. To prove density, choose exhaustion cutoffs η_R equal to one on $t \le R$, zero on $t \ge 2R$, with $|d\eta_R| \le C/R$. For $h \in H^1$,

$$\|h - \eta_R h\|_{H^1}^2 \le C \int_{t \ge R} (|h|^2 + |\nabla h|^2) + CR^{-2} \int_{R \le t \le 2R} |h|^2 \longrightarrow 0. \tag{5.10}$$

For a compactly supported tensor, choose a finite partition of unity subordinate to bundle charts whose closures are compact. Each partitioned coefficient extends by zero to an H^1 function on a Euclidean chart. Convolution with a compactly supported mollifier converges in H^1; choose the mollifier radius smaller than the distance from the partition support to the chart boundary. Summing these approximants gives smooth compactly supported tensors converging in H^1. Thus the closed form has domain H^1 and defines the Friedrichs realization. Since T is bounded, (5.4) passes to this domain. For a vector in the operator kernel, $Q_g(h) = \langle Eh, h\rangle = 0$; hence $\int T|h|^2 = 0$, and positivity of T gives $h = 0$.

For $0 \ne \chi \in C_c^\infty((1,2))$, put $h_R = \chi(t/R)g$. Since $\nabla g = 0$ and $\mathring{R}g = \mathrm{Ric} = 0$,

$$\frac{Q_g(h_R)}{\|h_R\|_2^2} = \frac{\int R^{-2}|\chi'(t/R)|^2 a^k b^k \, dt}{\int |\chi(t/R)|^2 a^k b^k \, dt} = O(R^{-2}). \tag{5.11}$$

The asymptotic comparison in Lemma 5.3 justifies the last bound. Therefore $\inf \sigma(E) = 0$.

At a point of the zero section, $T_x X = V \oplus W$, with $\dim V = k+1$ and $\dim W = k$. The common limit of K_{0P} and K_{PP} is β_0, the fibre sectional curvature tends to 1, and the mixed curvatures tend to $-\beta_0$. There are no mixed curvature components beyond those in (4.1). Thus the curvature operator on $\Lambda^2 V$, $\Lambda^2 W$, and $V \otimes W$ is multiplication by $\beta_0, 1, -\beta_0$, respectively. Each coefficient is nonzero. With $J_{XY}Z = \langle Y, Z\rangle X - \langle X, Z\rangle Y$, the mixed endomorphisms are nonzero multiples of $J_{v_i w_A}$. For distinct i, j and distinct A, B,

$$[J_{v_i w_A}, J_{v_j w_A}] = -J_{v_i v_j}, \qquad [J_{v_i w_A}, J_{v_i w_B}] = -J_{w_A w_B}.$$

These endomorphisms and the mixed ones span $\mathfrak{so}(V \oplus W)$. Curvature endomorphisms lie in the restricted holonomy algebra [2], so that algebra is $\mathfrak{so}(2k+1)$. The manifold is simply connected. The full holonomy therefore equals its restricted holonomy, namely $SO(2k+1)$. ☐

Propositions 5.2 and 5.4 and Lemma 5.3 prove Theorem A.

6. Nonlinear stability under Ricci–DeTurck flow

Fix $k \ge 5$ and write $g_* = g_k$. In this section and the next, r is distance from the compact singular orbit, $\rho = (1+r^2)^{1/2}$, and t denotes flow time. All norms, covariant derivatives and measures use the fixed background unless another metric is specified. For $0 < \eta < 1$, put

$$\mathcal{X}_k = L^2(S^2T^*M_k, g_k) \cap C_b^{2,\eta}(S^2T^*M_k, g_k), \qquad \|h\|_{\mathcal{X}_k} = \|h\|_2 + \|h\|_{C_b^{2,\eta}}. \tag{6.1}$$

The uniform Hölder norm is defined in a bounded-geometry atlas. Set

$$W(g, g_k)^a = g^{ij}\left(\Gamma(g)_{ij}^a - \Gamma(g_k)_{ij}^a\right), \tag{6.2}$$

$$\partial_t g = -2\,\mathrm{Ric}_g + \mathcal{L}_{W(g,g_k)} g. \tag{6.3}$$

Theorem 6.1 (Nonlinear attraction). *For every $k \ge 5$ and $0 < \eta < 1$, there is $\varepsilon_k > 0$ such that every smooth $h_0 \in \mathcal{X}_k$ with $\|h_0\|_{\mathcal{X}_k} < \varepsilon_k$ gives a solution of* (6.3), *$g(0) = g_k + h_0$, for all $t \ge 0$. The metrics are* *complete, uniformly equivalent to g_k, and have uniformly bounded curvature. Writing $h(t) = g(t) - g_k$, one has*

$$\|h(t)\|_2^2 + a_k \int_0^t \int_{M_k} \left(|\nabla h(s)|^2 + \rho^{-2}|h(s)|^2\right) dv_{g_k}\, ds \le \|h_0\|_2^2 \tag{6.4}$$

for some $a_k > 0$. For every integer $j \ge 0$,

$$\lim_{t\to\infty} \left(\|h(t)\|_2 + \sum_{a=0}^{j} \|\nabla^a h(t)\|_\infty \right) = 0. \tag{6.5}$$

Convergence holds in $\mathcal{X}_k$, and g_k is Lyapunov stable in that space. There are diffeomorphisms φ_t, $\varphi_0 = \mathrm{Id}$, such that $\widehat{g}(t) = \varphi_t^* g(t)$ is a complete immortal Ricci flow with initial metric $g_k + h_0$. Its pullbacks by φ_t^{-1} converge as in (6.5).

Theorem 6.2 (Ancient rigidity). *For every $k \ge 5$, there is $b_k > 0$ such that any smooth ancient solution of* (6.3)*, in the complete bounded-geometry local parabolic class, satisfying*

$$\sup_{t\le 0} \|g(t) - g_k\|_\infty < b_k, \qquad \sup_{t\le 0} \|g(t) - g_k\|_2 < \infty, \tag{6.6}$$

is identically g_k.

We prove these assertions under an abstract coercivity hypothesis. Let (M^n, g_*) be complete, connected, noncompact and Ricci-flat, with positive injectivity radius and bounded covariant derivatives of curvature of every order. Suppose $\rho \ge 1$ is smooth, proper, comparable to $1 + d_{g_*}(o, \cdot)$, and has bounded gradient. Assume

$$|\operatorname{Rm}_{g_*}| \le C_R \rho^{-2}, \qquad Q(u) \ge a\mathcal{D}_*(u), \qquad \mathcal{D}_*(u) = \|\nabla u\|_2^2 + \|\rho^{-1} u\|_2^2, \quad a > 0, \tag{6.7}$$

for every compactly supported smooth symmetric tensor. Here Q is the form of $E = \nabla^*\nabla - 2\mathring{R}_{g_*}$. Cutoff approximation and smoothing in compact coordinate charts extend this inequality to H^1 tensors.

The local input is short-time existence, uniqueness in the uniformly controlled parabolic class, and smoothing for the Ricci–DeTurck equation about a complete bounded-curvature Ricci-flat background [40, Lemma 3.2 and Section 3.1]. That local result does not require an ALE end. Its consequence used here is that, for sufficiently small b, a solution with $\|g - g_*\|_\infty < b$ satisfies

$$\|\nabla^j (g(t) - g_*)\|_\infty \le C_j b \qquad (j \ge 1) \tag{6.8}$$

away from the initial time by a fixed positive amount. On translated time intervals the constants are independent of the translation: one restarts the fixed short-time estimate on an interval ending at the specified time. Such a uniformly small solution continues across any finite time. The estimate up to the initial time is supplied separately by Lemma 6.3. No global ALE stability conclusion is used.

For $0 < T \le 1$, let $\mathscr{Y}_T$ be the space of sections with uniformly continuous coordinate coefficients on $M \times [0, T]$ and norm

$$\|f\|_{\mathscr{Y}_T} = \sup_{0\le t\le T} \|f(t)\|_{C_b^{0,\eta}}.$$

Let $\mathscr{X}_T$ consist of sections of class $C^{1,2}$ whose time derivative and spatial derivatives through order two have uniformly continuous coordinate coefficients, with norm

$$\|u\|_{\mathscr{X}_T} = \sup_{0\le t\le T} \|u(t)\|_{C_b^{2,\eta}} + \sup_{0\le t\le T} \|\partial_t u(t)\|_{C_b^{0,\eta}}. \tag{6.9}$$

The norms use the fixed bounded-geometry atlas and its tensor trivializations. Uniform convergence of the indicated coefficients and derivatives proves completeness. This definition does not require continuity of $t \mapsto u(t)$ in $C_b^{2,\eta}$.

Lemma 6.3 (Initial-time Hölder estimate). *Let (M, g_*) be complete without boundary, with positive injectivity radius and bounded covariant derivatives of curvature of every order. Suppose* $\mathrm{Ric}_{g_*} = 0$ *and fix $0 < \eta < 1$. There are $\delta_* > 0$ and $C_* > 0$, depending only on this background, the fixed atlas and η, with the following property. For smooth $h_0 \in C_b^{2,\eta}(S^2T^*M)$ satisfying $\|h_0\|_{C_b^{2,\eta}} < \delta_*$, the bounded Ricci–DeTurck solution with initial metric $g_* + h_0$ is defined on $[0,1]$ and satisfies*

$$\|h\|_{\mathscr{X}_1} \le C_* \|h_0\|_{C_b^{2,\eta}}, \qquad h = g - g_*. \tag{6.10}$$

The constants are independent of higher derivatives of h_0. No L^2 hypothesis, curvature decay, or coercivity assumption is needed.

Proof. First consider $P = \partial_t + E$, where $E = \nabla^*\nabla - 2\mathring{R}_{g_*}$. The scalar Cauchy estimate in [52, Theorem 5.1.9], stated with its initial term also in [53, Appendix A, Theorem A.3], gives the following uniform tensor estimate:

$$Pu = f, \quad u(0) = u_0 \quad \Longrightarrow \quad \|u\|_{\mathscr{X}_T} \le C_L\big(\|u_0\|_{C_b^{2,\eta}} + \|f\|_{\mathscr{Y}_T}\big), \qquad 0 < T \le 1. \tag{6.11}$$

We give the localization and existence argument. Choose nested coordinate balls of fixed radii and cutoffs equal to one on the smaller balls. The principal symbol of E is $g_*^{ij}\xi_i\xi_j\,\mathrm{Id}$; its eigenvalues are uniformly bounded above and below. The coordinate coefficients of P and the cutoff derivatives have uniform Hölder bounds. Extend each principal coefficient matrix to $\mathbb{R}^n$ by interpolation with a fixed positive matrix outside the larger ball. Ellipticity is preserved. Apply the cited scalar estimate to every coefficient of the localized tensor, placing the connection terms, curvature terms, and cutoff commutators on the right. These terms contain at most one derivative of u. Taking the supremum over the atlas gives

$$\|u\|_{\mathscr{X}_T} \le C\left(\|u_0\|_{C_b^{2,\eta}} + \|f\|_{\mathscr{Y}_T} + \sup_{0\le t\le T}\|u(t)\|_{C_b^{1,\eta}}\right).$$

The uniform chart interpolation inequality

$$\|u\|_{C_b^{1,\eta}} \le \varepsilon\|u\|_{C_b^{2,\eta}} + C_\varepsilon\|u\|_\infty$$

absorbs the highest norm after choosing $C\varepsilon < 1/2$. The regularized norm $(|u|^2 + \varepsilon^2)^{1/2}$ satisfies the scalar comparison inequality for the connection heat equation, with potential bounded by $2\|\mathring{R}_{g_*}\|_\infty$. Letting $\varepsilon \downarrow 0$ gives

$$\sup_{0\le t\le T}\|u(t)\|_\infty \le e^{CT}\left(\|u_0\|_\infty + T\|f\|_{L^\infty(M\times[0,T])}\right).$$

One may obtain the solution by solving on a smooth compact exhaustion, with initial value u_0 and time-independent lateral value u_0. This comparison is uniform over the exhaustion. The localized Cauchy estimates are used only on balls whose larger ball avoids the lateral boundary, and include their initial slices. To make their constants independent of the exhausting domain, choose nested radii $r_1 < r_2 < r_3$ in one fixed chart. The cutoff estimates and interpolation give, for the local highest-order norm $M(r)$,

$$M(r) \le \vartheta M(s) + C_\vartheta(s-r)^{-N}\left(\|u_0\|_{C_b^{2,\eta}} + \|f\|_{\mathscr{Y}_T} + \|u\|_{L^\infty}\right) \qquad (r_1 \le r < s \le r_2),$$

where N and the constants are uniform over the atlas. Iterate with $r_j = r_2 - (r_2 - r_1)2^{-j}$ and choose $\vartheta 2^N < 1$. The geometric series bounds $M(r_1)$ by the displayed data; the term $\vartheta^j M(r_j)$ tends to zero for each smooth exhaustion solution because the closed ball of radius r_2 lies in the ball of radius r_3, away from the lateral boundary. Compactness on each such cylinder gives a diagonal limit with the same estimates; corner compatibility at the auxiliary lateral boundary is not used. The equation gives the asserted bound for $\partial_t u$. Uniform spatial Hölder bounds and the local Cauchy estimates give uniform continuity of the indicated coefficients, also at $t = 0$. The scalar comparison applied to the difference proves uniqueness. For $T < 1$, extend f constantly from T to 1 and restrict the solution. Thus C_L is independent of T. This proves (6.11) and defines its linear solution map $\mathscr{S}(u_0, f)$.

In nondivergence form the Ricci–DeTurck equation reads

$$Ph = \mathcal{R}(h) = \big((g_* + h)^{ij} - g_*^{ij}\big)\nabla_i\nabla_j h + \mathcal{B}_0(h)(\nabla h, \nabla h) + \mathcal{C}_0(h)(\mathrm{Rm}_{g_*}, h, h). \tag{6.12}$$

The last two coefficient contractions are smooth for $g_* + h > 0$ and have bounded derivatives in a fixed small coordinate ball about zero. The formula follows from the connection-difference expression for the DeTurck vector; see [40, Section 3.1, the coordinate display following (9)]. In particular no forcing independent of h occurs, since g_* is Ricci-flat. The identity for the derivative of an inverse matrix and the inequality

$$[vw]_{\eta,b} \le \|v\|_\infty [w]_{\eta,b} + \|w\|_\infty [v]_{\eta,b}$$

give constants $C_N, r_* > 0$ such that

$$\begin{aligned} \|\mathcal{R}(u)\|_{\mathscr{Y}_1} &\le C_N \|u\|_{\mathscr{X}_1}^2, \\ \|\mathcal{R}(u) - \mathcal{R}(v)\|_{\mathscr{Y}_1} &\le C_N(\|u\|_{\mathscr{X}_1} + \|v\|_{\mathscr{X}_1})\|u - v\|_{\mathscr{X}_1} \end{aligned} \tag{6.13}$$

when both norms are at most r_*. The radius is also chosen so that all the corresponding metrics are uniformly positive definite. For example the principal term is bounded by $C\|u\|_{C_b^{0,\eta}}\|\nabla^2 u\|_{C_b^{0,\eta}}$; the gradient term by $C\|\nabla u\|^2_{C_b^{0,\eta}}$; and the curvature term by $C\|\mathrm{Rm}_{g_*}\|_{C_b^{0,\eta}}\|u\|^2_{C_b^{0,\eta}}$. Taking differences gives the second estimate by the same product inequality.

Put $d = \|h_0\|_{C_b^{2,\eta}}$ and $R_0 = 2C_L d$. Decrease δ_* until $R_0 \le r_*$ and $C_L C_N R_0 \le 1/4$. The map

$$u \longmapsto \mathscr{S}(h_0, \mathcal{R}(u))$$

maps the closed R_0-ball in $\mathscr{X}_1$ into itself: its norm is at most $C_L d + C_L C_N R_0^2 \le 3R_0/4$. Its Lipschitz constant is at most $2C_L C_N R_0 \le 1/2$. The contraction theorem gives a solution with norm at most $2C_L d$. Shrinking δ_* once more places this solution in the bounded uniqueness class of [40, Lemma 3.2], so it is the local solution used above. Positive-time regularity follows by differentiating the uniformly parabolic equation and applying the local estimates on interior cylinders. Thus (6.10) holds with $C_* = 2C_L$. The argument uses boundedness of the large Hölder norm up to the initial slice, not strong continuity in that norm. ☐

Lemma 6.4. *There is a background constant C such that, if $u \in L^2$ is a smooth tensor and $\|\nabla u\|_\infty \le L$, then*

$$\|u\|_\infty \le C\left(\|u\|_2 + L^{n/(n+2)}\|u\|_2^{2/(n+2)}\right). \tag{6.14}$$

Proof. Bounded geometry gives $\operatorname{Vol} B(x,s) \ge v s^n$ for $0 < s \le 1$, with $v > 0$ independent of x. If $A = |u(x)| > 0$ and $L > 0$, put $s = \min\{1, A/(2L)\}$. Integration along minimizing geodesics gives $|u| \ge A/2$ on $B(x,s)$, and therefore

$$\|u\|_2^2 \ge \frac{v}{4} A^2 \min\{1, (A/(2L))^n\}.$$

For $A \ge 2L$ this bounds A by $2v^{-1/2}\|u\|_2$. Otherwise it bounds A^{n+2} by $2^{n+2}v^{-1}L^n\|u\|_2^2$. The case $L = 0$ follows from the unit-ball bound. Taking the supremum over x proves the assertion. ☐

6.1. **Localized nonlinear energy.** In background derivatives, the equation for $h = g - g_*$ is

$$\partial_t h = -Eh + \mathcal{N}(h), \qquad \mathcal{N}(h) = \nabla_i\big(A^{ij}(h)\nabla_j h\big) + \mathcal{B}(h)(\nabla h, \nabla h) + \mathcal{C}(h)(\mathrm{Rm}_{g_*}, h, h). \tag{6.15}$$

The contractions are smooth and, for $|h|$ below a fixed constant,

$$|A(h)| \le C|h|, \qquad |D_h A(h)| + |\mathcal{B}(h)| + |\mathcal{C}(h)| \le C. \tag{6.16}$$

Indeed, the principal coefficient in the fixed-background coordinate formula [40, Section 3.1, the coordinate display following (9)] is g^{-1}. Subtracting the linear coefficient gives $g^{-1} - g_*^{-1} = O(h)$. Writing this term in divergence form adds a quadratic first-derivative term. The curvature expression has linear term $2\mathring{R}_{g_*}h$ and remainder bounded by $C|\mathrm{Rm}_{g_*}||h|^2$. No derivative of the background curvature is introduced.

Lemma 6.5. *There are $b_0 > 0$ and $C_0 < \infty$ such that, whenever $\|h\|_\infty \le b_0$, every time-independent compactly supported Lipschitz cutoff $0 \le \chi \le 1$ satisfies*

$$\frac{d}{dt}\|\chi h\|_2^2 + a\mathcal{D}_*(\chi h) \le C_0 \int_M |\nabla\chi|^2|h|^2\, dv_{g_*}. \tag{6.17}$$

The integrated inequality holds for exhausted cutoffs without compact support whenever its displayed terms are integrable.

Proof. The connection Laplacian gives

$$\int \langle Eh, \chi^2 h\rangle = Q(\chi h) - \int |\nabla\chi|^2|h|^2.$$

Put $b = \|h\|_\infty$. Integrating the first term of (6.15) once and using (6.16) gives

$$\begin{aligned}|\langle \mathcal{N}(h), \chi^2 h\rangle| &\le Cb \int \left(\chi^2|\nabla h|^2 + \chi|\nabla\chi||h||\nabla h| + \rho^{-2}\chi^2|h|^2\right) dv_{g_*}\\ &\le C_1 b\left(\mathcal{D}_*(\chi h) + \int |\nabla\chi|^2|h|^2\, dv_{g_*}\right).\end{aligned}$$

For the second line use $\chi\nabla h = \nabla(\chi h) - d\chi \otimes h$ and $2xy \le x^2 + y^2$. Testing with $\chi^2 h$ therefore gives

$$\frac{1}{2}\frac{d}{dt}\|\chi h\|_2^2 + (a - C_1 b)\mathcal{D}_*(\chi h) \le (1 + C_1 b)\int |\nabla\chi|^2|h|^2.$$

Choose $C_1 b_0 \le a/2$ and multiply by two. Smooth approximation proves the Lipschitz-cutoff assertion. Exhaustion proves the last assertion when its terms are integrable. □

Proposition 6.6. *On an interval on which $\|h(t)\|_\infty \le b_0$ and $h(0) \in L^2$,*

$$\|h(t)\|_2^2 + a\int_s^t \mathcal{D}_*(h(\tau))\, d\tau \le \|h(s)\|_2^2 \qquad (0 \le s \le t). \tag{6.18}$$

Proof. To establish integrability before using the global test, take $\chi_R = e^{-\rho/R}$. Completeness and Ricci-flatness give at most polynomial volume growth. The local bounds and this exponential weight make all finite-time integrals finite. First multiply χ_R by an outer compact cutoff; its additional errors vanish as its support exhausts M. Since $|\nabla\chi_R| \le CR^{-1}\chi_R$, Lemma 6.5 gives

$$E_R'(t) + a\mathcal{D}_*(\chi_R h(t)) \le CR^{-2}E_R(t), \qquad E_R = \|\chi_R h\|_2^2.$$

Hence $E_R(t) \le e^{Ct/R^2}\|h(0)\|_2^2$. Let $R \to \infty$ to obtain $h(t) \in L^2$. The integrated right side is bounded by $CtR^{-2}e^{Ct/R^2}\|h(0)\|_2^2 \to 0$. Fatou's lemma gives the inequality with $s = 0$, since the differentiated cutoffs converge locally to one. Restarting at s proves the general case. Local parabolic continuity and the same approximation give the inequalities at the endpoints. □

7. Global convergence and ancient rigidity

Proposition 7.1. *Under* (6.7)*, sufficiently small smooth data in $L^2 \cap C_b^{2,\eta}$ give a complete global solution. The equilibrium is Lyapunov stable in this intersection norm.*

Proof. Fix $0 < b < b_0$ below the local parabolic threshold. By Lemma 6.3, if $\|h_0\|_{C_b^{2,\eta}} < \min\{\delta_*, b/(2C_*)\}$, then

$$\sup_{0\le t\le 1}\|h(t)\|_{C_b^{2,\eta}} \le C_*\|h_0\|_{C_b^{2,\eta}} < b/2. \tag{7.1}$$

Suppose $T > 1$ is a first exit from $\|h\|_\infty < b$. The global energy inequality gives $\|h(t)\|_2 \le \|h_0\|_2$ for $t \le T$. Smoothing on translated intervals gives $\|\nabla h(t)\|_\infty \le Cb$ for $t \ge 1/2$. Thus

$$\|h(t)\|_\infty \le C\left(\|h_0\|_2 + b^{n/(n+2)}\|h_0\|_2^{2/(n+2)}\right) \qquad (1 \le t \le T).$$

Choose the initial L^2 norm so that this is less than $b/2$, a contradiction. Uniform ellipticity and the restarted derivative bounds exclude a finite-time loss of the local solution. Uniform equivalence to g_* gives completeness, and the order-two bounds give bounded curvature.

For Lyapunov stability choose b arbitrarily small first. Bounds through order three on $t \geq 1$ give a $C_b^{2,\eta}$ bound $C_\eta b$ by spatial interpolation. On $0 \leq t \leq 1$, use (7.1). Given a target intersection norm $\varepsilon > 0$, first choose b so that $C_\eta b < \varepsilon/2$, then take $\|h_0\|_{C_b^{2,\eta}} < \varepsilon/(2C_*)$ as well as the preceding smallness conditions. Choose the initial L^2 norm below $\varepsilon/2$ and below the first-exit threshold. Its value cannot increase by (6.18). These choices bound the full intersection norm for all $t \geq 0$, without a continuity assertion in the large Hölder space at $t = 0$. □

Lemma 7.2. *The preceding solution satisfies*

$$\lim_{R\to\infty} \sup_{t\geq 0} \int_{\{\rho\geq R\}} |h(t)|^2 \, dv_{g_*} = 0. \tag{7.2}$$

Proof. Let $\xi_R = 0$ on $\rho \leq R$, $\xi_R = 1$ on $\rho \geq 2R$, with $0 \leq \xi_R \leq 1$ and $|\nabla \xi_R|^2 \leq C\rho^{-2}\mathbf{1}_{\{R<\rho<2R\}}$. Apply (6.17) with an outer compact cutoff. On a finite interval its additional error is bounded by $CtL^{-2}\|h_0\|_2^2$ and tends to zero as $L \to \infty$. Omitting the nonnegative dissipation gives

$$\int_{\{\rho\geq 2R\}} |h(t)|^2 \leq \int_{\{\rho\geq R\}} |h_0|^2 + C\int_0^\infty \int_{\{\rho\geq R\}} \rho^{-2}|h(s)|^2 \, dv_{g_*} \, ds. \tag{7.3}$$

The first term tends to zero by $h_0 \in L^2$. The second tends to zero by dominated convergence and (6.18). The bound is independent of t. □

Proposition 7.3. *The preceding solution satisfies* $\|h(t)\|_2 \to 0$ *and* $\|\nabla^j h(t)\|_\infty \to 0$ *for every* $j \geq 0$*, as well as convergence in* $C_b^{j,\eta}$ *for every* $0 < \eta < 1$.

Proof. The nonnegative function $t \mapsto \|\rho^{-1}h(t)\|_2^2$ is integrable. Choose $t_i \to \infty$ at which it tends to zero. For fixed R,

$$\int_{\{\rho\leq 2R\}} |h(t_i)|^2 \leq 4R^2\|\rho^{-1}h(t_i)\|_2^2 \longrightarrow 0.$$

The exterior integral is uniformly small by Lemma 7.2. First let $i \to \infty$, then $R \to \infty$. This gives $\|h(t_i)\|_2 \to 0$, and monotonicity in (6.18) gives the limit at all times. The gradient bound and Lemma 6.4 imply $\|h(t)\|_\infty \to 0$. For $j \geq 1$, chart interpolation with a uniform bound K_{j+1} on orders through $j+1$ gives

$$\|\nabla^j h(t)\|_\infty \leq C_j\left(\|h(t)\|_\infty + \|h(t)\|_\infty^{1/(j+1)} K_{j+1}^{j/(j+1)}\right) \longrightarrow 0.$$

To obtain the Hölder estimate, use the fixed bounded-geometry atlas. For the coordinate coefficients of a tensor T, let $A = \|T\|_\infty$ and $L = C(\|\nabla T\|_\infty + \|T\|_\infty)$, where C is uniform over that atlas. On each smaller coordinate ball, the coefficient difference at points of distance d is bounded by $\min\{2A, Ld\}$. For $0 < \eta < 1$, splitting into $d \leq 2A/L$ and $d \geq 2A/L$ gives

$$[T]_{\eta,b} \leq C_\eta \|T\|_\infty^{1-\eta}(\|\nabla T\|_\infty + \|T\|_\infty)^\eta. \tag{7.4}$$

The assertion is immediate when $A = 0$ or $L = 0$. Apply the inequality to $T = \nabla^j h(t)$. Both norms on the right tend to zero by the preceding estimates, so the order-j Hölder seminorm tends to zero. □

Proposition 7.4. *A smooth ancient solution in the stated local parabolic class, with* $\sup_{t\leq 0}\|h(t)\|_\infty < b_0$ *and* $\sup_{t\leq 0}\|h(t)\|_2^2 \leq K < \infty$*, is zero.*

Proof. The energy inequality on $[s, 0]$, followed by $s \to -\infty$, gives $\int_{-\infty}^0 \mathcal{D}_*(h(t))\, dt \leq K/a$. Choose $s_i \to -\infty$ with $\|\rho^{-1}h(s_i)\|_2^2 \to 0$. Let $\eta_R = 1$ on $\rho \leq R$, $\eta_R = 0$ on $\rho \geq 2R$, with $|\nabla \eta_R|^2 \leq C\rho^{-2}\mathbf{1}_{\{R<\rho<2R\}}$. Integrating (6.17) from s_i to zero gives

$$\|\eta_R h(0)\|_2^2 \leq \|\eta_R h(s_i)\|_2^2 + C\int_{s_i}^0 \int_{\{R<\rho<2R\}} \rho^{-2}|h|^2 \, dv_{g_*} \, dt.$$

For fixed R the first term is at most $4R^2\|\rho^{-1}h(s_i)\|_2^2 \to 0$. Therefore

$$\|\eta_R h(0)\|_2^2 \le C\int_{-\infty}^0 \int_{\{\rho\ge R\}} \rho^{-2}|h|^2\,dv_{g_*}\,dt.$$

The right side tends to zero as $R\to\infty$, proving $h(0)=0$. Replacing zero by any terminal time proves the assertion. No backward-uniqueness assertion is used. □

Corollary 7.5 (Stationary rigidity in the fixed gauge). *A stationary solution of* (6.3) *whose difference from g_* lies in L^2 and has sufficiently small uniform norm equals g_*.*

Proof. Write the stationary metric as g_*+h. Choose smooth functions $0\le\eta_R\le 1$, equal to one on $\{\rho\le R\}$ and zero on $\{\rho\ge 2R\}$, with $|d\eta_R|\le C/R$. The localized inequality (6.17) applies on the compact support of η_R. Its time derivative vanishes, so, for sufficiently small $\|h\|_\infty$,

$$a\,\mathcal{D}_*(\eta_R h) \le C\int_M |d\eta_R|^2|h|^2\,dv_{g_*} \le CR^{-2}\|h\|_2^2. \tag{7.5}$$

Fix $L>0$ and let $R\to\infty$ with $R\ge L$. The left side is at least $a\int_{\{\rho\le L\}}\rho^{-2}|h|^2\,dv_{g_*}$. This integral is zero for every L, and $h=0$. □

Proof of Theorems 6.1 and 6.2. Theorem A and Lemma 5.3 give (6.7) for g_k. The compact core is smooth. Put $a_\infty=\sqrt{(k-1)/(2k-1)}$ and, for $s\in[1/4,4]$, define $a_R(s)=R^{-1}a(Rs)$ and $b_R(s)=R^{-1}b(Rs)$. Lemma 5.3 implies $(a_R,b_R)\to(a_\infty s, a_\infty s)$ in C^1 on this interval. The rescaled functions satisfy

$$\begin{aligned} a_R'' &= (k-1)\frac{1-(a_R')^2}{a_R} - k\frac{a_R' b_R'}{b_R},\\ b_R'' &= (k-1)\frac{1-(b_R')^2}{b_R} - k\frac{a_R' b_R'}{a_R}. \end{aligned} \tag{7.6}$$

Both denominators have a positive lower bound independent of large R. The right sides therefore converge in C^0. Differentiating these equations inductively proves convergence in every C^j norm on smaller annuli. Curvature and its covariant derivatives are smooth expressions in these coefficients, their inverses, and finitely many derivatives. Scaling back gives $|\nabla^j\operatorname{Rm}|=O(r^{-2-j})$.

Choose finitely many coordinate neighborhoods covering the unit annulus of the limiting smooth-link cone, with their closures inside the enlarged annulus. Smooth convergence and the inverse function theorem for the geodesic exponential map give a common positive normal-coordinate radius on smaller neighborhoods. A geodesic of length less than the distance to the enlarged annulus boundary cannot leave it. These neighborhoods consequently give uniform injectivity radii in the rescaled metrics. After scaling back, the injectivity radius on $\{R\le r\le 2R\}$ is bounded below by cR. The smooth compact core supplies a positive bound on the remaining region. Thus the background has bounded geometry of all orders.

Apply Propositions 7.1 and 7.3 for the forward assertions, and Proposition 7.4 for the ancient assertion. The energy statement is (6.18). For the Ricci-flow assertion, solve

$$\partial_t\varphi_t = -W(g(t),g_k)\circ\varphi_t, \qquad \varphi_0=\operatorname{Id}.$$

The vector field and its required derivatives are bounded on every finite time interval. Completeness gives diffeomorphisms for all finite times, and differentiation gives $\partial_t(\varphi_t^*g(t))=-2\operatorname{Ric}_{\varphi_t^*g(t)}$. Completeness and curvature bounds are preserved by pullback, while $(\varphi_t^{-1})^*\widehat{g}(t)=g(t)$. □

Corollary 7.6 (Equal-factor dynamical dichotomy). *For every $k\ge 2$, the normalized complete Ricci-flat Böhm metric on $\mathbb{R}^{k+1}\times S^k$ has holonomy $SO(2k+1)$. For $k=2,3,4$ it has infinitely many negative L^2 Einstein modes and the nonlinear ancient instabilities of* [3, Main Theorems I–II]. *For $k\ge 5$, Theorems 6.1 and 6.2 give attraction and bounded-L^2 ancient rigidity in their stated neighborhoods.*

Proof. The core curvature calculation in Proposition 5.4 uses the initial expansion, not the high-dimensional barrier. Its eigenvalues on $\Lambda^2 V$, $\Lambda^2 W$, and $V \otimes W$ are $(k-1)/(k+1)$, 1, and $-(k-1)/(k+1)$, all nonzero for $k \geq 2$. The curvature endomorphisms therefore generate the full orthogonal holonomy algebra; simple connectivity gives the group assertion. For $k = 2, 3, 4$, the parameter in [3, Main Theorems I–II] satisfies $2k \leq 8$. Their Lichnerowicz sign is opposite to E, so their positive modes are the negative modes here. Their ancient families retain the little-Hölder topology specified in that theorem. The remaining assertions follow from the two theorems proved above. □

No time-decay rate or convergence of φ_t as $t \to \infty$ is asserted. The ALE conclusions in [40, 41, 42] concern different ends; the cones here are nonflat. The ALF and generalized AF results of [43] have different end hypotheses as well.

Part 2. Gluing of compact caps and Einstein volumes

8. Smooth cap families at the collapsing orbit

Fix $p, q \geq 2$ and $d = p + q$. Let (Q^q, h_Q) be closed and connected, with $\mathrm{Ric}_{h_Q} = (q-1)h_Q$. In this part we use a one-ended Einstein metric

$$g = dt^2 + a(t)^2 \gamma_p + b(t)^2 h_Q, \qquad \mathrm{Ric}_g = \lambda g. \tag{8.1}$$

Only the Ricci tensor of h_Q enters its tangential equations. With $A = a'/a$, $B = b'/b$, and $H = pA + qB$, the tangential equations are

$$A' = (p-1)a^{-2} - HA - \lambda, \tag{8.2}$$

$$B' = (q-1)b^{-2} - HB - \lambda. \tag{8.3}$$

The constraint is

$$H^2 - pA^2 - qB^2 = p(p-1)a^{-2} + q(q-1)b^{-2} - (d-1)\lambda. \tag{8.4}$$

The general singular-orbit initial-value problem is treated in [17]. We give the local construction in this doubly warped class, including dependence on the Einstein constant. The uniqueness below is asserted only for this class and these normalized initial values.

Lemma 8.1. *For λ in any fixed compact interval, there is a common $t_0 > 0$ and a unique even analytic pair $f, g = O(t^2)$ such that*

$$a(t) = te^{f(t)}, \qquad b(t) = e^{g(t)} \tag{8.5}$$

satisfies (8.2)–(8.4) *on* $(0, t_0]$. *The pair depends analytically on λ. The metric extends smoothly over the zero section of $D^{p+1} \times Q$.*

Proof. Let $\mathcal{D} = t\partial_t$. Substitution in the tangential equations gives

$$(\mathcal{D}+1)(\mathcal{D}+2p-2)f + q\mathcal{D}g = (p-1)(e^{-2f} - 1 + 2f) - p(\mathcal{D}f)^2 - q\mathcal{D}f\mathcal{D}g - \lambda t^2, \tag{8.6}$$

$$\mathcal{D}(\mathcal{D}+p-1)g = ((q-1)e^{-2g} - \lambda)t^2 - p\mathcal{D}f\mathcal{D}g - q(\mathcal{D}g)^2. \tag{8.7}$$

For $F(t) = \sum_{j\geq 1} F_j t^{2j}$ define

$$\|F\|_{0,R} = \sum_{j\geq 1} |F_j| R^{2j}, \qquad \|F\|_{1,R} = \sum_{j\geq 1} (1+j)|F_j| R^{2j}. \tag{8.8}$$

The first norm is submultiplicative after including constant terms, and $\|\mathcal{D}F\|_{0,R} \leq 2\|F\|_{1,R}$. Put $L_f = (\mathcal{D}+1)(\mathcal{D}+2p-2)$ and $L_g = \mathcal{D}(\mathcal{D}+p-1)$. Their coefficients on t^{2j} are $(2j+1)(2j+2p-2)$ and $2j(2j+p-1)$, respectively. They are bounded below by $c(1+j)^2$, $j \geq 1$. Thus

$$\|L_f^{-1}F\|_{1,R} + \|L_g^{-1}F\|_{1,R} \leq C\|F\|_{0,R}. \tag{8.9}$$

The triangular inverse of the left side of (8.6)–(8.7) is

$$(F,G) \longmapsto \big(L_f^{-1}(F - q\mathcal{D}L_g^{-1}G),\ L_g^{-1}G\big). \tag{8.10}$$

It is bounded from the product 0-norm to the product 1-norm. Equivalently, $(\mathcal{D}+a)^{-1}F(t) = \int_0^1 s^{a-1}F(st)\,ds$; for $a=0$ this is used only when $F = O(t^2)$.

Let $\mathcal{X}_R$ be the product of the two complete weighted sequence spaces in (8.8), with norm $\|(f,g)\| = \|f\|_{1,R} + \|g\|_{1,R}$. For $\|f\|_{0,R} \le 1$,

$$\|e^{-2f} - 1 + 2f\|_{0,R} \le \sum_{j\ge 2} \frac{2^j \|f\|_{0,R}^j}{j!} \le 2e^2\|f\|_{0,R}^2,$$

$$\|\mathcal{D}f\,\mathcal{D}g\|_{0,R} \le 4\|f\|_{1,R}\|g\|_{1,R}.$$

On the ball $\|(f,g)\| \le MR^2 \le 1$, the right side of the equations consequently has 0-norm at most $C_0R^2 + C_1M^2R^4$. The nonlinear difference has norm at most $C_2(R^2+MR^2)(\|\delta f\|_{1,R} + \|\delta g\|_{1,R})$. Choose $M > 2CC_0$, with C the inverse bound, and then choose $R>0$ so that the image lies in that ball and its Lipschitz constant is less than $1/2$. The same estimates hold in the complexified series spaces and for λ in a fixed small complex neighborhood of the prescribed compact interval. Iterating the contraction from zero gives holomorphic functions of λ which converge uniformly in the series norm. The limit is therefore holomorphic; its restriction to real λ is real analytic. Moreover,

$$\|(I - D\mathcal{T})^{-1}\| \le \sum_{j\ge 0} \|D\mathcal{T}\|^j \le 2. \tag{8.11}$$

Writing the equation as $U = \mathcal{T}(\lambda, U)$ gives

$$\partial_\lambda U = (I - D_U\mathcal{T})^{-1}\partial_\lambda \mathcal{T}.$$

The inverse is bounded by (8.11); Cauchy's inequalities in a smaller complex parameter neighborhood bound every further parameter derivative. Absolute convergence on a smaller t-disc also bounds each spatial derivative. Coefficient comparison in the triangular equations gives uniqueness among even analytic solutions with the prescribed initial values.

It remains to verify the radial equation. For a solution of the tangential equations let

$$C = H^2 - pA^2 - qB^2 - p(p-1)a^{-2} - q(q-1)b^{-2} + (d-1)\lambda. \tag{8.12}$$

Put $P = pA^2 + qB^2$ and $S_o = p(p-1)a^{-2} + q(q-1)b^{-2}$. The tangential equations give

$$H' = S_o - H^2 - d\lambda, \quad P' = 2p(p-1)Aa^{-2} + 2q(q-1)Bb^{-2} - 2HP - 2\lambda H,$$

$$S_o' = -2p(p-1)Aa^{-2} - 2q(q-1)Bb^{-2}.$$

Thus $C' = 2HH' - P' - S_o' = -2HC$. The leading t^{-2} terms cancel in (8.12); the even expansions make C bounded. A nonzero solution of the defect equation is a nonzero constant times $a^{-2p}b^{-2q}$ and behaves as t^{-2p}, a contradiction. Hence the constraint holds. The trace of the tangential equations and the constraint give $H' = -\lambda - pA^2 - qB^2$, which is the radial Einstein equation. For $x \in \mathbb{R}^{p+1}$ put $t = |x|$ and $A_*(t^2) = e^{2f(t)}$. The normal part of the metric is

$$dt^2 + a^2\gamma_p = A_*(|x|^2)\sum_i dx_i^2 + \frac{1 - A_*(|x|^2)}{|x|^2}\left(\sum_i x_i\,dx_i\right)^2.$$

Since $A_*(0) = 1$, its displayed quotient is analytic in $|x|^2$. The fibre coefficient $e^{2g(t)}$ is analytic in $|x|^2$ and positive. The metric therefore extends smoothly and is positive definite at $x = 0$. ☐

Corollary 8.2. *Let g_s be the cap with Einstein constant d and noncollapsing radius s. After rescaling by s^{-2}, its normalized functions are those of Lemma 8.1 with parameter $\lambda = ds^2$. On every fixed compact interval on which the Ricci-flat normalized solution is regular, the functions converge with all spatial derivatives at rate $O_R(s^2)$, and their derivatives with respect to* $\log s$ *have the same rate.*

Proof. Let $\lambda_s = ds^2$ and $\widetilde{a}_s(t) = s^{-1}a_s(st)$, $\widetilde{b}_s(t) = s^{-1}b_s(st)$. Constant metric scaling changes the Einstein constant from d to ds^2; the initial values of these functions are those of Lemma 8.1. On a common initial interval that lemma gives

$$\|Y(\lambda_s) - Y(0)\|_{C^j} \le C_j s^2, \qquad \|s\partial_s Y(\lambda_s)\|_{C^j} = \|2ds^2\partial_\lambda Y(\lambda_s)\|_{C^j} \le C_j s^2,$$

where $Y = (a, b, a', b')$ and j is fixed. Let $[t_0, T]$ be a regular compact interval of the Ricci-flat solution. There $Y' = \mathcal{F}(Y, \lambda)$ is smooth, and a, b have positive lower bounds. Choose a compact neighborhood of $Y(0, [t_0, T])$ on which $\|D_Y\mathcal{F}\| + \|\partial_\lambda\mathcal{F}\| \le C_T$. Up to the first exit from that neighborhood,

$$|Y_s(t) - Y_0(t)| \le |Y_s(t_0) - Y_0(t_0)| + C_T \int_{t_0}^{t} (|Y_s - Y_0| + ds^2)\, dr.$$

Gronwall's inequality bounds the right side by $C_T' s^2$ and excludes an exit for small s. The derivative $Z_s = s\partial_s Y_s$ satisfies

$$Z_s' = D_Y\mathcal{F}(Y_s, ds^2)Z_s + 2ds^2\partial_\lambda\mathcal{F}(Y_s, ds^2),$$

so the same integral inequality gives $|Z_s| \le C_T'' s^2$. Repeated differentiation of these two equations bounds each fixed spatial derivative. The two C^j bounds hold on $[t_0, T]$ for every fixed j. □

9. The volume-radius equations and the Ricci-flat asymptotic

In the rest of Part II, $\lambda = d$ and $4 \le d \le 8$, unless a Ricci-flat comparison is specified. Define

$$a = a_0 r e^{qx/d}, \quad b = b_0 r e^{-px/d}, \quad a_0^2 = \frac{p-1}{d-1}, \quad b_0^2 = \frac{q-1}{d-1}, \quad y = x', \quad z = ry, \quad \sigma = \log r. \tag{9.1}$$

Put

$$\kappa = \frac{pq}{d^2(d-1)}, \qquad K(x) = \frac{pe^{-2qx/d} + qe^{2px/d}}{d}, \qquad v_0 = a_0^p b_0^q. \tag{9.2}$$

Then $K(0) = 1$, $K'(0) = 0$, $K''(0) = 4pq/d^2 > 0$, and $K(x) \ge 1$ by convexity.

Lemma 9.1. *Before the first zero of H, $r' > 0$ and*

$$r'^2 + r^2 = K(x) + \kappa z^2, \tag{9.3}$$

$$x_\sigma = z/f, \qquad z_\sigma = -(d-1)z + \frac{d-1}{f}(e^{-2qx/d} - e^{2px/d}), \tag{9.4}$$

$$f = \sqrt{K(x) + \kappa z^2 - r^2} = r'. \tag{9.5}$$

The shape energy

$$\mathcal{A} = \frac{1}{2}z^2 + \frac{K(x) - 1}{2\kappa} \tag{9.6}$$

satisfies the identities

$$\mathcal{A}' = -(d-1)\frac{r'}{r}z^2, \qquad r'^2 + r^2 = 1 + 2\kappa\mathcal{A}, \qquad r'' = -r - \frac{pq}{d^2}\frac{z^2}{r}. \tag{9.7}$$

Here the prime in (9.7) *is d/dt.*

Proof. From (9.1), $A = r'/r + (q/d)y$, $B = r'/r - (p/d)y$, and $H = dr'/r$. Substitute these into (8.4) to obtain (9.3). Subtracting (8.3) from (8.2) gives

$$y' = -d(r'/r)y + (d-1)r^{-2}(e^{-2qx/d} - e^{2px/d}). \tag{9.8}$$

The derivative of $z = ry$ is

$$z' = -(d-1)\frac{r'}{r}z + \frac{d-1}{r}(e^{-2qx/d} - e^{2px/d}).$$

Division by $\sigma' = r'/r > 0$ gives (9.4). Finally $K'(x)/(2\kappa) = -(d-1)(e^{-2qx/d} - e^{2px/d})$. Since $x' = z/r$, differentiation gives

$$\mathcal{A}' = zz' + \frac{K'(x)}{2\kappa}\frac{z}{r} = -(d-1)\frac{r'}{r}z^2 + \frac{d-1}{r}z(e^{-2qx/d} - e^{2px/d}) + \frac{K'(x)z}{2\kappa r}.$$

The last two terms cancel. Also $K + \kappa z^2 = 1 + 2\kappa\mathcal{A}$. Differentiating $r'^2 + r^2 = 1 + 2\kappa\mathcal{A}$ and dividing by $2r' > 0$ gives $r'' + r = -\kappa(d-1)z^2/r$. All functions extend continuously to a minimal orbit with $r > 0$, giving the same identity there. □

For the Ricci-flat normalized cap, the same definitions give $r'^2 = K(x) + \kappa z^2$ and (9.4) with $f = \sqrt{K(x) + \kappa z^2}$.

Lemma 9.2. *The normalized Ricci-flat cap exists for all $t \geq 0$, is complete, and satisfies $U = (x, z) \to 0$ as $r \to \infty$. Moreover,*

$$a(t)/t \to a_0, \quad b(t)/t \to b_0, \quad a'(t) \to a_0, \quad b'(t) \to b_0.$$

Proof. Near the core, $r = Ct^{p/d}(1 + O(t^2))$, $C > 0$, so $r' > 0$. The constraint gives $r' = \sqrt{K + \kappa z^2} \geq 1$, and $d\mathcal{A}/d\sigma = -(d-1)z^2$. Since K is proper, the trajectory after any fixed positive time remains in a compact sublevel of $\mathcal{A}$. Its vector field is smooth with denominator at least one, so it exists for all $\sigma \geq \sigma_0$. Also $1 \leq r' \leq C$ and $dt/d\sigma = e^\sigma/r' \geq C^{-1}e^\sigma$, so $t \to \infty$. Smoothness of the compact core and fibre, and unbounded radial distance, give completeness.

The energy identity gives $\int_{\sigma_0}^\infty z^2\, d\sigma < \infty$. The derivatives $z_\sigma, z_{\sigma\sigma}$ are bounded on the compact shape set. Put $L = \max\{1, \sup|z_\sigma|\}$. If $|z(\sigma_i)| \geq \epsilon > 0$ for a sequence $\sigma_i \to \infty$, then $|z| \geq \epsilon/2$ on each interval of radius $\epsilon/(2L)$ about σ_i. Select a pairwise disjoint subsequence of these intervals. Each contributes at least $\epsilon^3/(4L)$ to $\int z^2\, d\sigma$, contradicting its finiteness. Hence $z \to 0$.

Put $K_0 = \max\{1, \sup|z_{\sigma\sigma}|\}$. If $|z_\sigma(\sigma_i)| \geq \epsilon > 0$, then on $[\sigma_i, \sigma_i + \epsilon/(2K_0)]$ the derivative has constant sign and absolute value at least $\epsilon/2$. Consequently

$$\left|z\big(\sigma_i + \epsilon/(2K_0)\big) - z(\sigma_i)\right| \geq \epsilon^2/(4K_0),$$

contrary to $z \to 0$. Therefore $z_\sigma \to 0$. The shape equation now implies $e^{-2qx/d} - e^{2px/d} \to 0$. This function is strictly decreasing in x and vanishes only at zero, so $x \to 0$. Consequently $r' \to 1$, $r/t \to 1$, and

$$a' = a_0 e^{qx/d}(r' + qz/d), \qquad b' = b_0 e^{-px/d}(r' - pz/d)$$

give the stated limits. □

Lemma 9.3. *Suppose* $4 \leq d \leq 8$. *For the normalized Ricci-flat cap, there is a vector $c \neq 0$ such that*

$$U_{\mathrm{RF}}(r) = r^{A_0}c + O(r^{-2\alpha}), \qquad A_0 = \begin{pmatrix} 0 & 1 \\ -2(d-1) & -(d-1) \end{pmatrix}, \tag{9.9}$$

where α, β are given by (1.8). *The same remainder estimate holds after every fixed number of $r\partial_r$ derivatives.*

Proof. Linearization of the autonomous Ricci-flat shape equation gives $U_\sigma = A_0U + N_0(U)$ with $N_0(U) = O(|U|^2)$ and all derivatives smooth near zero. The eigenvalues of A_0 are $-\alpha \pm i\beta$. Choose a real invertible matrix converting its linear part to $-\alpha I + \beta\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. In the resulting polar coordinates (ϱ, θ),

$$\varrho_\sigma = -\alpha\varrho + O(\varrho^2), \qquad \theta_\sigma = \beta + O(\varrho). \tag{9.10}$$

The trajectory is not the zero trajectory, because one factor collapses at its smooth initial orbit. Uniqueness prevents ϱ from vanishing at finite σ. For large σ_0, $\varrho(\sigma) \le \varrho(\sigma_0)e^{-\alpha(\sigma-\sigma_0)/2}$. Therefore $\int_{\sigma_0}^{\infty} \varrho < \infty$. Write the error in ϱ_σ/ϱ as e_1 and that in θ_σ as e_2, where $|e_i| \le C\varrho$. Integration gives

$$e^{\alpha\sigma}\varrho(\sigma) = e^{\alpha\sigma_0}\varrho(\sigma_0)\exp\left(\int_{\sigma_0}^{\sigma} e_1\right), \tag{9.11}$$

$$\theta(\sigma) - \beta\sigma = \theta(\sigma_0) - \beta\sigma_0 + \int_{\sigma_0}^{\sigma} e_2. \tag{9.12}$$

The amplitude has a finite strictly positive limit and the phase has a finite limit. These formulas first give $\varrho = O(e^{-\alpha\sigma})$; the tails of both error integrals are then $O(e^{-\alpha\sigma})$. Replacing amplitude and phase by their limits changes the vector by $O(e^{-2\alpha\sigma})$. Returning to U proves (9.9), with $c \neq 0$. For the remainder $V(\sigma) = U(\sigma) - e^{A_0\sigma}c$,

$$V' = A_0 V + N_0(U), \qquad V = O(e^{-2\alpha\sigma}), \quad N_0(U) = O(e^{-2\alpha\sigma}).$$

The equation for U first gives $\partial_\sigma^j U = O(e^{-\alpha\sigma})$ for each fixed j, by induction. Since $N_0(0) = DN_0(0) = 0$, the chain rule then gives $\partial_\sigma^j N_0(U) = O(e^{-2\alpha\sigma})$. Differentiation of the equation for V proves $\partial_\sigma^j V = O(e^{-2\alpha\sigma})$ for each j. This reproduces the polar argument of [34, Lemma 3.3] in the present variables. □

10. Differentiation with respect to the cap radius

Fix $r_0 \in (0,1)$. In $0 < r \le r_0$, the positive-Einstein shape equation is

$$U_\sigma = A(r)U + N(r,U), \qquad A(r) = \begin{pmatrix} 0 & (1-r^2)^{-1/2} \\ -2(d-1)(1-r^2)^{-1/2} & -(d-1) \end{pmatrix}. \tag{10.1}$$

For U in a fixed neighborhood of zero,

$$|N(r,U)| \le C|U|^2, \qquad |D_U N(r,U)| \le C|U|. \tag{10.2}$$

The constants are uniform on $0 < r \le r_0$. In particular, there is no forcing term independent of U: the zero solution is the exact sine-cone shape.

Lemma 10.1. *The linear equation in* (10.1) *has a fundamental matrix*

$$\Phi(r) = r^{A_0}(I + O(r^2)), \qquad \|\Phi(r)\Phi(\rho)^{-1}\| \le C(r/\rho)^{-\alpha}, \qquad \|\Phi(r)^{-1}\| \le Cr^{\alpha} \tag{10.3}$$

for $0 < \rho \le r \le r_0$.

Proof. Choose a fixed real invertible matrix P such that $P^{-1}A_0P = -\alpha I + \beta J$, $J^2 = -I$ and $J^\top = -J$. It follows that

$$\|r^{A_0}\| \le Cr^{-\alpha}, \qquad \|r^{-A_0}\| \le Cr^{\alpha}.$$

Writing $\Phi = r^{A_0}Y$ gives

$$Y_\sigma = C(r)Y, \qquad C(r) = r^{-A_0}(A(r) - A_0)r^{A_0}, \qquad \|C(r)\| \le C_1 r^2.$$

Define Y by the Volterra equation

$$Y(r) = I + \int_0^r C(\rho)Y(\rho)\,\frac{d\rho}{\rho}.$$

Its successive integrals have norms at most $(C_1r^2/2)^j/j!$. They converge uniformly, and

$$\|Y(r)\| \le e^{C_1r^2/2}, \qquad \|Y(r) - I\| \le e^{C_1r^2/2} - 1 = O(r^2).$$

The inverse equation $Z_\sigma = -ZC(r)$, $Z(0) = I$, has the same construction. Differentiating ZY gives $ZY = I$. Thus Y^{-1} is uniformly bounded. Finally

$$\Phi(r)\Phi(\rho)^{-1} = r^{A_0}Y(r)Y(\rho)^{-1}\rho^{-A_0}, \qquad \Phi(r)^{-1} = Y(r)^{-1}r^{-A_0}.$$

The preceding estimates prove all assertions. □

Proposition 10.2. *Let U_s be the positive-Einstein cap with noncollapsing radius s. It reaches $r = r_0$ for all sufficiently small s. At that section,*

$$U_s(r_0) = \Phi(r_0)s^{-A_0}c + o(s^\alpha), \tag{10.4}$$

$$s\partial_s U_s(r_0) = -\Phi(r_0)s^{-A_0}A_0 c + o(s^\alpha), \tag{10.5}$$

where $c \neq 0$ is the normalized cap vector in (9.9). *Parameter derivatives here and below are taken at fixed r.*

Proof. Choose a large fixed R. The section $r = sR$ corresponds after rescaling to volume radius R. The Ricci-flat section is transverse, since its r' is positive. Corollary 8.2 and the implicit function theorem for that section give

$$U_s(sR) = U_{\mathrm{RF}}(R) + O_R(s^2) \tag{10.6}$$

including a derivative in $\log s$ along the moving section. For R large, Lemma 9.3 makes these data small.

The integral equation and Lemma 10.1 give, as long as the solution stays in the fixed neighborhood,

$$\begin{aligned} |U_s(r)| &\le C(r/(sR))^{-\alpha}|U_s(sR)| \\ &\quad + C\int_{sR}^{r}(r/\rho)^{-\alpha}|U_s(\rho)|^2\frac{d\rho}{\rho}. \end{aligned} \tag{10.7}$$

Use the norm $M_s(r) = \sup_{sR\le\rho\le r}(\rho/s)^\alpha|U_s(\rho)|$. For fixed R and small s, (10.6) bounds the first term of the resulting inequality by C_0, independently of R after increasing C_0. The integral term is at most $C_1R^{-\alpha}M_s(r)^2$. Choose R so that $4C_0C_1R^{-\alpha} < 1/2$ and $2C_0R^{-\alpha}$ lies strictly inside the chosen neighborhood of $U = 0$. If M_s first reached $2C_0$, then

$$2C_0 \le C_0 + 4C_1R^{-\alpha}C_0^2 < \tfrac{3}{2}C_0,$$

a contradiction. Thus $M_s \le 2C_0$ until r_0; this bound also precludes exit from the neighborhood. Consequently

$$|U_s(r)| \le C(s/r)^\alpha \qquad (sR \le r \le r_0). \tag{10.8}$$

The positive square root defining f remains bounded below by a positive constant depending on r_0; the section is therefore reached.

Let $W_s = s\partial_s U_s$ at fixed r. Differentiating the equation gives

$$(W_s)_\sigma = (A(r) + D_U N(r, U_s))W_s. \tag{10.9}$$

The derivative of (10.6) along $r = sR$ is

$$W_s(sR) = -(U_s)_\sigma(sR) + O_R(s^2). \tag{10.10}$$

Let $Y_s(r) = \Phi(r)^{-1}W_s(r)$. Its equation is

$$(Y_s)_\sigma = \Phi^{-1}D_U N(r, U_s)\Phi Y_s, \qquad \|\Phi^{-1}D_U N\Phi\| \le C|U_s|.$$

By (10.8),

$$\int_{sR}^{r_0}|U_s|\,\frac{dr}{r} \le CR^{-\alpha}, \qquad |Y_s(r)| \le |Y_s(sR)|e^{CR^{-\alpha}}.$$

Equation (10.10) gives $|Y_s(sR)| \le Cs^\alpha$. The bound $\|\Phi(r)\| \le Cr^{-\alpha}$ proves $|W_s(r)| \le C(s/r)^\alpha$.

For the original equation the weighted nonlinear error is

$$\int_{sR}^{r_0}\|\Phi(r)^{-1}\|\,|N(r, U_s(r))|\frac{dr}{r} \le Cs^{2\alpha}\int_{sR}^{r_0} r^{-\alpha}\frac{dr}{r} \le Cs^\alpha R^{-\alpha}. \tag{10.11}$$

The differentiated error, containing $D_U N\, W_s$, has the same bound. At the initial section, (9.9) and (10.6) give

$$\Phi(sR)^{-1}U_s(sR) = s^{-A_0}c + O(s^\alpha R^{-\alpha}) + O_R(s^{\alpha+2}). \tag{10.12}$$

Here $\Phi(sR)^{-1} = (I + O_R(s^2))(sR)^{-A_0}$ and matrix multiplication in the leading term uses $(sR)^{-A_0}R^{A_0} = s^{-A_0}$. The vector fields in the positive-Einstein and Ricci-flat equations at $r = sR$ differ by $O_R(s^2)$. Equations (10.10) and (9.9) therefore give

$$\Phi(sR)^{-1}W_s(sR) = -s^{-A_0}A_0c + O(s^\alpha R^{-\alpha}) + O_R(s^{\alpha+2}).$$

Let $E_{0,s}$ and $E_{1,s}$ be the differences between the two sides of (10.4) and (10.5). Variation of constants and (10.11) show that there are $R_1, C > 0$ such that, for every fixed $R \geq R_1$,

$$\limsup_{s\downarrow 0} s^{-\alpha}(|E_{0,s}| + |E_{1,s}|) \leq CR^{-\alpha}. \tag{10.13}$$

The constant C is independent of R; the admissible upper bound on s may depend on R. The left side of (10.13) is independent of R. Letting $R \to \infty$ proves both limits. □

Lemma 10.3. *Every positive-Einstein cap considered above reaches a unique first orbit with $H = 0$. Its hitting map is smooth in the initial cap parameter. Near the sine-cone, the map from the section $r = r_0$ to this minimal-orbit section is a local diffeomorphism.*

Proof. Write $A = a'/a$ and $B = b'/b$ on a fixed cap. The constraint gives

$$H' = -d - pA^2 - qB^2 = -d - \frac{H^2}{d} - \frac{pq}{d}(A - B)^2. \tag{10.14}$$

In particular, while $H \geq 0$,

$$\frac{d}{dt}\arctan(H/d) = -1 - \frac{pq(A - B)^2}{d^2 + H^2} \leq -1. \tag{10.15}$$

Fix $t_0 > 0$ before the first zero. On each interval where $A < 0$, (8.2) gives $A' \geq -d$; hence on $[t_0, T]$ with $H \geq 0$,

$$A(t) \geq \min\{A(t_0), 0\} - d(T - t_0).$$

Equation (8.3) gives $B(t) \geq \min\{B(t_0), 0\} - d(T - t_0)$ by the same integration on intervals where $B < 0$. Since $H(t) \leq H(t_0)$, $pA = H - qB$ and $qB = H - pA$ give upper bounds for both functions. Integration of $a'/a = A$, $b'/b = B$ bounds a, b above and away from zero. The regular ODE thus continues on every finite interval on which $H \geq 0$. By (10.15), H must reach zero in finite time. The limit of $\arctan(H/d)$ at zero is $\pi/2$. Near the collapsing orbit $A - B = t^{-1} + O(t)$, so the last integrand in (10.15) is strictly positive on an interval. Its hitting time T_s satisfies

$$T_s = \frac{\pi}{2} - \int_0^{T_s} \frac{pq(A - B)^2}{d^2 + H^2}\,dt < \frac{\pi}{2}.$$

At the hitting point $H' \leq -d < 0$. The implicit function theorem gives smooth dependence of T_s on the cap parameter, and strict decrease of H gives uniqueness of the zero.

For the local transition near the sine-cone put $\omega = r'$. The state (r, ω, x, y) lies on

$$\mathscr{C} = \{\omega^2 + r^2 - K(x) - \kappa r^2 y^2 = 0\}. \tag{10.16}$$

Its vector field is

$$r' = \omega, \quad \omega' = -r - \frac{pq}{d^2}ry^2, \quad x' = y, \quad y' = -d\frac{\omega}{r}y + (d - 1)r^{-2}(e^{-2qx/d} - e^{2px/d}).$$

At $(1, 0, 0, 0)$, the derivative of the defining function in r equals two. Thus $\mathscr{C}$ is a smooth three-manifold there. The initial section $r = r_0$, $\omega > 0$, is transverse, as is the final section $\omega = 0$, where $\omega' = -1$. The flow and its time-reversed flow define inverse smooth maps between neighborhoods in these two sections. Their coordinates are (x, y), proving the local diffeomorphism assertion. Write $\mathcal{P}_{xy}$ for this map. The initial state in Proposition 10.2 is instead $U = (x, z)$, where $z = r_0 y$ on the initial section. Consequently the map acting on that state is

$$J_{r_0}(x, z) = (x, z/r_0), \qquad \mathcal{P} = \mathcal{P}_{xy} \circ J_{r_0}, \qquad D\mathcal{P}(0) = D\mathcal{P}_{xy}(0)\begin{pmatrix} 1 & 0 \\ 0 & r_0^{-1} \end{pmatrix}. \tag{10.17}$$

Since $r_0 > 0$, J_{r_0} and $D\mathcal{P}(0)$ are invertible. □

11. The equatorial equation and gluing of caps

Let $\Gamma_-(s) = (x, y)$ be the data on the first minimal orbit of an a-collapsing cap. For a round fibre let $\Gamma_+(s)$ be the corresponding data for a b-collapsing cap. Both use the outward radial normal. The sine-cone corresponds to $(x, y) = (0, 0)$.

11.1. The Wronskian in the full dimensional range. Linearization on the sine-cone gives

$$u'' + d \cot t\, u' + 2(d-1)\csc^2 t\, u = 0, \tag{11.1}$$

where primes denote the geodesic derivative.

Lemma 11.1. *For every $4 \le d \le 8$, there are constants $\mu, \nu > 0$ and $\delta_d \in \mathbb{R}$, depending only on d, with*

$$\mu\nu - \delta_d^2 = \beta^2. \tag{11.2}$$

For each cap type there is $C \in \mathbb{C} \setminus \{0\}$ and a common pair $U, V \in \mathbb{C}$, independent of the cap parameter, such that

$$\Gamma(s) = \operatorname{Re}\left[C s^{\alpha - i\beta}\begin{pmatrix} U \\ V \end{pmatrix}\right] + o(s^\alpha), \tag{11.3}$$

$$s\Gamma'(s) = \operatorname{Re}\left[(\alpha - i\beta) C s^{\alpha - i\beta}\begin{pmatrix} U \\ V \end{pmatrix}\right] + o(s^\alpha). \tag{11.4}$$

In particular,

$$s\Gamma'(s) = \mathcal{B}_d \Gamma(s) + o(s^\alpha), \quad \mathcal{B}_d = \begin{pmatrix} \alpha + \delta_d & -\mu \\ \nu & \alpha - \delta_d \end{pmatrix}, \quad c_1 s^\alpha \le |\Gamma(s)| \le c_2 s^\alpha. \tag{11.5}$$

Proof. The indicial roots of (11.1) are $-\alpha \pm i\beta$. The normalized Frobenius solution is $u(t) = t^{-\alpha + i\beta}(1 + O(t^2))$, with differentiated remainders. To specify the analytic coefficients, write the equation as

$$t^2 u'' + P(t) t u' + Q(t) u = 0, \qquad P(t) = dt \cot t, \quad Q(t) = 2(d-1) t^2 \csc^2 t. \tag{11.6}$$

The functions P, Q extend as even holomorphic functions near zero. Substitution of $t^{-\alpha + i\beta} \sum_{j \ge 0} a_j t^{2j}$, $a_0 = 1$, gives the coefficient $4j(j + i\beta)$ for a_j, $j \ge 1$. Choose $r_1 > 0$ strictly smaller than a common disk of holomorphy of P and Q. Cauchy's coefficient estimate bounds the absolute value of each t^{2j} coefficient of P and Q by $C_0 r_1^{-2j}$. The recurrence therefore gives

$$|a_j| \le \frac{C_1}{j^2} \sum_{k=0}^{j-1} (k+1) r_1^{-2(j-k)} |a_k|.$$

For $0 < R < r_1$, set $b_j = |a_j| R^{2j}$ and $q = (R/r_1)^2$. If $b_0, \dots, b_{j-1} \le 1$, then $b_j \le C_1 j^{-1} \sum_{\ell=1}^{j} q^\ell \le C_1 q/(1-q)$. Choose R with $C_1 q/(1-q) < 1$. Induction gives $b_j \le 1$. Thus the Frobenius series converges absolutely for $|t| < R$ and termwise differentiation is valid on every smaller disk. The coefficients of the original differential equation are smooth on each compact subinterval of $(0, \pi)$, so its linear initial-value theorem continues this solution to $\pi/2$. Since the coefficients are real,

$$\frac{d}{dt}\left(\sin^d t\ \operatorname{Im}(\overline{u(t)} u'(t))\right) = 0.$$

Its limit at zero is β, since $d - 2\alpha - 1 = 0$. Define

$$U = u(\pi/2), \quad V = u'(\pi/2), \quad \mu = |U|^2, \quad \nu = |V|^2, \quad \delta_d = \operatorname{Re}(\overline{U} V). \tag{11.7}$$

It follows that

$$\operatorname{Im}(\overline{U} V) = \beta > 0. \tag{11.8}$$

Thus $U, V \neq 0$ and (11.2) holds. The real and imaginary parts of u form a real fundamental pair. We identify its normalization with the transfer in Proposition 10.2. Put

$$s_+ = -\alpha + i\beta, \qquad e_+ = (1, s_+)^\top, \qquad C = c_x - i\frac{c_z + \alpha c_x}{\beta}, \quad c = (c_x, c_z)^\top. \tag{11.9}$$

Then $A_0 e_+ = s_+ e_+$, $c = \mathrm{Re}(Ce_+)$, and

$$s^{-A_0} c = \mathrm{Re}\big(Cs^{\alpha - i\beta} e_+\big).$$

The real-linear map $c \mapsto C$ is injective; hence $C \neq 0$.

For $r = \sin t$ with $0 < t < \pi/2$, define

$$\psi(r) = \big(u(\arcsin r),\ ru'(\arcsin r)\big)^\top. \tag{11.10}$$

Equation (11.1) gives $\psi_\sigma = A(r)\psi$. The Frobenius expansion implies $\psi(r) = r^{s_+}(e_+ + O(r^2))$. Since $\Phi(r) = r^{A_0}(I + O(r^2))$, $\Phi(r)^{-1}\psi(r) \to e_+$. This vector is constant, because both Φ and ψ solve the same linear equation. Thus $\psi(r) = \Phi(r)e_+$.

Let $\mathcal{P}$ be the hitting map in (10.17). At the sine-cone, the first variation of $K(x)$ vanishes and the terms involving y in the radial equation and constraint are quadratic. The first variations of r and ω therefore vanish for a pure shape variation starting at $r = r_0$. The first variation of the hitting time is zero as well, since $\omega' = -1$ at the final section. The remaining linear equation is (11.1). Consequently

$$D\mathcal{P}(0)\,\Phi(r_0)e_+ = (U, V)^\top. \tag{11.11}$$

By smoothness, write $\mathcal{P}(Y) = D\mathcal{P}(0)Y + \mathcal{R}(Y)$, with $|\mathcal{R}(Y)| \leq C_0|Y|^2$ and $|D\mathcal{R}(Y)| \leq C_0|Y|$ near zero. In Proposition 10.2, both $U_s(r_0)$ and $s\partial_s U_s(r_0)$ are $O(s^\alpha)$. The remainder and its logarithmic derivative after composition with $\mathcal{P}$ are therefore $O(s^{2\alpha})$. Substituting (11.9) and (11.11) proves (11.3)–(11.4).

Set $w = Cs^{\alpha - i\beta}U$. Since $V/U = (\delta_d + i\beta)/\mu$, the leading real state satisfies

$$x = \mathrm{Re}\, w, \qquad y = \frac{\delta_d \,\mathrm{Re}\, w - \beta \,\mathrm{Im}\, w}{\mu}.$$

Differentiation in $\log s$ gives the matrix in (11.5). The real map from $(\mathrm{Re}\, w, \mathrm{Im}\, w)$ to (x, y) is invertible and $|w| = |CU|s^\alpha$, which proves both norm bounds. □

A continuous angle lift of Γ satisfies

$$\frac{d\theta}{d\log s} = \frac{\nu x^2 - 2\delta_d xy + \mu y^2}{x^2 + y^2} + o(1). \tag{11.12}$$

The numerator is positive definite by (11.2). Thus its angular velocity is bounded above and below by positive constants for sufficiently small s, and $\theta(s) \to -\infty$ as $s \downarrow 0$. Normal reversal is

$$\mathsf{S}(x, y) = (x, -y). \tag{11.13}$$

It reverses angular orientation, but need not preserve an adapted rotation norm when $\delta_d \neq 0$.

Lemma 11.2 (Opposite-winding intersection). *Let two nonvanishing plane curves have angle parametrizations $\theta \in (-\infty, a]$ and $\psi \in [b, \infty)$, with positive continuous radii $R_1(\theta), R_2(\psi)$. Suppose $R_1(\theta) \to 0$ as $\theta \to -\infty$, and $R_2(\psi) \to 0$ as $\psi \to \infty$. Then they have common points tending to zero, with both parameters tending to their respective infinite endpoints.*

Proof. For sufficiently large integers N, define

$$F_N(\theta) = R_1(\theta) - R_2(\theta + 2\pi N), \qquad b - 2\pi N \leq \theta \leq a.$$

At the left endpoint $F_N < 0$, and at the right endpoint $F_N > 0$. Choose a zero θ_N and put $\psi_N = \theta_N + 2\pi N$. Equal angles modulo 2π and equal radii give a common point. If a subsequence of θ_N were bounded below, positivity of R_1 on a compact interval would give a positive lower bound, while $R_2(\psi_N) \to 0$. Thus $\theta_N \to -\infty$ and the common radius tends to zero. A bounded subsequence of ψ_N would then contradict positivity of R_2 on a compact interval. Hence $\psi_N \to \infty$. □

Proposition 11.3 (General small matchings). *For a round fibre there are $s_j, t_j \downarrow 0$ such that*

$$\Gamma_-(s_j) = \mathsf{S}\Gamma_+(t_j) = Z_j \neq 0, \qquad |Z_j| \asymp s_j^\alpha \asymp t_j^\alpha. \tag{11.14}$$

They give smooth Einstein metrics on S^{d+1}. For every (Q, h_Q) in Theorem B, there are $s_j \downarrow 0$ with $\Gamma_-(s_j) = (x_j, 0)$, $x_j \neq 0$. Even reflection gives a smooth Einstein metric on $S^{p+1} \times Q$.

Proof. The two outward cap curves satisfy (11.12) and have radii tending to zero. The reflected second curve has the opposite orientation. Apply Lemma 11.2; a subsequence makes both scales strictly decreasing. The norm bounds in (11.5) give their comparison. At a minimal orbit,

$$r^2 = \frac{K(x)}{1 - \kappa y^2}, \qquad A = \frac{q}{d} y, \qquad B = -\frac{p}{d} y. \tag{11.15}$$

Equality after reversal therefore gives equal induced metrics and opposite second fundamental forms. In a signed normal coordinate, (a, b, a', b') agree, and uniqueness of the regular Einstein ODE identifies all derivatives across the seam. The product boundary identification gives

$$(D^{p+1} \times S^q) \cup_{S^p \times S^q} (S^p \times D^{q+1}) \cong S^{p+q+1}.$$

For a general fibre, only the a-collapsing cap is used. Its angle crosses infinitely many values in $\pi\mathbb{Z}$. At each crossing $y = 0$, $x \neq 0$, and (11.15) gives $A = B = 0$. Even reflection satisfies the same ODE and is smooth across the boundary. The identity double of $D^{p+1} \times Q$ is $S^{p+1} \times Q$. □

11.2. **The even-dimensional transfer and its exact constants.**

Lemma 11.4. *For $d = 5, 7$, there are nonzero vectors v_-, v_+ and positive numbers μ, ν, with $\mu\nu = \beta^2$, such that*

$$\Gamma_\pm(s) = e^{B_d \log s} v_\pm + o(s^\alpha), \tag{11.16}$$

$$s\Gamma_\pm'(s) = B_d \Gamma_\pm(s) + o(s^\alpha), \tag{11.17}$$

$$B_d = \begin{pmatrix} \alpha & -\mu \\ \nu & \alpha \end{pmatrix}. \tag{11.18}$$

More precisely,

$$B_5 = \begin{pmatrix} 2 & -5/8 \\ 32/5 & 2 \end{pmatrix}, \qquad B_7 = \begin{pmatrix} 3 & -7/16 \\ 48/7 & 3 \end{pmatrix}. \tag{11.19}$$

Proof. Use (11.1) on $r = \sin t$. Set $\tau = \log\tan(t/2)$ and $u = (\cosh \tau)^\alpha f$. Since $d\tau/dt = 1/\sin t$ and $\sin t = \operatorname{sech} \tau$, differentiation gives

$$f'' + (\beta^2 + \alpha(\alpha + 1)\operatorname{sech}^2 \tau) f = 0. \tag{11.20}$$

The derivatives in (11.20) are with respect to τ. For an integer $\ell \geq 1$,

$$\left[\partial_\tau^2 + \beta^2 + \ell(\ell+1)\operatorname{sech}^2 \tau\right](\partial_\tau - \ell \tanh \tau) = (\partial_\tau - \ell \tanh \tau)\left[\partial_\tau^2 + \beta^2 + \ell(\ell-1)\operatorname{sech}^2 \tau\right]. \tag{11.21}$$

This identity follows by distributing the derivatives and using $(\tanh \tau)' = \operatorname{sech}^2 \tau$. Starting with $f_0 = e^{i\beta\tau}$, define

$$f_\ell = \frac{(\partial_\tau - \ell \tanh \tau) f_{\ell-1}}{i\beta + \ell}. \tag{11.22}$$

Then f_α solves (11.20) and is asymptotic to $e^{i\beta\tau}$ at $-\infty$. Let $U_\ell = f_\ell(0)$ and $V_\ell = f_\ell'(0)$. Equation (11.22) and the preceding differential equation give

$$U_\ell = \frac{V_{\ell-1}}{i\beta + \ell}, \qquad V_\ell = (i\beta - \ell) U_{\ell-1}, \qquad U_0 = 1, \quad V_0 = i\beta. \tag{11.23}$$

The recurrence implies

$$V_\ell \overline{U_\ell} = \frac{i\beta - \ell}{-i\beta + \ell} U_{\ell-1} \overline{V_{\ell-1}} = -\overline{V_{\ell-1} \overline{U_{\ell-1}}}.$$

Since $V_0 \overline{U}_0 = i\beta$, induction gives $V_\ell \overline{U_\ell} = i\beta$ for all ℓ. Define

$$\mu = |U_\alpha|^2, \qquad \nu = |V_\alpha|^2. \tag{11.24}$$

Since $V_\alpha \overline{U_\alpha} = i\beta$ and $\beta > 0$, both numbers are positive and $\mu\nu = |V_\alpha \overline{U_\alpha}|^2 = \beta^2$. For the required two values,

$$|U_2|^2 = \frac{\beta^2 + 1}{\beta^2 + 4}, \qquad |U_3|^2 = \frac{\beta^2(\beta^2 + 4)}{(\beta^2 + 1)(\beta^2 + 9)}.$$

Thus $\beta^2 = 4$, $\alpha = 2$ gives $\mu = 5/8$, $\nu = 32/5$; $\beta^2 = 3$, $\alpha = 3$ gives $\mu = 7/16$, $\nu = 48/7$.

At the equator the state of the complex solution of (11.1) is (U_α, V_α), since $d\tau/dt = 1$ and $(\cosh\tau)^\alpha$ has value one and derivative zero there. A real linearized cap state is the real part of $Cs^{\alpha - i\beta}(U_\alpha, V_\alpha)$ for some $C \neq 0$. Multiplication by $\alpha - i\beta$ acts on its real state by (11.18): the identity $V_\alpha / U_\alpha = i\beta/\mu$ gives the first row, and $\mu\nu = \beta^2$ gives the second. Apply $\mathcal{P} = \mathcal{P}_{xy} \circ J_{r_0}$ from (10.17) to the (x, z)-state in Proposition 10.2. Then $\mathcal{P}(U) = D\mathcal{P}(0)U + \mathcal{R}(U)$, with $|\mathcal{R}(U)| \le C|U|^2$ and $|D\mathcal{R}(U)| \le C|U|$. Since $|U_s(r_0)| + |s\partial_s U_s(r_0)| \le Cs^\alpha$,

$$|\mathcal{R}(U_s(r_0))| + |s\partial_s \mathcal{R}(U_s(r_0))| \le Cs^{2\alpha}.$$

Its derivative $D\mathcal{P}(0)$ is the linear transfer just computed. Thus the errors are $o(s^\alpha)$ as asserted. Invertibility of the transfer and $c \neq 0$ imply $v_\pm \neq 0$. $\square$

Proposition 11.5. *For every pair in* (1.12)*, there are constants $S_-, S_+ > 0$ and a sequence of smooth gluings producing spheres with cap radii*

$$s_j = S_- e^{-\pi j/\beta}(1 + o(1)), \qquad t_j = S_+ e^{-\pi j/\beta}(1 + o(1)). \tag{11.25}$$

Their minimal-orbit data satisfy

$$\mathcal{R}(\Gamma_-(s_j)) = R_* e^{-\pi\alpha j/\beta}(1 + o(1)), \quad R_* > 0, \qquad \mathcal{R}(x, y)^2 = \nu x^2 + \mu y^2. \tag{11.26}$$

Proof. In the coordinates $(X, Y) = (\sqrt{\nu}\, x, \sqrt{\mu}\, y)$, the matrix B_d is $\alpha I + \beta\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. Choose continuous real lifts of the angles for small positive cap radius. Equations (11.16)–(11.17) imply

$$\mathcal{R}_\pm(s) = c_\pm s^\alpha(1 + o(1)), \qquad \frac{d\log\mathcal{R}_\pm}{d\log s} = \alpha + o(1), \tag{11.27}$$

$$\theta_\pm(s) = \beta\log s + \theta_\pm^0 + o(1), \qquad \frac{d\theta_\pm}{d\log s} = \beta + o(1), \tag{11.28}$$

where $c_\pm > 0$. In particular $\mathcal{R}_\pm$ are strictly increasing near zero.

Normal reversal is $\mathsf{S}(x, y) = (x, -y)$. A gluing of the two caps requires $\Gamma_-(s) = \mathsf{S}\Gamma_+(t)$. The ranges of the two strictly increasing radius functions both contain an interval $(0, \epsilon)$. Equality of radii therefore has, for every sufficiently small s, a unique solution $t = t(s)$. Differentiating that equality gives

$$\frac{d\log t}{d\log s} = \frac{d\log\mathcal{R}_-(s)/d\log s}{d\log\mathcal{R}_+(t)/d\log t} = 1 + o(1).$$

Consequently

$$t(s) = \left(\frac{c_-}{c_+}\right)^{1/\alpha} s(1 + o(1)), \qquad \frac{d\log t}{d\log s} = 1 + o(1). \tag{11.29}$$

The remaining equation is $\theta_-(s) + \theta_+(t(s)) \in 2\pi\mathbb{Z}$. Its left side equals $2\beta\log s + \theta_* + o(1)$ and has derivative $2\beta + o(1) > 0$ in $\log s$. It tends to $-\infty$ as $s \downarrow 0$. For every sufficiently large integer j, that function therefore has exactly one preimage of $-2\pi j$. Its equation gives

$$\log s_j = -\frac{\pi j}{\beta} - \frac{\theta_*}{2\beta} + o(1), \qquad s_j = e^{-\theta_*/(2\beta)} e^{-\pi j/\beta}(1 + o(1)).$$

Equation (11.29) gives the corresponding positive constant S_+ for t_j. Substituting in (11.27) gives $R_* = c_- S_-^\alpha > 0$ and (11.26).

At a minimal orbit the constraint gives

$$r^2 = \frac{K(x)}{1 - \kappa y^2}, \qquad A = \frac{q}{d} y, \quad B = -\frac{p}{d} y. \tag{11.30}$$

For small data, $1 - \kappa y^2 > 0$. Hence equality of (x, y) after normal reversal identifies the induced metrics and opposite second fundamental forms. Use a signed normal coordinate across the common orbit. Equality of the induced metric gives the same a, b; reversal of the outward normal gives the same signed derivatives a', b'. The regular first-order Einstein system for (a, b, a', b') has a smooth vector field because $a, b > 0$ at that orbit. Uniqueness identifies both extensions on a common collar, and therefore all normal derivatives agree. The glued metric is smooth and Einstein. The two cores are $D^{p+1} \times S^q$ and $S^p \times D^{q+1}$, glued by their product boundary identification. This is the boundary of $D^{p+1} \times D^{q+1}$ with its corner rounded, which is diffeomorphic to S^{d+1}. □

12. The cap volume and its leading coefficient

Let $\Omega = \omega_p \operatorname{Vol}(Q, h_Q)$ and let V_∞ be as in (1.9). The orbit volume is $\Omega v_0 r^d$. Let $V_-(s)$ denote the volume of an a-collapsing cap up to the first minimal orbit. The notation $V_+(s)$ is used for the second cap type when the fibre is round; then $V_\infty = V_{p,q}$.

Lemma 12.1. *For each available cap type,*

$$\lim_{s \downarrow 0} V_\pm(s) = \frac{V_\infty}{2}. \tag{12.1}$$

Proof. Write $r_{\max}$ for the radius at the minimal orbit. The energy is nonnegative. From (9.7), $1 \le r_{\max}^2 = 1 + 2\kappa \mathcal{A}_{\max}$. At a fixed large rescaled section $r = sR$, Lemma 9.3 and Corollary 8.2 give $\limsup_{s \to 0} \mathcal{A}(sR) \le C R^{-2\alpha}$. Since $\mathcal{A}$ decreases until the minimal orbit, $\limsup_{s \to 0} r_{\max}^2 \le 1 + C R^{-2\alpha}$. Let $R \to \infty$ to obtain $r_{\max} \to 1$.

The inequality $r'' \le -r$ implies, on the increasing branch,

$$r'^2 \ge r_{\max}^2 - r^2. \tag{12.2}$$

Indeed, for $t \le T_s$, where $r'(T_s) = 0$,

$$r'(t)^2 + r(t)^2 - r_{\max}^2 = -\int_t^{T_s} 2r'(s)(r''(s) + r(s))\, ds \ge 0,$$

since $r' \ge 0$ and $r'' + r \le 0$ on this interval. Consequently

$$V_\pm(s) = \Omega v_0 \int_0^{r_{\max}} \frac{r^d}{r'}\, dr \le \Omega v_0 r_{\max}^d \int_0^1 \frac{u^d}{\sqrt{1 - u^2}}\, du. \tag{12.3}$$

The improper integral at the maximum is finite; near the core the original smooth volume is finite. For a lower bound fix R and $\delta > 0$. Between $r = sR$ and $r = 1 - \delta$, $r'^2 \le 1 - r^2 + 2\kappa \mathcal{A}(sR)$. Hence

$$V_\pm(s) \ge \Omega v_0 \int_{sR}^{1-\delta} \frac{r^d}{\sqrt{1 - r^2 + 2\kappa \mathcal{A}(sR)}}\, dr. \tag{12.4}$$

Take $s \to 0$, then $R \to \infty$, then $\delta \downarrow 0$. Combine with (12.3) and use $u = \sin t$. This gives (12.1). □

Lemma 12.2. *For the constants of Lemma 11.1, put $C_0 = \Omega pq/d^2$ and*

$$Q(x, y) = \frac{v_0}{4\alpha}(-\nu x^2 + 2(\alpha + \delta_d)xy - \mu y^2). \tag{12.5}$$

Then

$$V_\pm(s) - \frac{V_\infty}{2} = -C_0 Q(\Gamma_\pm(s)) + o(s^{2\alpha}). \tag{12.6}$$

Proof. At a minimal orbit, (11.30) gives

$$v(x, y) = a^p b^q = v_0 \left(\frac{K(x)}{1 - \kappa y^2} \right)^{d/2} = v_0 + O(x^2 + y^2). \tag{12.7}$$

For a cap variation, the normal derivative of the boundary has been incorporated in the family pullback of Lemma 2.1. At the minimal orbit,

$$\begin{aligned} \langle \mathrm{II}, h^\top \rangle &= 2pA \, d \log a + 2qB \, d \log b \\ &= \frac{2pq}{d} y \, dx. \end{aligned} \tag{12.8}$$

The $d \log r$ terms cancel because $H = 0$. Equation (2.2), with $\Lambda = d$, becomes

$$dV_\pm = -C_0 v(x, y) y \, dx. \tag{12.9}$$

Differentiating (12.5) gives

$$DQ(Z)[\mathcal{B}_d Z] = v_0 y (\mathcal{B}_d Z)_x. \tag{12.10}$$

Multiply the row $\frac{v_0}{2\alpha}(-\nu x + (\alpha + \delta_d) y, (\alpha + \delta_d) x - \mu y)$ by $((\alpha + \delta_d) x - \mu y, \nu x + (\alpha - \delta_d) y)$; the result is $v_0((\alpha + \delta_d) xy - \mu y^2)$. Let $\ell = \log s$. Using (11.5), (12.7) and (12.9),

$$\frac{d}{d\ell} (V_\pm(s) + C_0 Q(\Gamma_\pm(s))) = o(e^{2\alpha\ell}). \tag{12.11}$$

Indeed the error from the differentiated cap is $o(s^\alpha) O(s^\alpha)$ and the error from $v - v_0$ is $O(s^{4\alpha})$. The expression in parentheses tends to $V_\infty/2$ by Lemma 12.1. For each $\epsilon > 0$, the absolute derivative in (12.11) is bounded by $\epsilon e^{2\alpha\ell}$ for sufficiently negative ℓ. Integrating from $-\infty$ to $\log s$ bounds the error by $\epsilon s^{2\alpha}/(2\alpha)$. This proves (12.6). □

Proposition 12.3 (Volume of a matched pair). *For the sphere matchings and reflection doubles of Proposition 11.3, with minimal state $Z_j = (x_j, y_j)$,*

$$\mathrm{Vol}(g_j) - V_\infty = \frac{\Omega p q v_0}{2\alpha d^2} (\nu x_j^2 + \mu y_j^2) + o(|Z_j|^2) > 0 \tag{12.12}$$

for all sufficiently large j. For spheres, $\Omega = \omega_p \omega_q$, $V_\infty = V_{p,q}$; for reflection doubles, $y_j = 0$.

Proof. The two outward states at a sphere gluing are Z_j and $\mathsf{S} Z_j$. Their cap scales are comparable by (11.14), so the two remainders in (12.6) sum to $o(|Z_j|^2)$. The primitive in (12.5) satisfies

$$Q(Z) + Q(\mathsf{S} Z) = -\frac{v_0}{2\alpha} (\nu x^2 + \mu y^2). \tag{12.13}$$

The terms containing xy, including the term with δ_d, cancel. Adding the two cap formulas gives (12.12). For a reflection double the two volumes are equal and $\mathsf{S} Z_j = Z_j$, since $y_j = 0$, so the same identity applies. The diagonal coefficients are positive and $Z_j \neq 0$, $Z_j \to 0$. □

Proof of Theorem B. The smooth metrics are supplied by Proposition 11.3. Their positive volume excess tends to zero by Proposition 12.3. Choose a subsequence inductively so that each excess is less than one half its predecessor. The volumes are then pairwise distinct. For a closed n-manifold with $\mathrm{Ric}_g = dg$, the unit-volume metric $\widehat{g} = \mathrm{Vol}(g)^{-2/n} g$ has Einstein–Hilbert value

$$\mathcal{S}(\widehat{g}) = nd \ \mathrm{Vol}(g)^{2/n}.$$

The functional is locally constant on the full unit-volume Einstein moduli space [36, Theorem 3.6]; its local analytic structure is described in [28, 6]. Normalization is continuous and commutes with diffeomorphisms. Since $d > 0$, volume is locally constant on (1.7), and hence constant on each connected component. The selected metrics lie in different components. The same argument applies to their unit-volume representatives. No symmetry restriction is imposed on the moduli space. The dimensions $d = 4, 5, 6, 7, 8$ give the five spheres stated in the theorem. □

Proof of Theorem 1.1. In Böhm's notation [8, Sections 1–3], vanishing of the O'Neill tensor removes its terms from the two warping equations. Normalize the positive base Einstein constant to $q-1$; in particular, $q \geq 2$. The resulting equations are (8.2)–(8.4). The smooth-core conditions are $a(0) = 0$, $a'(0) = 1$, $b(0) = s$, $b'(0) = 0$; their geometric smoothness is the assertion in [8, Theorem 2.3].

Let P be the principal orbit with reference metric $g_S + g_Q$ and set $\Omega = \mathrm{Vol}(P, g_S + g_Q)$. The two summands have ranks p, q, so the volume density of $a^2 g_S + b^2 g_Q$ is $a^p b^q$ times the fixed reference density. The shape equations, differentiated transfer, Wronskian and minimal-boundary variation are unchanged with this value of Ω. The cap curve crosses $y = 0$ infinitely often. At a crossing $A = B = 0$; reflection gives a smooth Einstein metric on $\mathscr{E} \cup_{\partial\mathscr{E}} \mathscr{E}$. The computation in Proposition 12.3, with $y_j = 0$, gives positive excess tending to zero. Subsequence selection and local constancy, as in the proof of Theorem B, give the conclusion. No global product decomposition of P or $\mathscr{E}$ is assumed. □

Corollary 12.4 (The infinite families in Böhm's conjecture). *The connected-component assertion holds for the infinite families in* [8, Theorems 3.3, 3.4 and 3.6].

Proof. Theorem 1.1 covers the identity doubles of Theorem 3.3. For Theorem 3.4, the fixed positive Einstein base has dimension $q \geq 2$, the sphere parameter has $p \geq 2$, and $p + q \leq 8$; apply Theorem B. The sphere pairs in Theorem 3.6 are

$$(2,2), (2,3), (2,4), (3,3), (2,5), (3,4), (2,6), (3,5), (4,4),$$

which are precisely the unordered pairs $p, q \geq 2$ with $4 \leq p + q \leq 8$. Theorem B applies to each. □

Corollary 12.5 (Geometric rate for reflection doubles). *Order the reflection doubles by successive sufficiently small zeros of $y(s)$. There are constants $s_*, C_* > 0$ such that*

$$s_j = s_* e^{-\pi j/\beta}(1 + o(1)), \qquad \mathrm{Vol}(g_j) - V_\infty = C_* e^{-2\pi\alpha j/\beta}(1 + o(1)). \tag{12.14}$$

Their volumes are eventually strictly decreasing. This includes the disk-bundle doubles in Theorem 1.1.

Proof. Equations (11.3) and (11.4) give

$$s^{-\alpha} y(s) = \mathrm{Re}(CVe^{-i\beta \log s}) + o(1),$$

including one derivative in $\log s$. The leading sinusoid has simple zeros separated by π/β. On disjoint fixed small neighborhoods of those zeros its derivative has constant sign and a uniform positive lower bound in absolute value. Outside those neighborhoods its value is bounded away from zero. The C^1 error tends uniformly to zero for sufficiently negative $\log s$. Thus exactly one zero persists in each neighborhood, and no additional zero occurs. Its displacement tends to zero, giving $\log s_j = \log s_* - \pi j/\beta + o(1)$. At a leading zero of the second component, the first component is nonzero by (11.8). Its absolute leading coefficient is the same at consecutive zeros. Substituting in (12.12), with $y_j = 0$, proves the volume formula. The ratio of consecutive positive excesses tends to $e^{-2\pi\alpha/\beta} < 1$. □

Proof of Theorem 1.2. In this proof $Q = S^q$, $V_\infty = V_{p,q}$, and $\delta_d = 0$ by Lemma 11.4. At a gluing, $\Gamma_-(s_j) = Z_j$ and $\Gamma_+(t_j) = \mathsf{S}Z_j$. Therefore

$$Q(Z_j) + Q(\mathsf{S}Z_j) = -\frac{v_0}{2\alpha}(\nu x_j^2 + \mu y_j^2). \tag{12.15}$$

By Proposition 11.5, the two cap scales are comparable and the common radius is comparable to either scale to power α. Thus the two errors in (12.6) sum to $o(\mathcal{R}(Z_j)^2)$, and

$$\mathrm{Vol}(g_j) - V_{p,q} = \frac{C_0 v_0}{2\alpha}\mathcal{R}(Z_j)^2 + o(\mathcal{R}(Z_j)^2). \tag{12.16}$$

Now apply (11.26). The coefficient of $e^{-2\pi\alpha j/\beta}$ is $C_0 v_0 R_*^2/(2\alpha) > 0$. Set $\kappa_0 = 2\pi\alpha/\beta > 0$. The ratio of two consecutive volume excesses tends to $e^{-\kappa_0} < 1$. Hence there is j_0 such that

$$0 < \mathrm{Vol}(g_{j+1}) - V_{p,q} < \mathrm{Vol}(g_j) - V_{p,q} \qquad (j \geq j_0).$$

Replace g_j by g_{j+j_0}. In (1.13), replace $C_{p,q}$ by $C_{p,q}e^{-\kappa_0 j_0} > 0$. The asserted asymptotic is unchanged in form, and all volumes of the relabelled sequence are pairwise distinct.

For a metric g with $\mathrm{Ric}_g = dg$ put $\widehat{g} = \mathrm{Vol}(g)^{-2/n}g$. It has volume one, and its Einstein constant is $d\,\mathrm{Vol}(g)^{2/n}$. The normalized total scalar curvature of $\widehat{g}$ is therefore $nd\,\mathrm{Vol}(g)^{2/n}$. The local analytic Einstein premoduli theorem implies that this functional is locally constant on Einstein moduli; see [6, Corollary 12.52] and [36, Theorem 3.6]. All hypotheses apply: the manifold is closed and smooth and the quotient is by diffeomorphisms in the smooth topology. Thus volume is locally constant on (1.7). A locally constant real function is constant on each connected component. Distinct volumes consequently belong to distinct components.

The constants in (1.14) follow from

$$I_d := \int_0^{\pi} \sin^d t\,dt, \qquad I_1 = 2, \qquad I_d = \frac{d-1}{d} I_{d-2}, \tag{12.17}$$

obtained by integration by parts, and $\omega_j = 2\pi^{(j+1)/2}/\Gamma((j+1)/2)$. In particular $I_5 = 16/15$, $I_7 = 32/35$; insertion in (1.10) gives the three displayed values. □

Corollary 12.6. *Each of $\mathcal{E}_5(S^6)$ and $\mathcal{E}_7(S^8)$ has infinitely many connected components. For each sequence of Theorem 1.2 and fixed sufficiently small $\epsilon_0 > 0$, set $\delta_j = \mathrm{Vol}(g_j)/V_{p,q} - 1$. Then*

$$\#\{j : \epsilon \le \delta_j \le \epsilon_0\} = \frac{\beta}{2\pi\alpha}\log(1/\epsilon) + O(1) \qquad (\epsilon \downarrow 0). \tag{12.18}$$

The two sequences on S^8 can be restricted to tails belonging to disjoint sets of components.

Proof. Put $b = 2\pi\alpha/\beta > 0$. Equation (1.13) gives constants $A > 0$ and j_0 such that $|\log\delta_j + bj| \le A$ for $j \ge j_0$. Write $L = \log(1/\epsilon)$ and $L_0 = \log(1/\epsilon_0)$. Every integer $j \ge j_0$ satisfying $L_0 + A \le bj \le L - A$ is counted in (12.18); every counted integer satisfies $L_0 - A \le bj \le L + A$. The two intervals have lengths differing by $4A/b$. Counting integers changes either length by at most two. The omitted prefix has fixed finite length. Thus the count is $L/b + O(1)$, as asserted.

The ratio of the two limiting volumes on S^8 is $2\sqrt{2}/3 \ne 1$. Choose disjoint open intervals about those limits. Convergence places each sufficiently late sequence in its respective interval. Local constancy of volume on Einstein moduli precludes a connected component from containing metrics from both sequences. □

Remark 12.7. The count in (12.18) concerns the constructed sequence only. It is not an upper bound for the number of all Einstein components near the limiting volume. The argument neither identifies the full infinitesimal Einstein kernel nor proves isolation. The frequency β vanishes at $d = 9$; the proof does not give infinitely many components on S^{10}.

Part 3. Conical mass, entropy, and the topology of smooth fillings

13. Marked conical ends and the scalar flux

Throughout Part III, $n \ge 3$, M is connected, complete, oriented and smooth, with no boundary and exactly one end, and (1.16) holds in a fixed end identification. The link Z is closed and connected. A smooth positive function ρ on M is fixed with $\rho = r$ sufficiently far out. Different choices on a compact set give equivalent estimates below. Tensor norms and derivatives on the end in asymptotic statements use g_C unless a metric is specified.

Lemma 13.1. *The mass limit* (1.15) *exists and is finite.*

Proof. In cone covariant coordinates, let $A = \nabla^g - \nabla^C$. The exact connection-difference formula is

$$A^k{}_{ij} = \frac{1}{2} g^{k\ell}\big((\nabla_C)_i e_{j\ell} + (\nabla_C)_j e_{i\ell} - (\nabla_C)_\ell e_{ij}\big). \tag{13.1}$$

For vector fields X, Y, V, the curvature difference is

$$(R^g - R^C)(X,Y)V = (\nabla^C_X A)(Y,V) - (\nabla^C_Y A)(X,V) + A(X, A(Y,V)) - A(Y, A(X,V)).$$

Contract this identity with g^{-1}. The linear scalar-curvature term is (2.3) at the Ricci-flat metric g_C. Therefore

$$R_g = \mathrm{div}_C\, \beta_C(e) + \mathcal{Q}(e), \qquad |\mathcal{Q}(e)| \le C\big(|e||\nabla_C^2 e| + |\nabla_C e|^2 + r^{-2}|e|^2\big). \tag{13.2}$$

The last term accounts for quadratic contractions with the nonflat reference curvature. The bound is $O(r^{-2\tau-2})$, integrable because $2\tau > n-2$. The measures dv_g, dv_C are uniformly comparable on the end, so $R_g \in L^1(dv_C)$. Let $\mathfrak{f}(R) = \int_{\{r=R\}} \beta_C(e)(\partial_r)\, dA_C$. The divergence theorem gives

$$|\mathfrak{f}(R_2) - \mathfrak{f}(R_1)| \le \int_{r \ge R_1} |R_g|\, dv_C + C\int_{R_1}^{\infty} r^{n-3-2\tau}\, dr \longrightarrow 0 \quad (R_1 \to \infty),$$

uniformly for $R_2 \ge R_1$. Both tails tend to zero because $R_g \in L^1(dv_C)$ and $n - 3 - 2\tau < -1$. Thus $\mathfrak{f}(R)$ has a finite limit. $\square$

Definition 13.2. An admissible change of marking is a finite composition of an isometry of the exact cone preserving its radial coordinate and time-one maps of vector fields X satisfying

$$|\nabla_C^j X| = O(r^{1-\delta-j}) \quad (j \ge 0), \qquad \delta > \frac{n-2}{2}. \tag{13.3}$$

The vector fields may be extended arbitrarily over a fixed inner collar. Only the resulting end map is used.

Proposition 13.3. *The mass is unchanged by an admissible change of marking.*

Proof. An isometry of g_C preserving r preserves each flux integral. Let F_s be the local flow of a vector field satisfying (13.3). For large initial radius, $|d(r \circ F_s)/ds| \le Cr^{1-\delta}$, uniformly for $0 \le s \le 1$. Thus $r \circ F_s \asymp r$ on this interval, and the flow exists there outside a larger compact set. The differentiated flow equation gives the corresponding bounds on its spatial derivatives. Taylor's formula in integral form gives

$$F_1^* g_C = g_C + \mathcal{L}_X g_C + \int_0^1 (1-s) F_s^* \mathcal{L}_X^2 g_C\, ds,$$

$$F_1^* e = e + \int_0^1 F_s^* \mathcal{L}_X e\, ds.$$

Here $\mathcal{L}_X^2 g_C = O(r^{-2\delta})$ and $\mathcal{L}_X e = O(r^{-\tau-\delta})$, including one more derivative. Consequently

$$F_1^* g - g_C = e + \mathcal{L}_X g_C + O(r^{-2\delta}) + O(r^{-\tau-\delta}). \tag{13.4}$$

The fluxes of the last two terms are bounded by $CR^{n-2-2\delta} + CR^{n-2-\tau-\delta}$, which tends to zero.

For the linear term, commutation of covariant derivatives and $\mathrm{Ric}_C = 0$ give

$$\beta_C(\mathcal{L}_X g_C)_i = \nabla^j(\nabla_j X_i - \nabla_i X_j). \tag{13.5}$$

Put $A_{ji} = \nabla_j X_i - \nabla_i X_j$. On a closed hypersurface Σ, extend its unit normal geodesically and put $b(Y) = A(Y, \nu)$ for tangent Y. In a tangential orthonormal frame,

$$(\nabla^j A_{ji})\nu^i = \mathrm{div}_\Sigma(b^\sharp) - \sum_{a,b} \mathrm{II}_{ab} A(e_a, e_b) = \mathrm{div}_\Sigma(b^\sharp).$$

The normal–normal term is zero by antisymmetry; the displayed contraction with II is zero for the same reason. Integration over Σ_R gives zero. Thus each generating change preserves the limiting flux. After a change, the new decay rate is at least $\min\{\tau, \delta\}$, still greater than $(n-2)/2$. Applying the argument successively proves invariance under every finite composition in Definition 13.2. $\square$

Remark 13.4. No classification of all possible asymptotic coordinate changes is needed here. Theorem C concerns the specified marking, or its equivalence class under Definition 13.2. In particular it does not identify arbitrary AC charts without the required decay of their transition maps.

14. Homogeneous estimates and covering spaces

Put $q_* = 2n/(n-2)$. Let $\dot{H}^1(M)$ denote the completion of $C_c^\infty(M)$ in the norm $\|dv\|_2$.

Lemma 14.1. *There is $C < \infty$ such that*

$$\|\rho^{-1}v\|_2 + \|v\|_{q_*} \le C\|dv\|_2 \qquad (v \in C_c^\infty(M)). \tag{14.1}$$

If $\pi : \widehat{M} \to M$ is any Riemannian covering, finite or infinite, then

$$\|v\|_{L^{q_*}(\widehat{M})} \le C\|dv\|_{L^2(\widehat{M})} \qquad (v \in C_c^\infty(\widehat{M})). \tag{14.2}$$

The latter inequality also holds for spinors, with dv replaced by their covariant derivative.

Proof. On the exact cone end, integration of $(r^{n-2}v^2)'$ gives

$$(n-2)\int_R^\infty v^2 r^{n-3}\,dr = -R^{n-2}v(R)^2 - 2\int_R^\infty v v_r r^{n-2}\,dr. \tag{14.3}$$

Discard the nonpositive boundary term and apply Cauchy–Schwarz to get $\int_R^\infty v^2 r^{n-3} \le 4(n-2)^{-2}\int_R^\infty v_r^2 r^{n-1}$. Also, since v is compactly supported,

$$|v(R,z)|^2 \le \frac{R^{2-n}}{n-2}\int_R^\infty |v_r(r,z)|^2 r^{n-1}\,dr. \tag{14.4}$$

Metric and measure comparison transfer both inequalities to a sufficiently distant end of g. On its compact complement K, the inequality

$$\|v\|_{L^2(K)}^2 \le C\big(\|dv\|_{L^2(K)}^2 + \|v\|_{L^2(\partial K)}^2\big) \tag{14.5}$$

holds. Choose K connected. If (14.5) failed, there would be v_j with $\|v_j\|_{L^2(K)} = 1$ and $\|dv_j\|_{L^2(K)} + \|v_j\|_{L^2(\partial K)} \to 0$. A subsequence converges strongly in $L^2(K)$ and weakly in $H^1(K)$ to v. Its weak derivative is zero. Connectedness makes v constant. Continuity of the trace map gives zero trace for v, so $v = 0$, contrary to $\|v\|_{L^2(K)} = 1$. The trace in (14.5) is controlled by (14.4). This proves the Hardy assertion globally.

On every dyadic annulus $A_R = \{R < r < 2R\}$, rescaling by R^{-2} gives a family of metrics uniformly equivalent, with uniform local derivatives, to a metric on the fixed annulus of the compact cone link. Finitely many coordinate charts and the Euclidean Sobolev inequality give

$$\left(\int_{A_R} |v|^{q_*}\right)^{2/q_*} \le C\int_{A_{R/2,4R}} \big(|dv|^2 + R^{-2}|v|^2\big). \tag{14.6}$$

Sum over a finite-overlap dyadic covering and use $(\sum b_j)^{2/q_*} \le \sum b_j^{2/q_*}$. The weighted term is bounded by the Hardy inequality. On K, the local Sobolev inequality and (14.5) give the same bound. This proves (14.1).

For a compactly supported function on a covering, define

$$V(x) = \left(\sum_{y\in\pi^{-1}(x)} |v(y)|^2\right)^{1/2}. \tag{14.7}$$

The sum is locally finite and has compact support downstairs: a compact subset of the cover meets only finitely many sheets above each sufficiently small relatively compact trivializing neighborhood. The function V is locally Lipschitz. Pointwise Cauchy–Schwarz gives $|dV|^2 \le \sum_{y\in\pi^{-1}(x)} |dv(y)|^2$, and $q_* \ge 2$ gives $\sum |v(y)|^{q_*} \le (\sum |v(y)|^2)^{q_*/2}$. Integrating proves $\|v\|_{q_*} \le \|V\|_{q_*}$ and $\|dV\|_2 \le \|dv\|_2$. To justify differentiation at zeros, in a trivializing neighborhood replace V by $(\sum |v(y)|^2 + \epsilon^2)^{1/2} - \epsilon$ and let $\epsilon \downarrow 0$. The gradient bound is uniform. A finite partition of unity and convolution, as in (5.10), approximate the compactly supported V in H^1. Apply (14.1) and pass to the limit to obtain (14.2). For a spinor ψ, differentiating $(|\psi|^2 + \epsilon^2)^{1/2} - \epsilon$ gives gradient norm at most $|\nabla\psi|$ by Cauchy–Schwarz. Apply the scalar inequality and let $\epsilon \downarrow 0$. ☐

Lemma 14.2. *Let the geometry satisfy* (1.16), *and use* (1.22). *Then* Γ *is finite. The preimage of a sufficiently remote end in* $\widetilde{M}$ *has* $[G : i_*(\Gamma)]$ *components. Each has link* $\widehat{Z} = \widetilde{Z}/K$*, covering degree* d_Γ*, and the pulled-back asymptotic bounds. Under* (SC)*, the selected component has the specified compatible parallel-spinor cone.*

Proof. The complete lift of h_Z to its universal cover has dimension $n-1$ and Ricci tensor $(n-2)h_Z > 0$. Bonnet–Myers makes that cover compact, so $|\Gamma| < \infty$. Fix a basepoint in the end and identify the fibre of the universal cover of M with G. Lifts of loops from Z act on that fibre by right multiplication by $i_*(\Gamma)$. The orbits are the left cosets $g\, i_*(\Gamma)$. Each orbit is one connected component of the lifted end, and the subgroup of Γ defining its covering is $K = \ker i_*$. Its link is consequently $\widetilde{Z}/K$, and the covering has $d_\Gamma = |\Gamma/K|$ sheets. Local isometry transfers the asymptotic bounds unchanged. In particular each selected collar and each compact radial annulus in it is compact, even if $[G : i_*(\Gamma)]$ is infinite.

If $\widetilde{M}$ is spin, its spin structure is unique because $H^1(\widetilde{M}; \mathbb{Z}/2) = 0$. The induced spin structure on the selected end need not be the only one there. Condition (SC) specifies exactly that structure. Under (1.23), $K = 1$ and the lifted end is simply connected. Its spin structure is unique, so the spin structure induced from $\widetilde{M}$ is the required one. No uniqueness on $C(\widehat{Z})$ is assumed when $K \neq 1$. □

15. The spinorial mass identity

Complex spinor bundles carry their Hermitian metric, whose real part is used in quadratic identities. Clifford multiplication is denoted by a dot and satisfies

$$X \cdot Y + Y \cdot X = -2\langle X, Y\rangle, \qquad \mathrm{Re}\langle X \cdot Y \cdot \psi, \psi\rangle = -\langle X, Y\rangle |\psi|^2. \tag{15.1}$$

The Dirac operator is denoted by $\mathscr{D}$ to distinguish it from the radial shape difference used in Parts I and II. We use

$$\mathscr{D}^2 = \nabla^*\nabla + R/4. \tag{15.2}$$

The construction of the spinor bundle and (15.2) are as in [31]. The comparison of spinors under a change of metric is the fibre-isometric comparison of [10].

Lemma 15.1. *On an end with reference metric* $g_0 = g_C$ *and a unit parallel spinor* ψ_C*, let* ψ_0 *be its image under the fibre-isometric spinor comparison with* $g = g_C + e$*. Then*

$$|\psi_0|_g = 1, \qquad |\nabla^g \psi_0| = O(|\nabla_C e|), \qquad \mathscr{D}_g \psi_0 = -\frac{1}{4}\beta_C(e)^\sharp \cdot \psi_0 + O(|e||\nabla_C e|). \tag{15.3}$$

The difference between the corresponding Dirac boundary integral and one fourth of the mass flux tends to zero.

Proof. For $g_s = g_0 + sh$, choose the positive symmetric map $B_s = (\mathrm{Id} + sh^\sharp)^{-1/2}$ sending a g_0-orthonormal frame to a g_s-orthonormal frame, and its spin lift. In a g_0-normal frame at a point, differentiating the Levi-Civita connection and the frame gives the spin-connection coefficient $\frac{1}{8}(\nabla_j h_{ik} - \nabla_k h_{ij}) e_j \cdot e_k$. The variation of the directional derivative contributes $-\frac{1}{2}\nabla_{h^\sharp e_i}$. Clifford contraction therefore gives

$$(\mathscr{D})'(h)\psi = -\frac{1}{2}\sum_{i,j} h_{ij} e_i \cdot \nabla_{e_j}\psi + \frac{1}{4}(d\,\mathrm{tr}\, h - \mathrm{div}\, h)^\sharp \cdot \psi. \tag{15.4}$$

The Clifford reduction can be written without a convention-dependent cancellation. For an orthonormal frame,

$$e_i \cdot e_j \cdot e_k + e_k \cdot e_j \cdot e_i = 2\delta_{ik} e_j - 2\delta_{ij} e_k - 2\delta_{jk} e_i.$$

Using $h_{ik} = h_{ki}$, contraction of the spin-connection variation is

$$\frac{1}{8}\sum_{i,j,k}(\nabla_j h_{ik} - \nabla_k h_{ij}) e_i \cdot e_j \cdot e_k = \frac{1}{4}(d\,\mathrm{tr}\, h - \mathrm{div}\, h)^\sharp \cdot .$$

Together with the directional-derivative term, this is (15.4), with the sign in [10].

Increase the initial radius so that $|e|_{g_C} < 1/2$. Then $g_s = g_C + se$ is positive definite for $0 \le s \le 1$, with inverse metrics uniformly bounded. Apply (15.4) along this path. The initial covariant derivative of ψ_C vanishes. Formula (13.1) bounds the transported derivative by $Cs|\nabla_C e|$. Changes of inverse metrics and frames in the linear formula contribute $O(|e||\nabla_C e|)$. Integration in s yields (15.3). The comparison is fibre-isometric, so the norm assertion is exact.

For a compact region with outer normal ν, integration of (15.2) gives

$$\int \left(|\nabla\psi|^2 + \tfrac{R}{4}|\psi|^2 - |\mathscr{D}\psi|^2\right) = \int_\partial \mathrm{Re}\langle \nabla_\nu \psi + \nu \cdot \mathscr{D}\psi, \psi\rangle. \tag{15.5}$$

For ψ_0 at infinity, the real normal-derivative term is zero because its norm is one. Equations (15.1) and (15.3) give the other term as $\beta_C(e)(\partial_r)/4$, with error $O(|e||\nabla_C e|)$. The changes in normal and surface measure have the same error. Its integrated size is $O(R^{n-2-2\tau}) \to 0$. This proves the boundary estimate. $\square$

Lemma 15.2. *Let (N, g) be a connected complete spin manifold of dimension $n \ge 3$, with infinite volume and $R \ge 0$. Suppose $\|\xi\|_{2n/(n-2)} \le C\|\nabla\xi\|_2$ for compactly supported smooth spinors. Let $\mathcal{H}$ be their completion in*

$$\|\xi\|_{\mathcal{H}}^2 = \int_N \left(|\nabla\xi|^2 + \frac{R}{4}|\xi|^2\right). \tag{15.6}$$

Then $\mathscr{D} : \mathcal{H} \to L^2(N)$ is an isometric isomorphism. If a smooth spinor ψ_0 satisfies $\nabla\psi_0, \mathscr{D}\psi_0 \in L^2$ and $R^{1/2}\psi_0 \in L^2$, there is a unique $\chi \in \mathcal{H}$ for which $\Psi = \psi_0 + \chi$ is harmonic. It satisfies

$$\int_N \left(|\nabla\Psi|^2 + \frac{R}{4}|\Psi|^2\right) = \int_N \left(|\nabla\psi_0|^2 + \frac{R}{4}|\psi_0|^2 - |\mathscr{D}\psi_0|^2\right). \tag{15.7}$$

Proof. The Sobolev inequality identifies every Cauchy sequence in $\mathcal{H}$ with a unique L^{q_*} limit. Its gradients converge in L^2; on compact sets, where $L^{q_*} \subset L^2$, the limit has the corresponding weak gradient. The sequence $\sqrt{R}\,\chi_j$ is also Cauchy in L^2. On a compact set R is bounded, so local L^2 convergence of χ_j identifies this limit with $\sqrt{R}\,\chi$. An exhaustion therefore identifies the potential term on the whole manifold. In particular,

$$\|\chi\|_{\mathcal{H}}^2 = \|\nabla\chi\|_2^2 + \frac{1}{4}\|\sqrt{R}\,\chi\|_2^2. \tag{15.8}$$

Thus $\mathcal{H}$ is a space of distributions with the displayed quadratic norm. Integration of (15.2) on compactly supported spinors gives $\|\mathscr{D}\xi\|_2 = \|\xi\|_{\mathcal{H}}$, so the extended map is an isometry with closed range.

Suppose $\eta \in L^2$ is orthogonal to that range. For every compact test ξ, formal self-adjointness gives $\langle \mathscr{D}\eta, \xi\rangle = \langle \eta, \mathscr{D}\xi\rangle = 0$. The local elliptic estimate for a Dirac operator is

$$\|\eta\|_{H^{j+1}(U)} \le C_{j,U,V}\left(\|\mathscr{D}\eta\|_{H^j(V)} + \|\eta\|_{L^2(V)}\right), \qquad U \Subset V,$$

with the distributional interpretation for $j = 0$; see [31]. Iteration yields smoothness of η. Let θ_L be the Lipschitz distance cutoff equal to one on B_L, zero outside B_{2L}, with $|d\theta_L| \le 2/L$. Completeness makes its support compact. The compact H^1 version of (15.2), obtained by local convolution, gives

$$\int_N \left(|\nabla(\theta_L\eta)|^2 + \frac{R}{4}|\theta_L\eta|^2\right) = \int_N |d\theta_L|^2|\eta|^2 \le 4L^{-2}\|\eta\|_2^2 \longrightarrow 0. \tag{15.9}$$

Here $\mathscr{D}(\theta_L\eta) = d\theta_L \cdot \eta$ and Clifford multiplication by a one-form has squared norm $|d\theta_L|^2$. On every fixed compact set, (15.9) implies $\nabla\eta = 0$. The norm of a parallel spinor is constant, so infinite volume and $\eta \in L^2$ imply $\eta = 0$. The orthogonal complement of the closed range is zero; the range is all of L^2.

Set $\chi = -\mathscr{D}^{-1}(\mathscr{D}\psi_0)$. It belongs to $\mathcal{H} \subset L^{q_*}$. The sum Ψ is distributionally harmonic and hence smooth by the local estimate above. Put $\mathcal{B}(\xi, \eta) = \int (\langle\nabla\xi, \nabla\eta\rangle + R\langle\xi, \eta\rangle/4)$. For compact η, polarization of (15.2) gives $\mathcal{B}(\psi_0,\eta)=\langle\mathscr{D}\psi_0,\mathscr{D}\eta\rangle$. Both sides are continuous in $\mathcal{H}$, so the identity holds with $\eta=\chi$. Therefore

$$\mathcal{B}(\Psi,\Psi)=\mathcal{B}(\psi_0,\psi_0)-2\|\mathscr{D}\psi_0\|_2^2+\|\chi\|_{\mathcal{H}}^2=\mathcal{B}(\psi_0,\psi_0)-\|\mathscr{D}\psi_0\|_2^2,$$

which is (15.7). Uniqueness follows from the isometry of $\mathscr{D}$. □

Proposition 15.3. *Assume* (1.16), (SC), *and* $R_g\geq 0$. *On the selected compatible component of the lifted end, there is a smooth harmonic spinor* Ψ *on* $\widetilde{M}$ *for which*

$$d_\Gamma \mathfrak{m}_C(g)=4\int_{\widetilde{M}}\left(|\nabla\Psi|^2+\frac{R}{4}|\Psi|^2\right)\,dv_{\widetilde{g}}. \tag{15.10}$$

It is the sum of a cut-off transported unit cone spinor, supported on that end and its inner collar, and a spinor in $L^{q_*}(\widetilde{M})$.

Proof. By Lemma 14.2, the chosen component of the lifted end has compact link $\widehat{Z}$ and degree $d_\Gamma=|i_*(\Gamma)|$. Take a unit parallel cone spinor for its induced spin structure, as required by (SC). Transport it by the fibre-isometric comparison, multiply by a smooth radial cutoff equal to zero in an inner collar and one sufficiently far out, and extend by zero to the rest of $\widetilde{M}$. This gives a smooth reference spinor ψ_0. The transition collar is compact, and Lemma 15.1 gives

$$\nabla\psi_0,\ \mathscr{D}\psi_0\in L^2(\widetilde{M}),\qquad \int_{\widetilde{M}}R|\psi_0|^2<\infty. \tag{15.11}$$

Indeed the derivative tail is bounded by $C\int_{R_0}^\infty r^{n-3-2\tau}\,dr<\infty$, and the scalar term is supported on one finite-degree covering of the original end and a compact set. No summation over the other lifted ends occurs.

By Lemma 14.1, the hypotheses of Lemma 15.2 hold on $\widetilde{M}$. Choose the harmonic spinor $\Psi=\psi_0+\chi$ furnished by that lemma. Its correction belongs to L^{q_*} and its energy is (15.7). Apply (15.5) to ψ_0 on the truncated part of its selected end, including the inner collar on which it vanishes. The inner boundary integral is zero. By Lemma 15.1, the outer boundary integral is

$$\frac{1}{4}\int_{\widetilde{\Sigma}_R}\beta_{\widetilde{C}}(\widetilde{e})(\partial_r)\,dA_{\widetilde{C}}+o(1)=\frac{d_\Gamma}{4}\int_{\Sigma_R}\beta_C(e)(\partial_r)\,dA_C+o(1).$$

The absolute integrability in (15.11) permits passage to the limit in the interior integrals. The mass limit exists by Lemma 13.1. Substitution into (15.7) proves (15.10). □

Corollary 15.4. *Assume* (1.16) *and* (SC). *If* $R_g\geq 0$, *then* $\mathfrak{m}_C(g)\geq 0$. *If also* $\mathfrak{m}_C(g)=0$, *then* $\mathrm{Ric}_g=0$, $i_*:\Gamma\to G$ *is surjective,* $|G|=d_\Gamma<\infty$, *and the universal cover carries a nonzero parallel spinor. If* i_* *is injective, it is an isomorphism.*

Proof. Equation (15.10) has a nonnegative right side. If it vanishes, $\nabla\Psi=0$. The spinor is not zero: otherwise $\chi=-\psi_0$ would have unit norm at infinity on the chosen end, contrary to $\chi\in L^{q_*}$. For the curvature R^Σ of the spin connection, the contracted identity is

$$\sum_i e_i\cdot R^\Sigma(X,e_i)\Psi=-\tfrac{1}{2}\,\mathrm{Ric}(X)\cdot\Psi;$$

see [31, 15] for the curvature convention in (15.2). Since $\nabla\Psi=0$, one has $R^\Sigma(X,e_i)\Psi=0$, and hence $\mathrm{Ric}(X)\cdot\Psi=0$. The Clifford relation gives $|V\cdot\Psi|^2=|V|^2|\Psi|^2$. Since $|\Psi|$ is a nonzero constant, $\mathrm{Ric}(X)=0$ for every X. If another lifted end existed, the reference spinor would vanish there and $\chi=\Psi$ would have constant positive norm on an infinite-volume set. Again this contradicts $\chi\in L^{q_*}$. There is only one component of the lifted end. Lemma 14.2 gives $[G:i_*(\Gamma)]=1$. Hence i_* is surjective and $|G|=|i_*(\Gamma)|=d_\Gamma<\infty$. Under the additional injectivity hypothesis it is an isomorphism. □

Proposition 15.5 (Extension of compatible parallel spinors). *Assume* (1.16), (SC), $R_g \geq 0$ *and* $\mathfrak{m}_C(g) = 0$. *Let $\mathcal{P}_C$ be the complex vector space of parallel spinors on the compatible cone $C(\widehat{Z})$. Fix the selected lifted end and its fibre-isometric spinor identification. There is a complex-linear isometric map*

$$\mathcal{E} : \mathcal{P}_C \longrightarrow \{\text{parallel spinors on } \widetilde{M}\}. \tag{15.12}$$

It is characterized by the condition that $\mathcal{E}\psi$ minus a cut-off transport of ψ, supported on the selected end and a compact collar, belongs to $L^{q_*}(\widetilde{M})$.

Proof. Apply Lemma 15.2 to the cut-off transport of each $\psi \in \mathcal{P}_C$. The boundary calculation of Proposition 15.3 is homogeneous of degree two in ψ; its boundary value is $d_\Gamma \mathfrak{m}_C(g)|\psi|^2/4 = 0$. The resulting harmonic spinor is therefore parallel. The transport, the Dirac equation, and its inverse on the homogeneous space are complex-linear, proving linearity of $\mathcal{E}$.

A difference of two parallel spinors satisfying the asserted asymptotic condition is parallel and belongs to L^{q_*}. The chosen end has infinite volume, so that difference vanishes. This proves uniqueness, including independence of the cutoff used in a compact collar.

Let $A = |\mathcal{E}\psi|$ and $B = |\psi|$, both constant. On the selected end, the transported spinor has norm B. If $A \neq B$, the reverse triangle inequality bounds the norm of their difference below by the positive constant $|A - B|$, contradicting its L^{q_*} integrability. Hence $A = B$. Polarization of the Hermitian norm proves that $\mathcal{E}$ is isometric. □

Remark 15.6 (The induced spin structure is essential). It is not enough to assume that $\widetilde{M}$ is spin and that $C(\widetilde{Z})$ has a parallel spinor while discarding compatibility. LeBrun's scalar-flat ALE metrics on the total space of $\mathcal{O}(-\ell) \to \mathbb{CP}^1$, $\ell > 2$, have negative mass [47]; see the metric and group action in [48, Section 1.3, equations (1.16)–(1.17)]. For even $\ell \geq 4$ this total space is simply connected and spin. Indeed it retracts onto $\mathbb{CP}^1$, and its tangent bundle there is $T\mathbb{CP}^1 \oplus \mathcal{O}(-\ell)$, so $w_2 = (2-\ell)H \bmod 2 = 0$. The end has fundamental group $\mathbb{Z}/\ell\mathbb{Z}$, its map to the filling group is zero, and its universal covering cone is Euclidean. These facts do not supply a compatible parallel spinor on the induced lifted-end cone. Omitting that requirement would contradict Corollary 15.4 on these negative-mass metrics. No numerical normalization of their mass is used here.

16. Conformal deformation and the mass–energy inequality

We retain (1.16) and (SC), without assuming a sign for R_g. The notation Q_a is that of (1.17).

Lemma 16.1. *If $Q_4(v) \geq 0$ for all $v \in C_c^\infty(M)$, there is a unique $v \in \dot{H}^1(M)$ such that $u = 1 + v$ solves*

$$c_n \Delta_g u + R_g u = 0, \qquad c_n = \frac{4(n-1)}{n-2}. \tag{16.1}$$

It is smooth, positive, bounded above and away from zero, and tends uniformly to one. For every

$$\frac{n-2}{2} < s < \min\{\tau, n-2\}, \tag{16.2}$$

one has $\nabla^j(u-1) = O(r^{-s-j})$ for every fixed j.

Proof. The scalar curvature is bounded on the compact core and is $O(r^{-\tau-2})$ on the end by (13.2). Together with $R \in L^1$ this gives $R \in L^{n/2} \cap L^{2n/(n+2)}$. Lemma 14.1 makes the form $\int Rvw$ and the functional $\int Rv$ continuous on $\dot{H}^1$. Moreover

$$Q_{c_n}(v) = Q_4(v) + (c_n - 4)\|dv\|_2^2 \geq \frac{4}{n-2}\|dv\|_2^2. \tag{16.3}$$

Apply the Riesz representation theorem to the equivalent Hilbert inner product Q_{c_n}. There is a unique $v \in \dot{H}^1$ satisfying $Q_{c_n}(v, \varphi) = -\int R\varphi$ for compact tests. This is (16.1). On nested coordinate balls $B' \Subset B$, the interior elliptic estimate gives

$$\|v\|_{W^{2,a}(B')} \leq C_{a,B',B}\big(\|R(1+v)\|_{L^a(B)} + \|v\|_{L^a(B)}\big), \qquad 1 < a < \infty,$$

whenever the right side is finite; see [21, Chapter 9]. Initially $v \in L^{q_*}_{\mathrm{loc}}$ by (14.1). The coefficients, including R, are smooth on B. Repeated application of the displayed estimate and the Sobolev embedding $W^{2,a} \hookrightarrow L^b$, $1/b = 1/a - 2/n$ when $2a < n$, raises the integrability exponent above $n/2$ after finitely many steps. At equality $2a = n$, every finite exponent is available on a smaller ball. The resulting $W^{2,a}$ bound gives local Hölder continuity. The interior Schauder estimate and differentiation of the equation then give $v \in C^\infty(M)$.

The boundary value at infinity follows from an annular estimate. On a dyadic annulus rescaled to unit size, the equation for v is uniformly elliptic, with bounded smooth coefficients, a bounded potential, and source bounded by $CR^{-\tau}$. Its local estimate is

$$\sup_{A_R} |v| \le C\left(R^{-(n-2)/2}\|v\|_{L^{q_*}(A_{R/2,4R})} + R^{-\tau}\right). \tag{16.4}$$

The estimate follows by putting $w = |v| + \|\text{source}\|_\infty$ in the unit annulus, using regularized absolute values, and testing its weak subsolution inequality with $\eta^2 w^{a-1}$, $a \ge 2$. Cauchy–Schwarz yields $\int |d(\eta w^{a/2})|^2 \le Ca^2 \int (|d\eta|^2 + 1)w^a$. Local Sobolev and cutoffs between nested annuli give $\|w\|_{an/(n-2),U'} \le (Ca^2/\operatorname{dist}(U', \partial U)^2)^{1/a}\|w\|_{a,U}$. Set $\chi = n/(n-2) > 1$, $a_j = q_*\chi^j$, and choose nested subannuli $U_j \searrow U_\infty$ with consecutive distances bounded below by $c2^{-j}$. Iteration gives a factor at most

$$\prod_{j\ge 0}(Ca_j^2 4^j)^{1/a_j} < \infty,$$

because $\sum_{j\ge0}(1 + j + \log a_j)/a_j < \infty$. Thus $\|w\|_{L^\infty(U_\infty)} \le C\|w\|_{L^{q_*}(U_0)}$. Under $g \mapsto R^{-2}g$, the L^{q_*} norm is multiplied by $R^{-n/q_*} = R^{-(n-2)/2}$ and the rescaled source is $O(R^{-\tau})$. This is (16.4). These are also the local estimates of [21, Chapters 8–9]. Since $v \in L^{q_*}$, (16.4) implies $v \to 0$ uniformly.

The negative part u_- is compactly supported and belongs to H^1. Testing (16.1) with it gives $Q_{c_n}(u_-) = 0$. Equation (16.3) implies $u_- = 0$. On a relatively compact coordinate ball let $K \ge \max\{0, \sup R/c_n\}$. The function $w = -u \le 0$ satisfies

$$(\operatorname{div}\nabla - K)w = (K - R/c_n)u \ge 0.$$

This operator is uniformly elliptic and its zeroth-order coefficient is nonpositive. If u vanished at an interior point, w would attain its nonnegative maximum zero. The strong maximum principle [21, Theorem 3.5] would make w constant on that ball. Thus the zero set of u is both open and closed. Connectedness and $u \to 1$ exclude it; hence $u > 0$. Smoothness on a compact core and convergence to one give positive upper and lower bounds globally.

Choose s as in (16.2). On the end,

$$(c_n\Delta_g + R)r^{-s} = c_n s(n-2-s)r^{-s-2} + o(r^{-s-2}). \tag{16.5}$$

Its leading coefficient is positive, and its order dominates $|R| = O(r^{-\tau-2})$. Increase a constant C so that $Cr^{-s} \pm v \ge 0$ on a fixed inner boundary and $(c_n\Delta + R)(Cr^{-s} \pm v) \ge 0$ on the exterior. For $\epsilon > 0$, add ϵu to either expression. At a sufficiently distant outer boundary it is nonnegative because $v \to 0$ and $u \to 1$. Division by the positive solution u converts the operator to a divergence-form operator without a zeroth-order term:

$$(c_n\Delta + R)(uw) = -c_n u^{-1}\operatorname{div}(u^2\nabla w). \tag{16.6}$$

On the finite annulus put $z = (Cr^{-s} \pm v + \epsilon u)/u$. It is nonnegative on both boundary components and satisfies $-\operatorname{div}(u^2\nabla z) \ge 0$. Testing with its negative part gives $\int u^2|\nabla z_-|^2 \le 0$. Hence $z_- = 0$ and $Cr^{-s} \pm v + \epsilon u \ge 0$. The inner radius is fixed; the outer radii tend to infinity. For each fixed point the resulting inequality holds, and letting $\epsilon \downarrow 0$ proves $Cr^{-s} \pm v \ge 0$. This proves $|v| \le Cr^{-s}$. Let v_R denote the pullback to a fixed annulus equipped with $R^{-2}g$. All coefficients of the rescaled operator, with any prescribed finite number of derivatives, are uniformly bounded, and its ellipticity constant is bounded below. Its source is $-R^2R_g = O(R^{-\tau})$ in those norms. The interior Schauder estimate [21, Theorem 6.2], followed by differentiation of the equation, gives, for nested fixed annuli $A' \Subset A$,

$$\|v_R\|_{C^{j,1/2}(A')} \le C_j\left(\|v_R\|_{C^0(A)} + R^{-\tau}\right) \le C'_j R^{-s}.$$

Rescaling the j covariant derivatives multiplies their norms by R^{-j}. Thus $\nabla^j v = O(r^{-s-j})$ for every fixed j. $\square$

Lemma 16.2. *For the function in Lemma 16.1, define $g^\circ = u^{4/(n-2)}g$. It is complete and scalar-flat, satisfies the conical hypotheses with decay above $(n-2)/2$, and*

$$\ell := \lim_{R\to\infty} \int_{\Sigma_R} \partial_\nu u \, dA_g = c_n^{-1} \int_M R_g u \, dv_g, \tag{16.7}$$

$$Q_{c_n}(u) = c_n \ell, \tag{16.8}$$

$$\mathfrak{m}_C(g^\circ) = \mathfrak{m}_C(g) - c_n \ell. \tag{16.9}$$

Proof. Conformal scalar curvature obeys

$$R_{u^{4/(n-2)}g} = u^{-(n+2)/(n-2)}(c_n \Delta_g u + R_g u). \tag{16.10}$$

Thus g° is scalar-flat. Positivity and two-sided bounds for u make it smooth and complete. Its topology, spin universal cover and end inclusion are unchanged. Equation (16.2) and the differentiated bounds give the required conical rate.

Integrate (16.1) over the compact region M_R bounded by Σ_R. Since $Ru \in L^1$, its flux has limit (16.7). The boundary error in multiplying by u is

$$\left| \int_{\Sigma_R} (u-1) \partial_\nu u \, dA_g \right| \le C R^{n-2-2s} \longrightarrow 0. \tag{16.11}$$

The Dirichlet energy is finite because $u - 1 \in \dot{H}^1$; the scalar term is absolutely integrable because u is bounded and $R_g \in L^1$. Multiplication of (16.1) by u gives

$$\int_{M_R} (c_n |du|^2 + R_g u^2) \, dv_g = c_n \int_{\Sigma_R} u \, \partial_\nu u \, dA_g.$$

Use (16.11) and (16.7), then let $R \to \infty$. This proves (16.8).

On the end, writing $v = u - 1$,

$$g^\circ - g_C = e + \frac{4}{n-2} v g_C + O(v^2) + O(ve). \tag{16.12}$$

For a scalar v, $\beta_C(v g_C) = -(n-1)dv$. Therefore the linear contribution in (16.12) is $-c_n \int_{\Sigma_R} v_r \, dA_C$. Its difference from $-c_n \int \partial_\nu u \, dA_g$ is bounded by $CR^{n-2-s-\tau} \to 0$. The two remainder fluxes are bounded by $CR^{n-2-2s} + CR^{n-2-s-\tau} \to 0$. Take limits and use (16.7) to obtain (16.9). $\square$

Proposition 16.3. *Under the hypotheses of Theorem C, $\lambda_{\mathrm{nc}}(g) \le \mathfrak{m}_C(g)$. If equality holds, both are zero and* (1.24) *holds. Whenever Q_4 is nonnegative on compact tests, the conformal solution above gives the exact identity*

$$\mathfrak{m}_C(g) - Q_4(u) = \frac{4}{n-2} \int_M |du|^2 \, dv_g + \frac{4}{d_\Gamma} \int_{\widetilde{M}} |\nabla^{g^\circ} \Psi^\circ|^2 \, dv_{\widetilde{g}^\circ}, \tag{16.13}$$

where Ψ° is the harmonic spinor of Proposition 15.3 for the scalar-flat conformal metric.

Proof. If $Q_4(v) < 0$ for a compactly supported v, then

$$Q_4(1 + tv) = t^2 Q_4(v) + 2t \int Rv + \int R \longrightarrow -\infty \quad (|t| \to \infty). \tag{16.14}$$

Thus $\lambda_{\mathrm{nc}} = -\infty \le \mathfrak{m}_C$, since Lemma 13.1 gives a finite mass. Otherwise Lemmas 16.1 and 16.2 apply. Their conformal factor tends to one with differentiated decay. Thus the marked limiting cone, the induced spin structure, and (SC) are unchanged by the conformal deformation. The scalar-flat case of (15.10) gives $\mathfrak{m}_C(g^\circ) = 4 d_\Gamma^{-1} \int |\nabla^{g^\circ} \Psi^\circ|^2$. Use (16.9), (16.8), and $Q_{c_n}(u) = Q_4(u) + (c_n - 4) \int |du|^2$ to obtain (16.13).

To compare with the defining infimum, choose cutoffs χ_R equal to one on $\{\rho \le R\}$, zero on $\{\rho \ge 2R\}$, with $|d\chi_R| \le C/\rho$. The cutoff error satisfies

$$\|d(v - \chi_R v)\|_2^2 \le 2 \int_{\rho \ge R} |dv|^2 + C \int_{R \le \rho \le 2R} \rho^{-2} v^2 \longrightarrow 0$$

by (14.1). Thus $\chi_R v \to v$ in homogeneous energy. Since u, v are bounded and $R_g \in L^1$, the scalar terms converge by dominated convergence. Hence

$$\lambda_{\mathrm{nc}}(g) \le \lim_{R\to\infty} Q_4(1 + \chi_R v) = Q_4(u) \le \mathfrak{m}_C(g). \tag{16.15}$$

If equality holds, both nonnegative terms in (16.13) vanish. Thus $du = 0$ and $u \to 1$ gives $u = 1$. Equation (16.1) implies $R_g = 0$, and then $Q_4(1) = 0$ while $Q_4 \ge 0$ on every admissible function; thus $\lambda_{\mathrm{nc}} = 0$ and $\mathfrak{m}_C = 0$. By Corollary 15.4, $\mathrm{Ric}_g = 0$, the end homomorphism is surjective, and its target is finite. Proposition 15.5 extends all compatible parallel cone spinors. Under injectivity the end homomorphism is an isomorphism. □

Lemma 13.1, Propositions 13.3, 15.5 and 16.3, and Corollary 15.4 prove the mass and spinor-extension statements of Theorem C. Its entropy assertion is Theorem 18.3.

For a metric with a marked end, let

$$\mathscr{A}_C = \left\{ w \in C^\infty(M;\mathbb{R}) : \begin{array}{l} \text{for some } s > (n-2)/2, \\ |\nabla_C^j (w-1)| = O(r^{-s-j}) \quad (j \ge 0) \end{array} \right\}. \tag{16.16}$$

Following [23, Section 8], write

$$\lambda_{\mathrm{nc}}^{\mathrm{HHS}}(g) = \inf_{w \in \mathscr{A}_C} \int_M \left(4|dw|_g^2 + R_g w^2\right) dv_g \tag{16.17}$$

whenever these integrals and the infimum are defined, with extended real values allowed. No sign condition on the real test functions is imposed. The following comparison requires no spin hypothesis.

Lemma 16.4. *Suppose that g is uniformly equivalent to g_C on its marked end and $R_g \in L^1(M, dv_g)$. Then every test function in $\mathscr{A}_C$ has finite energy, and*

$$\lambda_{\mathrm{nc}}^{\mathrm{HHS}}(g) = \inf_{v \in C_c^\infty(M)} Q_4(1+v) = \lambda_{\mathrm{nc}}(g). \tag{16.18}$$

Proof. Let $w = 1 + v \in \mathscr{A}_C$, and fix its decay exponent $s > (n-2)/2$. Uniform equivalence of the metrics gives

$$\int_{r \ge R} |dv|_g^2 \, dv_g \le C \int_R^\infty r^{n-3-2s} \, dr = \frac{C}{2s + 2 - n} R^{n-2-2s}. \tag{16.19}$$

The function w is bounded, so $R_g w^2$ is integrable. Choose $\chi_R = 1$ on the compact core and on $r \le R$, $\chi_R = 0$ on $r \ge 2R$, with $|d\chi_R|_g \le C/R$. Put $w_R = 1 + \chi_R v$. Then $w_R - 1 \in C_c^\infty(M)$ and

$$\begin{aligned} \|d(w_R - w)\|_2^2 &\le 2 \int_{r \ge R} |dv|_g^2 \, dv_g + 2 \int_{R \le r \le 2R} |d\chi_R|_g^2 |v|^2 \, dv_g \\ &\le C R^{n-2-2s} \longrightarrow 0. \end{aligned} \tag{16.20}$$

Furthermore $|w_R| \le 1 + \|v\|_\infty$ and $w_R \to w$ pointwise. Dominated convergence gives $\int R_g w_R^2 \, dv_g \to \int R_g w^2 \, dv_g$. Thus $Q_4(w_R) \to Q_4(w)$, which proves $\lambda_{\mathrm{nc}}(g) \le Q_4(w)$ and then $\lambda_{\mathrm{nc}}(g) \le \lambda_{\mathrm{nc}}^{\mathrm{HHS}}(g)$. Every compactly supported perturbation of one belongs to $\mathscr{A}_C$, so the reverse inequality follows by inclusion of the test classes. The argument remains valid if either infimum is $-\infty$. □

The equality compares two classes of functions tending to one; neither infimum is the bottom of an L^2 realization.

17. The large-scale entropy and the mass flux

In this section only the geometric assumptions (1.16) are imposed. In particular, neither a spin structure nor a sign for R_g is assumed. The functionals and constants are those of (1.18)–(1.19) and (1.20).

Lemma 17.1 (Radial component of the mass form). *On the marked end put*

$$H = \operatorname{tr}_C e, \qquad a = e(\partial_r, \partial_r), \qquad \zeta_A = e(\partial_r, \partial_A).$$

Here ∂_A denotes a coordinate vector field on Z; thus ζ is a coordinate one-form on Z, not its orthonormal rescaling. Then

$$\beta_C(e)(\partial_r) = a_r + \frac{n}{r}a - H_r - \frac{1}{r}H + r^{-2}\operatorname{div}_{h_Z}\zeta. \tag{17.1}$$

Proof. Write $e_{AB} = e(\partial_A, \partial_B)$. The cone connection satisfies $\Gamma^r_{AB} = -r(h_Z)_{AB}$ and $\Gamma^C_{Ar} = r^{-1}\delta^C_A$. Hence

$$(\nabla^C_A e)_{Br} = (\nabla^{h_Z}_A \zeta)_B + r(h_Z)_{AB}a - r^{-1}e_{AB}.$$

Since $r^{-2}h_Z^{AB}e_{AB} = H - a$, contraction gives

$$(\operatorname{div}_C e)_r = a_r + r^{-2}\operatorname{div}_{h_Z}\zeta + \frac{n-1}{r}a - \frac{1}{r}(H - a).$$

Subtracting $\partial_r H$ proves (17.1). □

Lemma 17.2 (Gaussian average of the flux). *Let*

$$f(r) = \int_{\{r\}\times Z} \beta_C(e)(\partial_r)\, dA_C, \qquad \varphi_T = \frac{r^2}{4T}. \tag{17.2}$$

For a fixed sufficiently large R,

$$\begin{aligned} &\int_{r>R} e^{-\varphi_T}\left(\frac{rH_r + H}{2} - \frac{r^2 a}{4T}\right) dv_C \\ &= -\frac{1}{2}e^{-R^2/(4T)}R^n \int_Z a(R,z)\, dv_{h_Z} - \frac{1}{2}\int_R^\infty r e^{-r^2/(4T)} f(r)\, dr. \end{aligned} \tag{17.3}$$

Consequently, the left side of (17.3), *divided by T, converges to $-\mathfrak{m}_C(g)$ as $T \to \infty$.*

Proof. Multiply (17.1) by $r^n e^{-r^2/(4T)}$ and integrate over Z and $r > R$. The integral of the h_Z-divergence is zero because Z is closed. The terms containing a obey

$$\int_R^\infty e^{-r^2/(4T)}(r^n a_r + nr^{n-1}a)\, dr = -e^{-R^2/(4T)}R^n a(R) + \frac{1}{2T}\int_R^\infty e^{-r^2/(4T)} r^{n+1} a\, dr.$$

The boundary term at infinity vanishes by the polynomial decay of a. Rearrangement gives (17.3).

By Lemma 13.1, f is bounded on $[R, \infty)$ and tends to $\mathfrak{m}_C(g)$. The kernels

$$k_T(r) = \frac{r}{2T}e^{-r^2/(4T)}\mathbf{1}_{(R,\infty)}(r)$$

have total integral $e^{-R^2/(4T)} \to 1$. For each fixed $A > R$, $\int_R^A k_T\, dr \to 0$. Given $\epsilon > 0$, choose A so that $|f(r) - \mathfrak{m}_C(g)| < \epsilon$ for $r > A$. The integral of $k_T|f - \mathfrak{m}_C(g)|$ on (R, A) tends to zero, and its integral on (A, ∞) is at most ϵ. Thus $\int k_T f \to \mathfrak{m}_C(g)$. The first term on the right side of (17.3) is bounded independently of T and disappears after division by T. □

Lemma 17.3 (Normalized Gaussian test functions). *Put $J = dv_g/dv_C$ on the end. Fix a positive smooth function v which is $J^{-1/2}$ outside a compact set. Choose a smooth nonnegative proper function Φ equal to $r^2/4$ there. Define*

$$Z_T = \int_M e^{-\Phi/T}v^2\, dv_g, \qquad u_T = Z_T^{-1/2}e^{-\Phi/(2T)}v. \tag{17.4}$$

Then $Z_T = N_T + O(1)$, $\|u_T\|_2 = 1$, and $\mu(g,T) \le \mathcal{W}(g, u_T, T)$ for each $T > 0$.

Proof. The normalization follows from the definition. On the end, $v^2 dv_g = dv_C$ exactly. Also

$$\int_{C(Z)} e^{-r^2/(4T)}\, dv_C = \mathrm{Vol}(Z)\, 2^{n-1}\Gamma(n/2)T^{n/2} = \theta(4\pi T)^{n/2} = N_T. \tag{17.5}$$

The differences between the two integrals are confined to a fixed compact set and a fixed cone ball. Their integrands are bounded uniformly for $T \geq 1$. This proves $Z_T - N_T = O(1)$.

The function v and its inverse are bounded, and $|dv| = O(r^{-\tau-1})$. For fixed T, the functions u_T, du_T, $R_g u_T^2$ and $u_T^2 \log u_T^2$ have integrable absolute values of the orders required by (1.18). Indeed, each end integral is bounded by a polynomial in r times $e^{-r^2/(4T)}$. Take $0 \leq \chi_A \leq 1$, equal to one on $\{\rho \leq A\}$ and zero on $\{\rho \geq 2A\}$, with $|d\chi_A| \leq C/A$. Normalize $\chi_A u_T$ in L^2. These functions converge to u_T in H^1 and in scalar-curvature energy. For the logarithmic term, use

$$(\chi_A u_T)^2 \log(\chi_A u_T)^2 = \chi_A^2 u_T^2 \log u_T^2 + u_T^2 \chi_A^2 \log \chi_A^2.$$

The first term converges by dominated convergence. The absolute value of $s^2 \log s^2$ is bounded on $0 \leq s \leq 1$; the second term is supported outside $\{\rho \leq A\}$ and its integral tends to zero. Normalization tends to one. Taking the limit of these compact-test values proves the assertion about μ. □

Proposition 17.4 (Coefficient of the Gaussian test family). *For each fixed choice in Lemma 17.3,*

$$\lim_{T\to\infty} \frac{N_T}{T}\big(\mathcal{W}(g, u_T, T) - \log\theta\big) = Q_4(v) - \mathfrak{m}_C(g). \tag{17.6}$$

Proof. Set $P_T = e^{-\Phi/T}$. Direct differentiation of (17.4) gives

$$\begin{aligned}\mathcal{W}(g, u_T, T) - \log\theta = \frac{1}{Z_T}\int_M P_T\bigg[&T(4|dv|^2 + R_g v^2) - 4v\langle dv, d\Phi\rangle \\ &+ \frac{v^2}{T}|d\Phi|^2 + \frac{\Phi}{T}v^2 - 2v^2\log v - nv^2\bigg] dv_g + \log\frac{Z_T}{N_T}.\end{aligned} \tag{17.7}$$

The first integral term divided by T converges to $Q_4(v)$: $\int |dv|^2 < \infty$, v is bounded, and $R_g \in L^1$. All remaining terms over a fixed compact set are $O(1)$ for $T \geq 1$.

On the end let $L = \log J$. Since $v = J^{-1/2}$, the remaining integrand in (17.7), expressed with the measure $e^{-r^2/(4T)} dv_C$, is

$$rg^{-1}(dr, dL) + \frac{r^2}{4T}g^{-1}(dr, dr) + \frac{r^2}{4T} + L - n.$$

The corresponding cone expression is $r^2/(2T) - n$. Its integral over the entire cone is zero, by differentiating (17.5) with respect to T. Removing a fixed cone ball changes that integral by $O(1)$. The difference is therefore

$$A_T = rg^{-1}(dr, dL) + L + \frac{r^2}{4T}\big(g^{-1}(dr, dr) - 1\big). \tag{17.8}$$

Choose the inner radius so that $|e|_C \leq 1/2$. Taylor's formula on the finite-dimensional positive-definite matrix cone, applied to $L = \frac{1}{2}\log\det(I + g_C^{-1}e)$ and to the inverse, yields

$$\begin{aligned} L &= \tfrac{1}{2}H + O(|e|^2), \qquad dL = \tfrac{1}{2}dH + O(|e||\nabla_C e|),\\ g^{-1}(dr, dr) &= 1 - a + O(|e|^2). \end{aligned} \tag{17.9}$$

The Taylor constants are uniform for $|e|_C \leq 1/2$. Substitution in (17.8) gives

$$A_T = \frac{rH_r + H}{2} - \frac{r^2 a}{4T} + \mathcal{R}_T, \qquad |\mathcal{R}_T| \leq C(1 + r^2/T)r^{-2\tau}. \tag{17.10}$$

The estimate includes all radial and mixed components of e; (17.1) was derived without a coordinate condition on them. For $T \geq 1$,

$$\frac{1}{T}\int_{r>R} e^{-r^2/(4T)}|\mathcal{R}_T|\, dv_C \leq C\int_{r>R} r^{-2\tau-2}\left[\frac{r^2}{T} + \left(\frac{r^2}{T}\right)^2\right] e^{-r^2/(4T)}\, dv_C \longrightarrow 0. \tag{17.11}$$

The bracket times the exponential is uniformly bounded and converges to zero at each fixed r. The function $r^{-2\tau-2}$ is integrable because $2\tau > n-2$. This proves the limit by dominated convergence.

By Lemma 17.2, the integral of the linear part of (17.10), divided by T, tends to $-\mathfrak{m}_C(g)$. Finally $N_T/Z_T \to 1$. Since $Z_T - N_T = O(1)$ and $N_T \to \infty$, $|\log(Z_T/N_T)| \le C/N_T$ for large T, and hence

$$\frac{N_T}{T}\log(Z_T/N_T) = O(T^{-1}) \longrightarrow 0. \tag{17.12}$$

Combining these limits in (17.7) proves (17.6). □

Proof of Theorem 1.3. First the infimum defining λ_{nc} may be restricted to positive competitors. For $w-1 \in C_c^\infty(M)$ set

$$w_\epsilon = \frac{\sqrt{w^2+\epsilon^2}}{\sqrt{1+\epsilon^2}}.$$

This is smooth, positive, and equal to one off $\operatorname{supp}(w-1)$. The derivative satisfies $|dw_\epsilon| \le |dw|$. On the regular part of $\{w=0\}$, the zero set is a hypersurface and has measure zero; on its remaining part $dw = 0$. Thus $dw = 0$ almost everywhere on the zero set. It follows that $|dw_\epsilon|^2 \to |dw|^2$ almost everywhere. The scalar terms converge by dominated convergence on the compact support and are unchanged elsewhere. Therefore $Q_4(w_\epsilon) \to Q_4(w)$.

Fix a positive competitor w, and take R beyond its nonconstant region. With $\chi_R = 1$ on $r \le R$ and $\chi_R = 0$ on $r \ge 2R$, define

$$v_R = w \quad \text{on the core}, \qquad v_R = \chi_R w + (1-\chi_R)J^{-1/2} \quad \text{on the end}.$$

It is smooth and positive. On the transition annulus $w = 1$, while $J^{-1/2}-1 = O(r^{-\tau})$ and its derivative is $O(r^{-\tau-1})$. Consequently

$$\|d(v_R - w)\|_2^2 \le CR^{n-2-2\tau} \longrightarrow 0, \qquad Q_4(v_R) \longrightarrow Q_4(w). \tag{17.13}$$

For the second assertion the gradient of w is zero on the altered region. The functions v_R are uniformly bounded and agree with w on expanding compact sets, so $R_g \in L^1$ gives convergence of the potential term. For each fixed R, Lemma 17.3 and Proposition 17.4 imply

$$\limsup_{T\to\infty} \frac{N_T}{T}(\mu(g,T) - \log\theta) \le Q_4(v_R) - \mathfrak{m}_C(g).$$

First let $R \to \infty$, then take the infimum over w. This gives (1.21). If $\lambda_{\mathrm{nc}} = -\infty$, the same argument holds for every prescribed real upper bound, and the upper limit is $-\infty$.

For the unscaled assertion, fix a single v as in Lemma 17.3. Equation (17.6) and $T/N_T \to 0$ give $\mathcal{W}(g, u_T, T) \to \log\theta$. Thus $\nu(g) \le \log\theta$. □

Corollary 17.5 (Strict deficit at large scale). *If $D = \mathfrak{m}_C(g) - \lambda_{\mathrm{nc}}(g) \in (0,\infty)$, then*

$$\mu(g,T) \le \log\theta - \frac{D}{2}\frac{T}{N_T} \tag{17.14}$$

for all sufficiently large T. If $\lambda_{\mathrm{nc}}(g) = -\infty$, the same conclusion holds with any specified $D > 0$, after increasing the lower bound on T.

Proof. Put $a(T) = (N_T/T)(\mu(g,T) - \log\theta)$. In the finite case, (1.21) gives $\limsup_{T\to\infty} a(T) \le -D < -D/2$. By the definition of the upper limit there is T_0 such that $a(T) \le -D/2$ for every $T \ge T_0$. Multiplication by $T/N_T > 0$ gives (17.14). If $\lambda_{\mathrm{nc}} = -\infty$, the same definition applies to every prescribed finite $D > 0$. □

18. Entropy rigidity and extension of parallel spinors

Lemma 18.1 (Asymptotic volume ratio). *Under* (1.16), *for each $o \in M$,*

$$\lim_{R\to\infty} \frac{\mathrm{Vol}_g B_g(o,R)}{v_n R^n} = \theta, \qquad v_n = \mathrm{Vol}(B_1(0) \subset \mathbb{R}^n). \tag{18.1}$$

No Ricci-curvature sign is required for this limit.

Proof. For each $\epsilon \in (0,1)$, choose an end radius R_ϵ such that $(1-\epsilon)g_C \le g \le (1+\epsilon)g_C$ for $r \ge R_\epsilon$. Join o to points of the fixed compact cross-section $r = R_\epsilon$ by curves of uniformly bounded length; compactness supplies the common bound. Extending these curves radially gives

$$d_g(o,x) \le C_\epsilon + \sqrt{1+\epsilon}\, r(x).$$

Conversely, a curve from the compact core to x has a final segment outside $r = R_\epsilon$. Its length is at least $\sqrt{1-\epsilon}(r(x) - R_\epsilon)$, by integration of $|dr|$. Taking an infimum over curves gives the opposite distance bound. Thus $d_g(o,x)/r(x) \to 1$ uniformly in the link, after $\epsilon \downarrow 0$.

Also $J \to 1$ uniformly. The cone volume of $\{r < S\}$ is $\mathrm{Vol}(Z)S^n/n = \theta v_n S^n$. The two distance bounds enclose a large metric ball between two such radial regions up to a fixed compact set. Dividing their volumes by $v_n R^n$, taking lower and upper limits, and then letting $\epsilon \downarrow 0$ proves (18.1). □

Lemma 18.2 (Sharp logarithmic Sobolev bound). *Let $n \ge 3$. If a complete connected noncompact n-manifold satisfies* $\mathrm{Ric}_g \ge 0$ *and has positive asymptotic volume ratio θ, then*

$$\int_M u^2 \log u^2 \, dv_g \le \frac{n}{2} \log\left(\frac{2}{\pi e n} \theta^{-2/n} \int_M |du|^2 \, dv_g \right), \qquad u \in C_c^\infty(M), \quad \|u\|_2 = 1. \tag{18.2}$$

Consequently $\mu(g,T) \ge \log\theta$ for every $T > 0$.

Proof. Equation (18.2) is the $p = 2$ case of [44, Theorem 1.2]. The stated dimension, completeness, connectedness, nonnegative Ricci curvature, and positive asymptotic volume ratio satisfy its hypotheses. The inequality applies to $|u|$ by smooth approximation, with unchanged energy bound, if the reference formulation is made for nonnegative functions.

Set $E = \int |du|^2$. A nonzero compactly supported L^2-normalized function on a connected noncompact manifold cannot have $E = 0$, so $E > 0$. Since $R_g \ge 0$, (18.2) gives

$$\mathcal{W}(g,u,T) \ge 4TE - \frac{n}{2}\log\left(\frac{2}{\pi e n}\theta^{-2/n}E\right) - n - \frac{n}{2}\log(4\pi T).$$

The derivative of the right side with respect to E is $4T - n/(2E)$ and its second derivative is $n/(2E^2) > 0$. Its unique minimum occurs at $E = n/(8T)$ and equals $\log\theta$. Taking the infimum over u proves the conclusion. □

Theorem 18.3 (Global entropy rigidity). *Under* (1.16) *and Definition 1.4,*

$$\nu(g) \le \log\theta, \qquad \nu(g) = \log\theta \iff \mathrm{Ric}_g = 0. \tag{18.3}$$

In the equality case, $\mathfrak{m}_C(g) = \lambda_{\mathrm{nc}}(g) = 0$, the map i_* is surjective, $|\pi_1(M)| = d_\Gamma < \infty$, and every compatible parallel cone spinor extends by the complex-linear isometry of Proposition 15.5.

Proof. The upper bound follows from Theorem 1.3 and does not use the spin hypothesis. If $\nu(g) = \log\theta$, its definition gives $\mu(g,T) \ge \log\theta$ for all T. Thus

$$0 \le \limsup_{T\to\infty} \frac{N_T}{T}(\mu(g,T) - \log\theta) \le \lambda_{\mathrm{nc}}(g) - \mathfrak{m}_C(g) \le 0,$$

where the last inequality is Proposition 16.3. In particular $\lambda_{\mathrm{nc}}(g) > -\infty$ and equality holds in the mass–energy inequality. Its equality statement and Proposition 15.5 give the asserted consequences.

Conversely, $\mathrm{Ric}_g = 0$ implies $\mathrm{Ric}_g \ge 0$. By Lemma 18.1 the asymptotic volume ratio is the positive number θ. The hypotheses of Lemma 18.2 hold, so $\nu(g) \ge \log\theta$. The already proved upper bound yields equality, and hence all its stated consequences. □

Corollary 18.4. *Under the hypotheses of Theorem 18.3,* $\mathrm{Ric}_g \geq 0$ *implies* $\mathrm{Ric}_g = 0$.

Proof. Equations (18.1) and (18.2) give $\nu(g) \geq \log\theta$. The opposite inequality is Theorem 1.3. Apply Theorem 18.3. □

Corollary 18.5. *A Ricci-flat metric satisfying* (1.16) *and Definition 1.4 has zero mass and finite fundamental group. The restricted spin holonomy fixes every extended compatible spinor. In particular its restricted orthogonal holonomy is not $SO(n)$. If i_* is injective, then it is an isomorphism.*

Proof. The equality conclusions of Theorem 18.3 give zero mass, finiteness, extension, and surjectivity. A parallel spinor is fixed by restricted spin holonomy. For orthonormal vectors X, Y, $(X \cdot Y)^2 = -\mathrm{Id}$. Thus a nonzero spinor cannot be annihilated by every infinitesimal rotation, as full $SO(n)$ restricted holonomy would require. The last assertion is injectivity together with surjectivity. □

These conclusions use the rate $\tau > (n-2)/2$ and the induced spin structure in Definition 1.4. They do not determine the restricted holonomy group from the cone alone and do not assert an extension theorem at an arbitrary positive conical rate. The proof of Theorem 1.3 establishes an upper limit, not an asymptotic equality for μ or existence of a minimizer. Neither the endpoint $2\tau = n-2$ nor a uniform power estimate for the remainder is included.

19. Ancient Ricci flows at the cone entropy

Let $(M, g(t))$, $-\infty < t \leq 0$, be a smooth complete ancient Ricci flow. Assume that $|\mathrm{Rm}|$ is uniformly bounded on $M \times [a, b]$ for every finite interval $[a, b] \subset (-\infty, 0]$. At a basepoint $(x_0, 0)$ write the conjugate heat kernel as

$$K(x_0, 0; y, -T) = (4\pi T)^{-n/2} e^{-F_T(y)}, \qquad \mathcal{N}_{x_0,0}(T) = \int_M F_T K(x_0, 0; \cdot, -T)\, dv_{g(-T)} - \frac{n}{2}. \tag{19.1}$$

The logarithmic potential F_T in this formula is not a soliton potential. Set

$$\mathcal{N}_\infty = \lim_{T\to\infty} \mathcal{N}_{x_0,0}(T). \tag{19.2}$$

The finite lower bound in the next statement is a separate hypothesis.

Theorem 19.1 (Stationary ancient flows). *Suppose* $(M, g(0))$ *satisfies* (1.16) *and Definition 1.4, with link volume ratio θ. If, for some basepoint,*

$$\log\theta \leq \mathcal{N}_\infty < \infty, \tag{19.3}$$

then

$$g(t) = g(0) \quad (t \leq 0), \qquad \mathrm{Ric}_{g(0)} = 0, \qquad \mathcal{N}_\infty = \log\theta. \tag{19.4}$$

The equality conclusions of Theorem 18.3 hold for $g(0)$.

Proof. The curvature hypothesis is a uniform bound on each full space–time strip of finite length, not merely a bound on compact spatial sets. Together with completeness, ancient existence, and the finite lower bound for (19.2), it is the hypothesis of [45, Theorem 1.14]. In [45, Section 2.1], the probability density p is nonnegative, has integral one, and satisfies $\sqrt{p} \in C_c^\infty(M)$. For a nonnegative smooth amplitude v with $\|v\|_2 = 1$, substituting $p = v^2$ gives

$$\overline{\mathcal{W}}(g, v^2, T) = \int_M \left\{ T(4|dv|^2 + Rv^2) - v^2 \log v^2 \right\} dv_g - n - \frac{n}{2}\log(4\pi T) = \mathcal{W}(g, v, T). \tag{19.5}$$

Here $0\log 0 = 0$; the energy at zeros is defined through the smooth square root. Thus these two specified test classes are identified. Restricting our class to nonnegative smooth amplitudes does not change the infimum. Indeed, for a real compactly supported amplitude u, choose $\chi \in C_c^\infty(M)$, $\chi \geq 0$, equal to one on a neighbourhood of $\operatorname{supp} u$. The functions $u_\epsilon = (u^2 + \epsilon^2\chi^2)^{1/2}$ are smooth, $\|u_\epsilon\|_2^2 = 1 + \epsilon^2\|\chi\|_2^2$, and $|du_\epsilon| \leq |du| + \epsilon|d\chi|$. After normalization their energies and entropies converge to those of u, by dominated convergence on a fixed compact set; here $du = 0$ almost everywhere on $\{u = 0\}$. The cited theorem therefore gives

$$\inf_{t\le 0} \nu(g(t)) = \mathcal{N}_\infty. \tag{19.6}$$

In particular

$$\log\theta \le \mathcal{N}_\infty \le \nu(g(0)) \le \log\theta,$$

where the last inequality is Theorem 1.3. Hence $\nu(g(0)) = \log\theta$, and Theorem 18.3 makes $g(0)$ Ricci-flat and gives its topological and spinorial conclusions.

Fix $a < 0$. The static flow $\bar{g}(t) = g(0)$ and the given flow are complete and have uniformly bounded curvature on $[a, 0]$; they agree at $t = 0$. Kotschwar's backward uniqueness theorem [46, Theorem 1] therefore gives $g(t) = \bar{g}(t)$ on this interval. Since $a < 0$ was arbitrary, the equality holds for every $t \le 0$. The equality of the entropies was already obtained above. □

The value in (19.3) is the cone value: for $p_T = N_T^{-1} e^{-r^2/(4T)}$, equation (17.5) gives

$$-\log((4\pi T)^{n/2} p_T) = \frac{r^2}{4T} + \log\theta, \qquad \int_{C(Z)} \frac{r^2}{4T}\, p_T \, dv_C = \frac{n}{2}. \tag{19.7}$$

Its Nash entropy is consequently $\log\theta$. This calculation does not identify the backward Nash entropy of an arbitrary flow having one conical spatial slice. In particular no perturbative smallness or symmetry is imposed in Theorem 19.1, but the hypothesis (19.3) is not removed.

20. Expanding solitons and instantaneous smoothing

A cone is called non-Euclidean if its metric completion is not isometric to Euclidean space. A smooth filling has no orbifold points. The end identification in this section is part of the data.

Lemma 20.1. *Let (M^n, g, F) be a complete gradient expanding soliton,*

$$\mathrm{Ric}_g = \nabla^2 F - \tfrac{1}{2} g, \tag{20.1}$$

with one end C^2-asymptotic at positive rate to a Ricci-flat cone over a closed link. Then the metric approaches that cone exponentially in potential coordinates, $R_g \in L^1$, and $R_g \ge 0$.

Proof. In this proof put $\Delta_0 = \operatorname{div}\nabla$ and $\Delta_F = \Delta_0 + \langle \nabla F, \nabla\cdot\rangle$. Taking the trace and divergence of (20.1), and using the contracted Bianchi identity, gives

$$\Delta_0 F = n/2 + R, \qquad \tfrac{1}{2} dR = d\Delta_0 F + \mathrm{Ric}(\nabla F, \cdot), \qquad dR = -2\,\mathrm{Ric}(\nabla F, \cdot).$$

It follows that

$$d\big(|\nabla F|^2 + R - F\big) = 2\nabla^2 F(\nabla F, \cdot) + dR - dF = 0.$$

Add a constant to F so that

$$|\nabla F|^2 + R = F. \tag{20.2}$$

This normalization does not change the soliton equation.

The given C^2 convergence at positive rate implies $r^2|\mathrm{Ric}_g| \to 0$ on the end and bounded curvature on M. Choose a compact set containing the complement of that end and enlarge it so that $|\mathrm{Ric}| \le 1/4$ outside it. There $g/4 \le \nabla^2 F \le 3g/4$. For a minimizing unit-speed geodesic from a fixed basepoint, all points after a fixed parameter lie outside the chosen compact set. Integration of these Hessian bounds along the geodesic gives

$$c_1 d_g(o, x)^2 - C_1 \le F(x) \le C_2 d_g(o, x)^2 + C_2.$$

The constants are uniform because initial values and first derivatives on the fixed ball are bounded. Thus F is proper and bounded below. Equation (20.2) gives $|\nabla F| = O(r)$ and shows that $2\sqrt{F}$ is comparable to the original cone radius outside a compact set.

The hypotheses of [16, Theorem 3.4 and Remark 3.5] now apply. For some $c > 0$, that theorem gives, for every $j \geq 0$,

$$|\nabla^j \operatorname{Ric}_g| \leq C_j(1+r)^{N_j} e^{-cr^2} \tag{20.3}$$

with finite constants C_j, N_j. We record the effect of the normalization (20.2) on the metric coordinates. Set $r = 2\sqrt{F}$ and transport points along $\nabla F/|\nabla F|^2$. Then

$$g = \left(1 - \frac{4R}{r^2}\right)^{-1} dr^2 + h_r.$$

On tangent vectors to an F-level, differentiating the pulled-back metric and using (20.1) gives

$$\partial_r h_r = \frac{4r}{r^2 - 4R} \operatorname{Ric}^\top + \frac{2r}{r^2 - 4R} h_r,$$

$$\partial_r(r^{-2}h_r) = \frac{4}{r(r^2 - 4R)} \operatorname{Ric}^\top + \frac{8R}{r^3(r^2 - 4R)} h_r. \tag{20.4}$$

The differentiated decay in (20.3) makes the right side integrable with exponential decay in the rescaled norms. Integration from r to infinity gives the exponential convergence of $r^{-2}h_r$, together with all spatial derivatives; the radial coefficient has the same property. The given conical limit identifies the limiting link up to cone isometry. In particular, scalar curvature is integrable.

The scalar variation of Ricci flow, applied to the soliton, gives

$$\Delta_F R = -R - 2|\operatorname{Ric}|^2, \qquad \Delta_0 F = n/2 + R. \tag{20.5}$$

The first identity can equivalently be obtained by taking one more divergence of $dR = -2\operatorname{Ric}(\nabla F, \cdot)$:

$$\Delta_0 R = -\langle \nabla R, \nabla F\rangle - 2\langle \operatorname{Ric}, \nabla^2 F\rangle = -\langle \nabla R, \nabla F\rangle - 2|\operatorname{Ric}|^2 - R.$$

If R is negative somewhere, its exponential decay makes its negative minimum attained. At that minimum $\nabla R = 0$ and $\Delta_0 R \geq 0$. With $v = -R > 0$, (20.5) gives $0 \leq v - 2|\operatorname{Ric}|^2 \leq v - 2v^2/n$. Hence $0 \leq R_- \leq n/2$ everywhere.

Put $v = R_-$. Convex smooth approximations to $s \mapsto (-s)_+$, followed by their distributional limit, give

$$\Delta_F v \geq -v + 2|\operatorname{Ric}|^2 \mathbf{1}_{\{R<0\}}. \tag{20.6}$$

Choose nonnegative radial cutoffs χ_R tending to one, supported in $r \leq 2R$, equal to one for $r \leq R$, with $|d\chi_R| \leq C/R$ and $|\Delta_0 \chi_R| \leq C/R^2$ on the end. Integration by parts yields

$$\int \chi_R \Delta_F v = \int v \Delta_0 \chi_R - \int \chi_R v \Delta_0 F - \int v\langle d\chi_R, dF\rangle.$$

The absolute value of the sum of the first and last terms is at most $C\int_{R\leq r\leq 2R} v(R^{-2} + R^{-1}|dF|)\, dv_g$. It tends to zero by (20.3), polynomial volume growth, and $|dF| = O(r)$. Dominated convergence in the remaining terms gives $\int \Delta_F v = -\frac{n}{2}\int v + \int v^2$. Therefore

$$\begin{aligned} 0 &\leq \left(1 - \frac{n}{2}\right)\int v + \int v^2 - 2\int_{\{R<0\}} |\operatorname{Ric}|^2 \\ &\leq \frac{n-2}{n}\int v(v - n/2) \leq 0. \end{aligned} \tag{20.7}$$

If v were nonzero, connectedness and $v \to 0$ would give a nonempty open set on which $0 < v < n/2$. Its contribution to the last integral would be strictly negative. Thus $v = 0$, proving $R_g \geq 0$. □

Theorem 20.2. *Let $C(Z)$ be a non-Euclidean Ricci-flat cone whose universal covering cone has a nonzero parallel spinor. If an oriented smooth complete gradient expanding soliton has one end C^2-asymptotic at positive rate to this cone, then*

$$\ker[\pi_1(Z) \longrightarrow \pi_1(M)] \neq \{1\} \quad \text{or} \quad w_2(\widetilde{M}) \neq 0. \tag{20.8}$$

Proof. Suppose both alternatives fail. The universal cover of the oriented filling is then spin, and the end homomorphism is injective. By Lemma 20.1, in a potential-coordinate marking the conical rate is arbitrarily large, $R_g \in L^1$ and $R_g \geq 0$. The flux in (1.15) is zero. Theorem C gives $\mathrm{Ric}_g = 0$ and identifies $\pi_1(M)$ with the finite group $\pi_1(Z)$.

Let $\pi : \widetilde{M} \to M$ be the finite universal covering, and set $\widetilde{g} = \pi^* g$, $\widetilde{F} = F \circ \pi$. The covering is proper, so $\widetilde{F}$ is proper and has a minimum o. The lifted metric is complete and $\widetilde{\nabla}^2 \widetilde{F} = \widetilde{g}/2$. Along every unit-speed geodesic from o, $\widetilde{F} = \widetilde{F}(o) + t^2/4$. If $x \neq o$ were a critical point, a minimizing geodesic from o to x would give $\langle \widetilde{\nabla} \widetilde{F}, \dot{\gamma} \rangle = t/2 > 0$ at x. Thus o is the unique critical point. The Hessian equation also gives $d(|\widetilde{\nabla}\widetilde{F}|^2 - \widetilde{F}) = 0$. Evaluation at o therefore yields $|\widetilde{\nabla}\widetilde{F}|^2 = \widetilde{F} - \widetilde{F}(o)$. On $\widetilde{M} \setminus \{o\}$ put $r = 2\sqrt{\widetilde{F} - \widetilde{F}(o)}$. Then $dr = 2r^{-1} d\widetilde{F}$ and

$$|\widetilde{\nabla} r| = 1, \qquad \widetilde{\nabla}^2 r = r^{-1}(\widetilde{g} - dr^2). \tag{20.9}$$

Near o, geodesic polar coordinates exist and r is geodesic distance. For $0 < a < b$, the set $\{a \leq r \leq b\}$ is compact by properness. The flow of $\widetilde{\nabla} r$, whose derivative of r is one, identifies its level sets. Under these identifications, the tangential metric h_r satisfies $\partial_r h_r = 2r^{-1} h_r$. Thus $\widetilde{g} = dr^2 + r^2 h$ with h independent of r. Smoothness at o gives $h = \lim_{r \downarrow 0} r^{-2} h_r = \gamma_{n-1}$ on the unit tangent sphere. The flow exists across each finite interval of positive radii, and completeness extends the geodesics from o to every radius. Hence $(\widetilde{M}, \widetilde{g})$ is Euclidean space. Every deck transformation preserves $\widetilde{F}$, so it fixes o. A deck transformation of a connected covering which fixes a point is the identity. The deck group is therefore trivial. Hence M and its asymptotic cone are Euclidean, contrary to the hypothesis. □

Definition 20.3. An admissible instantaneous smoothing in this paper is a smooth Ricci flow $(M, g(t))$, $0 < t < T$, with the following properties, as in [23, Section 8]: each positive-time metric is complete with bounded curvature; as $t \downarrow 0$ the flow converges to the given cone in the Gromov–Hausdorff sense everywhere and smoothly on each compact subset away from the vertex; and

$$\liminf_{t \downarrow 0} \lambda_{\mathrm{nc}}^{\mathrm{HHS}}(g(t)) \geq 0. \tag{20.10}$$

Here $\lambda_{\mathrm{nc}}^{\mathrm{HHS}}$ is the functional (16.17), in the prescribed end identification; the integrals defining it and its infimum are required to be well-defined at each positive time. The manifold is a fixed oriented smooth filling, with a single conical end and the end homomorphism determined by that identification. When a positive-time metric is uniformly conical and has integrable scalar curvature, Lemma 16.4 identifies this functional with (1.17). No such identification is used without those hypotheses.

Corollary 20.4. *Any admissible instantaneous smoothing of the cone in Theorem 20.2 satisfies* (20.8).

Proof. Suppose both alternatives in (20.8) fail. Theorem 8.2 of [23], applied with the hypotheses in Definition 20.3, gives an exponentially conical metric g_* on the same smooth filling such that $\lambda_{\mathrm{nc}}^{\mathrm{HHS}}(g_*) > 0$. Its end identification induces the same homomorphism of fundamental groups, up to basepoint conjugacy. The cone is Ricci-flat, so the exponential metric estimates give $R_{g_*} \in L^1$ and $\mathfrak{m}_C(g_*) = 0$. By Lemma 16.4, $\lambda_{\mathrm{nc}}(g_*) = \lambda_{\mathrm{nc}}^{\mathrm{HHS}}(g_*) > 0$. The spin-cover and end-injectivity hypotheses have not changed. Theorem C therefore gives $\lambda_{\mathrm{nc}}(g_*) \leq \mathfrak{m}_C(g_*) = 0$, a contradiction. No scalar-curvature sign is assumed for g_*. □

Corollary 20.5. *Let h_{FH} be an inhomogeneous nearly Kähler metric on the standard S^6 constructed by Foscolo–Haskins, normalized by* $\mathrm{Ric} = 5h_{\mathrm{FH}}$. *If an oriented seven-manifold admits either a smooth gradient expander as in Theorem 20.2 or an admissible smoothing of its cone, then it contains an embedded 2-sphere with oriented normal bundle ν satisfying*

$$\langle w_2(\nu), [S^2] \rangle = 1. \tag{20.11}$$

In particular, no such filling can have $\pi_2(M) = 0$.

Proof. Foscolo–Haskins [18] supply the inhomogeneous nearly Kähler metric. Its normalized cone has holonomy contained in G_2 and therefore a parallel spinor. It is not the Euclidean cone, since its link is not the round sphere. The link satisfies $\pi_1(S^6) = \{1\}$, so its end homomorphism is injective. Thus Theorem 20.2 and Corollary 20.4 give $w_2(\widetilde{M}) \neq 0$.

Let $\pi : \widetilde{M} \to M$ denote the covering. Since $H_1(\widetilde{M}; \mathbb{Z}) = 0$, the universal coefficient theorem [25, Theorem 3.2, p. 195] gives

$$H^2(\widetilde{M}; \mathbb{Z}/2) \cong \mathrm{Hom}(H_2(\widetilde{M}; \mathbb{Z}), \mathbb{Z}/2). \tag{20.12}$$

Thus there is $\xi \in H_2(\widetilde{M}; \mathbb{Z})$ with $\langle w_2(T\widetilde{M}), \xi\rangle = 1$. The Hurewicz theorem [25, Theorem 4.32, p. 366] applies to the simply connected manifold and gives a map $f : S^2 \to \widetilde{M}$ with $f_*[S^2] = \xi$. The covering differential identifies $T\widetilde{M}$ with $\pi^* TM$. For $f_0 = \pi \circ f$, naturality gives

$$\langle f_0^* w_2(TM), [S^2]\rangle = \langle f^* w_2(T\widetilde{M}), [S^2]\rangle = 1. \tag{20.13}$$

Smooth approximation [27, Chapter 2, Section 2] replaces f_0 by a smooth map in its homotopy class.

In $J^1(S^2, M)$, the rank-one and rank-zero strata have respective codimensions $(2-1)(7-1) = 6$ and $2 \cdot 7 = 14$. The jet transversality theorem [22, Chapter II, Theorem 4.9] permits an arbitrarily small smooth perturbation transverse to both. Their codimensions exceed $\dim S^2 = 2$, so the resulting map is an immersion. Apply the multijet transversality theorem [22, Chapter II, Theorem 4.13] on $S^{2(2)} = (S^2 \times S^2) \setminus \{(x,x) : x \in S^2\}$ to the equality condition in $M \times M$. That condition has codimension seven, whereas $\dim S^{2(2)} = 4$. Its transverse inverse image is empty. The two transversality conditions can be imposed simultaneously: the immersion condition is open, and multijet transversality gives a dense residual set within that open set. The resulting immersion $j : S^2 \to M$ is injective. A continuous injection from compact S^2 to Hausdorff M is a homeomorphism onto its image; hence j is a smooth embedding. All perturbations may be chosen in a sufficiently small C^0 neighborhood of the compact image, so short geodesics give the required homotopies.

The normal bundle ν is oriented by the orientations of M and S^2. The identity $j^* TM = TS^2 \oplus \nu$ and the Whitney sum formula give $j^* w_2(TM) = w_2(TS^2) + w_2(\nu) = w_2(\nu)$, because the Euler number of TS^2 is two and $w_2(TS^2)$ is its reduction modulo two. Equation (20.13), which is preserved by homotopy, proves (20.11). A nullhomotopic map pulls back every positive-degree cohomology class to zero, so j represents a nonzero class in $\pi_2(M)$. □

21. The three asymptotic hypotheses

The preceding results distinguish full tensor coercivity, fixed-Einstein-constant volume, and a spinorial scalar-curvature boundary invariant. The classes of cones need not coincide.

Proposition 21.1. *A Ricci-flat cone over a nontrivial product of two positive-dimensional round spheres has no nonzero parallel spinor.*

Proof. Let X, Y be orthonormal cone-tangential vectors belonging to different factors. The product link has zero mixed curvature. The cone curvature endomorphism on their plane is $-r^{-2} X \wedge Y$, with no other components. Its spin-curvature action is $\pm(2r^2)^{-1} X \cdot Y$. The sign depends on the wedge-endomorphism convention and has no effect on its square. Since $X \perp Y$ and both are unit,

$$(X \cdot Y)^2 = -\mathrm{Id}, \qquad \left(R^{\Sigma}(X, Y)\right)^2 = -\frac{1}{4r^4}\,\mathrm{Id}.$$

Thus the spin-curvature action is invertible. A parallel spinor must be annihilated by every curvature endomorphism, so it must vanish. □

Consequently the mass theorem is not an implication from Theorem A or Theorem B. Conversely, its spinorial hypotheses do not establish full-tensor stability of an arbitrary filling. Theorems A and C concern noncompact manifolds; they give no compact Ricci-flat metric of full orthogonal holonomy. Theorem B concerns connected components, not full nondegeneracy. The global entropy comparison is one-sided, and its rigidity assertion retains (SC). The stationary ancient-flow theorem assumes the backward Nash-entropy bound (19.3); it does not follow from spatial conicality alone. The alternatives in (20.8) remain necessary conditions only. They neither construct a smoothing in the remaining topological classes nor exclude all weak notions of singular Ricci flow.

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