UNIQUENESS, MORSE INDEX, AND NULLITY OF CARLOTTO–SCHULZ MINIMAL HYPERSURFACES

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Abstract. For the Carlotto–Schulz minimal embeddings of $\mathbb{S}^{n-1} \times \mathbb{S}^{n-1} \times \mathbb{S}^1$ in $\mathbb{S}^{2n}$, we prove uniqueness and nondegeneracy of the normalized monotone closing profile. The first positive Laplace eigenvalue is $2n-1$, and its eigenspace consists of ambient coordinate functions. The full Morse index is 27 for $n=2$ and $(n^3+9n^2+11n+3)/3$ for $n \geq 3$; the nullity is n^2+2n, and every Jacobi field is induced by a rotation. The scalar argument also applies to graphical doubles with unequal spherical factors. Uniqueness for $n \geq 3$ follows from a monotone Jacobi-residual ratio and compact continuation. The kernel theorem gives local rigidity and, for nearby bumpy ambient metrics, minimal hypersurfaces of every index from the base index through that index plus n^2+2n. In $\mathbb{S}^4$ there are at least twenty-one such three-tori.

Contents

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1. Introduction

Symmetry reduces several minimal-hypersurface constructions to a geodesic problem on an orbit space. In the reduction of Hsiang and Lawson [10], the volume of a principal orbit supplies the weight in the reduced variational problem. Carlotto and Schulz [2, Theorem 1.1] used such a reduction to construct, for every integer $n \geq 2$, an embedded minimal hypersurface

$$\mathbb{S}^{n-1} \times \mathbb{S}^{n-1} \times \mathbb{S}^1 \;\longrightarrow\; \mathbb{S}^{2n}.$$

Their profile starts on the factor-exchange axis and meets the equator orthogonally; reflection gives a closed curve. The existence argument does not determine the number of normalized closing parameters. Nor does it determine the first Laplace eigenspace, the full Morse index, or the space of Jacobi fields. We determine these quantities for the normalized monotone construction.

1.1. First eigenvalues and index estimates.

For a minimal immersion $X : M^d \to \mathbb{S}^{d+1}$, the coordinate equation $\Delta X = dX$ gives a scalar eigenvalue at d; see [25] and the derivation in section 2. It does not place that eigenvalue first. Yau's conjecture asserts that $\lambda_1 = d$ when M is closed and embedded [30]. The lower bound of Choi–Wang [5] and the refinements in [6, 31] concern general embedded minimal hypersurfaces. Zeng's preprint [32] claims the unrestricted equality, whereas Yu [31] formulates that equality as a conjecture and proves a lower bound. Neither a validation nor a refutation of Zeng's proposed proof is used here. Our first-eigenvalue argument is independent of that claim: a positive comparison on the profile circle determines the radial spectrum, and equality in the angular quadratic forms identifies the entire first eigenspace.

The index arguments also have a substantial antecedent. Montiel and Ros [18, Lemmas 12–13] compare the spectrum of a Schrödinger operator with spectra on a partition of its domain. Carlotto, Schulz and Wiygul developed this method with symmetry in [3, Proposition 3.1] and for compact Lie group actions in [4, Proposition 2.2]. In [4, Remark 3.13], they report numerical evidence that the $O(2) \times O(2)$-invariant index of the Carlotto–Schulz three-torus is three. The index-growth theorem of that paper concerns Hsiang's hypersurfaces, not an exact full-index count for the present family.

Perdomo [22, Theorems 3.3, 4.3 and 5.1] gives the spherical-harmonic reduction, the full-index lower bound n^2+4n+3, and a scalar criterion for Yau's equality. The citation refers to the second arXiv version, which corrects the harmonic multiplicities. His subsequent paper [23, Conjecture 1.1] proposes the values 27 for $n = 2$ and $(n^3 + 9n^2 + 11n + 3)/3$ for $n \geq 3$, and reports numerical tests of the index through $n = 100$. The reciprocal-volume Jacobi eigenfunction occurs in [23, Lemma 3.1]; its unequal-factor counterpart is given in [24, Theorem 2.2]. Neither the predicted polynomial nor these eigenfunctions is introduced here.

Exact index and nullity are known for other distinguished spherical families. Kapouleas and Wiygul [12] determine the index $2g + 3$ and nullity 6 of Lawson's surfaces $\xi_{g,1}$, $g \geq 2$, and exclude Jacobi fields other than those induced by ambient Killing fields. Kapouleas and Zou [13, Theorem 1.2] obtain index $2\gamma + 5$ and nullity 6 for their equatorial doublings of genus $\gamma = m + 1$, for all sufficiently large m. These results concern minimal surfaces in $\mathbb{S}^3$, rather than the hypersurfaces studied here. They supply close antecedents for the simultaneous determination of negative spectrum, kernel, and local rigidity.

The additional conclusions are the uniqueness of the normalized monotone profile, the identification of the entire first Laplace eigenspace, and the exclusion of every Jacobi mode not present in the stated index and kernel. The index is taken on all normal variations, not only the invariant ones. The finite calculations used for its remaining angular sectors enclose the exact closing solution and cover every integer dimension.

1.2. **The class of profiles.** All spheres in this paper have sectional curvature one. Put

$$p = n - 1, \qquad d = 2p + 1 = 2n - 1. \tag{1.1}$$

The letter p is also allowed to be real in the planar ordinary differential equation. Geometric assertions are made only when p is an integer.

Definition 1.1. A normalized monotone closing profile for $p \geq 1$ is a solution (r, θ, α) on $[0, \ell]$ of

$$r' = \cos\alpha, \qquad \theta' = \frac{\sin\alpha}{\sin r}, \qquad \alpha' = \frac{2p\cos\alpha\cot 2\theta}{\sin r} - (2p+1)\cot r\sin\alpha, \tag{1.2}$$

with initial values

$$r(0) = r_0 \in (0, \pi/2), \quad \theta(0) = \pi/4, \quad \alpha(0) = -\pi/2, \tag{1.3}$$

such that

$$0 < r < \pi/2, \quad 0 < \theta < \pi/4, \quad -\pi/2 < \alpha < 0 \quad (0 < t < \ell), \tag{1.4}$$

and

$$r(\ell) = \pi/2, \qquad \alpha(\ell) = 0, \qquad \theta(\ell) > 0. \tag{1.5}$$

The trajectory first meets the union $\{r = \pi/2\} \cup \{\alpha = 0\}$ at $t = \ell$.

For an integer p, reflection gives functions a, b, c of period $T = 4\ell$ and an embedding

$$X(y, z, t) = (a(t)y, b(t)z, c(t)), \qquad (a, b, c) = (\sin r\cos\theta, \sin r\sin\theta, \cos r). \tag{1.6}$$

We denote its image by M_n. The initial hypersurface $\{\theta = \pi/4\}$ and the direction of traversal are fixed by definition 1.1. Lai and Wei [15, Theorem 1.2] construct two nonisometric constant-mean-curvature hypertori of the same topological type when the mean curvature is sufficiently small and negative. Their assertion concerns nonzero mean curvature. It neither contradicts nor proves the uniqueness statement below, which is restricted to minimal normalized monotone profiles.

1.3. **Statements.** We take $\Delta = -\operatorname{div}\nabla$. On a two-sided minimal hypersurface in $\mathbb{S}^{d+1}$ the operator representing the second variation of area is

$$J = \Delta - |A|^2 - d. \tag{1.7}$$

Its index is the number of strictly negative eigenvalues, counted with multiplicity; its nullity is the dimension of its kernel. Scalar eigenvalues are indexed by $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots$.

Theorem 1.2 (Monotone uniqueness). *For every real $p \geq 2$ there is exactly one profile in definition 1.1. Its closing equation is nondegenerate with respect to the initial radius. The same assertions hold for $p = 1$. The proof for $p \geq 2$ is analytic; the proof for $p = 1$ uses the interval exclusion in section 5.*

Theorem 1.3 (First eigenspace). *For every $n \geq 2$,*

$$\lambda_1(M_n) = 2n - 1, \qquad E_{2n-1}(M_n) = \operatorname{span}_{\mathbb{R}}\{x_1|_{M_n}, \ldots, x_{2n+1}|_{M_n}\}. \tag{1.8}$$

The dimension of this eigenspace is $2n + 1$. On invariant functions, the first positive eigenvalue is simple and the next eigenvalue is strictly greater than $4n - 3$.

Theorem 1.4 (Full index and nullity). *For every integer $n \geq 2$,*

$$I_n := \operatorname{ind}(M_n) = \begin{cases} 27, & n = 2, \\ (n^3 + 9n^2 + 11n + 3)/3, & n \geq 3, \end{cases} \qquad \operatorname{nul}(M_n) = n^2 + 2n. \tag{1.9}$$

Every Jacobi field is the normal component of an ambient infinitesimal rotation. The index and nullity calculation uses the finite enclosures specified in proposition 9.1.

A bumpy metric has no nonzero Jacobi fields on any closed immersed minimal hypersurface. The genericity theorem used for this term is White's theorem [29, Theorem 2.1].

Theorem 1.5 (Nearby ambient metrics). *Fix $n \geq 2$ and $0 < \alpha < 1$. There is a neighborhood of the round metric in $C^{6,\alpha}$ such that every smooth metric in that neighborhood has a nearby embedded minimal hypersurface diffeomorphic to M_n with index I_n. For every bumpy metric in the neighborhood, every integer in*

$$[I_n, I_n + n^2 + 2n] \cap \mathbb{Z} \tag{1.10}$$

is the full index of such a hypersurface. If b_j is defined by

$$\sum_{j=0}^{n^2+2n} b_j t^j = (1 + t + \cdots + t^n) \prod_{r=1}^{n} \frac{1 - t^{n+1+r}}{1 - t^r}, \tag{1.11}$$

and $e_{-1} = 0$, $e_j = (\sum_{i=0}^{j} b_i) \bmod 2 \in \{0, 1\}$, then at least

$$\frac{b_j + e_{j-1} + e_j}{2} \tag{1.12}$$

distinct embedded hypersurfaces have index $I_n + j$. For $n = 2$, their total number is at least twenty-one.

The neighborhood may depend on n. It can be restricted to positively sectionally curved metrics. The count is of embedded images, not of parametrizations and not necessarily of ambient isometry classes.

1.4. **The arguments and their scope.** The uniqueness proof uses flattened coordinates in which the weighted length has a separated potential. For two Neumann test functions ϕ_1, ϕ_2, the functions $-\mathscr{J}\phi_i$ are positive in the open quarter. Their ratio is strictly monotone for every real factor parameter $p \geq 2$, which excludes a Neumann kernel. Uniform bounds for the intercepts and the angle denominator make the set of closing solutions compact. Continuation from an isotropic Gaussian problem then gives uniqueness and the invariant index. For $p = 1$, an analytic comparison series bounds the entire initial range; finite interval inclusions exclude the two exterior ranges and isolate one nondegenerate closing in the remaining interval.

The scalar argument is independent of the higher Jacobi calculation. The balanced difference of two orbit radii solves a scalar equation whose potential is strictly greater than the coordinate eigenvalue. Its two zeros impose two independent trace conditions on the profile circle. A positive-square identity and the min–max principle give the radial gap; positive references a, b give a lower bound of d in every nonconstant angular sector. The same criterion applies to the graphical doubles of Firester–Tsiamis [7, Theorem 1.1], as stated in corollary 6.2. Their existence theorem, not a new construction of unequal-factor hypersurfaces, is the input to that application.

For the Jacobi operator, differentiation identifies the radial scalar spectrum with the first mixed spectrum after a shift. Factor exchange and a two-dimensional quadratic-form inequality give the strict lower bound $J_{j,k} > 3/2$ for $j, k \geq 1$, $(j,k) \neq (1,1)$. The higher single-factor sectors have positive Dirichlet realizations on a half-profile. Their remaining inertia is therefore the inertia of an outward-flux form on two endpoint values. The general fixed-boundary and Dirichlet-to-Neumann index decomposition has an antecedent in Tran's work [26]. In the present problem, exact geometric solutions permit division of the transfer numerators by their vanishing dimension factor before integration. The resulting regular equations give one finite set of parameter enclosures for all dimensions, rather than an asymptotic assertion with an unspecified cutoff.

The longitudinal principal curvature is determined by the trace identity $k_t = -p(k_a + k_b)$. The missing factor in the printed formula of [23, Section 2], and the reciprocal-density identity, are also recorded in [16]. We derive the geometric formulas without attributing that error to another author's calculations. Completed-square and geometric-crossing arguments also occur in [20]; no positivity theorem for its Ricci-flat tensor operator is assumed for the scalar operator here.

The full kernel is the tangent space to the compact rotation orbit. Reduction of the area functional along a nondegenerate critical manifold is in the setting introduced by Bott [1]; the varying-metric elliptic framework is due to White [28]. Equivariant nondegeneracy and local rigidity in a different minimal surface setting are treated in [14]. Our application uses the computed full index and kernel. An explicit Hessian congruence gives the index of every nearby solution, and the ordered double cover of the rotation orbit gives the multiplicities in theorem 1.5. Its cohomology uses the projective-bundle module theorem and the real Grassmannian Schubert basis [8, 21]; the count does not require a perfect Morse function.

Monotone uniqueness is not a classification of arbitrary invariant closed profiles. In particular, it does not prove [2, Conjecture 3.7], which concerns every embedded minimal three-torus in $\mathbb{S}^4$. Nor is theorem 1.3 a proof of unrestricted Yau. The nearby-metric statements concern distinct embedded images; no pairwise nonisometry assertion is made for hypersurfaces of equal index.

The finite interval lemmas propositions 5.3 and 9.1 have computer-assisted proofs. Their differential equations, Taylor inclusion theorem, rational arithmetic identities, and parameter-covering implications are given in Appendices A and B. The complete rational partitions and step enclosures accompany the article and are part of these proofs. The general enclosure method has an antecedent in [19]. No approximate profile replaces an exact closing solution, and no unquantified remainder in the dimension parameter is used.

2. Profile geometry and scalar operators

2.1. **The induced metric and curvature.** Let $p, q \geq 1$ be integers, $d = p + q + 1$, and let $\gamma = (a, b, c) : \mathbb{R}/T\mathbb{Z} \to \mathbb{S}^2$ be unit speed, with $a, b > 0$. For $y \in \mathbb{S}^p$ and $z \in \mathbb{S}^q$, set

$$X(y, z, t) = (a(t)y, b(t)z, c(t)), \qquad w = a^p b^q. \tag{2.1}$$

The tangent directions belonging to the two spherical factors and the profile direction are mutually orthogonal. Hence

$$g = dt^2 + a^2 g_{\mathbb{S}^p} + b^2 g_{\mathbb{S}^q}, \quad dv_g = w\, dt\, dv_{\mathbb{S}^p}\, dv_{\mathbb{S}^q}, \quad \Delta = L_w + a^{-2}\Delta_{\mathbb{S}^p} + b^{-2}\Delta_{\mathbb{S}^q}, \tag{2.2}$$

where $L_w = -w^{-1}(w\,\cdot\,')'$.

In spherical coordinates, choose

$$\begin{aligned} \nu_1 &= \cos r \sin\alpha \cos\theta + \cos\alpha \sin\theta, \\ \nu_2 &= \cos r \sin\alpha \sin\theta - \cos\alpha \cos\theta, \\ \nu_3 &= -\sin r \sin\alpha. \end{aligned} \tag{2.3}$$

Then $\nu = (\nu_1 y, \nu_2 z, \nu_3)$ is a unit normal. We use the shape endomorphism $S = d\nu$, so that reversing the normal reverses its eigenvalues but not $|A|^2$. In the two spherical directions its eigenvalues are ν_1/a and ν_2/b. When $p = q$, direct differentiation in the profile direction gives

$$\begin{aligned} k_a &= \csc r \cos\alpha \tan\theta + \cot r \sin\alpha, \\ k_b &= \cot r \sin\alpha - \csc r \cos\alpha \cot\theta, \\ k_t &= \alpha' + \cot r \sin\alpha. \end{aligned} \tag{2.4}$$

Their multiplicities are $p, p, 1$. The equation $pk_a + pk_b + k_t = 0$ is exactly (1.2), and

$$k_t = -p(k_a + k_b), \qquad |A|^2 = p(k_a^2 + k_b^2) + p^2(k_a + k_b)^2. \tag{2.5}$$

The factor p in the first equality is necessary. The displayed longitudinal formula in [23, Section 2] omits it when $p > 1$. We use (2.4)–(2.5), derived from the trace equation. This corrects the displayed formula without attributing an error to the numerical conclusions of that paper.

Lemma 2.1. *For a minimal immersion* (2.1),

$$L_w a = da - p/a, \qquad L_w b = db - q/b, \qquad L_w c = dc. \tag{2.6}$$

For its Gauss map ν, every constant ambient vector e, and every constant skew-symmetric ambient matrix Q,

$$J\langle \nu, e\rangle = -d\langle \nu, e\rangle, \qquad J\langle QX, \nu\rangle = 0. \tag{2.7}$$

In particular, the radial coefficients in (2.3) *satisfy*

$$J_{1,0}\nu_1 = -d\nu_1, \qquad J_{0,1}\nu_2 = -d\nu_2, \qquad J_{0,0}\nu_3 = -d\nu_3, \tag{2.8}$$

where $J_{j,k} = L_w + j(j + p - 1)a^{-2} + k(k + q - 1)b^{-2} - |A|^2 - d$.

Proof. At a point choose a geodesic orthonormal tangent frame. The Euclidean trace of the second derivative of X is the sum of the spherical mean curvature term, which vanishes, and $-dX$. Thus $\Delta X = dX$. The coordinate functions on $\mathbb{S}^p$ have eigenvalue p, and those on $\mathbb{S}^q$ have eigenvalue q. Substitution in (2.2) proves (2.6).

Write D for the Euclidean connection and $S_{ij} = \langle D_{e_i}\nu, e_j\rangle$. At the chosen point,

$$D_{e_i}\nu = S_{ij}e_j, \qquad D_{e_i}e_j = -S_{ij}\nu - \delta_{ij}X.$$

It follows that

$$\Delta\nu = -\sum_{i,j}(\nabla_i S_{ij})e_j + |S|^2\nu + (\operatorname{tr} S)X. \tag{2.9}$$

The Codazzi identity in the unit sphere is $\nabla_i S_{jk} = \nabla_j S_{ik}$. Its contraction gives $\sum_i \nabla_i S_{ij} = \nabla_j \operatorname{tr} S = 0$. Hence $\Delta\nu = |S|^2\nu$ and the first identity in (2.7) follows by pairing with e. Substitution of the three angular types of ν in (2.2) gives (2.8).

For a normal variation with speed f, differentiation of the metric and shape endomorphism gives

$$\dot g_{ij} = 2fS_{ij}, \qquad \dot S^i{}_j = -(\nabla^2 f)^i{}_j - f(S^2)^i{}_j - f\delta^i{}_j, \qquad \frac{d}{ds}\operatorname{tr} S = Jf. \tag{2.10}$$

To verify the shape formula, use fixed local coordinates for the immersion and write $X_i = \partial_i X$. Differentiating $\langle \nu, \nu\rangle = 1$, $\langle \nu, X\rangle = 0$, and $\langle \nu, X_i\rangle = 0$ gives $\dot\nu = -\nabla f - fX$ and $\dot X_i = f_i\nu + fS_i{}^k X_k$. At a geodesic coordinate origin,

$$\begin{aligned}\dot S_{ij} &= \langle D_i\dot\nu, X_j\rangle + \langle D_i\nu, \dot X_j\rangle = -\nabla_i\nabla_j f - fg_{ij} + f(S^2)_{ij},\\ \dot g^{ij} &= -2fS^{ij}.\end{aligned}$$

Raising an index therefore gives the middle formula in (2.10). Its trace, with $\Delta f = -\operatorname{tr}\nabla^2 f$, is $\Delta f - (|S|^2 + d)f = Jf$. For $X_s = e^{sQ}X$, the trace vanishes for every s. The tangential component of the variation differentiates the zero function $\operatorname{tr} S$ and contributes zero. Its normal speed $f = \langle QX, \nu\rangle$ therefore satisfies $Jf = 0$.

□

2.2. **Reflection and embeddedness.** For a normalized closing, the transformations $(r,\theta) \mapsto (r, \pi/2 - \theta)$ and $(r,\theta) \mapsto (\pi - r, \theta)$ preserve the weighted orbit length. The endpoint tangents are perpendicular to $\{\theta = \pi/4\}$ and $\{r = \pi/2\}$, respectively. Uniqueness for the nonsingular geodesic initial-value problem identifies the reflected arcs with smooth continuations. In terms of a, b, c,

$$\begin{aligned} &a(-t) = b(t), && b(-t) = a(t), && c(-t) = c(t),\\ &a(2\ell - t) = a(t), && b(2\ell - t) = b(t), && c(2\ell - t) = -c(t).\end{aligned} \tag{2.11}$$

The four open arcs lie in distinct chambers distinguished by the signs of $a - b$ and c. Each arc is injective because r and θ are strictly monotone there. Their only common points are the prescribed adjacent endpoints. Hence the closed profile is embedded. If two values of (1.6) agree, taking the norms of the first two blocks and the value of the last coordinate first identifies the profile point. Positivity of a, b then identifies y, z. Thus X is an embedding.

2.3. **Quadratic forms and trace conditions.** On a compact interval, a smooth positive weight is bounded above and below by positive constants. The form

$$Q_V(f,g) = \int w(f'g' - Vfg) \tag{2.12}$$

on H^1, or on any of its closed zero-trace subspaces, is closed after adding a sufficiently large multiple of the $L^2(w)$ norm. The embedding of H^1 into $L^2(w)$ is compact: bounded H^1 sets have continuous representatives with $|f(t) - f(s)| \le \|f'\|_{L^2}|t-s|^{1/2}$ and a uniform supremum bound, so a uniformly converging subsequence exists. The associated self-adjoint realizations have discrete spectra and satisfy min–max. The same argument applies to the circle with periodic endpoint data.

Lemma 2.2 (Positive reference identity). *Let $\mathcal{J} = L_w - V$, and first suppose that u is smooth and strictly positive on the closed interval I. For every smooth f,*

$$Q_V(f,f) = \int wu^2\left(\frac{f}{u}\right)'^2 + \int w\frac{\mathcal{J}u}{u}f^2 + \left[w\frac{u'}{u}f^2\right]_{\partial I}. \tag{2.13}$$

The identity also holds by endpoint limits if u is positive only in the interior, has simple zeros at any zero endpoint, and f has zero trace at each such endpoint. The corresponding boundary term then vanishes. It also vanishes at an endpoint where $u \ne 0$ and $u' = 0$. A lower bound obtained in this way extends to the corresponding H^1 form domain.

Proof. For $f = u\psi$,

$$w(f')^2 = wu^2(\psi')^2 + (wuu'\psi^2)' - u(wu')'\psi^2.$$

Integrating proves (2.13). Near a simple zero A, $u(t) = \kappa(t-A) + O((t-A)^2)$ with $\kappa \neq 0$. For smooth f of zero trace, $f = O(t-A)$ and $w(u'/u)f^2 = O(t-A)$. For density, extend an H^1 function across the endpoints by reflection and mollify. If the prescribed zero endpoints are t_i, choose fixed smooth functions χ_i satisfying $\chi_i(t_j) = \delta_{ij}$ and subtract $\sum_i f_k(t_i)\chi_i$ from each mollification f_k. Continuity of point evaluation on H^1 makes these corrections converge to zero in H^1. The corrected functions have the required traces. Passing to the limit in the resulting quadratic-form inequality proves the assertion; equality of singular integrals is not needed for this passage. □

Lemma 2.3 (Finite-codimension comparison)**.** *Let $\mathcal{V}$ be a closed subspace of codimension r in a form domain, and suppose $Q(f,f) \geq c\|f\|^2$ on $\mathcal{V}$. Then its rth eigenvalue, indexed from zero, is at least c.*

Proof. Let E_r be the span of eigenfunctions with indices $0, \dots, r$. The quotient map $E_r \to \mathcal{D}(Q)/\mathcal{V}$ has target of dimension r and domain of dimension $r+1$. Its kernel contains $0 \neq f \in E_r \cap \mathcal{V}$. Expanding f in an orthonormal eigenbasis gives

$$c\|f\|^2 \leq Q(f,f) = \sum_{j=0}^{r} \lambda_j |f_j|^2 \leq \lambda_r \|f\|^2.$$

Division by $\|f\|^2 > 0$ proves $\lambda_r \geq c$. The evaluation functionals used below are independent: smooth functions supported in disjoint neighborhoods of their evaluation points prescribe their values independently. Thus their kernels have the asserted codimensions. □

Lemma 2.4 (Ground states and separated boundaries)**.** *For a regular scalar operator $L_w - V$ on a compact interval, each separated Dirichlet or Neumann eigenspace is one-dimensional. Its first eigenfunction is strictly positive in the interior, after a choice of sign. For the Neumann–Neumann and Neumann–Dirichlet spectra, denoted by λ_j^{NN} and λ_j^{ND},*

$$\lambda_j^{\mathrm{NN}} < \lambda_j^{\mathrm{ND}} < \lambda_{j+1}^{\mathrm{NN}}. \tag{2.14}$$

Proof. At the left endpoint each separated boundary condition leaves exactly one free initial value. Existence and uniqueness for the second-order equation therefore bound an eigenspace dimension by one. A minimizer of the first Rayleigh quotient exists by form compactness. Replacing it by its absolute value preserves its norm and energy. The weak Euler equation makes the replacement smooth. A nonnegative nonzero solution cannot vanish in the interior: its derivative would vanish there as well, contradicting uniqueness. This proves positivity. The Neumann–Dirichlet form domain is a codimension-one subspace of the Neumann–Neumann form domain. Min–max gives the two inequalities in (2.14) with $\leq$. Equality would be a common spectral value. Since solutions satisfying the left Neumann condition form a line, a common eigenfunction would have both value and normal derivative zero at the right endpoint, a contradiction. □

3. The weighted planar problem

3.1. **Euclidean coordinates on the orbit space.** For a quarter-profile introduce

$$X = -\log\tan(r/2), \qquad Y = \pi/4 - \theta. \tag{3.1}$$

Then $\sin r = \operatorname{sech} X$, $\cos r = \tanh X$, and

$$dr^2 + \sin^2 r\, d\theta^2 = \operatorname{sech}^2 X (dX^2 + dY^2).$$

The orbit density is proportional to $\sin^{2p} r \sin^p(2\theta)$. Multiplication by the additional factor $\sin r$ converting spherical arclength into Euclidean arclength gives

$$\omega_p = e^{-\Phi_p}, \qquad \Phi_p(X,Y) = d\log\cosh X - p\log\cos 2Y. \tag{3.2}$$

The omitted factor is $2^{-p} > 0$. Multiplication of a variational functional by this constant preserves its critical points and the inertia of its Hessian.

Let σ be Euclidean arclength and $s = \alpha + \pi/2$. Put

$$A(X) = d\tanh X, \qquad B(Y) = 2p\tan 2Y. \tag{3.3}$$

The tangent, outward normal and curvature are

$$T = (-\sin s, \cos s), \quad N = (\cos s, \sin s), \quad K = A(X)\cos s + B(Y)\sin s, \tag{3.4}$$

and

$$X_s = -\frac{\sin s}{K}, \qquad Y_s = \frac{\cos s}{K}, \qquad s_\sigma = K. \tag{3.5}$$

The initial and final points are $(X_0, 0)$ and $(0, Y_0)$. All quantities X, Y, K are positive in the open quarter, and K is positive at both endpoints.

To verify the Euler equation, vary a plane curve normally by fN. Since $T_\sigma = -KN$, the first variation of $\int e^{-\Phi}\,d\sigma$ is $\int \omega(K - \Phi_N)f$. Thus stationarity is $K = \Phi_N$, which is (3.4). Reflecting the quarter in the coordinate axes gives a smooth oval with positive curvature and total turning 2π. Its tangent angle increases once through a full turn; every direction has a unique supporting tangent. The enclosed region Ω is convex. In particular, its two coordinate vertices give

$$X\cos s + Y\sin s \geq \max\{X_0\cos s, Y_0\sin s\}. \tag{3.6}$$

Lemma 3.1 (Elementary comparisons)**.** *One has* $3 < \pi < 22/7$*, hence* $\pi^2 < 10$*, and* $\tan(5/4) < 25/8$*.*

Proof. Since $\sin x < x$ for $x > 0$, substitution of $x = \pi/6$ gives $\pi > 3$. Polynomial division and integration give

$$0 < \int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,dx = \frac{22}{7} - \pi. \tag{3.7}$$

Indeed the quotient is $x^6 - 4x^5 + 5x^4 - 4x^2 + 4$ and the remainder is -4. Its polynomial integral is $22/7$, and $4\int_0^1(1+x^2)^{-1}dx = \pi$. The inequality $484/49 < 10$ gives the next assertion. Let

$$s_+(x) = x - x^3/6 + x^5/120, \qquad c_-(x) = 1 - x^2/2 + x^4/24 - x^6/720.$$

The alternating series bounds give $\sin x \leq s_+(x)$ and $\cos x \geq c_-(x)$ on $[0, 3/2]$. At $u = 5/4$,

$$c_-(u) = \frac{929495}{2949120} > 0, \qquad \frac{5}{2}c_-(u) - \frac{s_+(u)}{u} = \frac{33047}{1179648} > 0. \tag{3.8}$$

Thus $\tan u < 5u/2 = 25/8$. □

3.2. **Width estimates.** The following estimates apply to every closing quarter, without a uniqueness assumption.

Lemma 3.2. *For* $p \geq 1$*,*

$$X_0 \leq \frac{\pi}{4\sqrt{p}}, \qquad Y_0 < \frac{7}{8\sqrt{p}}. \tag{3.9}$$

For $p \geq 2$*,*

$$X_0 Y_0 \geq \frac{\pi}{2(12p+1)}. \tag{3.10}$$

Proof. Write the quarter as $Y = Y(X)$ starting at the vertical-axis point. From (3.5),

$$Y_{XX} = (1 + Y_X^2)(A(X)Y_X - B(Y)). \tag{3.11}$$

Here $Y_X < 0$, $A \geq 0$ and $B \geq 4pY$. Consequently $Y_{XX} + 4pY \leq 0$ while $Y > 0$, with $Y_X(0) = 0$. Let $v(X) = Y_0\cos(2\sqrt{p}X)$. Up to its first zero,

$$(vY_X - v_X Y)' = v(Y_{XX} + 4pY) \leq 0.$$

It vanishes initially, so Y/v is nonincreasing. Thus Y cannot remain positive beyond $\pi/(4\sqrt{p})$; this proves the first bound.

For $x \geq 0$,

$$\tanh x \geq \frac{x}{1 + x^2/3}. \tag{3.12}$$

Indeed, the difference after multiplication by $\cosh x(1 + x^2/3)$ is $F(x) = (1 + x^2/3)\sinh x - x\cosh x$. Its derivative is $(x/3)(x\cosh x - \sinh x) \geq 0$, because the expression in parentheses has derivative $x \sinh x \geq 0$ and vanishes at zero. The first width bound now gives

$$A(X) \geq \frac{dX}{1 + \pi^2/(48p)} \geq 2pX. \tag{3.13}$$

The second inequality is equivalent to $1 \geq \pi^2/24$.

Write the quarter instead as $X = X(Y)$ from $(X_0, 0)$. Its equation is

$$X_{YY} = (1 + X_Y^2)(B(Y)X_Y - A(X)) \leq 4pYX_Y - 2pX. \tag{3.14}$$

The factor $1 + X_Y^2$ may be removed in this upper bound because the quantity it multiplies is negative. Put $z = 2\sqrt{p}Y$. Then $X_{zz} - zX_z + X/2 \leq 0$. Let

$$h'' - zh' + h/2 = 0, \qquad h(0) = 1, \quad h'(0) = 0. \tag{3.15}$$

Its entire even series $h = \sum c_k z^{2k}$ has $c_1 = -1/4$, $c_2 = -1/32$, and

$$c_{k+1} = \frac{2k - 1/2}{(2k+2)(2k+1)} c_k < 0 \quad (k \geq 1).$$

Thus $h(z) \leq 1 - z^2/4 - z^4/32$ and $h(7/4) \leq -481/8192 < 0$. Before its first zero,

$$\left[e^{-z^2/2}(hX_z - h_z X)\right]' = e^{-z^2/2} h(X_{zz} - zX_z + X/2) \leq 0.$$

The quotient X/h is therefore nonincreasing. The first zero of X occurs no later than that of h, proving $Y_0 < 7/(8\sqrt{p})$.

Assume $p \geq 2$. Equations (3.9) and (3.12), with $\pi^2 < 10$, give

$$\tanh X \geq \frac{48}{53} X > \frac{9}{10} X, \qquad \tan 2Y \leq 5Y. \tag{3.16}$$

For the second inequality, $2Y < 7/(4\sqrt{2}) < 5/4$. The derivative of $\tan u/u$ has the sign of $u\sec^2 u - \tan u$, whose derivative is $2u\sec^2 u \tan u > 0$. Lemma 3.1 therefore gives $\tan 2Y \leq 5Y$.

By total turning, $K = \nabla\Phi \cdot N$, and the divergence theorem,

$$2\pi = \int_\Omega \Delta_{\mathbb{R}^2} \Phi_p.$$

The integrand $A'(X) + B'(Y)$ is positive. Bounding the integral by the coordinate rectangle gives

$$\begin{aligned} 2\pi &\leq 4Y_0 d \tanh X_0 + 8pX_0 \tan 2Y_0 \\ &\leq 4X_0Y_0(12p+1), \end{aligned} \tag{3.17}$$

which proves (3.10). $\square$

3.3. **The planar Jacobi form.** For an even separated potential $\Phi(X,Y)$, set

$$\mathscr{J} = -\frac{d^2}{d\sigma^2} + \Phi_\sigma \frac{d}{d\sigma} - \big(K^2 + \Phi_{NN}\big), \qquad L_\omega = -\omega^{-1}(\omega \, \cdot \, ')'. \tag{3.18}$$

Its quadratic form is the second variation of weighted length. To verify it without a convention ambiguity, for $\gamma_u = \gamma + ufN$,

$$|\partial_\sigma \gamma_u| = 1 + uKf + \tfrac{1}{2}u^2(f_\sigma)^2 + O(u^3),$$

and

$$e^{-\Phi(\gamma_u)} = \omega\{1 - u\Phi_N f + \tfrac{1}{2}u^2(\Phi_N^2 - \Phi_{NN})f^2 + O(u^3)\}.$$

Using $K = \Phi_N$ gives

$$Q(f) = \int \omega\{f_\sigma^2 - (K^2 + \Phi_{NN})f^2\}. \tag{3.19}$$

The axes are reflection lines. Normal variations with endpoints free on the axes have Neumann conditions. Equivalently, those conditions will be obtained directly from the shooting variation in lemma 4.3.

Lemma 3.3. *If $\Phi_X = A(X)$ and $\Phi_Y = B(Y)$, then*

$$\mathscr{J}(\cos s) = -A' \cos s, \quad \mathscr{J}(\sin s) = -B' \sin s, \quad L_\omega X = A, \quad L_\omega Y = B. \tag{3.20}$$

Proof. Differentiation gives $K_\sigma = \Phi_{TN} + K\Phi_T$, $N_\sigma = KT$ and $T_\sigma = -KN$. In $\mathscr{J}(\cos s)$ the terms containing K^2 cancel; the remainder is $\Phi_{TN} \sin s - \Phi_{NN} \cos s = -A' \cos s$. In $\mathscr{J}(\sin s)$ the remainder is $-\Phi_{TN} \cos s - \Phi_{NN} \sin s = -B' \sin s$. Also $X_\sigma = -\sin s$, $Y_\sigma = \cos s$ and $\Phi_T = -A \sin s + B \cos s$. Substitution into $L_\omega = -\partial_\sigma^2 + \Phi_T \partial_\sigma$ gives the last two identities. $\square$

4. Analytic uniqueness for $p \geq 2$

4.1. **A residual-ratio criterion.** For (3.2), the function $H = Y \cos s$ is positive inside the quarter and has simple zeros at its endpoints. By lemma 3.3,

$$\mathscr{J} H = (B - A'Y) \cos s + 2K \sin s \cos s \geq (2p-1)H. \tag{4.1}$$

Here $B \geq 4pY$ and $A' \leq d = 2p+1$. The identity in lemma 2.2, now with weight ω, makes the Dirichlet–Dirichlet form strictly positive.

Lemma 4.1. *Suppose the Dirichlet form of* (3.18) *is strictly positive and its Neumann ground eigenvalue is negative. Let Neumann functions ϕ_1, ϕ_2 satisfy $\mathscr{J}\phi_i = -P_i$, where $P_i > 0$ in the interior. If P_1/P_2 is strictly increasing, the Neumann kernel is zero.*

Proof. A nonzero kernel function f is orthogonal to the positive ground eigenfunction and therefore changes sign. Its zeros are simple by initial-value uniqueness. It cannot have two interior zeros: restriction between them and extension by zero would give a nonzero Dirichlet function of quadratic form zero. It cannot vanish at an endpoint, since it has Neumann data there. Thus it has exactly one zero s_0. Green's identity gives $\int \omega f P_i = 0$. But

$$fP_2 \left(\frac{P_1}{P_2} - \frac{P_1(s_0)}{P_2(s_0)} \right)$$

has one strict sign on both sides of s_0, up to the choice of sign of f. Its integral is nonzero, contradicting the two equalities. $\square$

In the present problem the constant function has negative quadratic form, since $K^2 + \Phi_{NN} > 0$. Thus the ground eigenvalue in lemma 4.1 is indeed negative. Choose

$$\phi_1 = A(X) \cos s, \qquad \phi_2 = B(Y) \sin s. \tag{4.2}$$

Their derivatives vanish at the endpoints. For example, $(A\cos s)_\sigma = -A'\sin s\cos s - AK\sin s$; at the initial endpoint $\sin s = 0$, and at the final endpoint $\cos s = 0$, $A(0) = 0$. The derivative of $B\sin s$ vanishes because $B(0) = 0$ initially and $\cos s = 0$ finally. The product rule gives

$$P_1 = \sin^2 s(2KA' + A''\cos s), \qquad P_2 = \cos^2 s(2KB' + B''\sin s). \tag{4.3}$$

Put $x = \tanh X$, $y = \tan 2Y$, and $t = \tan s$. Since $A'' = -2xA'$ and $B'' = 4yB'$, both residuals are positive in the interior and

$$\mathcal{R} := \frac{P_1}{P_2} = \frac{d}{2}\frac{1-x^2}{1+y^2}\,t^2\frac{x+yt}{dx+(d+1)yt}. \tag{4.4}$$

Proposition 4.2. *For every closing quarter with real $p \geq 2$, $\mathcal{R}_s > 0$ on $0 < s < \pi/2$. Consequently its Neumann Jacobi kernel is zero.*

Proof. Set $D_0 = dx + (d-1)yt$. Equations (3.5) give

$$x_s = -\frac{(1-x^2)t}{D_0}, \qquad y_s = \frac{2(1+y^2)}{D_0}, \qquad t_s = 1+t^2. \tag{4.5}$$

The logarithmic derivative is

$$\begin{aligned}(\log\mathcal{R})_s = &-\frac{2xx_s}{1-x^2} - \frac{2yy_s}{1+y^2} + \frac{2t_s}{t} + \frac{x_s + ty_s + yt_s}{x+yt}\\ &-\frac{dx_s + (d+1)(ty_s + yt_s)}{dx+(d+1)yt}.\end{aligned} \tag{4.6}$$

Substitution of (4.5), followed by multiplication by the positive quantity $tD_0(x+yt)(dx+(d+1)yt)$, gives

$$\begin{aligned}\mathcal{P}_d = {}&2(d^2-1)t^5y^3 + (6d^2+d+1)t^4xy^2 + (6d^2+3d+3)t^3x^2y\\ &+ 2(d-3)(d+1)t^3y^3 + 2d(d+1)t^2x^3 + (6d^2-9d-7)t^2xy^2\\ &+ d(6d-5)tx^2y + 2d^2x^3 - t^3y - 2t^2x.\end{aligned} \tag{4.7}$$

An unexpanded identity fixing this algebraic calculation is

$$\begin{aligned}\mathcal{P}_d = {}&[2(1+t^2)D_0 + (2xt-4y)t](x+yt)(dx+(d+1)yt)\\ &- t[2x(1+y^2)t + xy(1+t^2)D_0 + y(1-x^2)t^2].\end{aligned}$$

For $d \geq 5$, comparison of the eight positive monomials in (4.7) yields

$$\mathcal{P}_d \geq \frac{24}{25}d^2(1+t^2)(x+yt)^3 - t^2(2x+yt). \tag{4.8}$$

In their displayed order, the eight coefficient differences are

$$\begin{gathered}\frac{26d^2}{25} - 2, \quad \frac{78d^2}{25} + d + 1, \quad \frac{78d^2}{25} + 3d + 3, \quad \frac{2(d-5)(13d+15)}{25},\\ \frac{26d^2}{25} + 2d, \quad \frac{78}{25}(d-5)^2 + \frac{111}{5}(d-5) + 26, \quad \frac{d(78d-125)}{25}, \quad \frac{26d^2}{25}.\end{gathered} \tag{4.9}$$

Every entry is nonnegative; no coefficient estimate depends on a numerical profile.

Let $c_* = 24/25$ and $m_* = 9/10$. The support inequality (3.6), (3.16), and $\tan 2Y \geq 2Y$ give

$$x\cos s + y\sin s \geq m_*\max\{X_0\cos s, Y_0\sin s\}. \tag{4.10}$$

By (3.10),

$$c_*d^2m_*^2(X_0^2 + Y_0^2) \geq c_*d^2m_*^2\frac{\pi}{12p+1} \geq \frac{486\pi}{625} > 2. \tag{4.11}$$

The middle inequality uses $(2p+1)^2 \geq 12p+1$ for $p \geq 2$; the last uses $\pi > 3$. Both inequalities $c_* d^2 m_*^2 X_0^2 \leq 2\sin^2 s$ and $c_* d^2 m_*^2 Y_0^2 \leq 2\cos^2 s$ cannot hold. Multiplication of the violated inequality by the complementary trigonometric square and (4.10) imply

$$c_* d^2 (x\cos s + y\sin s)^2 > 2\sin^2 s\cos^2 s.$$

Equivalently, $c_* d^2(1+t^2)(x+yt)^2 > 2t^2$. Since $2x+yt \leq 2(x+yt)$, (4.8) gives $\mathcal{P}_d > 0$. This proves monotonicity. The last assertion follows from lemma 4.1 and (4.1). □

4.2. **Shooting and its Jacobi field.**

Lemma 4.3. *For an even separated potential, suppose a closing quarter has $K > 0$ on its closed interval. Let $\mathcal{F}(z) = X(\pi/2; z)$ be the closing residual for initial values $(X, Y) = (z, 0)$. Its derivative vanishes if and only if the Neumann Jacobi operator has a nonzero kernel.*

Proof. The fixed-angle equation is analytic on a neighborhood of the compact trajectory. If $V = \partial_z\gamma$ and $f = V \cdot N$, then $V_s = \partial_z(K^{-1})T$ and $N_s = T$, so $f_s = V \cdot T$. The normal speed of this family of weighted geodesics solves $\mathscr{J} f = 0$. At the initial point $f = 1$ and $f_\sigma = 0$. At the final point $T = (-1, 0)$, hence

$$f_\sigma(\pi/2) = -K(\pi/2)\mathcal{F}'(z). \tag{4.12}$$

If the derivative vanishes, f is a nonzero Neumann kernel function. Conversely, every Neumann solution is determined by its initial value, since its initial derivative is zero. A nonzero one is a multiple of f. This proves the equivalence. In the physical coordinates, $X_0 = -\log\tan(r_0/2)$ has derivative $-1/\sin r_0$. At the closing endpoint, $r_t = 1$ and $\alpha_t = 2p\cot(2\theta(\ell)) > 0$. Differentiating the two alternative endpoint conditions gives

$$F'(r_0) = -\frac{2p\cot(2\theta(\ell))}{\sin r_0}\mathcal{F}'(X_0), \tag{4.13}$$

where F is the angle measured at the moving equator intersection. Thus nondegeneracy is equivalent in both conventions. □

Lemma 4.4 (Compact families of simple roots)**.** *Let $[a, b]$ be a compact interval and let $\mathcal{Z}$ be a compact subset of $\mathbb{R} \times [a, b]$. Suppose that near each $(z_0, t_0) \in \mathcal{Z}$ there is a C^1 function $F(z, t)$ such that $\mathcal{Z} = \{F = 0\}$ locally and $F_z(z_0, t_0) \neq 0$. Then the number of points of $\mathcal{Z}$ over t is finite and independent of t.*

Proof. The implicit function theorem expresses $\mathcal{Z}$ near each of its points as one graph over a relatively open parameter interval. A fiber is therefore discrete. It is also compact, hence finite. Fix t_0 and choose disjoint graph neighborhoods of all the points in its fiber. After shrinking the parameter interval, every one of these graphs meets each nearby fiber exactly once. There are no further points in nearby fibers. Otherwise a sequence of such points with parameter tending to t_0 would, by compactness, have a subsequence converging to a point in the fiber at t_0, and would eventually belong to one of the chosen neighborhoods. The cardinality is thus locally constant. A locally constant integer-valued function on $[a, b]$ is constant. □

4.3. **The Gaussian starting problem.**

Lemma 4.5. *For $a_0 = 2$ and every $b_0 \in [2, 4]$, the potential $\Phi = (a_0X^2 + b_0Y^2)/2$ has exactly one positive closing quarter. Every such closing is nondegenerate.*

Proof. The function $H = Y\cos s$ satisfies

$$\mathscr{J} H = (b_0 - a_0)Y\cos s + 2K\sin s\cos s > 0.$$

The ratio $\mathscr{J} H/H$ extends to positive endpoint values $b_0 - a_0 + 2a_0^2X_0^2$ and $3b_0 - a_0$. Thus the Dirichlet form is strictly positive. The Neumann tests $X\cos s, Y\sin s$ have positive residuals $2K\sin^2 s, 2K\cos^2 s$, with ratio $\tan^2 s$. Lemma 4.1 excludes a Neumann kernel; constants have negative form because the Hessian of Φ is positive.

The graph equations and cosine comparison give

$$X_0 \le \frac{\pi}{2\sqrt{b_0}}, \quad Y_0 \le \frac{\pi}{2\sqrt{a_0}}, \quad X_0Y_0 \ge \frac{\pi}{2(a_0+b_0)}. \tag{4.14}$$

For the last bound use $(a_0+b_0)\,\mathrm{Area}(\Omega) = 2\pi$ and $\mathrm{Area}(\Omega) \le 4X_0Y_0$. These inequalities keep both intercepts bounded above and away from zero. Let $\eta > 0$ be a common lower bound for both intercepts, supplied by (4.14). The support inequality gives

$$K \ge 2(X\cos s + Y\sin s) \ge 2\eta \max\{\cos s, \sin s\} \ge \sqrt{2}\eta.$$

Thus the trajectories lie in a compact subset on which the angle equation has no zero denominator. All derivatives of that vector field are bounded there. From any sequence of closings and parameters, compactness of the initial data and continuous dependence on a compact time interval give a convergent subsequence of solutions. The endpoint equalities pass to the limit. Its interior satisfies $X_s < 0$ and $Y_s > 0$, so it remains a closing quarter. The set of closings over $b_0 \in [2,4]$ is compact.

At $a_0 = b_0 = 2$, rotation of the curve has normal speed $\psi = X\sin s - Y\cos s$. Rotational invariance gives $\mathscr{J}\psi = 0$, and ψ vanishes at both endpoints. Dirichlet positivity implies $\psi = 0$. Since $\gamma\cdot T = -\psi$, the radius is constant. The equation $K = 2\gamma\cdot N$ then gives radius $1/\sqrt{2}$. This quarter-circle exists and is unique.

Every closing is a simple root by lemma 4.3. Apply lemma 4.4 to this compact root set. Its fiber at $b_0 = 2$ consists of the quarter-circle; every fiber therefore has one point. □

4.4. **Proper continuation in the dimensional parameter.**

Set $\delta = 1/p$, $\bar{X} = \sqrt{p}X$, $\bar{Y} = \sqrt{p}Y$. The rescaled potential is

$$\Phi_\delta(\bar{X},\bar{Y}) = \left(\frac{2}{\delta}+1\right)\log\cosh(\sqrt{\delta}\bar{X}) - \frac{1}{\delta}\log\cos(2\sqrt{\delta}\bar{Y}). \tag{4.15}$$

Its apparent singularity at zero is removable; expansion of the even analytic logarithms gives the limit $\bar{X}^2 + 2\bar{Y}^2$. For every closing with $0 < \delta \le 1/2$, lemma 3.2 gives

$$\bar{X}_0 \le \pi/4, \quad \bar{Y}_0 \le 7/8, \quad \bar{X}_0\bar{Y}_0 \ge \pi/25. \tag{4.16}$$

Thus $\bar{X}_0 \ge 8\pi/175$ and $\bar{Y}_0 \ge 4/25$. Also $2\sqrt{\delta}\bar{Y} \le 7/(4\sqrt{2}) < \pi/2$. In rescaled coordinates the forces are

$$\bar{A}_\delta = (2+\delta)\frac{\tanh(\sqrt{\delta}\bar{X})}{\sqrt{\delta}} \ge 2\bar{X}, \qquad \bar{B}_\delta = 2\frac{\tan(2\sqrt{\delta}\bar{Y})}{\sqrt{\delta}} \ge 4\bar{Y},$$

with their continuous values at zero. Put $\eta = \min\{8\pi/175, 4/25\}$. Along a closing, the support inequality therefore gives $\bar{K} \ge \sqrt{2}\eta > 0$. The trigonometric denominator and $\bar{K}$ are bounded away from zero on these trajectories. The analytic vector field is consequently defined on a neighborhood of their compact containing set. The solution-limit argument of lemma 4.5 applies there and proves compactness of the whole closing set over $0 \le \delta \le 1/2$.

For $\delta > 0$, proposition 4.2 and lemma 4.3 make every root simple. At zero the same follows from lemma 4.5. By lemma 4.4, the cardinality is constant. The Gaussian problem has one root at zero; hence every parameter has one root. This proves theorem 1.2 for every real $p \ge 2$.

4.5. **The Neumann index along the continuation.**

Proposition 4.6. *For every real $p \ge 2$, the Neumann realization of $\mathscr{J}$ on its unique closing quarter has index one and zero nullity. Its second eigenvalue is strictly positive.*

Proof. The nondegeneracy established in proposition 4.2 and lemma 4.5 holds along both continuations used above. Their weights, potentials and interval lengths depend continuously on the parameter. Pulling each arclength interval back to $[0,1]$ gives forms

$$Q_s(f) = \int_0^1 \left(a_s|f'|^2 - b_s|f|^2\right), \qquad \|f\|_s^2 = \int_0^1 h_s|f|^2, \tag{4.17}$$

with a_s, h_s uniformly positive and all coefficients continuous in the parameter in the supremum norm. On a compact parameter interval, there are constants $a_*, h_*, h^* > 0$ and $b^* < \infty$ such that $a_s \geq a_*$, $h_* \leq h_s \leq h^*$ and $|b_s| \leq b^*$. The min–max principle and the Neumann spectrum on $[0,1]$ give

$$\lambda_j(Q_s) \geq \frac{a_*}{h^*}\pi^2 j^2 - \frac{b^*}{h_*}. \tag{4.18}$$

After adding a fixed positive constant to the Rayleigh quotient, supremum-norm convergence of the coefficients bounds nearby forms above and below by factors tending to one. Min–max therefore makes each fixed eigenvalue continuous. The bound (4.18) excludes negative spectrum escaping to unbounded index. In the absence of a zero eigenvalue, the negative index is locally constant.

At the isotropic Gaussian quarter, the radius is $1/\sqrt{2}$, $\Phi = X^2 + Y^2$, and ω is constant along the curve. Since $\sigma = s/\sqrt{2}$,

$$\mathscr{J} = -2\partial_s^2 - 4, \qquad \lambda_j^{\mathrm{NN}} = 8j^2 - 4 \quad (j \geq 0). \tag{4.19}$$

There is precisely one negative eigenvalue and no zero. Constancy of the index first along $b_0 \in [2,4]$ and then along $\delta \in [0, 1/2]$ proves the assertion. Rescaling the planar coordinates multiplies weighted length by a positive constant and rescales normal speeds by a positive constant; it does not change the inertia of its second variation. ∎

5. The circle-factor closing problem

The residual-ratio argument above requires $p \geq 2$. This section first proves a global initial-range bound by an analytic comparison. A finite interval exclusion then determines the entire closing set.

Lemma 5.1 (A logarithmic comparison)**.** *Let h be the even solution on $(-1,1)$ of*

$$(1-z^2)h'' - 2zh' + \frac{9}{16}h = 0, \qquad h(0) = 1, \quad h'(0) = 0. \tag{5.1}$$

Then

$$h(z) \leq 1 + \frac{207}{1024}\log(1-z^2) \qquad (0 \leq z < 1), \tag{5.2}$$

and $h(\sin(3/2)) < 0$.

Proof. The coefficients $h(z) = \sum_{k\geq 0} c_k z^{2k}$ satisfy

$$c_0 = 1, \quad c_1 = -\frac{9}{32}, \qquad c_{k+1} = \frac{2k(2k+1) - 9/16}{(2k+2)(2k+1)}c_k. \tag{5.3}$$

The ratio tends to one, so the series converges for $|z| < 1$ and termwise differentiation verifies the equation. Initial-value uniqueness identifies it with h. For $k \geq 1$, put $a_k = -kc_k$. The recurrence gives

$$a_k = \frac{9}{32}\prod_{j=1}^{k-1}\left(1 - \frac{9}{32j(2j+1)}\right). \tag{5.4}$$

For numbers $0 \leq b_j \leq 1$, induction gives $\prod(1-b_j) \geq 1 - \sum b_j$. Moreover

$$\sum_{j=1}^{k-1}\frac{1}{j(2j+1)} < \sum_{j=1}^{\infty}\frac{1}{j(j+1)} = 1.$$

Thus $a_k \geq 207/1024$. Summing $h = 1 - \sum_{k\geq 1} a_k z^{2k}/k$ and using $\sum_{k\geq 1} z^{2k}/k = -\log(1-z^2)$ proves (5.2).

The alternating sine estimate gives

$$\sin(3/2) > \frac{71493}{71680} > \frac{997}{1000}. \tag{5.5}$$

Also

$$e = \sum_{k=0}^{\infty} \frac{1}{k!} \leq \frac{8}{3} + \frac{1}{24}\sum_{j=0}^{\infty} 5^{-j} = \frac{87}{32} < \frac{11}{4}.$$

Since $5991 \cdot 161051 = 964856541 < 1024000000$,

$$1 - \sin^2(3/2) < \frac{5991}{10^6} < \left(\frac{4}{11}\right)^5 < e^{-5}.$$

Therefore $h(\sin(3/2)) < 1 - 5 \cdot 207/1024 = -11/1024 < 0$. ☐

Lemma 5.2. *Every normalized monotone closing for $p = 1$, with $z_0 = \pi/2 - r_0$, satisfies*

$$\frac{1}{25} < z_0 < \frac{4}{5}. \tag{5.6}$$

Proof. By lemma 3.2, $X_0 \leq \pi/4$. The elementary hyperbolic bound and $\pi^2 < 10$ give $3\tanh X \geq 72X/29 > 9X/4$. The graph equation therefore implies

$$X_{YY} - 2\tan 2Y\, X_Y + \frac{9}{4}X \leq 0. \tag{5.7}$$

Substitution of $z = \sin 2Y$ in its equality gives (5.1). The integrating factor of (5.7) is $\cos 2Y$. Before the first zero of the positive comparison solution, its weighted Wronskian with X is nonincreasing from zero. Hence the quotient of X by that solution is nonincreasing, and X reaches zero no later than the comparison solution. Lemma 5.1 gives

$$Y_0 < 3/4. \tag{5.8}$$

The alternating bounds at $3/2$ give

$$\cos(3/2) > \frac{359}{5120}, \qquad \sin(3/2) < \frac{1281}{1280}, \qquad \tan(3/2) < \frac{5124}{359} < 15.$$

Monotonicity of $\tan(2Y)/Y$ yields $\tan 2Y_0 < 20Y_0$. Using the rectangle estimate before its final simplification gives $2\pi < 172 X_0 Y_0$. Therefore

$$X_0 Y_0 > \pi/86, \qquad X_0 > 2\pi/129 > 2/43.$$

Since $z_0 = \arcsin(\tanh X_0)$, its derivative in X_0 is $\operatorname{sech} X_0 < 1$ for $X_0 > 0$. Thus $z_0 < X_0 \leq \pi/4 < 4/5$. Conversely,

$$z_0 > \tanh(2/43) \geq \frac{258}{5551} > \frac{1}{25}.$$

This proves the claim. ☐

Proposition 5.3 (Finite shooting exclusion)**.** *Every initial value $z_0 \in [1/25, 337/1000]$ exits through the side $r = \pi/2$ before $\alpha = 0$. Every initial value $z_0 \in [17/50, 4/5]$ exits through $\alpha = 0$ before $r = \pi/2$. The interval $[337/1000, 17/50]$ contains exactly one monotone closing and its closing derivative is nonzero.*

Proof. The endpoint inequalities below follow from the enclosure lemma of section A, applied to the following initial-value problems. The rational initial intervals and the containing boxes for successive time intervals belong to the finite enclosure families of section B.2. The passage from those finite inequalities to the exact solution is proposition A.2. Set $x = \tanh X$, $y = \tan 2Y$, $S = \sin s$, $C = \cos s$. In Euclidean arclength the equation is polynomial:

$$\begin{aligned} x' &= -(1-x^2)S, & y' &= 2(1+y^2)C, \\ S' &= (3xC+2yS)C, & C' &= -(3xC+2yS)S, \end{aligned} \tag{5.9}$$

with initial data $(\sin z_0, 0, 0, 1)$. Let $I_1, \ldots, I_{47}$ and $K_1, \ldots, K_{62}$ be the closed rational intervals in the two families of section B.2. Their unions contain $[1/25, 337/1000]$ and $[17/50, 4/5]$, respectively. For initial values in each I_i, the final enclosure satisfies $x < 0 < C$; for initial values in each K_i, it satisfies $C < 0 < x$. Every intervening solution enclosure satisfies

$$-1 < x < 1, \qquad \big(y > 0 \text{ and } S > 0\big) \quad \text{or} \quad \big(x > 0,\ C > 0,\ 3xC + 2yS > 0\big). \tag{5.10}$$

Initially $y = S = 0$. In the second region, $y' > 0$ and $S' > 0$; in the first region both quantities are positive by definition. Thus $y, S > 0$ after the initial time. Consequently $x' < 0$ throughout. At a zero of C, its derivative is $-2yS^2 < 0$, so C cannot cross zero from negative to positive. A final enclosure $x < 0 < C$ therefore implies that $x = 0$ occurred before $C = 0$. A final enclosure $C < 0 < x$ implies the opposite order. This proves the two strict first-exit assertions.

On the central interval use $R = \pi/2 - r$, $U = \pi/4 - \theta$ and s as independent variable. Put $c_R = \cos R$, $s_R = \sin R$, $c_U = \cos 2U$, $s_U = \sin 2U$ and

$$B = \frac{2Ss_U}{c_U c_R} + \frac{3Cs_R}{c_R}.$$

The rational differential equations are

$$\begin{aligned} (c_R)_s &= s_R S/B, & (s_R)_s &= -c_R S/B, \\ (c_U)_s &= -2s_U C/(c_R B), & (s_U)_s &= 2c_U C/(c_R B), \\ S_s &= C, & C_s &= -S. \end{aligned} \tag{5.11}$$

The initial values are $(\cos z_0, \sin z_0, 1, 0, 0, 1)$. To differentiate in z_0, set $q_R = \partial R/\partial z_0$ and $q_U = \partial U/\partial z_0$, with initial values $(1, 0)$, and

$$\begin{aligned} B_R &= \frac{2S(s_U/c_U)s_R + 3C}{c_R^2}, \qquad B_U = \frac{4S}{c_U^2 c_R}, \qquad \dot B = B_R q_R + B_U q_U, \\ (q_R)_s &= S\dot B/B^2, \qquad (q_U)_s = \frac{Cs_R}{c_R^2 B} q_R - \frac{C}{c_R B^2}\dot B. \end{aligned} \tag{5.12}$$

All denominators are positive on the accepted enclosures. At the exact endpoint $s = \pi/2$, the endpoint enclosures give

$$\begin{aligned} s_R(337/1000) &\in (-0.001108855, -0.001108854), \\ s_R(17/50) &\in (0.000916863, 0.000916864), \\ q_R(z_0) &\in (0.383, 0.895) \quad (337/1000 \le z_0 \le 17/50). \end{aligned} \tag{5.13}$$

The derivative of s_R is $c_R q_R > 0$. The intermediate value theorem and strict monotonicity therefore give one and only one root in the central interval. At the root, $R_s = -S/B < 0$, $U_s = C/(c_R B) > 0$ and the positive cosine branches give the required quarter chamber. The root is nondegenerate. Combined with the outside exclusions and lemma 5.2, this proves the proposition. ☐

This completes theorem 1.2. The use of real $p \ge 2$ in the analytic proof and the exceptional calculation at $p = 1$ cover every integer $p \ge 1$. No assertion is needed for $1 < p < 2$.

6. The first Laplace eigenspace

We first give a criterion that does not require equal factor dimensions.

Proposition 6.1 (Two-zero criterion)**.** *For a minimal immersion* (2.1)*, suppose*

$$h = \frac{a}{\sqrt{p}} - \frac{b}{\sqrt{q}} \tag{6.1}$$

has exactly two zeros on the profile circle. Then its invariant scalar spectrum satisfies

$$\mu_0 = 0, \qquad \mu_1 = d \text{ with multiplicity one}, \qquad \mu_2 \geq d + \min_t \frac{\sqrt{pq}}{ab} \geq d + 2\sqrt{pq}. \tag{6.2}$$

The first full eigenspace is exactly the ambient coordinate space, and has dimension $d+2$.

Proof. By lemma 2.1,

$$L_w h = \left(d + \frac{\sqrt{pq}}{ab}\right) h. \tag{6.3}$$

A zero is simple: zero value and derivative would force the solution to vanish identically. Requiring a periodic H^1 function to vanish at the two zeros has codimension two. On its two complementary intervals apply lemma 2.2 with the positive reference $|h|$. Lemma 2.3 gives the asserted bound for μ_2. The inequality $2ab \leq a^2 + b^2 \leq 1$ gives its last term.

The function c is nonconstant. If it were constant, its equation would give $c = 0$, so $a = \cos\theta$, $b = \sin\theta$ with a periodic real-valued $\theta \in (0, \pi/2)$. Unit speed would imply $\theta'^2 = 1$, contradicting an extremum of θ. Thus c supplies an eigenvalue d. Since $\mu_2 > d$, it is the simple first positive invariant eigenvalue.

Let $\alpha_j = j(j+p-1)$ and $\beta_k = k(k+q-1)$. The scalar angular operator is $L_{j,k} = L_w + \alpha_j/a^2 + \beta_k/b^2$. For the positive periodic reference a, lemma 2.2 gives

$$\int w\{(f')^2 + (p/a^2 - d)f^2\} = \int wa^2 \left(\frac{f}{a}\right)'^2. \tag{6.4}$$

Hence $L_{1,0} \geq d$, with equality precisely for multiples of a. The equation for b gives $L_{0,1} \geq d$, with equality precisely for multiples of b. If $j \geq 1$,

$$L_{j,k} = L_{1,0} + (\alpha_j - p)/a^2 + \beta_k/b^2.$$

The additional potential is nonnegative and is strictly positive unless $(j,k) = (1,0)$. The remaining sectors follow from $L_{0,1}$. Choose orthonormal bases $Y_{j,\alpha}$ and $Z_{k,\beta}$ of spherical harmonics. Every $f \in L^2(M)$ has the expansion

$$f = \sum_{j,k,\alpha,\beta} f_{jk\alpha\beta}(t) Y_{j,\alpha}(y) Z_{k,\beta}(z), \qquad \|f\|^2 = \sum_{j,k,\alpha,\beta} \|f_{jk\alpha\beta}\|^2_{L^2(w)}. \tag{6.5}$$

The orthogonal projection onto each term commutes with the weak Laplacian, by (2.2). For an eigenfunction of eigenvalue λ, every nonzero projected coefficient therefore satisfies $L_{j,k} f_{jk\alpha\beta} = \lambda f_{jk\alpha\beta}$. The preceding inequalities exclude each nonlisted sector separately. They also give a finite list of possible sectors at any fixed λ, since $a, b \leq 1$ and $L_{j,k} \geq \alpha_j + \beta_k$. Thus an eigenfunction need not be a single product, and no additional summand at or below d is omitted.

The eigenspace at d is therefore

$$\operatorname{span}_{\mathbb{R}}\{c, ay_1, \ldots, ay_{p+1}, bz_1, \ldots, bz_{q+1}\}. \tag{6.6}$$

The three angular types are orthogonal, the coordinates in each factor are independent, and a, b are positive. This proves the dimension. $\square$

For a normalized monotone closing, $a-b$ has exactly two simple zeros on the full circle. On the first open quarter it is positive; the reflections (2.11) give its signs on the other quarters. Furthermore $ab<1/2$ everywhere: equality requires both $a=b$ and $c=0$, which never coincide. Compactness makes the maximum strictly less than $1/2$. Applying proposition 6.1 with $p=q$ proves theorem 1.3, including the strict bound $\mu_2>4n-3$.

Corollary 6.2 (Graphical doubles). *Let $p,q\geq 1$, $d=p+q+1$, and $0<\tau_-<\tau_+<1$. Suppose $F\in C^\infty((\tau_-,\tau_+))\cap C^0([\tau_-,\tau_+])$ is positive in the interior and vanishes at both endpoints. Suppose the arcs*

$$\gamma_\pm(\tau)=\frac{(\sqrt{1-\tau^2},\tau,\pm F(\tau))}{\sqrt{1+F(\tau)^2}} \tag{6.7}$$

join to a smooth regular closed curve, and that the associated immersion (2.1)*, parametrized by arclength, is minimal. Then*

$$\lambda_1=d,\qquad E_d=\operatorname{span}_{\mathbb{R}}\{c,ay_1,\dots,ay_{p+1},bz_1,\dots,bz_{q+1}\},\qquad \dim E_d=d+2.$$

The invariant eigenvalue d is simple, and the next invariant eigenvalue is at least $d+2\sqrt{pq}$. In particular these conclusions hold for the bi-orthogonal graphical doubles of [7, Theorem 1.1]*, for every $p,q\geq 1$.*

Proof. On either open arc the balanced radius is

$$h=\frac{\varphi(\tau)}{\sqrt{1+F(\tau)^2}},\qquad \varphi(\tau)=\frac{\sqrt{1-\tau^2}}{\sqrt{p}}-\frac{\tau}{\sqrt{q}},\qquad \varphi'(\tau)=-\frac{\tau}{\sqrt{p}\sqrt{1-\tau^2}}-\frac{1}{\sqrt{q}}<0.$$

Thus h is not identically zero, and its numerator has the unique zero $\tau_*=\sqrt{q/(p+q)}$ in $(0,1)$. Integrating (6.3) over the profile circle gives

$$\int_0^T w\left(d+\frac{\sqrt{pq}}{ab}\right)h\,dt=0.$$

Its coefficient is strictly positive. Hence h takes both signs, and strict monotonicity of φ implies $\tau_-<\tau_*<\tau_+$. Each arc contributes one zero; they are distinct because $F(\tau_*)>0$ gives opposite nonzero last coordinates. They are simple either by uniqueness for (6.3), or by differentiating the preceding formula at τ_*. Thus proposition 6.1 applies.

For the examples of [7], the positive-height homogeneous graph intersects the unit sphere in (6.7); this is their equation (2.3). Their free-boundary closing construction, in the proof of Theorem 1.1, supplies the smooth minimal double and the two interior endpoint parameters. It supplies the hypotheses just used, not an index formula or a uniqueness assertion for unequal factors. □

7. The lower Jacobi sectors

In this section $p=n-1$ is an integer and the profile is a normalized monotone closing. Derivatives are in physical arclength t.

7.1. A geometric factorization.

Lemma 7.1. *Put $v=(\log w)'$ and $D=d/dt$ on $L^2(w\,dt)$. Then*

$$|A|^2=v'+p/a^2+p/b^2,\qquad L_w=D^*D,\qquad J_{1,1}+d=DD^*, \tag{7.1}$$

where $\operatorname{Dom}D=\operatorname{Dom}D^*=H^1(\mathbb{R}/T\mathbb{Z})$ and $D^*=-D-v$. The two products have domain $H^2(\mathbb{R}/T\mathbb{Z})$, with the values and first derivatives matched at the endpoints.

Proof. The profile position, tangent and normal form an orthonormal basis of $\mathbb{R}^3$, so $a^2+a'^2+\nu_1^2=b^2+b'^2+\nu_2^2=1$. The radial Ricci formula and the minimal Gauss equation give

$$-p\frac{a''}{a}-p\frac{b''}{b}=d-1-k_t^2.$$

Therefore

$$\begin{aligned} v' + p/a^2 + p/b^2 &= -(d-1) + k_t^2 + p(1-a'^2)/a^2 + p(1-b'^2)/b^2 \\ &= k_t^2 + p\nu_1^2/a^2 + p\nu_2^2/b^2 = |A|^2. \end{aligned}$$

Periodic integration by parts gives $D^* = -D - v$. Multiplication of these first-order operators gives the last two identities. □

For every $\mu > 0$, D maps $\ker(D^*D - \mu)$ isomorphically onto $\ker(DD^* - \mu)$, with inverse $\mu^{-1}D^*$. The kernels at zero are, respectively, $\mathbb{R}$ and $\mathbb{R}w^{-1}$. Thus, with multiplicities,

$$\operatorname{spec}(J_{1,1}) = \operatorname{spec}(L_w) - d. \tag{7.2}$$

By theorem 1.3, the radial spectrum of $J_{1,1}$ consists of a simple negative value $-d$, a simple zero, and values greater than $2p$. Its negative function is w^{-1} and its zero function is c'. The reciprocal-volume negative modes have the antecedent [23, Lemma 3.1]; (7.2) identifies the rest of this radial spectrum.

7.2. **All other mixed sectors.**

Lemma 7.2 (Exchange estimate)**.** *Let a measure-preserving involution τ preserve the domain and the values of a nonnegative quadratic form E, and let $\gamma > 0$. Suppose $E(o) \geq \gamma\|o\|^2$ for every odd function o. Let $a, b > 0$, $a \circ \tau = b$ and $a^2 + b^2 \leq 1$. For $A_0 > \gamma/2$,*

$$E(f) + A_0 \int a^{-2} f^2 \geq (A_0 + \gamma/2)\|f\|^2. \tag{7.3}$$

If the underlying space is compact and a, b are continuous, the inequality has a strictly positive additional multiple of $\|f\|^2$.

Proof. Write $f = e + o$ as its even and odd parts. Invariance gives orthogonality for the norm and the form, so $E(f) \geq \gamma\|o\|^2$. Average the potential with its transform by τ. If

$$T_0 = \frac{A_0}{2}(a^{-2} + b^{-2}), \qquad V_0 = \frac{A_0}{2}(a^{-2} - b^{-2}),$$

the remainder after subtracting the right side of (7.3) is bounded below by the integral of the form with matrix

$$\begin{pmatrix} T_0 - A_0 - \gamma/2 & V_0 \\ V_0 & T_0 - A_0 + \gamma/2 \end{pmatrix}.$$

Its first diagonal entry is at least $A_0 - \gamma/2 > 0$, since $a^{-2} + b^{-2} \geq 4$. Its determinant is

$$\frac{A_0^2(1 - a^2 - b^2)}{a^2b^2} + A_0^2 - \gamma^2/4 > 0. \tag{7.4}$$

The matrix is positive definite. On a compact space its least eigenvalue has a positive minimum, proving the last assertion. □

Multiplication by w^{-1} is unitary from $L^2(w^{-1}dt)$ to $L^2(wdt)$ and conjugates DD^* to $L_{w^{-1}}$. The latter has first positive eigenvalue d by (7.2). Reflection $t \mapsto -t$ preserves its measure and exchanges a, b. Every odd function has mean zero. Apply lemma 7.2 with $\gamma = d$ and $A_0 = p + 2$. Since

$$J_{2,1} = DD^* - d + (p+2)/a^2,$$

one obtains

$$J_{2,1} > 3/2, \qquad J_{1,2} > 3/2. \tag{7.5}$$

Increasing either angular degree adds a nonnegative potential. Thus all mixed sectors except $(1,1)$ have spectrum strictly above $3/2$.

Lemma 7.3 (Reflection characters)**.** *Let w, V be smooth functions of period 4ℓ, with $w > 0$, and suppose both are invariant under $t \mapsto -t$ and $t \mapsto 2\ell - t$. The periodic realization of $L_w - V$ is the orthogonal direct sum of the four separated problems on $[0, \ell]$:*

(R_0, R_ℓ)	$(+,+)$	$(+,-)$	$(-,+)$	$(-,-)$
boundary conditions	NN	ND	DN	DD.

Here $R_0 f(t) = f(-t)$ and $R_\ell f(t) = f(2\ell - t)$.

Proof. The two transformations preserve $w\,dt$. Their products in opposite orders differ by a translation of 4ℓ; hence they are commuting self-adjoint involutions on the periodic space. They preserve the quadratic form. The projections $\frac{1}{4}(1 + \varepsilon R_0)(1 + \eta R_\ell)$, with $\varepsilon, \eta \in \{1, -1\}$, are orthogonal and sum to the identity. An odd character has zero trace at the corresponding fixed point. For an eigenfunction, an even character has zero derivative there. Conversely, even reflection of a Neumann eigenfunction and odd reflection of a Dirichlet eigenfunction preserve both its value and its weighted first derivative at the seam. The reflected function is a weak solution on the circle. Its equation $(wf')' = -w(V + \lambda)f$ makes it smooth by successive differentiation. Restriction multiplies the squared norm and the quadratic form by $1/4$. It therefore preserves each eigenvalue and its multiplicity. For form-domain functions, only Dirichlet traces are imposed; Neumann conditions arise from the Euler equation. ☐

7.3. **The invariant sector.** Set

$$h = a - b, \qquad q = -\sin r \sin\alpha, \qquad \beta = \min_{[0,\ell]} p/(ab) > 2p. \tag{7.6}$$

For $0 < t < \ell$, differentiation of (1.2) gives

$$\begin{aligned} h' &= \cos r \cos\alpha(\cos\theta - \sin\theta) - \sin\alpha(\sin\theta + \cos\theta) > 0, \\ q' &= -2p\cos\alpha(\cos\alpha \cot 2\theta - \cos r \sin\alpha) < 0. \end{aligned} \tag{7.7}$$

The endpoint values are

$$h(0) = 0,\ h'(0) = \sqrt{2},\ h'(\ell) = 0, \quad q'(0) = 0,\ q(\ell) = 0,\ q'(\ell) < 0. \tag{7.8}$$

By lemma 2.1,

$$J_{0,0}(hq) = \frac{p}{ab} hq - 2h'q' > \beta hq \qquad (0 < t < \ell). \tag{7.9}$$

Its positive reference identity gives $\lambda_0^{\mathrm{DD}} \geq \beta$. The reference q gives the simple eigenvalue $\lambda_0^{\mathrm{ND}} = -d$ and $\lambda_1^{\mathrm{ND}} \geq \beta$ by codimension one. The positive Dirichlet–Neumann function $-c'$ satisfies $J_{1,1}(-c') = 0$ and therefore

$$J_{0,0}(-c') = -p(a^{-2} + b^{-2})(-c').$$

It gives one negative direction, and codimension one excludes a second. The constant function gives a negative Neumann–Neumann ground value; codimension two against the Dirichlet space gives $\lambda_2^{\mathrm{NN}} \geq \beta$.

We will use the following sign relation also to fix the nonzero Neumann eigenvalue.

Lemma 7.4. *Let $F(r_0)$ be the closing angle at the moving intersection with $r = \pi/2$, defined near a closing profile. If $\eta = \lambda_1^{\mathrm{NN}}(J_{0,0}; [0, \ell])$, then*

$$\operatorname{sgn}\eta = -\operatorname{sgn} F'(r_0). \tag{7.10}$$

Proof. The spherical profile frame is $T = \cos\alpha\, e_r + \sin\alpha\, e_\theta$, $\nu = \sin\alpha\, e_r - \cos\alpha\, e_\theta$. Let V be the derivative of the family with respect to its initial radius and put $u = -\langle V, \nu\rangle$. Linearizing minimality gives $J_{0,0}u = 0$, $u(0) = 1$, $u'(0) = 0$. At the final endpoint, write $\delta r, \delta\alpha$ for the fixed-time variations. Since $r'(\ell) = 1$, the moving endpoint has variation $-\delta r$. Hence

$$F' = \delta\alpha - \alpha'(\ell)\delta r.$$

Differentiation of u gives the same expression for $u'(\ell)$: there $T = e_r$, $\nu = -e_\theta$, and $\nu' = \alpha'(\ell)T$. Thus $u'(\ell) = F'$.

Let y_λ solve $J_{0,0}y_\lambda = \lambda y_\lambda$ with initial values $(1,0)$. Between the first two Neumann–Dirichlet eigenvalues, $-d$ and $\lambda_1^{\mathrm{ND}} > 0$, define

$$m(\lambda) = w(\ell)y_\lambda'(\ell)/y_\lambda(\ell).$$

Differentiating the equation and its Wronskian gives

$$m'(\lambda) = -\frac{\int_0^\ell wy_\lambda^2}{y_\lambda(\ell)^2} < 0. \tag{7.11}$$

At $-d$, $y_{-d} = q/q(0)$ and $y_{-d}'(\ell) < 0$. The undivided Wronskian identity yields

$$w(\ell)y_{-d}'(\ell)\,\partial_\lambda y_\lambda(\ell)|_{-d} = \int_0^\ell wy_{-d}^2 > 0.$$

Thus $y_\lambda(\ell) < 0$ immediately above $-d$, and throughout $(-d, \lambda_1^{\mathrm{ND}})$ because no Dirichlet eigenvalue intervenes. The Neumann value η is the unique zero of m there. Interlacing follows from the codimension-one form inclusion; equality would give the same nonzero solution both endpoint conditions, which initial-value uniqueness excludes. Since $y_0 = u$, $\operatorname{sgn} m(0) = -\operatorname{sgn} F'$. Strict decrease gives $\operatorname{sgn} m(0) = \operatorname{sgn}\eta$, proving (7.10). □

Proposition 7.5 (Invariant index)**.** *For every integer $p \geq 1$, the invariant Jacobi index is three and the invariant kernel is zero. For a normalized monotone closing, before using the sign of the second Neumann eigenvalue, one has*

$$i_{0,0} = 3 + \mathbf{1}_{\{\eta<0\}}, \qquad \dim\ker J_{0,0} = \mathbf{1}_{\{\eta=0\}}, \qquad \eta = \lambda_1^{\mathrm{NN}}(J_{0,0}). \tag{7.12}$$

For $p \geq 2$ this conclusion follows by analytic continuation of the planar form. For $p = 1$ it uses the positive initial derivative in proposition 5.3.

Proof. In the flattened orbit coordinates, spherical arclength is $dt = \operatorname{sech} X\,d\sigma$, and

$$a^p b^p\,dt = 2^{-p}e^{-\Phi_p}\,d\sigma. \tag{7.13}$$

A Euclidean normal speed f has spherical normal speed $u = \operatorname{sech} X\,f$. The two functionals in (7.13) agree for every nearby profile, not only for solutions. At a stationary profile their second variations consequently satisfy

$$Q_{J_{0,0}}(\operatorname{sech} X\,f) = 2^{-p}Q_{\mathscr{J}}(f). \tag{7.14}$$

The identity is first obtained for smooth speeds that are constant in a collar of each endpoint. Their even reflections are smooth, and the endpoints of the corresponding normal variations remain on the reflection axes. Such speeds are dense in H^1: first approximate by a smooth function, then replace that function by its endpoint value in collars of length ε, using cutoffs with first derivative bounded by C/ε. The value error is $O(\varepsilon)$ and the derivative error is bounded on a set of length $O(\varepsilon)$; both tend to zero in the required norms. Both Hessians are continuous in H^1, so (7.14) extends to the entire form domain. The terms involving second derivatives of the change of variables vanish because the first variation is zero. Multiplication by $\operatorname{sech} X$ and the smooth arclength change are invertible on H^1. At both endpoints $(\operatorname{sech} X)_\sigma = \operatorname{sech} X \tanh X \sin s = 0$, so they also preserve the Neumann zero-mode conditions.

By proposition 4.6, the physical quarter Neumann problem has index one and no kernel when $p \geq 2$. By lemma 7.3, the other three boundary characters have counts $1, 1, 0$ and no kernel by (7.9) and the comparisons preceding lemma 7.4. Thus

$$i_{0,0} = 3, \qquad \dim\ker J_{0,0} = 0 \qquad (p \geq 2). \tag{7.15}$$

For $p = 1$, the same comparisons give (7.12). In the central problem of proposition 5.3, let $z_0 = \pi/2 - r_0$ and $G(z_0) = R(\pi/2; z_0)$. That proposition gives $G'(z_0) > 0$ at its unique root. At the closing endpoint

$R_s = -1/B$, where $B > 0$. Differentiation of $R(s(z_0), z_0) = 0$ gives $s'(z_0) = BG'(z_0)$. Since $F = s - \pi/2$ and $dz_0/dr_0 = -1$, one obtains

$$F'(r_0) = -BG'(z_0) < 0. \tag{7.16}$$

Lemma 7.4 gives $\eta > 0$. The four characters therefore give index three and no kernel also for $p = 1$. □

7.4. **The first single-factor sectors.** On the upper half-profile $[-\ell, \ell]$, the invariant Dirichlet problem splits into the quarter ND and DD problems. Its ground value is $-d$ and its second eigenvalue is at least β. Since $J_{1,0} = J_{0,0} + p/a^2$, its second Dirichlet value is positive. The ambient-rotation field has radial factor b', so $J_{1,0}b' = 0$ with Dirichlet endpoints. It is not identically zero. Thus zero is the simple Dirichlet ground value; b' has no interior zero. Its value at the midpoint is negative. Repeating the argument with the other factor gives

$$a' > 0, \qquad b' < 0 \qquad (-\ell < t < \ell). \tag{7.17}$$

The profile normal satisfies $\nu' = k_t\gamma'$. In the first quarter, $k_t = q'/c' > 0$ by (7.7); reflection gives this positivity on the upper half. Hence $\nu_1' = k_t a' > 0$. The endpoint values are $\nu_1(-\ell) = -\cos\theta(\ell)$ and $\nu_1(\ell) = \sin\theta(\ell)$; it has exactly one zero. It has Neumann data and solves $J_{1,0}\nu_1 = -d\nu_1$. Cutting at its one zero and using lemma 2.2 makes the shifted Neumann form $J_{1,0} + d$ nonnegative on a codimension-one subspace. The sign-changing function is not the positive ground state, so the first two Neumann values are $< -d$ and $-d$.

The third Neumann eigenvalue is at least the Dirichlet ground value zero by codimension two. Equality is impossible. Indeed, the span of the first three Neumann eigenfunctions would intersect the Dirichlet domain nontrivially. Nonnegativity of the Dirichlet form would force such a vector to have zero form, hence to be a zero eigenfunction. Its negative Neumann components would vanish, so it would also satisfy Neumann data. A solution with value and derivative zero at an endpoint vanishes identically, a contradiction. The third Neumann value is therefore positive.

The periodic sector is the orthogonal sum of these Neumann and Dirichlet realizations. We have proved

$$\lambda_0(J_{1,0}) < -d, \quad \lambda_1(J_{1,0}) = -d, \quad \lambda_2(J_{1,0}) = 0 \text{ with multiplicity one}, \quad \lambda_3(J_{1,0}) > 0. \tag{7.18}$$

The other factor has the identical count by factor exchange. In particular, for $p = 1$,

$$\lambda_1(J_{j,0}) \geq -3 + \frac{j^2 - 1}{a_{\max}^2} > 0 \qquad (j \geq 2). \tag{7.19}$$

Thus every higher single-factor degree has at most one negative radial value in this dimension.

Corollary 7.6. *For every normalized monotone closing,*

$$\lambda_2(J_{j,0}) > 0 \quad (j \geq 2), \qquad \lambda_1(J_{j,0}) > 5 \quad (j \geq 3). \tag{7.20}$$

Thus a degree-two single-factor sector has at most two nonpositive radial eigenvalues, and every single-factor degree at least three has at most one.

Proof. The operator difference from $J_{1,0}$ is multiplication by $(\alpha_j - p)/a^2 > 0$. Min–max and (7.18) give the first inequality. For $j \geq 3$, $\alpha_j - p \geq 2p + 6$; hence

$$\lambda_1(J_{j,0}) \geq -d + \frac{\alpha_j - p}{a_{\max}^2} > -d + 2p + 6 = 5.$$

□

The mixed-sector and first single-factor conclusions hold for every normalized monotone closing. In particular, their proofs do not use the higher single-factor computation below.

8. Higher single-factor sectors and endpoint forms

Let $I = [-\ell, \ell]$ be the upper half-profile and set

$$\mathcal{J}_\xi = L_w + \xi/a^2 - |A|^2 - d, \qquad \xi \geq \alpha_2 = 2p + 2. \tag{8.1}$$

The positive last normal coordinate q has simple Dirichlet zeros and satisfies $J_{0,0}q = -dq$. Therefore lemma 2.2 gives

$$Q_\xi(f, f) = \int_I wq^2 \left(\frac{f}{q}\right)'^2 + \int_I w(\xi/a^2 - d)f^2 \geq \left(\frac{\xi}{a_{\max}^2} - d\right) \|f\|_{L^2(w)}^2 > 0 \tag{8.2}$$

first for smooth zero-trace functions. By approximation, the lower bound extends to every $0 \neq f \in H_0^1(I)$. Its Dirichlet realization is invertible.

For $z = (z_-, z_+) \in \mathbb{R}^2$, let h_z solve

$$\mathcal{J}_\xi h_z = 0, \qquad h_z(-\ell) = z_-, \quad h_z(\ell) = z_+. \tag{8.3}$$

To obtain existence, take any smooth extension of the endpoints and subtract the Dirichlet inverse of its residual. Uniqueness follows from (8.2). Define the symmetric boundary matrix M_ξ by

$$z^\mathsf{T} M_\xi \zeta = Q_\xi(h_z, h_\zeta). \tag{8.4}$$

Green's identity gives

$$M_\xi z = \left(-w(-\ell)h_z'(-\ell),\ w(\ell)h_z'(\ell)\right)^\mathsf{T}. \tag{8.5}$$

Lemma 8.1. *The periodic radial index and nullity of $J_{j,0}$, $j \geq 2$, equal the negative index and nullity of M_{α_j}. Moreover $M_\xi' > 0$.*

Proof. For every $f \in H^1(I)$, write $f = f_0 + h_z$, $f_0 \in H_0^1(I)$. The weak equation for h_z makes the cross term zero. Hence

$$Q_\xi(f, f) = Q_\xi(f_0, f_0) + z^\mathsf{T} M_\xi z. \tag{8.6}$$

The map $H_0^1(I) \oplus \mathbb{R}^2 \to H^1(I)$, $(f_0, z) \mapsto f_0 + h_z$, is an isomorphism of the form domains. The first term is positive definite. Hence every negative subspace projects injectively to a negative subspace of the finite form, and every negative subspace of the finite form lifts by $z \mapsto h_z$. The two maximal dimensions coincide. The kernel identity follows as well: a Neumann zero mode has the form h_z and its outward flux is $M_\xi z$. Equatorial reflection splits the periodic problem into this Neumann realization and the positive Dirichlet realization, proving the first assertion.

For fixed z, differentiate (8.3). The derivative $\dot{h}_z$ has zero trace and solves $\mathcal{J}_\xi \dot{h}_z = -a^{-2}h_z$. Differentiating $Q_\xi(h_z, h_z)$ gives

$$z^\mathsf{T} M_\xi' z = \int_I wa^{-2}h_z^2 > 0 \qquad (z \neq 0), \tag{8.7}$$

because the terms containing $\dot{h}_z$ vanish by the weak equation. □

8.1. The transfer determinant.

Let T_ξ be the transfer matrix on I in state (u, u_t) for $\mathcal{J}_\xi u = 0$ and write

$$T_\xi = \begin{pmatrix} A_T & B_T \\ C_T & D_T \end{pmatrix}.$$

Its Wronskian is $\det T_\xi = w(-\ell)/w(\ell) > 0$. Also $B_T > 0$: the solution with initial state $(0, 1)$ is positive immediately after the left endpoint; a first further zero, with zero extension outside its sign interval, would contradict (8.2). Solving for the initial derivative from the two endpoint values gives

$$M_\xi = \frac{1}{B_T}\begin{pmatrix} w(-\ell)A_T & -w(-\ell) \\ -w(-\ell) & w(\ell)D_T \end{pmatrix}, \qquad \det M_\xi = w(-\ell)w(\ell)\frac{C_T}{B_T}. \tag{8.8}$$

Thus the determinant sign is the sign of C_T.

Proposition 8.2 (Sufficient angular signs)**.** *If $p \geq 2$ and*

$$(C_T)_{\alpha_2} < 0, \qquad (C_T)_{\alpha_3} < 0, \qquad (C_T)_{\alpha_4} > 0, \tag{8.9}$$

then $i_{2,0} = i_{3,0} = 1$ and $i_{j,0} = 0$ for $j \geq 4$; these sectors have no kernel. For $p = 1$, the signs $(C_T)_{16} < 0$, $(C_T)_{25} > 0$ give one negative radial eigenvalue in degrees $2, 3, 4$, none in later degrees, and no kernel in these sectors.

Proof. For $p \geq 2$, the matrices at α_2, α_3 have negative determinant and hence exactly one positive and one negative eigenvalue. The positive eigenvalue remains positive as ξ increases by lemma 8.1. Positive determinant at α_4 therefore makes both eigenvalues positive. For $\xi > \alpha_4$, the matrix $M_\xi - M_{\alpha_4}$ is positive definite. The same holds at all α_j, $j > 4$, because $\alpha_j > \alpha_4$.

For $p = 1$, (7.19) excludes a second nonpositive radial eigenvalue in every degree $j \geq 2$. Negative determinant at degree four gives a negative ground value there, hence also in degrees two and three by strict angular monotonicity. Positive determinant at degree five cannot correspond to two negative eigenvalues, so it gives positivity. Larger degrees add a strictly positive potential. The strict determinant signs and the excited-eigenvalue bound exclude zero modes at the listed degrees; the ground values in degrees two and three are strictly negative because that of degree four is negative. ☐

8.2. **Concavity and its limitation.**

Proposition 8.3. *On $\xi \geq \alpha_2$, the boundary form satisfies*

$$M'_\xi > 0, \qquad M''_\xi < 0. \tag{8.10}$$

These inequalities alone do not determine the angular parameters at which its negative index changes.

Proof. Let G_ξ be the positive Dirichlet inverse and fix nonzero boundary values z. Differentiation gives $\dot{h}_z = -G_\xi(a^{-2}h_z)$. Differentiating (8.7) once more yields

$$z^{\mathsf{T}} M''_\xi z = -2\langle a^{-2}h_z, G_\xi(a^{-2}h_z)\rangle_{L^2(w)} < 0. \tag{8.11}$$

Positivity of the inverse is a consequence of (8.2). The right side is strictly negative because h_z is nonzero.

For the final assertion, choose $a > 0$ and $u, v > 0$, and consider

$$N(\xi) = \operatorname{diag}\bigl(\sqrt{\xi + a} - u, \sqrt{\xi + a} - v\bigr), \qquad \xi > -a. \tag{8.12}$$

Then $N' > 0$ and $N'' < 0$, whereas its two zero parameters are $u^2 - a$ and $v^2 - a$. By increasing a first, these may be prescribed at any two finite parameters in a given interval. Thus monotonicity and concavity do not imply the signs in (8.9). ☐

9. Uniform spectral inequalities in the dimension parameter

The change of variable $\delta = 1/p$ places the angular equations on a compact parameter interval. We first define rational differential equations for every real $\delta \in [0, 1]$. At $\delta = 1/p$ with p a positive integer, these are the normalized profile and Jacobi equations. For other values of δ, all identities below are identities of rational functions restricted to the invariant set (9.7); no manifold of nonintegral dimension is used. Two exact solutions give regular equations for the transfer numerators divided by δ. Their values at $\delta = 0$ are defined by these regular equations.

9.1. **Regular variables.** Put

$$\delta = p^{-1}, \quad \varepsilon = \sqrt{\delta}, \quad s = \alpha + \pi/2, \quad r = \pi/2 - \varepsilon R, \quad \theta = \pi/4 - \varepsilon U, \quad t = \varepsilon\tau. \tag{9.1}$$

Let $L = \pi/2$, $S = \sin s$, $C = \cos s$, and

$$c_R = \cos(\sqrt{\delta}R), \quad s_R = \frac{\sin(\sqrt{\delta}R)}{\sqrt{\delta}}, \quad c_U = \cos(2\sqrt{\delta}U), \quad s_U = \frac{\sin(2\sqrt{\delta}U)}{\sqrt{\delta}}. \tag{9.2}$$

The values at zero are $1, R, 1, 2U$; each function is analytic in δ. Define

$$B = \frac{2Ss_U}{c_U c_R} + (2+\delta)\frac{Cs_R}{c_R}, \qquad Z = 2\left(\frac{Ss_R}{c_R} - \frac{Cs_U}{c_U c_R}\right). \tag{9.3}$$

The profile equations become

$$R_s = -S/B, \qquad U_s = C/(c_R B), \qquad s_\tau = B, \quad R(0) = z, \quad U(0) = 0. \tag{9.4}$$

The closing equation is $G(z,\delta) = R(L; z, \delta) = 0$. The equivalent rational initial-value system is

$$\begin{aligned} (c_R)_s &= \delta s_R S/B, & (s_R)_s &= -c_R S/B, \\ (c_U)_s &= -2\delta s_U C/(c_R B), & (s_U)_s &= 2c_U C/(c_R B), \\ S_s &= C, & C_s &= -S. \end{aligned} \tag{9.5}$$

The parameter δ is constant along each solution. Without imposing any relation among the six state coordinates, differentiation gives

$$\begin{aligned} \partial_s(c_R^2 + \delta s_R^2) &= 2c_R \frac{\delta s_R S}{B} - 2\delta s_R \frac{c_R S}{B} = 0, \\ \partial_s(c_U^2 + \delta s_U^2) &= -4c_U \frac{\delta s_U C}{c_R B} + 4\delta s_U \frac{c_U C}{c_R B} = 0, \\ \partial_s(S^2 + C^2) &= 2SC - 2CS = 0. \end{aligned} \tag{9.6}$$

Their initial values are one. Thus every exact solution with the stated initial data satisfies

$$c_R^2 + \delta s_R^2 = 1, \qquad c_U^2 + \delta s_U^2 = 1, \qquad S^2 + C^2 = 1. \tag{9.7}$$

For use away from this invariant set, define the three defects

$$e_R = c_R^2 + \delta s_R^2 - 1, \qquad e_U = c_U^2 + \delta s_U^2 - 1, \qquad e_s = S^2 + C^2 - 1. \tag{9.8}$$

They are first integrals of the rational equation, whether or not their constant values are zero. Appendix A.6 gives the curvature and reference identities with these defects retained. Positive c_R, c_U and continuous phases recover (9.2); different branches of an inverse sine are not introduced.

For the shooting derivative write $q_R = R_z$, $q_U = U_z$. Put

$$B_R = \frac{2S(s_U/c_U)\delta s_R + (2+\delta)C}{c_R^2}, \qquad B_U = \frac{4S}{c_U^2 c_R}, \qquad D_B = B_R q_R + B_U q_U. \tag{9.9}$$

The exact variational equation is

$$(q_R)_s = SD_B/B^2, \qquad (q_U)_s = \frac{C\delta s_R}{c_R^2 B} q_R - \frac{C}{c_R B^2} D_B, \quad (q_R, q_U)(0) = (1, 0). \tag{9.10}$$

In particular $\partial_z s_R(L) = c_R(L) q_R(L)$. The displayed formula for B_R uses $e_R = 0$ to differentiate s_R/c_R; the formula for B_U uses $e_U = 0$ to differentiate s_U/c_U. Neither differentiation uses $e_s = 0$. The sensitivity is with respect to the prescribed initial curve, which lies in $e_R = e_U = e_s = 0$; it is not a derivative in six independent initial state coordinates.

9.2. **The normalized Jacobi equation.** Set

$$\mathsf{A} = \frac{2}{c_R^2 c_U^2}, \qquad \mathsf{A}_o = -\mathsf{A} s_U. \tag{9.11}$$

For $\delta > 0$, the substitution (9.1) gives $a^2 = c_R^2(1 + \varepsilon s_U)/2$ and $b^2 = c_R^2(1 - \varepsilon s_U)/2$. The relation $e_U = 0$ gives $(1 - \varepsilon s_U)(1 + \varepsilon s_U) = c_U^2$, and therefore $a^{-2} = \mathsf{A} + \varepsilon \mathsf{A}_o$ and $b^{-2} = \mathsf{A} - \varepsilon \mathsf{A}_o$. This algebraic step uses neither $e_R = 0$ nor $e_s = 0$. Define the following rational functions, including at $\delta = 0$:

$$\begin{aligned} \mathsf{K} &= 2\left(\frac{Ss_U}{c_Uc_R} + \frac{Cs_R}{c_R}\right), \\ \mathcal{S} &= \frac{S^2}{c_R^2}\left(\frac{4}{c_U^2} - 2\right) + \frac{4\delta CSs_Rs_U}{c_R^2c_U} + \frac{2\delta C^2s_R^2}{c_R^2} + \mathsf{K}^2. \end{aligned} \tag{9.12}$$

For $\delta = 1/p > 0$ with integer p, substitution in (2.4) identifies $\mathcal{S} = \delta|A|^2$ and the longitudinal principal curvature as K/ε. Here is the algebra in this identification. Put

$$u = \frac{Sc_U}{c_R(1+\varepsilon s_U)} - \varepsilon\frac{Cs_R}{c_R}, \qquad v = -\varepsilon\frac{Cs_R}{c_R} - \frac{S(1+\varepsilon s_U)}{c_Rc_U}.$$

Using only $e_U = 0$, one obtains

$$\begin{aligned} u + v &= -\varepsilon\mathsf{K}, \\ u^2 + v^2 &= \frac{S^2}{c_R^2}\left(\frac{4}{c_U^2} - 2\right) + \frac{4\delta CSs_Rs_U}{c_R^2c_U} + \frac{2\delta C^2s_R^2}{c_R^2}, \\ u^2 + v^2 + \delta^{-1}(u+v)^2 &= \mathcal{S}. \end{aligned} \tag{9.13}$$

For arbitrary real $\delta > 0$, (9.13) is an algebraic identity, not an invocation of a geometric Gauss equation. Its final expression is regular at $\delta = 0$. Define

$$\begin{aligned} M &= 2\mathsf{A} - \mathcal{S} - (2+\delta), \\ H_j &= M + [j - 2 + j(j-1)\delta]\mathsf{A}, \\ O_j &= [j + j(j-1)\delta]\mathsf{A}_o. \end{aligned} \tag{9.14}$$

For every real $\delta \in [0,1]$, define P_j by the following initial-value problem. At $\delta = 1/p$ with integer p, $\delta\alpha_j = j + j(j-1)\delta$, so it is exactly the zero-energy equation for $\delta J_{j,0}$ in state (u, u_τ):

$$(P_j)_s = \frac{1}{B}\begin{pmatrix} 0 & 1 \\ H_j + \varepsilon O_j & -Z \end{pmatrix} P_j, \qquad P_j(0) = \mathrm{Id}. \tag{9.15}$$

The positive change from t to τ does not alter an index or a transfer-entry sign.

9.3. **Even and odd columns.** For $\varepsilon \neq 0$, define

$$E_j(\delta) = \frac{P_j(\varepsilon) + P_j(-\varepsilon)}{2}, \qquad W_j(\delta) = \frac{P_j(\varepsilon) - P_j(-\varepsilon)}{2\varepsilon}. \tag{9.16}$$

The profile equation and its initial values are analytic in δ, and the coefficient of ε in (9.15) changes sign under $\varepsilon \mapsto -\varepsilon$. Analytic parameter dependence gives removable values at $\varepsilon = 0$. Both expressions in (9.16) are analytic functions of $\delta = \varepsilon^2$, and $P_j = E_j + \varepsilon W_j$. Set

$$F_j = \frac{1}{B}\begin{pmatrix} 0 & 1 \\ H_j & -Z \end{pmatrix}, \qquad G_j = \frac{1}{B}\begin{pmatrix} 0 & 0 \\ O_j & 0 \end{pmatrix}. \tag{9.17}$$

Then

$$(E_j)_s = F_jE_j + \delta G_jW_j, \qquad (W_j)_s = F_jW_j + G_jE_j, \quad E_j(0) = \mathrm{Id}, \quad W_j(0) = 0. \tag{9.18}$$

No Taylor truncation in δ is involved.

Changing ε to $-\varepsilon$ exchanges a, b and leaves the even profile quantities fixed. Let $R_0 = \mathrm{diag}(1, -1)$. Reflection of the quarter transfer gives the upper-half transfer

$$P_j(\varepsilon)R_0P_j(-\varepsilon)^{-1}R_0.$$

If the second rows of E_j, W_j at L are (C_e, D_e) and (C_o, D_o), its $(2,1)$-entry has positive Wronskian denominator and numerator

$$\mathcal{N}_j = 2(C_eD_e - \delta C_oD_o). \tag{9.19}$$

For example, the numerator before the even–odd decomposition is $C_+D_- + D_+C_-$; expansion gives (9.19).

9.4. **Exact division in degree two.** Write $\mathcal{D} = B\partial_s$ along the rational equation; it is $d/d\tau$ when $\delta > 0$. Put $N_0 = (2+\delta)s_R(0) > 0$ and define

$$f_2 = \frac{c_R S}{N_0}. \tag{9.20}$$

For all six state coordinates, before imposing any of the three first integrals, direct substitution gives

$$\mathcal{D}(c_R S) + Z c_R S = (2+\delta)s_R(S^2 + C^2). \tag{9.21}$$

Apply $\mathcal{D}$ and use $\mathcal{D}s_R = -c_R S$ and $\mathcal{D}(S^2 + C^2) = 0$. It follows that

$$\mathcal{D}^2(c_R S) + Z\mathcal{D}(c_R S) + (\mathcal{D}Z)c_R S = -(2+\delta)c_R S(S^2 + C^2).$$

On $e_R = e_U = e_s = 0$, the rational identity (A.14) gives $\mathcal{D}Z = \mathcal{S} - 2\mathsf{A}$. The preceding equation is therefore $\mathcal{D}^2 f_2 + Z\mathcal{D}f_2 = Mf_2$. This proves the mixed reference equation for every real $\delta \in [0,1]$, without using an integer-dimensional factorization. For integer p, it agrees with the geometric multiple of $-c_\tau/\varepsilon$. At $s = 0$ one has $S = 0$, $C = c_U = 1$, $s_U = 0$ and $c_R B = (2+\delta)s_R$, whence the initial state is $(0,1)$. Differentiating (9.20) gives $(f_2)_\tau = (\delta s_R S^2 + c_R CB)/N_0$. At $s = L$, where $S = 1$ and $C = 0$, this gives

$$(f_2)_\tau(L) = \frac{\delta s_R(L)}{(2+\delta)s_R(0)}, \tag{9.22}$$

which is zero on the closing profile. The initial rectangles satisfy $(2+\delta)s_R(0) > 0$.

Write the second even column as

$$E_2^{(2)} = \begin{pmatrix} f_2 \\ (f_2)_\tau \end{pmatrix} + \delta V_2.$$

The even degree-two potential differs from M by $2\delta\mathsf{A}$. Consequently the divided columns solve

$$\begin{aligned} (V_2)_s &= F_2 V_2 + G_2 W_2^{(2)} + \frac{1}{B}\begin{pmatrix} 0 \\ 2\mathsf{A}f_2 \end{pmatrix}, \\ (W_2^{(2)})_s &= F_2 W_2^{(2)} + G_2\left[\begin{pmatrix} f_2 \\ (f_2)_\tau \end{pmatrix} + \delta V_2\right], \end{aligned} \tag{9.23}$$

with zero initial columns. The first columns retain (9.18). At the exact closing root,

$$\frac{\mathcal{N}_2}{\delta} = 2\left[(E_2^{(1)})_2(V_2)_2 - (W_2^{(1)})_2(W_2^{(2)})_2\right]. \tag{9.24}$$

The differential equations and the right side are regular at $\delta = 0$.

9.5. **Exact division in degree three.** For every real $\delta \in [0,1]$, define the positive function

$$f_3(s) = \exp\left(-\int_0^s \frac{Z(v)}{B(v)}\,dv\right), \qquad (f_3)_s = -Zf_3/B, \qquad f_3(0) = 1. \tag{9.25}$$

For $\delta > 0$, differentiation of $\omega_\delta = (c_R^2 c_U)^{1/\delta}$ gives $\mathcal{D}\log\omega_\delta = Z$ directly from (9.5). Thus $f_3 = \omega_\delta(0)/\omega_\delta(s)$; at integer p this is $w(0)/w(s)$. The definition (9.25) has no singular exponent at $\delta = 0$. It gives $\mathcal{D}^2 f_3 + Z\mathcal{D}f_3 = -(\mathcal{D}Z)f_3$. On $e_R = e_U = e_s = 0$, (A.14) makes this $\mathcal{D}^2 f_3 + Z\mathcal{D}f_3 = [M + (2+\delta)]f_3$. This is the required reference equation for every real parameter. Its state is $(f_3, -Zf_3)$; at $s = 0$ it is $(1,0)$. At a closing endpoint, $s_R(L) = C(L) = 0$, so $Z(L) = 0$. The degree-three even forcing relative to that equation is

$$[(1+6\delta)\mathsf{A} - (2+\delta)]f_3 = \delta T_3 f_3, \qquad T_3 = \mathsf{A}(s_R^2 + c_R^2 s_U^2 + 6) - 1. \tag{9.26}$$

The factorization of the forcing uses precisely $e_R = e_U = 0$:

$$1 - c_R^2 c_U^2 = (1 - c_R^2) + c_R^2(1 - c_U^2) = \delta(s_R^2 + c_R^2 s_U^2).$$

It does not use $S^2 + C^2 = 1$. The latter identity was used, together with the two radius identities, to match the reference equation with the curvature-defined potential. Write

$$E_3^{(1)} = \begin{pmatrix} f_3 \\ -Zf_3 \end{pmatrix} + \delta V_3.$$

Then

$$\begin{aligned} (V_3)_s &= F_3 V_3 + G_3 W_3^{(1)} + \frac{1}{B}\begin{pmatrix} 0 \\ T_3 f_3 \end{pmatrix}, \\ (W_3^{(1)})_s &= F_3 W_3^{(1)} + G_3 \left[\begin{pmatrix} f_3 \\ -Zf_3 \end{pmatrix} + \delta V_3 \right], \end{aligned} \tag{9.27}$$

with zero initial columns. The second columns retain (9.18). On the exact closed branch,

$$\frac{\mathcal{N}_3}{\delta} = 2\left[(V_3)_2 (E_3^{(2)})_2 - (W_3^{(1)})_2 (W_3^{(2)})_2 \right]. \tag{9.28}$$

The right sides of (9.24) and (9.28) are also evaluated on trajectories that do not close when enclosing an entire initial rectangle. For such trajectories they denote only the right-hand sides of those formulas. The equalities with $\mathcal{N}_j/\delta$ are applied at the unique closing root, whose trajectory belongs to the same enclosure. Thus $R(L) = 0$ is used only on that trajectory, not on its entire enclosing rectangle.

Proposition 9.1 (Finite interval bounds)**.** *There is a common analytic closing branch in the rational system* (9.5) *on* $0 \le \delta \le 1$ *with* $0 < z < 1/2$. *On its specified enclosing rectangles* B, c_R, c_U *remain positive and* $G_z = q_R(L) > 0$. *On the branch,*

$$-30 < \mathcal{N}_2/\delta < -18, \qquad -40 < \mathcal{N}_3/\delta < -6, \qquad 15 < \mathcal{N}_4 < 115 \quad (0 < \delta \le 1/8). \tag{9.29}$$

At $\delta = 1/p$, $p = 2, \ldots, 7$, *the enclosures are as follows:*

p	$\mathcal{N}_2/\delta$	$\mathcal{N}_3/\delta$	$\mathcal{N}_4$
2	$(-99.35, -99.33)$	$(-337.66, -337.64)$	$(2956, 2957)$
3	$(-55.42, -55.41)$	$(-96.41, -96.40)$	$(807, 808)$
4	$(-41.90, -41.89)$	$(-52.13, -52.12)$	$(376, 377)$
5	$(-35.57, -35.55)$	$(-36.77, -36.75)$	$(227, 228)$
6	$(-31.94, -31.92)$	$(-29.56, -29.55)$	$(158, 159)$
7	$(-29.60, -29.58)$	$(-25.54, -25.52)$	$(120, 121)$

At $\delta = 1$ *the two highest-degree tests satisfy*

$$-60740 < \mathcal{N}_4 < -60380, \qquad 4700000 < \mathcal{N}_5 < 4760000. \tag{9.30}$$

Proof. Let $\Lambda_i \times Z_i$, $1 \le i \le 573$, be the rational parameter–initial-value rectangles of section B.1, with $Z_i = [z_i^-, z_i^+]$. The first 256 parameter intervals are consecutive closed intervals of length $1/2048$ whose union is $[0, 1/8]$. The remaining 317 have union $[1/8, 1]$. The rational enclosures specified in section B.1, applied through proposition A.2, give on each rectangle,

$$s_R(L; z_i^-, \delta) < 0 < s_R(L; z_i^+, \delta), \qquad q_R(L; z, \delta) > 0, \qquad c_R, c_U, B > 0. \tag{9.31}$$

Here the endpoint signs hold for every $\delta \in \Lambda_i$, the sensitivity sign holds for every $(z, \delta) \in Z_i \times \Lambda_i$, and the denominator signs hold on the whole time interval. Since $\partial_z s_R = c_R q_R > 0$, the intermediate value theorem gives exactly one root in Z_i for each $\delta \in \Lambda_i$.

For consecutive rectangles, $\sup \Lambda_i = \inf \Lambda_{i+1}$ and $\max(z_i^-, z_{i+1}^-) < \min(z_i^+, z_{i+1}^+)$. At the common parameter, the derivative is positive on $Z_i \cup Z_{i+1}$, an interval. Both zeros therefore coincide. The implicit function theorem gives analytic roots across each common endpoint, and these local roots define one analytic branch. For every finite reciprocal test, its initial interval is contained in a continuation interval at the exact value $\delta = 1/p$. It therefore encloses the same root.

At a root, the positive cosine branches, $R_s = -S/B$ and $U_s = C/(c_R B)$ identify a monotone closing in definition 1.1. All integer $p \geq 8$ correspond to a point of the first parameter range, and all remaining positive integers occur in the finite list. The exact divided formulas prove the determinant signs in (9.29); the remaining enclosures give the displayed finite values.

Finally, let $F(r_0)$ be the physical closing angle of lemma 7.4. Since $z = (\pi/2 - r_0)/\varepsilon$ and $R_s(L) = -1/B(L)$, differentiating the moving equator intersection gives

$$F'(r_0) = -B(L)G_z/\varepsilon < 0 \qquad (\delta > 0). \tag{9.32}$$

This agrees with (7.16) for $p = 1$. For $p \geq 2$, proposition 7.5 already determines the invariant index without this computational sign. The uniqueness theorem identifies the branch with every normalized monotone closing at integer parameters. □

10. The complete index and the Jacobi kernel

Proposition 9.1 supplies precisely the transfer signs in proposition 8.2; a positive Wronskian denominator does not alter them. The invariant count is proposition 7.5: its proof uses analytic continuation when $p \geq 2$ and the central interval shooting derivative when $p = 1$. No scalar-sector multiplicity is omitted: by (2.2), the full operator is the orthogonal sum of the radial operators on the complete spherical harmonic bases.

For $n \geq 2$, let $m_j = \dim \mathcal{H}_j(\mathbb{S}^{n-1})$. Then

$$m_0 = 1, \quad m_1 = n, \quad m_2 = \frac{n^2 + n - 2}{2}, \quad m_3 = \frac{n(n-1)(n+4)}{6}. \tag{10.1}$$

These follow by subtracting the dimension of homogeneous polynomials of degree $j - 2$ from that of degree j. The decomposition into harmonic polynomials and $|x|^2$ times lower-degree polynomials proves the subtraction formula.

Theorem 10.1 (Exact index decomposition)**.** *For every normalized monotone closing,*

$$\operatorname{ind} M_n = n^2 + 4n + i_{0,0} + 2\sum_{j\geq 2} m_j n_-(M_{\alpha_j}), \tag{10.2}$$

$$\operatorname{nul} M_n = n^2 + 2n + \dim\ker J_{0,0} + 2\sum_{j\geq 2} m_j \dim\ker M_{\alpha_j}. \tag{10.3}$$

The sums are finite. In every dimension, $i_{0,0} = 3$ *and* $\ker J_{0,0} = 0$*. Hence*

$$\operatorname{ind} M_n \geq n^2 + 4n + 3, \qquad \operatorname{nul} M_n \geq n^2 + 2n. \tag{10.4}$$

Proof. The harmonic sectors are mutually orthogonal and exhaust the operator. By lemmas 7.1 and 7.2, the first mixed sector contributes n^2 negative and n^2 zero directions and every higher mixed sector is positive. The two first single-factor sectors contribute $4n$ negative and $2n$ zero directions. The invariant contribution is proposition 7.5. The remaining sectors are paired by factor exchange, and lemma 8.1 gives each radial inertia. This proves both identities.

For a fixed smooth profile, put $C_* = \max(|A|^2 + d)$. Since $L_w \geq 0$ and $a \leq a_{\max}$,

$$J_{j,0} \geq \alpha_j / a_{\max}^2 - C_*.$$

The right side tends to infinity with j. Only finitely many summands can therefore be nonzero. The lower bounds follow since every additional summand is nonnegative and $i_{0,0} \geq 3$. The index lower bound has the antecedent [22]; the exact formulas here identify each contribution before the finite spectral signs are applied. □

For $n \geq 3$, the preceding results give the following complete index count.

Angular sectors	Negative dimension	Kernel dimension
$(0,0)$	3	0
$(1,0),(0,1)$	$4n$	$2n$
$(1,1)$	n^2	n^2
$(2,0),(0,2)$	$2m_2$	0
$(3,0),(0,3)$	$2m_3$	0
All remaining sectors	0	0

Thus

$$\operatorname{ind}(M_n) = 3 + 4n + n^2 + 2m_2 + 2m_3 = \frac{n^3+9n^2+11n+3}{3}, \qquad n \geq 3. \tag{10.5}$$

For $n = 2$, every nonzero circle-harmonic degree has dimension two. There is one radial negative eigenvalue in each of degrees $2, 3, 4$ on either factor. The count is $3 + 8 + 4 + 12 = 27$.

The complete kernel is

$$\ker J = \operatorname{span}_{\mathbb{R}}\{b'y_i,\ a'z_i,\ c'y_iz_j : 1 \leq i,j \leq n\}. \tag{10.6}$$

The first two collections are induced by rotations between the last coordinate and one spherical block. A rotation between the two blocks has normal coefficient $a\nu_2 - b\nu_1 = c'$, up to the common normal orientation. All functions in (10.6) are independent: their angular types differ, their first spherical harmonics are independent, and none of a', b', c' is identically zero. Hence the nullity is $n^2 + 2n$, and all Jacobi fields are ambient rotations. This proves theorem 1.4.

Corollary 10.2. *Each M_n is locally rigid among minimal embeddings in the round sphere, modulo ambient rotations.*

Proof. The normal-graph minimal equation, restricted to the orthogonal complement of (10.6), has invertible derivative J. The implicit function theorem gives the unique zero graph there. The finite-dimensional rotation parameters impose the orthogonality conditions because their normal fields form the kernel basis. The slice construction and its uniqueness are proved in section 11; with the ambient metric fixed, it gives precisely the assertion. ☐

The index in (10.5) counts all angular representations. Monotone uniqueness identifies the object to which it applies. Neither this corollary nor theorem 1.2 excludes other minimal hypersurfaces outside the specified monotone construction.

11. Nearby ambient metrics

Fix $0 < \alpha < 1$. A hypersurface is regarded as an embedded image, not as a parametrized immersion. The topology on ambient metrics in this section is $C^{6,\alpha}$; all metrics in the conclusions are smooth. No invariance is imposed on a perturbed metric.

11.1. **Reduction along a compact isometry orbit.** Let (N, g_0) be closed, let a compact Lie group G act isometrically, and let $\Sigma \subset N$ be a closed embedded two-sided minimal hypersurface. Put $\mathcal{O} = G\Sigma$. The normal component of an infinitesimal isometry identifies $T_\Sigma\mathcal{O}$ with a subspace of $\ker J_\Sigma$.

Theorem 11.1. *Assume*

$$\ker J_\Sigma = T_\Sigma\mathcal{O}, \qquad \operatorname{ind} J_\Sigma = I, \qquad \dim \mathcal{O} = k. \tag{11.1}$$

There are neighborhoods of g_0 and $\mathcal{O}$ with the following property. For each smooth metric g in the first neighborhood, a smooth function $F_g : \mathcal{O} \to \mathbb{R}$ has critical points in bijection with the g-minimal hypersurfaces in the second neighborhood. At corresponding points,

$$\operatorname{ind} J_{\Sigma_{g,x}} = I + \operatorname{ind} \operatorname{Hess} F_g(x), \tag{11.2}$$

$$\operatorname{nul} J_{\Sigma_{g,x}} = \dim \ker \operatorname{Hess} F_g(x). \tag{11.3}$$

In particular, every such metric has a nearby minimal hypersurface of index exactly I. If the nearby minimal hypersurfaces are nondegenerate, then F_g is Morse.

Proof. Choose a unit normal along Σ and use normal graphs with respect to g_0. Let

$$K = \ker J_\Sigma, \qquad E = K^\perp$$

in the fixed space $L^2(\Sigma, dv_{g_0})$. The projection onto K has finite rank and smooth coefficients, and hence is bounded on every Hölder space used below. The restriction

$$J_\Sigma : E \cap C^{2,\alpha} \longrightarrow E \cap C^{0,\alpha} \tag{11.4}$$

is an isomorphism. Indeed, the self-adjoint compact-resolvent realization has range $K^\perp$, so it gives an L^2 solution for $C^{0,\alpha}$ data in E. Local elliptic estimates, patched by a finite coordinate cover, place that solution in $C^{2,\alpha}$. Injectivity holds by the definition of E, and the elliptic estimate together with the absence of a zero eigenvalue gives a bounded inverse.

Let $\mathcal{A}(g,u)$ be the area of the normal graph. Define its variational density relative to the fixed measure by

$$D_u\mathcal{A}(g,u)[v] = \int_\Sigma \mathscr{E}(g,u)v\, dv_{g_0}. \tag{11.5}$$

Thus $\mathscr{E}$ is the scalar mean curvature multiplied by the normal-speed and area-density factors of the graph. Those factors are smooth and nonzero for small graphs. At $(g_0, 0)$, $D_u\mathscr{E} = J_\Sigma$. The mean-curvature formula in local graph coordinates is a second-order quasilinear elliptic expression in u; its coefficients depend on g and its first derivatives. It defines the differentiable maps between the Hölder spaces in (11.4). This is the local elliptic framework considered in [28].

For $A \in G$, solve

$$\Pi_E \mathscr{E}(A^*g, u) = 0, \qquad u \in E \cap C^{2,\alpha}. \tag{11.6}$$

The derivative in u at g_0 is the isomorphism (11.4). The implicit function theorem gives a unique small solution $u(A,g)$. Use finitely many coordinate neighborhoods in the compact group to choose one neighborhood of g_0 and a uniform bound for these solutions. For a fixed smooth g, elliptic bootstrapping and parameter differentiation make $u(A,g)$ smooth in A and on Σ. Only finite differentiability is needed uniformly as g varies in the indicated metric topology.

Let $H \subset G$ be the stabilizer of Σ. Its action on normal functions includes the sign of the chosen normal when an element reverses the two sides. It preserves J_Σ, K, E and (11.5). Uniqueness in (11.6) therefore makes the hypersurface $A(\operatorname{graph} u(A,g))$ and its area well-defined over G/H. Set

$$F_g(AH) = \operatorname{Area}_g\big(A(\operatorname{graph} u(A,g))\big). \tag{11.7}$$

To justify the slice and the critical-point correspondence, choose k Lie-algebra elements whose normal components form a basis of K. For a nearby hypersurface, impose the k equations saying that its normal graph, after the inverse of these rotations, is orthogonal to that basis. The derivative in the rotation parameters is its Gram matrix in L^2, up to an overall sign. It is invertible. The finite-dimensional implicit function theorem gives a unique local choice of orbit point and graph in E. These local coordinates are compatible under H. A uniform neighborhood exists by compactness of G/H. There can be no additional nearby identifications outside the local charts: otherwise convergent orbit points would have a common limit, where local uniqueness applies.

In these coordinates, (11.6) makes the first variation vanish in all complementary directions. The orbit derivatives, together with those directions, still span the full normal-variation space: their projection onto K is an invertible perturbation of the Gram matrix above. Thus $dF_g = 0$ is equivalent to vanishing of the complete first variation. Conversely, every nearby minimal hypersurface can be put in this slice and then satisfies (11.6); its complementary graph is the unique one used in (11.7).

At a critical point, write the area Hessian in orbit and complementary coordinates as

$$\begin{pmatrix} A & B^* \\ B & C \end{pmatrix}.$$

The block C acts as a bounded symmetric form on $E_1 = E \cap H^1(\Sigma)$. Choose $c > 0$ so that

$$(u, v)_1 = Q_{J_\Sigma}(u, v) + c\langle u, v\rangle_{L^2}$$

is a positive inner product equivalent to the H^1 inner product. The Riesz representative T_0 of $Q_{J_\Sigma}|_{E_1}$ has eigenvalues $\lambda/(\lambda + c)$, where λ ranges over the nonzero Jacobi spectrum. They tend to one and none is zero; therefore

$$\gamma := \inf_{\lambda \neq 0} \left| \frac{\lambda}{\lambda + c} \right| > 0.$$

In the fixed graph coordinates, coefficient convergence gives $\|T - T_0\|_{E_1 \to E_1} < \gamma/2$ for the Riesz representative T of C, after shrinking the neighborhoods. For $0 \le s \le 1$,

$$T_0 + s(T - T_0) = T_0\big(\mathrm{Id} + sT_0^{-1}(T - T_0)\big)$$

is invertible by its norm-convergent Neumann series. Since $T_0 - \mathrm{Id}$ is compact, each operator on this path is the sum of $\mathrm{Id} + s(T - T_0)$ and $T_0 - \mathrm{Id}$. The first summand is positive: $\gamma \le 1$ and $\|s(T - T_0)\| < \gamma/2 \le 1/2$. Thus these operators are self-adjoint compact perturbations of positive isomorphisms. Their finite negative spectral subspaces therefore have constant dimension, since no eigenvalue can cross zero. Thus $C : E_1 \to E_1^*$ is invertible with index I. The argument is uniform along the compact orbit, whose reference operators are isometrically conjugate.

Differentiation of the complementary stationarity equation gives $u_x = -C^{-1}B$. The cross block B maps the finite-dimensional orbit tangent space into E_1^*. Consequently,

$$\operatorname{Hess} F_g = A - B^* C^{-1} B. \tag{11.8}$$

For an orbit variation ξ and a complementary variation v, the exact equality

$$\begin{aligned} &\langle A\xi, \xi\rangle + 2\langle B\xi, v\rangle + \langle Cv, v\rangle \\ &\quad = \langle C(v + C^{-1}B\xi), v + C^{-1}B\xi\rangle + \langle (A - B^*C^{-1}B)\xi, \xi\rangle \end{aligned} \tag{11.9}$$

is an invertible congruence of the forms. The right side is an orthogonal sum. It proves (11.2) and (11.3), including zero modes. The index is the full Jacobi index, because the normal-graph coordinates parametrize every nearby normal variation.

Compactness of $\mathcal{O}$ gives a point at which F_g attains its minimum. Its Hessian is nonnegative, so (11.2) gives index I. If every corresponding hypersurface is nondegenerate, (11.3) makes every critical point nondegenerate. Thus F_g is Morse. □

For the round metric the function in (11.7) is constant and its complementary graph is zero. The same slice proof therefore proves corollary 10.2. A smooth bumpy metric has no closed minimal immersed hypersurface with a nonzero Jacobi field. The class of such metrics is generic by [29, Theorem 2.1], applied with the trivial finite group. No equivariance of the perturbed hypersurfaces is required.

11.2. **The orbit of the Carlotto–Schulz hypersurface.** For $M_n \subset \mathbb{S}^{2n}$, the fields in (10.6) are the normal components of rotations between the two n-dimensional blocks and between either block and the residual line. They are independent, so the orbit has dimension

$$k_n = n^2 + 2n. \tag{11.10}$$

The connected group $SO(n) \times SO(n)$ fixes M_n. Since

$$\dim SO(2n+1) - k_n = n(n-1) = \dim(SO(n) \times SO(n)),$$

it is the full identity component of the stabilizer.

Every element of the full stabilizer normalizes this identity component. Its representation on $\mathbb{R}^{2n+1}$ has two inequivalent nontrivial real irreducible summands of dimension n and a single fixed line. Thus a normalizing orthogonal transformation fixes the line and preserves or exchanges the two summands. For $n = 2$ the same argument uses the two distinct real weight planes of $SO(2)^2$. Conversely, every such transformation of determinant one preserves M_n: orthogonal transformations within the summands preserve the spherical orbits, and factor exchange and reversal of c preserve the reflected profile. Therefore

$$\mathcal{O}_n = \big\{\{P, Q\} : \ P, Q \subset \mathbb{R}^{2n+1}, \ \dim P = \dim Q = n, \ P \perp Q\big\} . \tag{11.11}$$

The two planes are unoriented and unordered. This is a compact connected homogeneous manifold. By theorem 1.4, it satisfies all hypotheses of theorem 11.1 with $I = I_n$ and $k = k_n$.

11.3. **The ordered cover and its cohomology.** Let

$$\widetilde{\mathcal{O}}_n = \{(P, Q) : \dim P = \dim Q = n, \ P \perp Q\} \longrightarrow \mathcal{O}_n \tag{11.12}$$

forget the order. Exchange is free, so this is a two-sheeted covering. Projecting to $P \in \mathrm{Gr}_n(\mathbb{R}^{2n+1})$ identifies the fiber with $\mathbb{R}P^n$: the complementary line $(P \oplus Q)^\perp$ determines Q. More precisely, $\widetilde{\mathcal{O}}_n$ is the real projectivization of the rank-$(n+1)$ complementary tautological bundle over the Grassmannian.

We use cohomology with coefficients in $\mathbb{F}_2$. If x is the first Stiefel–Whitney class of the tautological line on this projectivization, its restrictions $1, x, \dots, x^n$ form a basis of the cohomology of each fiber. Each graded fiber group is finite-dimensional over $\mathbb{F}_2$, and these classes are defined on the total space. The Leray–Hirsch theorem [8, Theorem 4D.1, pp. 432–433] therefore applies. Its map is $(u_0, \dots, u_n) \mapsto \sum_{r=0}^{n} \pi^* u_r \smile x^r$, where π is the projective-bundle projection. It is an isomorphism of graded modules over the base cohomology and, in particular, of graded vector spaces:

$$H^*(\widetilde{\mathcal{O}}_n; \mathbb{F}_2) \cong \bigoplus_{r=0}^{n} H^{*-r}(\mathrm{Gr}_n(\mathbb{R}^{2n+1}); \mathbb{F}_2). \tag{11.13}$$

For completeness, the additive Schubert calculation follows from two precise results. Hatcher's cell decomposition [9, Proposition 1.17, pp. 33–34] gives one cell of $\mathrm{Gr}_n(\mathbb{R}^N)$ for every strictly increasing sequence $1 \le \sigma_1 < \cdots < \sigma_n \le N$, of dimension $\sum_i (\sigma_i - i)$. These decompositions are compatible under the inclusions in $\mathrm{Gr}_n(\mathbb{R}^\infty)$. Hatcher's universal cohomology calculation [9, Theorem 3.9, pp. 84–85] gives

$$H^*(\mathrm{Gr}_n(\mathbb{R}^\infty); \mathbb{F}_2) = \mathbb{F}_2[w_1, \dots, w_n], \qquad |w_i| = i.$$

In each degree, the number of its monomials equals the number of Schubert cells: both count partitions with at most n parts, after conjugating a partition when necessary. If r_j is the rank of the cellular boundary in degree j, the cellular dimension identity $\dim C_j = \dim H_j + r_j + r_{j+1}$ forces $r_j = 0$ in every degree. The finite Grassmannian is a CW subcomplex, so its mod-two cellular boundary also vanishes. The cochains dual to its cells consequently form an additive cohomology basis. At $N = 2n + 1$, their indices are precisely the partitions in an n-by-$(n + 1)$ rectangle. This is the additive statement used here; see also [21, Section 3] for its Grassmannian formulation. Thus

$$B_n(t) := \sum_{j=0}^{k_n} b_j t^j = (1 + t + \cdots + t^n) \binom{2n+1}{n}_t, \qquad \binom{2n+1}{n}_t = \prod_{r=1}^{n} \frac{1 - t^{n+1+r}}{1 - t^r}. \tag{11.14}$$

The projective-bundle calculation uses only the module statement in (11.13); no assertion of triviality of the bundle is made.

Every integer s between 0 and $n(n+1)$ is the size of a partition in that rectangle: write $s = q(n+1)+r$, where $0 \le r < n + 1$, and take q rows of length $n + 1$ followed, when $r > 0$, by one row of length r. When $q = n$ one has $r = 0$. Multiplication by $1 + t + \cdots + t^n$ now gives

$$b_j > 0 \quad (0 \le j \le k_n), \qquad B_n(1) = (n + 1) \binom{2n+1}{n} = \frac{(2n + 1)!}{(n!)^2}. \tag{11.15}$$

11.4. **Morse inequalities on the cover.** For a sufficiently close smooth bumpy metric g, let c_j be the number of critical points of the reduced function F_g of index j. The function is Morse on a compact manifold, so these numbers are finite. Its lift to $\widetilde{\mathcal{O}}_n$ has exactly $2c_j$ critical points of index j: the covering is a local diffeomorphism, and hence preserves the Hessian index. The sublevel attachment theorem [17, Theorem 3.2] attaches one cell of dimension equal to the index when a single nondegenerate critical value is crossed; a proof and an explicit identification of this theorem are given in [27, p. 3, theorem]. All sublevel bands here are compact. If several critical points have the same value, choose disjoint coordinate neighborhoods and perturb the function by small constants near the points, with cutoffs supported in these neighborhoods. The critical points and their Hessians are unchanged. On the compact union of the cutoff annuli, the original gradient has a positive minimum; choosing the perturbations smaller than this minimum introduces no critical point. Thus the critical values can be made distinct without changing any count. Apply the attachment theorem at each critical value and the no-critical-value deformation theorem [17, Theorem 3.1] between consecutive values. At each attachment, replace its map by a cellular approximation [8, Theorem 4.8, p. 349]; homotopic attaching maps give homotopy-equivalent adjunction spaces. Induction gives a finite CW model with $2c_j$ cells in degree j, and hence a finite cellular chain complex over $\mathbb{F}_2$ with those chain dimensions and homology dimensions b_j. No assertion that the differentials vanish is made for this Morse complex.

If q_j is the rank of its differential from degree $j+1$ to degree j, the elementary dimension formula for a finite chain complex is

$$2c_j = b_j + q_{j-1} + q_j, \qquad q_j \geq 0, \qquad q_{-1} = q_{k_n} = 0. \tag{11.16}$$

The weak inequality $2c_j \geq b_j > 0$ already implies $c_j \geq 1$. Together with (11.2), every full index $I_n, I_n+1, \ldots, I_n+k_n$ occurs.

The integer in (11.15) is even, since

$$(2n+1)\binom{2n}{n} = 2(2n+1)\binom{2n-1}{n-1}.$$

For the sharper estimate set

$$e_{-1} = 0, \qquad e_j = \left(\sum_{i=0}^{j} b_i\right) \bmod 2 \in \{0,1\}. \tag{11.17}$$

Reducing (11.16) modulo two and inducting in j gives $q_j \equiv e_j \pmod 2$, hence $q_j \geq e_j$. At the last degree, $e_{k_n} = 0$; this also follows from the even integer in (11.15). Therefore

$$c_j \geq \frac{b_j + e_{j-1} + e_j}{2}, \qquad \sum_j c_j \geq \frac{(2n+1)!}{2(n!)^2} + \sum_{j=0}^{k_n-1} e_j. \tag{11.18}$$

Each right-hand side in the first inequality is an integer.

For $n = 2$, direct multiplication in (11.14) gives

$$B_2(t) = 1 + 2t + 4t^2 + 5t^3 + 6t^4 + 5t^5 + 4t^6 + 2t^7 + t^8, \tag{11.19}$$

$$(e_0, \ldots, e_8) = (1,1,1,0,0,1,1,1,0). \tag{11.20}$$

It follows that

$$(c_0, \ldots, c_8) \geq (1,2,3,3,3,3,3,2,1) \tag{11.21}$$

coordinatewise. Since $I_2 = 27$, there are at least twenty-one nearby embedded minimal three-tori, with full indices between 27 and 35 and with the multiplicities in (11.21). This proves theorem 1.5.

The neighborhood of the round metric can be chosen to consist of metrics with positive sectional curvature: the curvature tensor and the sectional-curvature function on the compact unit Grassmann bundle vary continuously in the C^2 topology. The perturbation neighborhood may depend on n. The count concerns embedded images; no assertion is made that distinct hypersurfaces with equal index are pairwise nonisometric. All conclusions in this section use the complete kernel calculation, and hence the computer-assisted assertions of proposition 9.1.

Appendix A. Rational enclosures for initial-value problems

The finite families of rational boxes in propositions 5.3 and 9.1 satisfy the inclusion inequalities below. These inequalities imply existence on the prescribed intervals and the stated endpoint bounds. Appendix B describes the corresponding initial-value problems and parameter families. For the general theory of initial-value enclosures, see [19].

A.1. **Rational interval inclusions.** For closed intervals $A = [a_-, a_+]$ and $B = [b_-, b_+]$ with rational endpoints, their exact sum, product and reciprocal are

$$\begin{aligned} A + B &= [a_- + b_-, a_+ + b_+], \\ AB &= [\min_{\varepsilon,\eta\in\{-,+\}} a_\varepsilon b_\eta, \max_{\varepsilon,\eta\in\{-,+\}} a_\varepsilon b_\eta], \\ B^{-1} &= [b_+^{-1}, b_-^{-1}] \qquad (0 \notin B). \end{aligned} \tag{A.1}$$

The first identity follows from monotonicity of addition; the second from the extrema of a bilinear function on a rectangle; the third from monotonicity of $x \mapsto 1/x$ on either half-line. Any containing rational intervals may replace the intervals on the right. Induction on a rational expression therefore gives an enclosure of its range, provided none of its denominator intervals contains zero. Repeated occurrences of the same variable may enlarge the enclosure; they cannot invalidate containment.

All boxes below are products of closed rational intervals. The notation $F(\mathcal{B}, \Lambda)$ denotes a rational box containing the image of the rational map F on $\mathcal{B} \times \Lambda$. Thus it asserts containment, not equality with the generally nonrectangular image.

A.2. **Existence and a Taylor bound.** Adjoining S, C makes the differential equations autonomous. Consider

$$Y' = F(Y, \lambda), \qquad Y(0) \in \mathcal{Y}_0, \qquad \lambda \in \Lambda, \tag{A.2}$$

where the denominators of F do not vanish on a neighborhood of $\mathcal{B} \times \Lambda$.

Lemma A.1 (One-step enclosure)**.** *Let $h > 0$. Suppose*

$$\mathcal{Y}_0 + [0, h]F(\mathcal{B}, \Lambda) \subset \operatorname{int} \mathcal{B}. \tag{A.3}$$

Every solution of (A.2) *then exists uniquely on $[0, h]$ and lies in $\mathcal{B}$. Suppose A_k contains its normalized initial derivative $Y^{(k)}(0)/k!$, for every permitted initial value and parameter, and that B_{m+1} contains the normalized $(m + 1)$st derivative at the initial point of every solution initialized in $\mathcal{B}$. Then*

$$Y(h) \in \sum_{k=0}^{m} A_k h^k + B_{m+1} h^{m+1}. \tag{A.4}$$

For a positive interval $H = [h_-, h_+]$, the same conclusion holds for every $h \in H$, using h_+ in (A.3) *and interval operations in* (A.4).

Proof. On a neighborhood of the box, F is smooth and locally Lipschitz in Y. Fix one exact parameter and one exact initial value. At a first exit time $t \le h$,

$$Y(t) = Y(0) + \int_0^t F(Y(s), \lambda)\, ds \in \mathcal{Y}_0 + [0, h]F(\mathcal{B}, \Lambda).$$

The latter box is contained in the interior of $\mathcal{B}$, whereas a first exit must lie on its boundary. No such time exists. The range is bounded and stays in a compact subset of the domain of the smooth vector field. If a maximal solution time were at most h, boundedness of F would give a limiting point there; local existence from this point would extend the solution. Thus the solution exists on the whole interval.

Define rational vector-valued functions by

$$\mathcal{T}_0(y,\lambda) = y, \qquad \mathcal{T}_{k+1}(y,\lambda) = \frac{D_y \mathcal{T}_k(y,\lambda) F(y,\lambda)}{k+1}. \tag{A.5}$$

The chain rule proves inductively that $Y^{(k)}(t)/k! = \mathcal{T}_k(Y(t),\lambda)$. In particular, the normalized derivative of order $m+1$ at every $t \in [0,h]$ belongs to B_{m+1}. Taylor's theorem gives

$$Y(h) = \sum_{k=0}^{m} \frac{Y^{(k)}(0)}{k!} h^k + \frac{1}{m!} \int_0^h (h-t)^m Y^{(m+1)}(t)\, dt.$$

After extracting the normalized derivative, the positive kernel in the last integral has mass h^{m+1}. The integral therefore lies in $B_{m+1} h^{m+1}$, proving (A.4). Replacing h by a positive interval takes a containing interval of all these expressions; existence uses its upper endpoint. □

The normalized coefficients can also be obtained without successive differentiation. If $u(t) = \sum_{k \ge 0} u_k t^k$, then

$$\begin{aligned} (u+v)_k &= u_k + v_k, \qquad (uv)_k = \sum_{j=0}^{k} u_j v_{k-j}, \\ (u^{-1})_0 &= u_0^{-1}, \qquad (u^{-1})_k = -u_0^{-1} \sum_{j=1}^{k} u_j (u^{-1})_{k-j} \quad (k \ge 1), \\ Y_{k+1} &= F_k/(k+1). \end{aligned} \tag{A.6}$$

The reciprocal formula is the coefficient of degree k in $u\,u^{-1} = 1$. The last formula is the coefficient of degree k in the differential equation. Induction on k and on the finite rational expression for F proves that interval versions of these recurrences enclose the exact coefficients. For B_{m+1} the initial box is $\mathcal{B}$, not $\mathcal{Y}_0$.

A.3. **Induction over a finite partition.**

Proposition A.2. *Fix a parameter box Λ and rational boxes $\mathcal{Y}_0, \ldots, \mathcal{Y}_N$. For $0 \le j < N$, suppose a positive length interval H_j and a containing box $\mathcal{B}_j$ satisfy* (A.3), *with $(\mathcal{Y}_0, \mathcal{B}, h)$ replaced by $(\mathcal{Y}_j, \mathcal{B}_j, \sup H_j)$. Suppose coefficient boxes $A_{j,k}, B_{j,m_j+1}$ have the properties of lemma A.1, and*

$$\sum_{k=0}^{m_j} A_{j,k} H_j^k + B_{j,m_j+1} H_j^{m_j+1} \subset \mathcal{Y}_{j+1}. \tag{A.7}$$

For any fixed parameter, any permitted initial value, and any positive lengths $h_j \in H_j$, the solution exists up to $\sum_{j<N} h_j$, and its value there belongs to $\mathcal{Y}_N$.

Proof. At $j = 0$ the assertion follows from lemma A.1 and (A.7). Suppose it holds through the first j intervals. The terminal value is in $\mathcal{Y}_j$. Autonomy permits this value to be used as the initial value of the next interval. Applying the same lemma at the fixed parameter gives existence over the next length and containment in $\mathcal{Y}_{j+1}$. Uniqueness identifies this continuation with the original solution. Induction proves the assertion. □

The strict inclusion at each step consists of the rational inequalities

$$\inf \mathcal{I}_{j,i} - b_{j,i}^- > 0, \qquad b_{j,i}^+ - \sup \mathcal{I}_{j,i} > 0, \qquad \mathcal{I}_j = \mathcal{Y}_j + [0, \sup H_j] F(\mathcal{B}_j, \Lambda), \tag{A.8}$$

where $\mathcal{B}_{j,i} = [b_{j,i}^-, b_{j,i}^+]$. The coefficient inclusions in lemma A.1, the strict inequalities (A.8), and the terminal inclusion (A.7) are distinct hypotheses of proposition A.2. Each is imposed on the finite families of Appendix B.

A.4. **Initial values and the exact terminal angle.** For $q = \delta z^2$ the initial rectangles satisfy $|q| \le 1$. The convergent series

$$\cos(\sqrt{\delta} z) = \sum_{k \ge 0} \frac{(-q)^k}{(2k)!}, \qquad \frac{\sin(\sqrt{\delta} z)}{\sqrt{\delta}} = z \sum_{k \ge 0} \frac{(-q)^k}{(2k+1)!} \tag{A.9}$$

include their analytic values at $\delta = 0$. After retaining $0 \le k \le 16$, the ratio of consecutive omitted absolute terms is less than $1/2$. Twice the first omitted absolute term bounds each tail, with the factor $|z|$ in the second series. This encloses the exact initial data; the choice $\delta = 1$ also covers the exceptional initial value $\sin z_0$.

The terminal angle lies in the rational interval

$$\frac{7074237752028440}{4503599627370496} < \frac{\pi}{2} < \frac{7074237752028441}{4503599627370496}. \tag{A.10}$$

Put $A_N(x) = \sum_{k=0}^{N-1} (-1)^k x^{2k+1}/(2k+1)$. For even N and $0 < x < 1$, the alternating-series inequalities are

$$A_N(x) < \arctan x < A_N(x) + \frac{x^{2N+1}}{2N+1}.$$

If $a = \arctan(1/5)$ and $b = \arctan(1/239)$, the addition formula gives $\tan(2a) = 5/12$, $\tan(4a) = 120/119$, and $\tan(4a - b) = 1$. Since $0 < 4a - b < 4/5 < \pi/2$, one has $4a - b = \pi/4$. Define

$$\begin{aligned} L_\pi &= 8A_{32}(1/5) - 2\left(A_8(1/239) + \frac{239^{-17}}{17}\right), \\ U_\pi &= 8\left(A_{32}(1/5) + \frac{5^{-65}}{65}\right) - 2A_8(1/239). \end{aligned}$$

Exact rational comparison gives

$$\frac{7074237752028440}{4503599627370496} < L_\pi < \frac{\pi}{2} < U_\pi < \frac{7074237752028441}{4503599627370496},$$

which proves (A.10). In the spectral problems, the first 201 lengths equal $1/128$. The final length interval contains $\pi/2 - 201/128$ and has positive lower endpoint. Proposition A.2 therefore bounds the value at the exact angle $\pi/2$.

A.5. **The parameter families.** The dimension-dependent initial-value problems consist of (9.5), together with the sensitivity and matrix equations of section 9. Their parameter families are described in section B.1. For each of the 573 continuation rectangles, three initial-value problems are considered: the left initial value, the right initial value, and the entire initial interval. The eight problems at fixed reciprocal parameters have the same three choices of initial values. The exact initial columns are $(1, 0)$ and $(0, 1)$ for the two ordinary even columns; every odd or divided correction column is initially zero, and $f_3(0) = 1$. These are the initial conditions in (9.18) and the divided equations of section 9.

On the 256 rectangles covering $[0, 1/8]$, the terminal expressions (9.24), (9.28), and (9.19) give the bounds in (9.29). The remaining 317 rectangles continue the same simple closing root to $\delta = 1$; the enclosures at fixed reciprocal parameters give the bounds in proposition 9.1. The agreement at a shared parameter follows from positivity of the initial-value derivative on the union of the overlapping initial intervals, as proved there. No discretization of the dimension parameter replaces its closed interval.

The exceptional families in section B.2 consist of 47 and 62 initial intervals and the three central initial-value problems. Their finite unions cover the closed initial ranges in proposition 5.3. The regions (5.10) contain each trajectory up to the relevant endpoint and determine the order of the two first exits. The central sensitivity inequality gives uniqueness and nondegeneracy.

A.6. **Real-parameter curvature and reference identities.** Regard $\delta, c_R, s_R, c_U, s_U, S, C$ as independent real variables, with $Bc_Rc_U \neq 0$. Set $\mathcal{D}\delta = 0$ and define a derivation by

$$\begin{aligned} \mathcal{D}c_R &= \delta s_R S, & \mathcal{D}s_R &= -c_R S, \\ \mathcal{D}c_U &= -2\delta s_U C/c_R, & \mathcal{D}s_U &= 2c_U C/c_R, \\ \mathcal{D}S &= BC, & \mathcal{D}C &= -BS. \end{aligned} \tag{A.11}$$

Thus $\mathcal{D} = B\partial_s$ on solutions of (9.5), and $\mathcal{D}e_R = \mathcal{D}e_U = \mathcal{D}e_s = 0$ identically. No value of these three first integrals has yet been imposed.

For the quotients in Z one has

$$\mathcal{D}(s_R/c_R) = -\frac{S(1+e_R)}{c_R^2}, \qquad \mathcal{D}\left(\frac{s_U}{c_Uc_R}\right) = \frac{2C(1+e_U)}{c_U^2c_R^2} - \frac{\delta s_U s_R S}{c_U c_R^2}.$$

Substitute these expressions in (9.3). If

$$\mathcal{Z}_0 = B\mathsf{K} - \frac{2S^2}{c_R^2} - \frac{4C^2}{c_R^2c_U^2} + \frac{2\delta CSs_Rs_U}{c_R^2c_U}, \tag{A.12}$$

then the unrestricted rational identity is

$$\mathcal{D}Z = \mathcal{Z}_0 - \frac{2S^2}{c_R^2}e_R - \frac{4C^2}{c_R^2c_U^2}e_U. \tag{A.13}$$

Thus the shortened formula $Z_\tau = \mathcal{Z}_0$ requires both radius identities; it does not require the angular identity. Expanding (9.12) and (A.12), without imposing any first integral, gives

$$\mathcal{S} - \mathcal{Z}_0 - 2\mathsf{A} = \frac{4}{c_R^2c_U^2}e_s.$$

Combining this with (A.13) yields

$$\mathcal{S} - \mathcal{D}Z - 2\mathsf{A} = \frac{2S^2}{c_R^2}e_R + \frac{4C^2}{c_R^2c_U^2}e_U + \frac{4}{c_R^2c_U^2}e_s. \tag{A.14}$$

In particular, on $e_R = e_U = e_s = 0$,

$$\mathcal{S} = Z_\tau + 2\mathsf{A}, \qquad M = -Z_\tau - (2+\delta). \tag{A.15}$$

These are rational differential identities for every real parameter, including $\delta = 0$. Only when $\delta = 1/p$ with integer p do we also identify $\mathcal{S} = \delta|A|^2$ geometrically.

The reference calculations admit equally explicit defect formulas. For $f = c_RS$, differentiating (9.21) and using (A.14) gives

$$\mathcal{D}^2 f + Z\mathcal{D}f - Mf = f\left\{\frac{2S^2}{c_R^2}e_R + \frac{4C^2}{c_R^2c_U^2}e_U + \left(\frac{4}{c_R^2c_U^2} - (2+\delta)\right)e_s\right\}. \tag{A.16}$$

For f_3 as in (9.25), the difference between its equation and the curvature-defined reference equation is

$$\mathcal{D}^2 f_3 + Z\mathcal{D}f_3 - [M + (2+\delta)]f_3 = (\mathcal{S} - \mathcal{D}Z - 2\mathsf{A})f_3.$$

Finally, the divided forcing has the unrestricted algebraic identity

$$(1+6\delta)\mathsf{A} - (2+\delta) - \delta T_3 = -\frac{2(e_R + c_R^2e_U)}{c_R^2c_U^2}. \tag{A.17}$$

The precise substitutions are therefore as follows:

Identity or simplification	Zero defects used
B_R and B_U in (9.9)	e_R and e_U, respectively
Reciprocal radii and (9.13)	e_U
$\mathcal{D}Z = \mathcal{Z}_0$ in (A.13)	e_R, e_U
Curvature–drift equality (A.15)	e_R, e_U, e_s
Degree-two reference equation for f_2	e_R, e_U, e_s
Curvature-defined degree-three reference equation for f_3	e_R, e_U, e_s
Forcing factorization (9.26)	e_R, e_U
Even–odd equations and numerator (9.19)	None

The endpoint identities additionally use $S(L) = 1$, $C(L) = 0$ and, where stated, $s_R(L) = 0$. The last equality is imposed only on the closing trajectory.

An enclosing rectangular set need not lie in the common zero set of e_R, e_U, e_s. The rational equations in the interval inclusions are evaluated on the whole rectangle. The identities above compare their exact solutions with the geometric equations; they do not permit intersecting a rectangular enclosure with an unproved constraint or replacing an interval image by a pointwise simplification valid only on the invariant set without a containment proof for the chosen rational extension.

Appendix B. Finite interval bounds

Throughout this appendix, $L = \pi/2$. The differential equations are those of sections 5 and 9; rational enclosures are understood in the sense of section A.

B.1. **Dimension-parameter families.** Let $\Lambda_i \times Z_i$, $1 \le i \le 573$, be the rational rectangles used in proposition 9.1, where $Z_i = [z_i^-, z_i^+] \subset (0, 1/2)$. For $1 \le i \le 256$,

$$\Lambda_i = \left[\frac{i-1}{2048}, \frac{i}{2048}\right], \qquad \bigcup_{i=1}^{256} \Lambda_i = [0, 1/8], \qquad \bigcup_{i=257}^{573} \Lambda_i = [1/8, 1].$$

Consecutive parameter intervals have a common endpoint, and consecutive initial intervals satisfy

$$\max\{z_i^-, z_{i+1}^-\} < \min\{z_i^+, z_{i+1}^+\}.$$

Each rectangle is considered with the three initial sets $\{z_i^-\}$, $\{z_i^+\}$ and Z_i. The eight additional parameter problems comprise the six values $\delta = 1/p$, $2 \le p \le 7$, and the degree-four and degree-five problems at $\delta = 1$. Their initial intervals are contained in the corresponding continuation intervals. The total number of initial-value problems is therefore $3(573 + 8) = 1743$.

For the spectral equations, write $Y = (Y_0, \ldots, Y_{32})$ in the following order:

Coordinates	Variables
$0, \ldots, 7$	$c_R, s_R, c_U, s_U, S, C, q_R, q_U$
8	f_3
$9, \ldots, 16$	$E_2^{(1)}, W_2^{(1)}, V_2, W_2^{(2)}$
$17, \ldots, 24$	$V_3, W_3^{(1)}, E_3^{(2)}, W_3^{(2)}$
$25, \ldots, 32$	$E_j^{(1)}, W_j^{(1)}, E_j^{(2)}, W_j^{(2)}$

Each two-entry column is ordered as value and τ-derivative. In the last row, $j = 4$, except in the degree-five problem at $p = 1$. The three endpoint polynomials are

$$2(Y_{10}Y_{14} - Y_{12}Y_{16}), \quad 2(Y_{18}Y_{22} - Y_{20}Y_{24}), \quad 2(Y_{26}Y_{30} - \delta Y_{28}Y_{32}).$$

On a closing trajectory these are, respectively, $\mathcal{N}_2/\delta$, $\mathcal{N}_3/\delta$ and $\mathcal{N}_j$, with the removable values given by (9.24) and (9.28) at $\delta = 0$. On a trajectory which does not close, they denote only the displayed polynomial expressions.

The first eight initial coordinates are

$$(c_R(0), s_R(0), 1, 0, 0, 1, 1, 0), \qquad f_3(0) = 1.$$

The first and second even columns have initial values $(1, 0)$ and $(0, 1)$. Every odd column and every divided correction column has initial value zero. The continuation equations use only the first eight coordinates.

The Taylor order is 14. There are 201 consecutive intervals of length $1/128$ and one final positive length interval containing $L - 201/128$, with L enclosed by (A.10). The endpoint enclosures give (9.31), (9.29), the six finite-parameter bounds in proposition 9.1, and (9.30).

B.2. **Exceptional first-exit families.** The exterior initial sets are the two rational families $(I_i)_{i=1}^{47}$ and $(K_i)_{i=1}^{62}$ in proposition 5.3, with

$$[1/25, 337/1000] \subset \bigcup_{i=1}^{47} I_i, \qquad [17/50, 4/5] \subset \bigcup_{i=1}^{62} K_i.$$

On each interval, the four-component polynomial system (5.9) has initial value $(\sin z_0, 0, 0, 1)$, with z_0 ranging over the whole interval. Its enclosures have Taylor order 14 and lengths at most $1/1024$. They satisfy (5.10) throughout the trajectory. The terminal bounds are $x < 0 < C$ for I_i and $C < 0 < x$ for K_i.

The three central initial sets are $\{337/1000\}$, $\{17/50\}$ and $[337/1000, 17/50]$. The eight-component system consists of (5.11) and (5.12). Its enclosures have Taylor order 16 and lengths $1/256$, followed by a positive interval containing the remaining length to L. All denominator intervals are positive, and the terminal bounds are those of (5.13).

B.3. **Dyadic endpoint inclusions.** The finite endpoint values in the arithmetic of [11] are regarded as exact elements of

$$\mathbb{D} = \mathbb{Z}[1/2] = \{m2^{-k} : m \in \mathbb{Z},\ k \in \mathbb{N}_0\}.$$

For a bounded interval I, let $\widehat{I}$ denote any interval with dyadic endpoints that contains I. The interval operations used in the finite families satisfy

$$A + B \subset \widehat{A + B}, \qquad AB \subset \widehat{AB}, \qquad B^{-1} \subset \widehat{B^{-1}} \quad (0 \notin B).$$

Here the intervals on the left are the exact ranges in (A.1). Each denominator interval excludes zero. The initial trigonometric values are enclosed by (A.9) and its rational tail bounds; the terminal angle satisfies (A.10).

B.4. **Finite chains of enclosing boxes.** For each of the preceding initial-value problems, fix a parameter box Λ, initial and terminal boxes $\mathcal{Y}_0, \dots, \mathcal{Y}_N$, positive length intervals $H_0, \dots, H_{N-1}$, and containing boxes $\mathcal{B}_0, \dots, \mathcal{B}_{N-1}$. For every $j < N$, put

$$\mathcal{I}_j = \mathcal{Y}_j + [0, \sup H_j] F(\mathcal{B}_j, \Lambda).$$

The component inequalities are

$$b_{j,i}^- < \inf \mathcal{I}_{j,i} \le \sup \mathcal{I}_{j,i} < b_{j,i}^+, \qquad \mathcal{B}_{j,i} = [b_{j,i}^-, b_{j,i}^+].$$

Let $A_{j,k}$ contain $\mathcal{T}_k(\mathcal{Y}_j, \Lambda)$, $0 \le k \le m_j$, and let B_{j,m_j+1} contain $\mathcal{T}_{m_j+1}(\mathcal{B}_j, \Lambda)$, where $\mathcal{T}_k$ is defined in (A.5). The terminal condition is

$$\sum_{k=0}^{m_j} A_{j,k} H_j^k + B_{j,m_j+1} H_j^{m_j+1} \subset \mathcal{Y}_{j+1}.$$

Thus the hypotheses of proposition A.2 hold on each chain. For every fixed parameter and every permitted initial value, the exact solution belongs to $\mathcal{B}_j$ during the jth interval and to $\mathcal{Y}_{j+1}$ at its endpoint. The final box gives the terminal bounds stated above.

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